A closed form method for predicting the history of circular orbit altitude decay of a near-earth satellite including the effect of a dynamic atmosphere

John Copley Buchholtz

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Department: Mechanical and Aerospace Engineering

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A CLOSED FORM METHOD FOR PREDICTING THE HISTORY OF CIRCULAR ORBIT ALTITUDE DECAY OF A NEAR-EARTH SATELLITE INCLUDING THE EFFECT OF A DYNAMIC ATMOSPHERE

BY

JOHN COLEY BUCHHOLTZ, 1943 -

A THESIS

Presented to the Faculty of the Graduate School of the UNIVERSITY OF MISSOURI-ROLLA

In Partial Fulfillment of the Requirements for the Degree

MASTER OF SCIENCE IN AEROSPACE ENGINEERING

1971

Approved by

[Signatures]
ABSTRACT

A closed form expression for the time to decay between two circular orbit altitudes is developed. The long-term dynamic variation of high altitude atmospheric density is included. The density is approximated as a ratio of two quadratics with the numerator being a function of the time-related exospheric temperature and the denominator being a function of altitude. The 11-year cyclic variation of exospheric temperature is then represented as a piecewise linear function of time.

The analytic expression for circular orbit altitude decay rate is developed by describing the energy loss due to atmospheric drag. A point mass representation of the Earth's gravitational potential is assumed.
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I. INTRODUCTION

The prediction of the altitude decay and orbit lifetime of an Earth orbiting satellite is essential to the planning of many advanced missions since a sufficient orbit lifetime must be available to complete the mission. The rate of altitude decay and orbit lifetime are dependent upon the initial total orbital energy of the satellite and the history of energy loss.

Altitude decay histories may be determined by numerically integrating the equations of motion. However, for simulation of long-term trajectories, an excessive amount of computer time is required. Closed-form methods offer rapid and inexpensive prediction of orbit decay, and therefore, the sensitivity of the orbit decay to changes in satellite configuration, initial altitude, or atmospheric properties can be evaluated. The investigation of closed form methods for predicting circular orbit decay of near-Earth satellites has been selected for study.

Satellites in nearly circular orbits decay in a spiral manner such that the orbit remains nearly circular. Satellites in initially elliptic orbits tend to have two phases of decay. The first is a steady decrease in the apogee altitude with the perigee altitude remaining nearly constant. The decrease in apogee altitude continues until the orbit becomes nearly circular. The second phase is the spiral decay. The ability to describe circular orbit altitude decay is therefore of value to the mission planning for all near-Earth orbiting satellites.
The predominate perturbating acceleration affecting the energy loss of a near-Earth satellite is atmospheric drag. Atmospheric drag is directly proportional to the atmospheric density. An accurate representation of the density at high altitudes is therefore required. At altitudes above 300 km, large changes in the magnitude of density occur with both time and altitude. For example, the atmospheric density can change by an order of magnitude between 400 and 500 km, but at 500 km, the density can also change by an order of magnitude within a three-year period. This paper addresses the problem of circular orbit decay of a satellite accounting for the dynamic effects of the upper atmosphere.
II. REVIEW OF LITERATURE

The dynamic variation of high altitude atmospheric density (density represented as a function of both time and altitude) is not directly treated in closed form expressions, for circular orbit decay times, surveyed and developed in References (1-3). El'yasberg (3) recognizes the severe limitation of the assumption of a static representation of density (density represented as a function of altitude alone). Bruce (4) concludes, from computer results, that estimates of satellite lifetime based on a static representation of density may be different by a factor of 30 or more from estimates based on a dynamic variation of density.

The same differential equation relating the change in circular orbit altitude and time is derived in References (1-3). The major differences in the closed form expressions developed result from the mathematical treatment of the density variation with altitude and the variation of the square root or inverse square of orbital radius with altitude. Garcia (2) assumes a constant orbital period and expands the inverse square variation of radius in powers of altitude through the cube. Billik (1) and El'yasberg (3) incorporate the expression for orbital period within the expression for altitude decay rate and considers the square root of orbital radius constant. The two most common static density approximations are an exponential variation of density with altitude and a power-law relation with altitude.
Johnston (5) develops an integral relation for satellite lifetime including an approximation to the dynamic density. The density is expressed as a product of a function of time and an exponential function of altitude. However, the analytic form of the function of time is not given and an iterative procedure is required to solve for lifetime.

The variations of atmospheric density at high altitudes has been correlated with solar activity, geomagnetic activity, calendar date, angular position relative to the Earth and the Earth-Sun line, and altitude. All parameters with the exception of altitude are combined into one parameter, independent of altitude, which is referred to as exospheric temperature. Predicted levels of exospheric temperature for the years 1971 to 1977 in three-month intervals are available in Reference (6). For the years 1975 through 1985, predicted levels of exospheric temperature are presented in Reference (7) again in three-month intervals. The density magnitude versus altitude for various exospheric temperature levels are tabulated in Reference (8) for altitudes between 120 and 1000 km and exospheric temperatures ranging from 600 to 2100°K. Analytic expressions for determining exospheric temperature and for relating exospheric temperature and altitude to density are presented in Reference (9).
III. EQUATION DEVELOPMENT

The rate of change of circular orbit altitude due to atmospheric drag is derived in Appendix A by describing the loss in orbital energy with orbit travel. Three standard assumptions are employed to reduce the complexity of the equations of motion for long-term estimations of circular orbit altitude decay. The atmosphere is assumed to be spherical and non-rotating, the Earth's gravitational potential is represented as a point mass, and the acceleration due to drag is considered constant within one revolution of orbit travel. The expression for the rate of decay is;

$$\frac{dH}{dt} = -D \rho \sqrt{\mu r}$$  \hspace{1cm} (1)

All symbols are defined in the nomenclature on Page 25.

The magnitude of high altitude atmospheric density, $\rho$, is considered to be a function of both altitude, $H$, and the exospheric temperature, $T_\infty$. The exospheric temperature varies with time throughout the lifetime of a satellite. A representative long-term variation of predicted nominal levels of $T_\infty$, from References (6) and (7), is presented in Figure 1. The sinusoidal semiannual variation of $T_\infty$ is not included.

The static density representation used in previous closed form developments is equivalent to assuming a constant $T_\infty$ and is valid only for short time periods. Large differences in long-term estimates of decay times result if the proper average value of $T_\infty$ is not used. As examples, the time to decay from 500 to 450 km and from 450 to 400 km
Figure 1. Representative Long-Term Exospheric Temperature History
for different constant $T_\infty$ levels, shown in Table I, were obtained by numerically integrating equation 1. The density magnitudes were determined from the analytic expressions available in Reference (9). Note that exospheric temperature changes of only 100°K can result in a factor of 2 change in the time to decay 50 km.

The change in the square root of orbital radius is negligibly small when compared with the corresponding change in atmospheric density with a decrease in altitude. For example, a change in altitude of 200 km results in less than a 2% change in the square root of radius but the density changes by an order of magnitude or more. The orbital radius, $r$, in equation 1 is therefore assumed to be constant and equal to the initial orbital radius, $r_1$, as was assumed by El'yasberg (3). The ballistic coefficient is also assumed to be constant. The resulting differential equation, from equation 1, relating time and altitude is:

$$\rho (H, T_\infty (t))dt = - \frac{1}{D \sqrt{\mu r_1}} dH$$

(2)

Closed form solution of equation 2 requires the separation of the variables $H$ and $T_\infty$. The analytic expressions in Reference (9) relating $H$ and $T_\infty$ to $\rho$ cannot be separated. To allow the separation of the variables, the density is approximated as the following ratio of quadratics in $T_\infty$ and $H$:

$$\rho (H, T_\infty (t)) = \frac{A + BT_\infty + CT_\infty^2}{1 + EH + FH^2}$$

(3)

where, $A$, $B$, $C$, $E$, and $F$ are constants.
Table I

Time to Decay 50 km for Constant Exospheric Temperature

<table>
<thead>
<tr>
<th>Exospheric Temperature, °K</th>
<th>Time to Decay 50 km, Days</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Initial Altitude = 500 km</td>
</tr>
<tr>
<td>800</td>
<td>4627</td>
</tr>
<tr>
<td>900</td>
<td>2286</td>
</tr>
<tr>
<td>1000</td>
<td>1296</td>
</tr>
<tr>
<td>1100</td>
<td>818</td>
</tr>
<tr>
<td>1200</td>
<td>561</td>
</tr>
</tbody>
</table>
The approximate form for $\rho$ is justified by the shape of the density profiles shown in Figures 2 and 3. The shape of the density variation with exospheric temperature shown in Figure 2, from Reference (8), indicates that at a constant altitude, the density varies in a parabolic manner with $T_\infty$. The reciprocal of density varies in a parabolic manner with altitude at a constant level of $T_\infty$ as is shown in Figure 3. The validity of this form for $\rho$ is established in Section IV.

The generalized least square method developed by Penrose (10) and described in Appendix B can be used to estimate the constants (A, B, C, E, and F) in equation 3. The estimate of the constants will be the best approximation in the least squares sense to any selected number of known values of density given the corresponding values of $H$ and $T_\infty$. Tabular values of density versus $H$ and $T_\infty$ are given in Reference (8).

A piecewise linear relationship between time and exospheric temperature is used to approximate the long-term exospheric temperature profile such as the profile shown in Figure 1.

$$T_\infty = at + b, \ t_0 \leq t \leq t_1$$  \hspace{1cm} (4)

where $t_0$ and $t_1$ are the time limits of the linear approximation of $T_\infty$.

Substituting equation 4 into equation 3 completes the dynamic representation of density.

$$\rho \ (H, at + b) = \frac{(A + Bb + Cb^2) + (Ba + 2Cab)t + Ca^2t^2}{1 + EH + FH^2}$$  \hspace{1cm} (5)
Figure 2. Atmospheric Density vs. Exospheric Temperature
Figure 3. Reciprocal of Atmospheric Density vs. Altitude
Combining equations 2 and 5 yields:

\[
\left[ (A+Bb+Cb^2) + (Ba+2Cab)t+Ca^2t^2 \right]dt = \\
\frac{-(l+EH+Fh^2)}{D \sqrt{\mu r_1}} dH
\]

The right side of equation 6 is directly integrated from \(H_1\) and \(H_2\).

The left side of equation 6 is piecewise integrated from 0 to \(T\). For a linear representation of \(T_\infty\), the resulting expression in the cube of time to decay from altitude \(H_1\) to altitude \(H_2\) is;

\[
(A+Bb+Cb^2)T + \frac{(Ba+2Cab)T^2}{2} + \frac{Ca^2T^3}{3} = \\
\frac{1}{D \sqrt{\mu r_1}} \left[ (H_1-H_2) + \frac{E}{2}(H_1^2-H_2^2) + \frac{F}{3}(H_1^3-H_2^3) \right]
\]

The time to decay is then determined by factoring the cubic expression in \(T\) in equation 7. A standard procedure for factoring a cubic equation is presented in Reference (11).

The condition of constant exospheric temperature is a special case which corresponds to a static representation of density. For this condition, the left side of equation 7 reduces to a function of the first degree polynomial in \(T\) as the slope, \(a\), of the time variation of exospheric temperature is zero. Then,

\[
(A+Bb+Cb^2)T = \\
\frac{1}{D \sqrt{\mu r_1}} \left[ (H_1-H_2) + \frac{E}{2}(H_1^2-H_2^2) + \frac{F}{3}(H_1^3-H_2^3) \right]
\]

The closed form expression, equation 7, was developed to estimate the time to decay between two circular orbit altitudes. However,
equation 7 can be rearranged such that a third degree polynomial in \( H_1 \) or \( H_2 \) is formed. This allows either \( H_1 \) or \( H_2 \) to be solved closed form given the final or initial altitude, respectively, and the time span and corresponding exospheric temperature profile. Note also that the restriction of a piece-wise linear representation of \( T_\infty \) is not required to solve for either \( H_1 \) or \( H_2 \) closed form. Because of the form of the approximation to density, equation 3, the resulting terms in the polynomial in both \( H_1 \) or \( H_2 \) are independent of the time representation of \( T_\infty \). The development of closed form expressions for \( H_1 \) or \( H_2 \) based on approximate time representations of \( T_\infty \), other than a piece-wise linear representation, is suggested for future investigation.
IV. DISCUSSION

The closed form expression derived, equation 7, gives the time to decay between two circular orbit altitudes including the effect of the dynamic variation of atmospheric density. However, an understanding of both the simplifying assumptions employed in the closed form development and the uncertainties in the parameters affecting the drag force are required to enable accurate estimates.

A. ASSUMPTIONS WITHIN CLOSED FORM DEVELOPMENT

Five assumptions are employed within the development of the closed form expression. Three of these are the standard assumptions, employed within previous closed form developments, of a non-rotating atmosphere, point mass representation of the Earth's gravitational potential, and a constant drag force within one revolution of orbit travel. The other two enable the inclusion of the effect of dynamic variation of density. The density is first approximated as a function of \( H \) and \( T_\infty \) and then a piece-wise linear relationship of \( T_\infty \) with time is assumed.

The Earth's atmosphere does rotate, however, the rotational rate is much smaller than the inertial circular orbit velocity of a near-Earth satellite. By assuming a non-rotating atmosphere, the complexity of the equations of motion are considerably reduced as planar motion may be assumed.

The point mass representation of the Earth's gravitational potential does not significantly affect the decay time estimates for nearly
circular orbits. The gravitational potential of the Earth is actually comprised of the point mass term plus the zonal harmonics which represent latitude mass variations and the tesseral harmonics which represent longitude mass variations. The tesseral harmonics have a negligible cumulative effect on near-Earth satellite motion. The zonal harmonics predominately affect the transfer of orbital energy and thus change the shape and position of the orbit. However, for nearly circular orbits the resulting change in shape is small. Also, over long time periods the accumulative effect on satellite decay due to changes in position is small.

The assumption of a constant atmospheric drag force within one revolution of orbit travel implies that: 1) the shape of the Earth's atmosphere is spherical, 2) only small changes in circular orbit altitude occur within one revolution, and 3) forces other than drag and gravity are not significant. The shape of the atmosphere is affected by both the oblateness of the Earth and the unequal heating of the atmosphere by the sun. The effect of unequal heating is referred to as the diurnal (or day-night) variation. The diurnal variation is averaged in long-range predictions of exospheric temperature given in References (6) and (7) by assuming that an average density magnitude is obtained at 9 AM local solar time. The effect of the variation of atmospheric shape caused by flattening of the Earth at the poles is negligible for satellites in low inclination orbits. The second and third assumptions, implied by considering a constant drag force, bound the altitude regime
within which the closed form expressions are valid to altitudes between about 300 and 700 km. Above approximately 700 km, the perturbing force due to solar pressure becomes significant and must be considered in orbit decay estimates. At altitudes below about 300 km, large changes in altitude occur as a satellite begins to rapidly re-enter the Earth's atmosphere.

The atmospheric density representation, equation 3, was selected based on the shape of the density profiles as a function of $H$ and $T_\infty$. The constants (A, B, C, E, and F) are determined by a least square fit of known discrete density magnitudes as outlined in Appendix B. To illustrate the accuracy of the density approximation, the constants were computed for two sets of 42 discrete density magnitudes from Reference (8). The first set included density magnitudes for altitudes ranging from 500 to 450 km in 10 km increments at exospheric temperatures of 600, 700, 800, 900, 1000, 1100, and 1300°K. The second set included density magnitudes for altitudes from 450 to 400 km in 10 km increments at the same seven levels of $T_\infty$. The constants obtained from the least square fit of each set of density magnitudes are presented in Table II. The per cent deviation between the given density magnitudes and the corresponding values recomputed from the approximated density representation are shown in Tables III and IV for sets 1 and 2, respectively. The approximate expressions for density yield reasonably accurate density values for exospheric temperatures between 700 and 1300°K which covers the complete range of predicted nominal $T_\infty$. 
Table II

Example Sets of Constants in Representation of Density

$700 \leq T_\infty \leq 1300^\circ K$

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Units</th>
<th>Value of Constants</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Set 1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$450 \leq H \leq 500$ km</td>
</tr>
<tr>
<td>A</td>
<td>kg/km$^3$</td>
<td>$5.585199(10)^{-5}$</td>
</tr>
</tbody>
</table>
Table III

Errors in Density Using Approximate Representation of Set 1

(Actual Values of Density Are Given in Reference (8))

<table>
<thead>
<tr>
<th>Altitude, km</th>
<th>$T_\infty = 700^\circ K$</th>
<th>$T_\infty = 800$</th>
<th>$T_\infty = 900$</th>
<th>$T_\infty = 1000$</th>
<th>$T_\infty = 1100$</th>
<th>$T_\infty = 1300$</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>5.2</td>
<td>-5.2</td>
<td>-6.7</td>
<td>-3.1</td>
<td>0.9</td>
<td>5.4</td>
</tr>
<tr>
<td>490</td>
<td>7.5</td>
<td>-3.5</td>
<td>-6.3</td>
<td>-4.2</td>
<td>-1.2</td>
<td>1.6</td>
</tr>
<tr>
<td>480</td>
<td>10.6</td>
<td>-1.1</td>
<td>-5.4</td>
<td>-4.7</td>
<td>-2.9</td>
<td>-1.8</td>
</tr>
<tr>
<td>470</td>
<td>15.2</td>
<td>2.7</td>
<td>-3.1</td>
<td>-3.7</td>
<td>-3.1</td>
<td>-3.8</td>
</tr>
<tr>
<td>460</td>
<td>21.8</td>
<td>8.9</td>
<td>2.0</td>
<td>0.0</td>
<td>-0.6</td>
<td>-3.1</td>
</tr>
<tr>
<td>450</td>
<td>31.0</td>
<td>18.3</td>
<td>10.6</td>
<td>7.5</td>
<td>5.9</td>
<td>1.8</td>
</tr>
</tbody>
</table>
Table IV

Errors in Density Using Approximate Representation of Set 2

(Actual Values of Density Are Given in Reference (8))

<table>
<thead>
<tr>
<th>Altitude, km</th>
<th>$T_\infty = 700^\circ K$</th>
<th>$T_\infty = 800$</th>
<th>$T_\infty = 900$</th>
<th>$T_\infty = 1000$</th>
<th>$T_\infty = 1100$</th>
<th>$T_\infty = 1300$</th>
</tr>
</thead>
<tbody>
<tr>
<td>450</td>
<td>-13.2</td>
<td>-12.2</td>
<td>-6.7</td>
<td>-1.1</td>
<td>3.0</td>
<td>5.8</td>
</tr>
<tr>
<td>440</td>
<td>-9.3</td>
<td>-10.4</td>
<td>-6.7</td>
<td>-2.5</td>
<td>0.5</td>
<td>1.6</td>
</tr>
<tr>
<td>430</td>
<td>-4.8</td>
<td>-7.9</td>
<td>-6.3</td>
<td>-3.5</td>
<td>-1.6</td>
<td>-2.3</td>
</tr>
<tr>
<td>420</td>
<td>1.1</td>
<td>-4.0</td>
<td>-4.2</td>
<td>-3.0</td>
<td>-2.2</td>
<td>-4.7</td>
</tr>
<tr>
<td>410</td>
<td>9.5</td>
<td>2.7</td>
<td>0.8</td>
<td>0.6</td>
<td>0.2</td>
<td>-4.1</td>
</tr>
<tr>
<td>400</td>
<td>21.1</td>
<td>13.2</td>
<td>10.0</td>
<td>8.5</td>
<td>7.1</td>
<td>1.3</td>
</tr>
</tbody>
</table>
A polynomial representation of $T_\infty$ with time of degree $n$ results in a polynomial in orbit decay time of degree $2n+1$. As polynomials in $T$ through the third degree can be directly factored, polynomial representations of $T_\infty$ with time, other than a piece-wise linear variation, are not suggested if orbit decay time is to be solved closed form. The long-term variation of $T_\infty$ with time, shown in Figure 1, can be closely approximated by a piece-wise linear function of time.

B. PREDICTION OF PARAMETERS AFFECTING DRAG

Large discrepancies in orbit decay time estimates can result due to uncertainties in the long-term prediction of parameters affecting the magnitude of atmospheric drag force. Variations in the estimation of the ballistic coefficient, caused by uncertainties in the estimation of the drag coefficient, satellite area, and satellite mass can significantly affect decay time predictions. The greatest discrepancy in orbit decay time estimates in most cases, however, results from the inability to very accurately predict future long-term exospheric temperature (or density) levels.

The ballistic coefficient is defined as the product of the drag coefficient and satellite projected area divided by the satellite mass, $(C_D A_s / m)$. The standard values for $C_D$ based on projected satellite area is 2.2, and Moe (12) predicts deviations from the standard value to be no more than 15 per cent. In the estimation of decay times for use in mission planning of future satellites, the expected growth in
weight should be considered. The projected area is determined from the shape, size, and attitude of the satellite. Benson and others (13) discuss the large variation in decay time estimates which can result due to the uncertainties in the long-term attitude prediction of satellites with complex shapes.

The future trend in exospheric temperature is established based on past variations in solar activity. However, exact future magnitudes cannot be predicted, and therefore, $+2\sigma$ bounds are presented along with the expected nominal values in References (6) and (7). At high altitudes, differences between the nominal and $+2\sigma$ levels in long range predictions can result in a factor of 2 or more difference in atmospheric density.

C. COMPARISON OF RESULTS

Circular orbit decay times computed using the closed form expression, equation 7, are compared with decay times determined by numerically integrating the expression for altitude decay rate, equation 1. The two altitude regions of 500 to 450 km and 450 to 400 km are considered and the variation of $T_\infty$ with time is assumed to linear in all comparisons. The comparison of decay time estimates is presented to illustrate the sensitivity of the approximate representation of the dynamic atmosphere on estimates of orbit decay times.

The time to decay 50 km computed using the closed form expression is compared with the numerically integrated decay time in Table V for $T_\infty$ linearly decreasing $200^\circ$K in six years. Four initial levels of
Table V

Time to Decay 50 km For $T_\infty$ Linearily Decreasing 200°K in 6 Years

<table>
<thead>
<tr>
<th>Initial Altitude, km</th>
<th>Initial $T_\infty$, °K</th>
<th>Time to Decay 50 km, Days</th>
<th>Numerically Integrated Result</th>
<th>Error, Per Cent</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>1000</td>
<td>2043</td>
<td>2143</td>
<td>4.7</td>
</tr>
<tr>
<td>500</td>
<td>1100</td>
<td>962</td>
<td>986</td>
<td>2.4</td>
</tr>
<tr>
<td>500</td>
<td>1200</td>
<td>612</td>
<td>618</td>
<td>1.0</td>
</tr>
<tr>
<td>500</td>
<td>1300</td>
<td>432</td>
<td>436</td>
<td>0.9</td>
</tr>
<tr>
<td>450</td>
<td>1000</td>
<td>610</td>
<td>628</td>
<td>2.9</td>
</tr>
<tr>
<td>450</td>
<td>1100</td>
<td>392</td>
<td>394</td>
<td>0.5</td>
</tr>
<tr>
<td>450</td>
<td>1200</td>
<td>276</td>
<td>276</td>
<td>0.0</td>
</tr>
<tr>
<td>450</td>
<td>1300</td>
<td>205</td>
<td>208</td>
<td>1.4</td>
</tr>
</tbody>
</table>
$T_\infty$ along with two initial altitudes are considered. The ballistic coefficient is assumed to be constant and equal to $(10)^{-8} \text{ km}^2/\text{kg}$, and the constants ($A, B, C, E,$ and $F$) from Table II were used for both altitude regimes considered. The calculation of decay time for an initial altitude of 500 km and an initial $T_\infty$ level of $1000^\circ \text{K}$ is presented in Appendix C. The differences in the decay times between the closed form and numerically integrated calculations are small when compared with differences which can result due to uncertainties in the prediction of initial levels of $T_\infty$. 
V. CONCLUSIONS

A closed form expression is developed for estimating the time to decay between two circular orbit altitudes within the 300 to 700 km altitude range. The atmospheric density is approximated as a ratio of quadratics in $T_\infty$ and $H$. The constants within the approximation are determined by a least square fit of known discrete density magnitudes and can be evaluated to cover the complete range of expected nominal exospheric temperature levels in at least 50 km steps in altitude. The dynamic variation of density, which must be considered in estimating decay times, is included by representing $T_\infty$ as a piecewise linear function of time.

The example orbit decay times computed using the closed form expression are in excellent agreement with numerically integrated results. However, the accuracy of the closed form expression should not be considered to be the accuracy with which orbit decay times can be estimated. Uncertainties in the prediction of parameters affecting the drag force can result in large differences in decay time estimates. The closed form expression along with realistic predictions of $T_\infty$ and ballistic coefficient can be used to bound the range of possible orbit decay times and establish trends of decay time with launch date, vehicle configuration, and initial altitude.
NOMENCLATURE

A  Constant in equations 3, 5, 6, 7, and 8, kg/km$^3$

$A_s$  Satellite projected area, km$^2$

a  Slope of linear representation of $T_\infty$ with time, °K/day

B  Constant in equations 3, 5, 6, 7, and 8, kg/(km$^3$ - °K)

b  Constant within linear representation of $T_\infty$ with time, °K

C  Constant in equations 3, 5, 6, 7, and 8, kg/(km$^3$ - °K$^2$)

$C_D$  Drag coefficient

D  Ballistic coefficient ($C_D A_s/m$), km$^2$/kg

E  Constant in equations 3, 5, 6, 7, and 8, l/km

F  Constant in equations 3, 5, 6, 7, and 8, l/km$^2$

H  Altitude above the Earth, km

m  Satellite mass, kg

R  Radius of the Earth, km

r  Radius from center of Earth to satellite, km

T  Orbit decay time, days

$T_\infty$  Exospheric temperature, °K

t  Time, days
GREEK SYMBOLS

\( \mu \)  Gravitational constant of the Earth, \( \text{km}^3/\text{day}^2 \)

\( \rho \)  Atmospheric density, \( \text{kg/km}^3 \)

SUBSCRIPTS

1  refers to the initial value of altitude

2  refers to the final value of altitude


VITA

John Copley Buchholtz was born on October 7, 1943, in Annapolis, Maryland. He received his primary education in Hales Corners, Wisconsin and his secondary education in Greendale, Wisconsin. He was admitted to Iowa State University in Ames, Iowa in September of 1961 and received a B.S. degree in Aerospace Engineering in May of 1966. While attending Iowa State University, he participated in an engineering co-operative training program with McDonnell Douglas Corporation, St. Louis, Missouri.

He was employed at North American Rockwell Corporation from June 1966 to July 1967. He attended the University of Southern California, Graduate Engineering School during the fall semester of 1966. He has been employed at McDonnell Douglas Corporation in the Guidance and Control Mechanics Department since July 1967. He has been enrolled in the Graduate School of the University of Missouri-Rolla, St. Louis Graduate Engineering Center, since June 1968.
APPENDICES
APPENDIX A

DERIVATION OF ALTITUDE DECAY RATE

The rate of circular orbit altitude decay is derived from a description of the total initial orbital energy and the loss of orbit energy with orbit travel. The change in energy in one revolution due to atmospheric drag is;

\[ \Delta E = \oint \mathbf{F}_D \cdot \mathbf{V} \, dt \]  \hspace{1cm} (A1)

where,

\[ \mathbf{F}_D = -C_{DAS} \rho \frac{V_r \mathbf{V}_r}{2} \]  \hspace{1cm} (A2)

A spherical, non-rotating atmosphere is assumed which implies that \( \mathbf{V}_r = \mathbf{V} \). The change in energy is then;

\[ \Delta E = - \oint C_{DAS} \rho \frac{V^2 V}{2} \, dt \]  \hspace{1cm} (A3)

or equivalently,

\[ \Delta E = - \frac{\mu}{2} \int_0^{2\pi} C_{DAS} \rho \, d\theta \]  \hspace{1cm} (A4)

as;

\[ Vdt = rd\theta \]  \hspace{1cm} (A5)

and for circular orbits;

\[ V^2 = \frac{\mu}{r} \]  \hspace{1cm} (A6)

Within one revolution an average area and density is assumed. The final expression for the change in energy is;

\[ \Delta E = - \pi \mu C_{DAS} \rho \]  \hspace{1cm} (A7)
For a point mass representation of the Earth's gravitational potential, the total energy of a satellite in a circular orbit is;

\[ E = \frac{-m\mu}{2r} \]  

(A8)

The changes in energy in one revolution using equation A8 is;

\[ \Delta E = \frac{m\mu}{2r^2} \Delta r \]  

(A9)

Combining equations A6, A7, and A9 yields;

\[ \Delta r = \frac{-2 \pi C_D A_s \rho r^2}{m} \]  

(A10)

Because the change in radius in one revolution, \( \Delta r \), is small, the variation may be taken as the derivative, \( dr/dn \). Equation (A10) may be rewritten as;

\[ \frac{dr}{dn} = \frac{-2 \pi C_D A_s \rho r^2}{m} \]  

(A11)

To convert to the time domain, the orbital period is used.

\[ Pdn = dt \]  

(A12)

For a circular orbit, the period is expressed as,

\[ P = \frac{2 \pi}{\sqrt{\mu/r^3}} \]  

(A13)

Substituting equations A12 and A13 into equation A11 and noting that \( dr = dH \) yields;

\[ \frac{dH}{dt} = \frac{-C_D A_s \rho \sqrt{\mu r}}{m} \]  

(A14)

The ballistic coefficient is defined as,

\[ D = \frac{C_D A_s}{m} \]  

(A15)

Therefore, the rate of change of circular orbit altitude is,

\[ \frac{dH}{dt} = -D \rho \sqrt{\mu r} \]  

(A16)
APPENDIX B

GENERALIZED LEAST SQUARE METHOD

The generalized least square method is used to estimate the coefficients within the approximation for atmospheric density. The density approximation is;

\[ \rho (H, T_{\infty} (t)) = \frac{A + BT_{\infty} + CT_{\infty}^2}{1 + EH + FH^2} \]  

(B1)

The coefficients (A, B, C, E, and F) are to be estimated so as to fit the density data throughout the region of interest. The values of density are known for discrete values of altitude and exospheric temperature. Rewriting equation B1;

\[ A + (T_{\infty}j)B + (T_{\infty}^2j)C - (\rho_{i,j}H_i)E - (\rho_{i,j}H_i^2)F = \rho_{i,j} \]  

(B2)

where, \( \rho_{i,j} = \rho(H_i, T_{\infty}j) \)

\( i = 1, 2, \ldots \)

\( j = 1, 2, \ldots \)

For any number of known values of \( \rho_{i,j} \) and the corresponding values of \( H_i \) and \( T_{\infty}j \) the following matrix equation is written:

\[
\begin{bmatrix}
1 & T_{\infty}j & T_{\infty}^2j & - \rho_{i,j}H_i & - \rho_{i,j}H_i^2 \\
\end{bmatrix}
\begin{bmatrix}
A \\
B \\
C \\
E \\
F \\
\end{bmatrix}
= 
\begin{bmatrix}
\rho_{i,j} \\
\end{bmatrix}
\]  

(B3)

or equivalently;

\[
\begin{bmatrix}
Y \\
\end{bmatrix}
\begin{bmatrix}
X \\
\end{bmatrix}
= 
\begin{bmatrix}
Z \\
\end{bmatrix}
\]  

(B4)
Multiplying both sides of equation B4 by the transpose of \([Y]\) yields;

\[
[Y]^T [Y] [X] = [Y]^T [Z] \tag{B5}
\]

Taking the inverse of \(([Y]^T [Y])\) and multiplying both sides of B5 by the inverse yields the following matrix equation which may be used to determine the constants in equation B1.

\[
[X] = \begin{bmatrix} A \\ B \\ C \\ E \\ F \end{bmatrix} = ([Y]^T [Y])^{-1} [Y]^T [Z] \tag{B6}
\]

The generalized inverse matrix method described provides the best least squares fit to the discrete data as shown by Penrose (10).
APPENDIX C
SAMPLE CALCULATIONS

The time in days to decay from a circular orbit altitude of 500 km to an orbital altitude of 450 km is determined. The exospheric temperature is assumed to decrease linearly from 1000°K to 800°K over a six year period. The average projected area of the satellite is 400 m² and the mass of the satellite is 88,000 kg.

The closed form expression, equation 7, for the time to decay between two circular orbit altitudes is;

\[
\frac{1}{D \sqrt{\mu_r t}} \left[ (H_1 - H_2) + \frac{E}{2} (H_1^2 - H_2^2) + \frac{F}{3} (H_1^3 - H_2^3) \right]
\]

The constants (A, B, C, E, and F) are determined from the least square fit of the density values tabulated in Reference (8) for altitudes ranging from 450 to 500 km and exospheric temperatures ranging from 600 to 1300°K. The constant values in the approximate representation of density provide a good fit of the given density values in the 800 to 1000°K exospheric temperature range. The constants are, from Table II;

\[
\begin{align*}
A &= 5.585199 \times 10^{-5} \text{ kg/km}^3 \\
B &= -1.762227 \times 10^{-7} \text{ kg/(km}^3\text{-°K)} \\
C &= 1.421983 \times 10^{-10} \text{ kg/(km}^3\text{-°K}^2)
\end{align*}
\]
E = -4.485515 (10)^{-3} \text{ 1/km}
F = 5.116007 (10)^{-6} \text{ 1/km}^2

Known constants (C_D, R, and \mu) are:

C_D = 2.2
R = 6378.16 \text{ km}
\mu = 398601.2 (86400.)^2 \text{ km}^3/\text{day}^2

From the problem statement, the following information is known:

A_S = 400 \text{ m}^2
m = 88000 \text{ kg}
H_1 = 500 \text{ km}
H_2 = 450 \text{ km}

T_{\infty} = at+b, \ 0 \leq t \leq 2190. \text{ days}

a = -0.09132420^\circ \text{K/day}
b = 1000.^\circ \text{K}

The ballistic coefficient and the initial orbital radius are:

D = C_D A/m
= (2.2)(400)/88000 \text{ m}^2/\text{kg}
= 10^{-8} \text{ km}^2/\text{kg}

r_1 = R+H_1
= 6378.16+500.
= 6878.16 \text{ km}
All parameters required within equation Cl are now known. Substituting in equation Cl yields the following polynomial in T.

\[ 3.953165 \times 10^{-13} T^3 - 4.939447 \times 10^{-9} T^2 + 2.182759 \times 10^{-5} T = 2.734912 \times 10^{-2} \]  

or equivalently,

\[ T^3 - 1.249492 \times 10^4 T^2 + 5.521548 \times 10^7 T - 6.918284 \times 10^{10} = 0 \]

The cubic equation in T is factored using the standard procedure outlined in Reference (ll). Only one real root, \( T_1 \), is obtained and is;

\[ T_1 = 2043 \text{ days} \]

The numerically integrated result from Table V is 2143 days. The time to decay 50 km from an initial altitude of 500 km computed from the closed form expression is therefore within 4.7% of the numerically integrated result.

Note that for a constant exospheric temperature;

\[ (A + Bb + Cb^2) T = 2.734912 \times 10^{-2} \]

At \( T_\infty = 1000 \, ^\circ\text{K} = \text{constant} \);

\[ T = 1253 \text{ days} \]

at \( T_\infty = 800 \, ^\circ\text{K} = \text{constant} \);

\[ T = 4651 \text{ days} \]

The decay time estimates for constant exospheric temperatures of 1000 \(^\circ\text{K}\) and 800 \(^\circ\text{K}\) are within 3.3% and 0.5%, respectively, of the corresponding numerically integrated results presented in Table I.