A digital computer program for studying elasto-plastic structural behavior due to cyclic loading

Rameshchandra Chandulal Hazariwala

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A DIGITAL COMPUTER PROGRAM FOR STUDYING
ELASTOPLASTIC STRUCTURAL BEHAVIOR DUE TO CYCLIC LOADING

by

RAMESHCHANDRA CHANDULAL HAZARIWALA, 1942 -

A

Thesis

submitted to the faculty of

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1970

[Signatures]
ABSTRACT

A computer program has been developed to study the behavior of plane stress structures under cyclic loading. Such phenomena as thermal ratcheting, alternate plasticity, shake-down and the Bauschinger effect may be considered. The incremental theory of plasticity has been used. The program deals with realistic conditions such as nonlinear strain hardening, nonlinear temperature distribution and occurrence of both compressive and tensile plastic flow. The concept of an average material property has been used.

Thermal ratcheting of a beam subjected to a constant bending moment and a temperature cycle, has been studied in detail. The analysis shows analytically that the rate of plastic strain growth reduces with an increase in the number of loading cycles. Applications of the computer program have been discussed.

Further, the thermal ratcheting of a two bar model has been discussed considering the simplifying assumptions of linear strain hardening and the absence of compressive plastic flow.
ACKNOWLEDGEMENTS

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The author finally wishes to express his thanks to Mrs. Rosemary Smith for typing this thesis.
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\( \sigma \)
Stress, psi.

\( \varepsilon \)
Strain, inch/inch.

\( \varepsilon_T \)
Total mechanical strain in uniaxial case, inch/inch.

\( \varepsilon_e \)
Elastic component of total mechanical strain \( \varepsilon_T \), inch/inch.

\( \varepsilon_p \)
Plastic component of total mechanical strain \( \varepsilon_T \), inch/inch.

\( E \)
Modulus of elasticity, psi.

\( n \)
Shape parameter for the strain hardening characteristics of a stress-strain curve.

\( \sigma_{.7}, \sigma_{.85}, \text{ etc.} \)
Stresses determined from the intersection of the stress-strain curve by the lines of slopes 0.7E and 0.85E, respectively, drawn from the origin, psi.

\( \sigma_e \)
Equivalent stress, psi.

\( \varepsilon_{et} \)
Equivalent total mechanical strain, inch/inch.

\( \Delta \varepsilon_p \)
Increment of plastic strain in uniaxial case, inch/inch.

\( \varepsilon_p \)
Summation of increments of plastic strain in uniaxial case, inch/inch.
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<td>Stress in 'x' direction on an 'x' face, etc., psi.</td>
</tr>
<tr>
<td>$\varepsilon_{xx}$, $\varepsilon_{zz}$, etc.</td>
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<tr>
<td>$\mu$</td>
<td>Poisson's ratio.</td>
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<tr>
<td>$\alpha$</td>
<td>Coefficient of thermal expansion, in./in.$^\circ$F.</td>
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<td>$T$</td>
<td>Temperature above the reference temperature, $^\circ$F.</td>
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<td>$\Delta\varepsilon_x$, $\Delta\varepsilon_z$, etc.</td>
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<td>Equivalent modified total strain, inch/inch.</td>
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<td>$\Delta\sigma_e$</td>
<td>Increment of an equivalent stress $\sigma_e$, psi.</td>
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<tr>
<td>$h$</td>
<td>Half of the total height of the section, inch.</td>
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b
Half of the total width of the section, inch.

$T_0$
Reference temperature, °F.

$\sigma_0$
Yield stress at the reference temperature $T_0$, psi.

$S_1, S_2, \text{etc.}$
$\sigma_{xx}/\sigma_0, \sigma_{zz}/\sigma_0, \text{etc.}$

$E_0$
Modulus of elasticity at the reference temperature $T_0$, psi.

$\varepsilon_0$
Yield strain at the reference temperature $T_0$, $\sigma_0/E_0$, inch/inch.

$H$
Dimensionless quantity, $E/E_0$.

$\tau$
Dimensionless quantity, $\alpha T/\varepsilon_0$.

$\eta_x, \eta_z$
Dimensionless quantities indicating $X/b$ and $Z/h$ respectively.

$e_{xx}'$, $e_{zz}'$, etc.
Dimensionless strain in 'x' direction on an 'x' face, $\varepsilon_{xx}/\varepsilon_0$, etc.

$\Delta e^p_{xx}, \Delta e^p_{zz}$, etc.
Dimensionless increment of plastic strain in an 'x' direction on an 'x' face, etc.

$e^p_{xx}, e^p_{zz}$, etc.
Summation of $\Delta e^p_{xx}$, etc.

$P$
Axial load, psi.

$M$
Bending moment, lb. inch.

$t$
Time, hr.

$K$
Thermal conductivity, btu/hr.ft.°F.
Specific conductivity, ft/hr.

Specific heat, btu/lb.°F.

Density, lbs/ft.³

Actual stress at the end of the current load increment, psi.

Areas of the cross sections of the bars A and B, in.²

Mean stress, psi.

Stress in the bar B at the end of half of the temperature cycle, etc., psi.

Yield stress, psi.

\( \frac{\sigma_{A\frac{1}{2}}}{\sigma_y} \)

\( \frac{\sigma_{B\frac{1}{2}}}{\sigma_y} \)

Plastic strain growth, inch/inch.

Strain in the bar A at the end of half of the temperature cycle, etc., inch/inch.

Tangent modulus of elasticity, psi.

\( \tan^{-1} E \)

\( \tan^{-1} E_s \)

Total mechanical strain at the end of the K th cycle, inch/inch.

Total plastic strain growth at the end of the k th cycle, inch/inch.
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I. INTRODUCTION

The presence of time dependent temperature gradients in structures normally produces complex thermal stress distributions. These stresses combine with the stress due to external mechanical loads. Thus, the design of structures such as those for high speed flight, turbine blades, etc., where large thermal stresses may occur, requires a knowledge of the structural behavior under the combined effects of mechanical and thermal loads. To define the allowable loads and safety factors for design, several types of problems must be studied.

Parkes\(^{(1)}\)* listed the following types of structural behavior as a result of cyclic thermal loading.

- **Elastic**
  - The stresses always lie in the elastic range.
- **Shake-down**
  - The stresses are inelastic in the first cycle of the loading, but thereafter acquire a prestress which ensures that they remain elastic during the following cycles.
- **Alternate Plasticity**
  - There are both positive and negative plastic flow in each cycle of zero net value. A stable hysteresis loop is developed.

\*Numbers in parenthesis refer to the bibliography.
Incremental Collapse or Thermal Ratcheting

There is plastic flow in each cycle of non-zero net value, and therefore, growth of deformation. This growth can result in failure of the structure.

Sudden Collapse

This will occur if the sum of the product of the ultimate stress which has been attained and the cross-sectional area fails to balance the applied load.

Ablation

The member becomes hot enough that it may melt and suffer loss of material.

Further, when cyclic loading is involved, failure may occur by (1) the achievement of a specified amount of distortion, (2) the appearance of a first crack, (3) crack penetration of a specified amount, or (4) fracture of the cross-section. In some cases of cyclic loading, the condition that finally develops is one in which stress and strain alternate about a mean component. For thermal ratcheting, this mean component increases progressively and monotonically even though the magnitude of external loading does not increase in successive cycles. In such cases, the ultimate termination of the useful life of a structure may be due to excessive deformation.

In Appendix A the author has derived the equations for progressive thermal distortion (thermal ratcheting) for a simple two bar model. Even with the simplifying
assumptions of linear strain hardening and the absence of plastic flow in the reverse direction, the equations are complex.

The problem is further complicated by the occurrence of nonlinear strain hardening, tensile and compressive plastic flow and nonlinear temperature distribution. In addition, thermal shock which is the result of the transient nature of many thermal problems may occur. Such properties as specific heat and thermal conductivity, which do not enter directly into consideration for thermal stresses under known conditions of temperature, become important in thermal shock applications because these properties determine the temperature, the temperature gradients, and the rate of change of the gradients. Consideration of these factors complicates the problem even more.

The object of this work has been to develop a computer program to study the structural behavior and the pattern of thermal stresses and strains during cyclic thermal loading of two dimensional elastoplastic problems. The mechanical load which may be present may or may not be constant. The stress-strain curve has been assumed to have a nonlinear strain hardening characteristic. Nonlinear transient temperature distributions and hence the possibility of occurrence of thermal shock have been considered. It is noteworthy that in order to take into
account the change in stress-strain characteristics with an increase in temperature, the concept of an average property has been used. Also, account has been taken of the fact that a certain portion of the cross-section may undergo strain hardening, while other portions may remain elastic. Hence, the stress-strain states will be different at different locations in the same cross-section. The method of successive approximations and the incremental theory of plasticity have been used.
II. REVIEW OF LITERATURE

Under repeated thermal loading, it has been noted that four types of stress-strain systems may be set up, namely, permanent elasticity, shake-down, alternate plasticity and thermal ratcheting. E.W. Parkes, in 1954, investigated the behavior of simplified aircraft wings under repeated thermal stress. He considered an I section for his analysis. He assumed a linearly elastic perfectly plastic stress-strain curve and the absence of temperature gradients in the web and flanges. These assumptions, of course, minimize the algebraic work and tend to simplify the analysis. In cases involving a strain hardening material and the presence of temperature gradients, Parkes' results may be far from reality.

Further, the same author, in 1956, derived the expressions for incremental collapse of the I section taking into account the variation of yield stress. In the previous paper the impression was given that incremental collapse was a rather rare phenomenon which could only occur when the initial stresses due to applied mechanical loads were extremely high. He subsequently suggested that incremental collapse might be of much wider concern. The wing was assumed to carry a constant bending moment due to flight loads and to be initially at a zero stress-strain state at a reference temperature taken as zero. It was
then subjected to a prescribed cycle. Parkes, however, did not consider the temperature history and the external load variation. Both assumptions, that is, elastic perfectly plastic material and the absence of temperature gradients, were considered in the discussion.

In another paper in 1958, Parkes discussed a design philosophy for repeated thermal loading. He described conditions and zones using a curve of load versus temperature to describe the various behavior patterns of a two bar model. A conclusion of his investigation was that "Factors applied to stress or load are not satisfactory criteria of safety under repeated thermal loading, and it is possible that a statistical approach to safety, based on the expected life history of the structure, will have to be used."

In 1964, K.B. Ayers investigated the behavior patterns of I beams carrying pure bending moments and subjected to repeated thermal cycles. These fall into a number of patterns which are basically similar to those obtained by Parkes. Here, also, the basic assumption of a linearly elastic perfectly plastic stress-strain curve was used. The product $Ea$ was assumed to remain constant and the rate of change of temperature was not considered.

In 1959, D.R. Miller investigated the thermal stress ratchet mechanism in pressure vessels. He considered a three bar model assuming linear strain hardening
and also assuming no compressive plastic deformation
during the temperature cycle. He derived the equations
for thermal ratcheting through a graphical construction.

In 1959, G.H. Sprague and P.C. Huang(7) considered
the fact that structures are subject to nonlinear stress
patterns due to the presence of temperature gradients.
The significance of such nonlinear stress systems, when
combined with external loading, on the inelastic behavior
and buckling characteristic of structures was presented.
The effect of residual stresses, resulting from inelastic
behavior under nonlinear stress systems on subsequent
structural behavior was also considered. They remarked,

"In general, any of the four conditions as
described by Parkes and discussed previously can
occur at random, dependent on the specific load-
temperature conditions. (...) While inelastic
behavior appreciably minimizes the problems of
thermal stress for single applications of high
load level conditions, a detailed analysis
of the structural behavior under subsequent
loading must be made for design conditions in-
volving a repeated load-temperature spectrum.
Thus, in missile design, where relatively
few applications of high level loading are en-
countered, a greater degree of plasticity can
be considered than would be permissable in the
design of an aircraft with its long life."

In 1966, S.S. Manson(2) in his text, Thermal Stress
and Low Cycle Fatigue, limited his discussion primarily
to fatigue since, in his words, "..... the mechanism of
progressive distortion being as yet only poorly analyzed."
Though he admits that in dealing with ductile materials
it is to be expected that failure will be due to distortion and/or fatigue rather than to fracture upon the application of a single cycle (see page 307 of reference 2).

As a result of the remarks of Parkes, G.H. Sprague and P.C. Huang on the randomness of occurrence of these four important behaviors, it was decided to devise a computer program which would be capable of taking into account a stress-strain curve with nonlinear strain hardening characteristics, the occurrence of plastic flow in both directions, variation of external loads and the variation of material properties. The incremental method of plasticity was selected because it enables one to trace the history of externally applied loads and temperature variations.
III. UNIAXIAL AND MULTIAXIAL STRESS-STRAIN CURVE
FOR STRAIN HARDENING MATERIALS

A. Stress-Strain Curve for Strain Hardening Materials

A conventional stress-strain curve is shown in Fig. 1. The linear part of the curve extends to point A, which is called the proportional limit. It is in this range that the linear theory of elasticity is valid. Upon further increase of the load, the strain no longer increases linearly with the stress, but the material still remains elastic, that is, upon removal of the load, the specimen returns to its original length. This condition will prevail until a point B, called the elastic limit or yield point, is reached. In most materials there is very little difference between the proportional limit A and the elastic limit B. In this work, points A and B have been assumed to coincide. Beyond the elastic limit, permanent plastic deformation takes place. As the load is increased beyond the elastic limit, the strain increases at a greater rate than the stress. However, the specimen will not deform further unless the load is increased. This condition is called work hardening or strain hardening. The stress required for further plastic flow is called flow stress. Finally, a point C is reached where the load is a maximum. Beyond this point, called the ultimate or tensile strength, a complicated triaxial state of stress
exists in ductile materials.

If at any point between the elastic limit B and the maximum load point C the load is removed, unloading will take place along a line parallel to the elastic line, OB (see Fig. 1). Only a portion of the strain is thus recovered. The total mechanical strain is therefore considered as being made up of two parts, $\varepsilon_e$, the elastic component, and $\varepsilon_p$, the plastic component. That is,

$$\varepsilon_T = \varepsilon_e + \varepsilon_p$$  \hspace{1cm} (1)

In terms of stress, the total mechanical strain may be approximated as,

$$\varepsilon_T = \frac{\sigma}{E} + \left(\frac{\sigma}{B}\right)n$$  \hspace{1cm} (2)

where $n$ is a shape parameter of the strain hardening portion of the stress-strain curve and $B$ is a constant. This expression is only intended to provide a convenient method of representing the stress-strain curve for later numerical work. The determination of these two quantities is explained in the discussion that follows.

Ramburg and Osgood$^{(8)}$ suggested a formula for describing the stress-strain curve in terms of Young's modulus $E$, a secant yield stress $\sigma_{0.7}$ ($\sigma_{0.7}$ is taken as the stress determined from the intersection of the stress-strain curve by a line of slope 0.7E drawn from the origin (see
Fig. 2), and a parameter n which describes the shape of a stress-strain curve in the yield region. In order to determine n, another stress $\sigma_{.85}$ is used. The value of $\sigma_{.85}$ is determined from the intersection of the curve by a line of slope $0.85E$ drawn through the origin. The suggested relation is

$$\frac{E\varepsilon_T}{\sigma_{.7}} = \frac{\sigma}{\sigma_{.7}} + \left( \frac{\sigma}{\sigma_{.7}} \right)^n$$

(3)

The equation for the shape parameter n is

$$n = 1 + \frac{\log_e (17/7)}{\log_e (\sigma_{.7}/\sigma_{.85})}$$

(4)

Dividing both sides of Eq. (3) by $E/\sigma_{.7}$ gives

$$\varepsilon_T = \frac{\sigma}{E} + \left( \frac{\sigma}{\frac{n-1}{n} \sigma_{.7}} \right)^n$$

(5)

Equating Eqs. (2) and (5) gives

$$B = \frac{\sigma_{.7}}{n} \frac{1}{E^n}$$

(6)

From Eqs. (4) and (6), it can be concluded that a knowledge of $\sigma_{.7}$, $\sigma_{.85}$, and $E$ is necessary if the Ramburg-Osgood equation is to be used to define the stress-strain curve.
B. Behavior of Materials Under Uniaxial Stress

Figure 3 shows the stress-strain curve as the line OABC. If the material is continuously loaded until the stress at B is reached and is then unloaded, unloading proceeds along BDE. If the stress is reduced to D and then again increased, the path will be DBC and the material acts as a new material having a yield point B and a stress-strain curve EDBC. Also it is assumed, for this discussion, that the material has the same stress-strain curve in compression and tension. Thus GFEBGC represents a new stress-strain curve after the material has once been subjected to tensile or compressive stresses which bring the stress state to point D where D can be any point on the elastic line BF (see page 111 of reference 2).

Thus, for a uniaxially loaded material that has once been subjected to a plastic flow $\varepsilon_p$, the strain at any subsequent stress $\sigma$, whether loaded further (for example, $\sigma_c$) or unloaded (for example, $\sigma_D$) is

$$\varepsilon_T = \frac{\sigma}{E} + \varepsilon_p + \Delta\varepsilon_{pl}$$

where $\Delta\varepsilon_{pl}$ is any new plastic flow. In case of unloading from B, $\Delta\varepsilon_{pl}$ is obviously zero. Whether new plastic flow takes place depends on the final stress and a consideration of the new stress-strain curve CBDEFG. If the stress reaches the value $\sigma_c$, the increment in plastic strain is
If the stress reaches $\sigma_D$, there is no further plastic strain even though $\sigma_D$ may be high enough to have caused plastic strain in the initial condition of the material prior to the plastic strain $\varepsilon_p$. If the stress goes to $\sigma_G$, the strain becomes

$$\varepsilon_T = \frac{\sigma_G}{E} + \varepsilon_p - \Delta\varepsilon_{p2}$$

C. Behavior of Materials Under Multiaxial Stress

When multiaxial stresses are present, the criterion for further plastic flow is whether the equivalent stress exceeds a specified value. If for example, plastic flow has already occurred in a body and the equivalent stress is $\sigma_B$, as shown in Fig. 3, the material at that point must be regarded as having the relation between "equivalent stress, $\sigma_e$" and "equivalent total mechanical strain, $\varepsilon_{et}$" given by the curve FEBC. If the stress components are changed so that the equivalent stress is reduced, no further plastic flow takes place and the material unloads along the line BDE. For such loading, elastic strains are determined by Hooke's law. If, however, the stress components are changed so that the equivalent stress is increased to $\sigma_C$, further plastic flow takes place. Since the stress-strain curve is EDBC, the increased equivalent plastic strain is $\Delta\varepsilon_{pl}$. The relation between uniaxial stress-strain curve and the multiaxial stress-strain curve has been discussed in article V.
IV. THE CONCEPT OF AN AVERAGE STRESS-STRAIN CURVE

The stress-strain characteristics of materials change with changes in temperature. These changes depend upon the sensitivity of Young's modulus \( E \), and the stresses \( \sigma_{0.7} \) and \( \sigma_{0.85} \), to the changes in temperature. In reality, the stress-strain curve would be different at different locations in a structure and would vary with the temperature variations during the cycle. Further, the rate of change of the mechanical properties increases with an increase in temperature (Fig. 4). It is proposed that as an adequate compromise a single stress-strain curve at each location be chosen such that it represents the average properties for the temperature range experienced by the station. Hence, if each location experiences a different maximum temperature, each location will have a different stress-strain curve. Therefore, with prior knowledge of maximum temperature, temperature range and with curves such as shown in Fig. 4, the average values of \( E \), \( \sigma_{0.7} \) and \( \sigma_{0.85} \) can be calculated in order to define the stress-strain curve as

\[
\varepsilon = \frac{\sigma}{E} + \left(\frac{\sigma}{B}\right)^n
\]

for each station.
V. GENERAL RELATIONS FOR PLANE ELASTOPLASTIC PROBLEMS

A. The Equilibrium, Compatibility and Stress-Strain Relations for the Plane Stress Problems

\[
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0
\]

\[
\frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \sigma_{xz}}{\partial x} = 0 \quad (7)
\]

\[
\frac{\partial^2 \varepsilon_{xx}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial x^2} - 2 \frac{\partial^2 \varepsilon_{xz}}{\partial x \partial z} = 0 \quad (8)
\]

and

\[
\varepsilon_{xx} = \frac{1}{E} \{ (1 + \mu) \sigma_{xx} - \mu (\sigma_{xx} + \sigma_{zz}) \} + \alpha T + \varepsilon_{xx}^P + \Delta \varepsilon_{xx}^P
\]

\[
\varepsilon_{zz} = \frac{1}{E} \{ (1 + \mu) \sigma_{zz} - \mu (\sigma_{xx} + \sigma_{zz}) \} + \alpha T + \varepsilon_{zz}^P + \Delta \varepsilon_{zz}^P
\]

\[
\varepsilon_{yy} = \frac{1}{E} \{ - \mu (\sigma_{xx} + \sigma_{zz}) \} + \alpha T + \varepsilon_{yy}^P + \Delta \varepsilon_{yy}^P
\]

\[
\varepsilon_{xz} = \frac{\sigma_{xz}}{2G} + \varepsilon_{xz}^P + \Delta \varepsilon_{xz}^P \quad (9)
\]

where,

\( \sigma_{xx} \) is the stress in 'X' direction on an 'X' face,

\( \varepsilon_{xx} \) is the strain in 'X' direction on an 'X' face,
\( \mu \) is the Poisson's ratio,
\( \alpha \) is the coefficient of thermal expansion,
\( T \) is the temperature above the reference temperature,
\( \Delta \varepsilon_{xx}^p \) is the unknown plastic strain increment occurring during the current increment of loading,
\( \varepsilon_{xx}^p = \sum_{k=1}^{i-1} \Delta \varepsilon_{xx,k}^p \) is the plastic strain accumulated during the first \( i - 1 \) increments of loading (\( i \) is the current increment of load).

and

\( G \) is the shear modulus.

The Prandtl-Reuss relations are
\[
\Delta \varepsilon_{xx}^p = \frac{\Delta \varepsilon_{ep}}{2\sigma_e} (2\sigma_{xx} - \sigma_{zz})
\]
\[
\Delta \varepsilon_{zz}^p = \frac{\Delta \varepsilon_{ep}}{2\sigma_e} (2\sigma_{zz} - \sigma_{xx})
\]
\[
\Delta \varepsilon_{xz}^p = \frac{3\Delta \varepsilon_{ep}}{2\sigma_e} \sigma_{xz}
\]

(10-a)

and assuming volume constancy, we have
\[
\Delta \varepsilon_{yy}^p = -\Delta \varepsilon_{xx}^p - \Delta \varepsilon_{zz}^p
\]

(10-b)

The corresponding plastic strain-total strain relations are
where the primed quantities are the modified total strains

\[
\begin{align*}
\varepsilon'_{xx} &= \varepsilon_{xx} - \varepsilon^p_{xx} \\
\varepsilon'_{yy} &= \varepsilon_{yy} - \varepsilon^p_{yy} \\
\varepsilon'_{zz} &= \varepsilon_{zz} - \varepsilon^p_{zz} \\
\end{align*}
\]

and

\[
\begin{align*}
\Delta \varepsilon_{ep} &= \frac{2}{\sqrt{3}} \left[ (\Delta \varepsilon^p_{xx})^2 + (\Delta \varepsilon^p_{zz})^2 + \Delta \varepsilon^p_{xx} \Delta \varepsilon^p_{zz} + (\Delta \varepsilon^p_{xz})^2 \right]^{\frac{1}{2}} \\
\varepsilon'_{et} &= \sqrt{2} \left[ (\varepsilon'_{xx} - \varepsilon'_{zz})^2 + (\varepsilon'_{zz} - \varepsilon'_{yy})^2 + (\varepsilon'_{yy} - \varepsilon'_{xx})^2 \\
&\quad + 6 (\varepsilon'_{xz})^2 \right]^{\frac{1}{2}} \\
\sigma_e &= (\sigma_{xx}^2 + \sigma_{zz}^2 - \sigma_{xx} \sigma_{zz} + 3 \sigma_{xz}^2)^{\frac{1}{2}}
\end{align*}
\]
\( \varepsilon'_{et} \) denotes the equivalent modified total strain, and
\( \sigma_e \) is the equivalent stress.

**B. Relationship Between Equivalent Plastic Strain Increment \( \Delta \varepsilon_{ep} \), Equivalent Stress \( \sigma_e \) and Equivalent Modified Total Strain \( \varepsilon'_{et} \)**

The relation between \( \varepsilon'_{et}, \Delta \varepsilon_{ep} \) and \( \sigma_e \) is given by

\[
\varepsilon'_{et} = \frac{2}{3} (1 + \mu) \frac{\sigma_e}{E} + \Delta \varepsilon_{ep}
\]  

(14)

Referring to the uniaxial stress-strain curve (see Fig. 5), let \( \Delta \sigma_e \) be the increment in stress to which corresponds a plastic strain increment \( \Delta \varepsilon_{ep} \). Let \( \sigma_e \) be the stress at the end of the current load increment. Then \( \varepsilon_{et} \) is the sum of the plastic strain increments plus the total elastic strain multiplied by \( \frac{2}{3}(1 + \mu) \). Solving Eq. (14) for \( \Delta \varepsilon_{ep} \) results in

\[
\Delta \varepsilon_{ep} = \varepsilon'_{et} - \frac{2}{3} (1 + \mu) \frac{\sigma_e}{E}
\]  

(15)

The equivalent stress \( \sigma_e \) can now be eliminated from Eqs. (14) and (15). Let the stress preceding the current load increment be \( \sigma_{e,i-1} \), that is,
\[ \sigma_{e,i} = \sigma_{e,i-1} + \Delta \sigma_e \]

Then expanding \( \sigma_{e,i} \) in a Taylor series about \( \sigma_{e,i-1} \) gives

\[ \sigma_{e,i} = \sigma_{e,i-1} + \left( \frac{d\sigma_e}{d\varepsilon_{ep}} \right)_{\varepsilon_{ep} = 0} \Delta \varepsilon_{ep} \quad (16) \]

The higher order terms in \( \Delta \varepsilon_{ep} \) have been neglected. Substituting for \( \sigma_{e,i} \) into Eq. (15) and solving for \( \Delta \varepsilon_{ep} \) gives

\[ \Delta \varepsilon_{ep} = \frac{\varepsilon_{et} - \frac{2}{3} \left[ \frac{(1 + \mu)/E}{(1 + \mu/E) \left( \frac{d\sigma_e}{d\varepsilon_{ep}} \right)_{i-1}} \right]}{1 + \frac{2}{3} \left[ \frac{(1 + \mu)/E}{(1 + \mu/E) \left( \frac{d\sigma_e}{d\varepsilon_{ep}} \right)_{i-1}} \right]} \quad (17) \]

This equation is useful in determining the magnitude of the increment of plastic strain. For linear strain hardening, Eq. (17) is exact (see page 127 of ref. 10).
VI. GENERAL EQUATIONS FOR SYMMETRICAL BEAMS

Consider a beam of depth '2h' and width '2b' as shown in Fig. 6.

The stress-strain relations are given by equation (9). In order to give Eq. (9) a non-dimensional form (10) let

\[ S_1 = \frac{\sigma_{xx}}{\sigma_0}, \quad S_2 = \frac{\sigma_{zz}}{\sigma_0}, \quad S_3 = \frac{\sigma_{xz}}{\sigma_0} \]  

(18)

\[ S = S_1 + S_2 \]  

(19)

\[ H = \frac{E}{E_0}, \quad \tau = \frac{\alpha \Delta T}{\varepsilon_0}, \quad \eta_z = \frac{Z}{h}, \quad \eta_x = \frac{X}{b} \]  

(20)

\[ e_{xx} = \frac{\varepsilon_{xx}}{\varepsilon_0}, \quad e_{xx}^p = \frac{\varepsilon_{xx}^p}{\varepsilon_0}, \quad \Delta e_{xx} = \frac{\Delta \varepsilon_{xx}}{\varepsilon_0}, \]  

(18)

\[ e_{zz} = \frac{\varepsilon_{zz}}{\varepsilon_0}, \quad e_{zz}^p = \frac{\varepsilon_{zz}^p}{\varepsilon_0}, \quad \Delta e_{zz} = \frac{\Delta \varepsilon_{zz}}{\varepsilon_0}, \]  

(19)

\[ e_{xz} = \frac{\varepsilon_{xz}}{\varepsilon_0}, \quad e_{xz}^p = \frac{\varepsilon_{xz}^p}{\varepsilon_0}, \quad \Delta e_{xz} = \frac{\Delta \varepsilon_{xz}}{\varepsilon_0} \]  

(20)

where \( \sigma_0 \) is the yield stress at a reference temperature, \( T_0, \varepsilon_0 = \sigma_0 / E_0 \) is the yield strain at the reference temperature, and \( E_0 \) is the modulus of elasticity at the reference temperature.

Equation (9) can be written as,
\[ e_{xx} = (1 + \mu) \frac{S_1}{H} - \mu \frac{S_2}{H} + \tau + e_{xx}^p + \Delta e_{xx}^p \] (22)

\[ e_{yy} = - \mu \frac{S_2}{H} + \tau + e_{yy}^p + \Delta e_{yy}^p \] (23)

and

\[ e_{zz} = (1 + \mu) \frac{S_2}{H} - \mu \frac{S_1}{H} + \tau + e_{zz}^p + \Delta e_{zz}^p \] (24)

Also,

\[ \varepsilon_{xz} = \frac{\sigma_{xz}}{2G} + \varepsilon_{xz}^p + \Delta \varepsilon_{xz}^p \] (25)

where

\[ G = \frac{E}{2(1 + \mu)} \] (26)

Substituting Eq. (26) into Eq. (25) and then writing the result in non-dimensional form yields

\[ e_{xz} = (1 + \mu) \frac{S_3}{H} + e_{xz}^p + \Delta e_{xz}^p \] (27)

The compatibility equation for plane stress is

\[ \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial x^2} = 2 \frac{\partial^2 \varepsilon_{xz}}{\partial x \partial z} \]
In non-dimensional form, the compatibility equation can be written as

\[
\frac{\partial^2 \varepsilon_{xx}}{\partial \eta_z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial \eta_x^2} = 2 \frac{\partial^2 \varepsilon_{xz}}{\partial \eta_x \partial \eta_z} \quad (28)
\]

Now substituting Eqs. (22), (23), and (27) into Eq. (28) yields

\[
\frac{\partial^2}{\partial \eta_z^2} (1 + \mu) \left( \frac{S_1}{H} \right) \mu \frac{2}{\partial \eta_z^2} \left( \frac{S_2}{H} \right) + \frac{\partial^2 \tau}{\partial \eta_z^2} + \frac{\partial^2 e_{xx}^p}{\partial \eta_z^2} + \frac{\partial^2 \Delta e_{xx}^p}{\partial \eta_z^2} + \frac{\partial^2 \Delta e_{zz}^p}{\partial \eta_z^2} \]

\[
\frac{\partial^2}{\partial \eta_x^2} \left[ (1 + \mu) \left( \frac{S_2}{H} \right) - \mu \left( \frac{S_2}{H} \right) + \tau + e_{zz}^p \right]
\]

\[
= 2 \left( \frac{\partial^2}{\partial \eta_x \partial \eta_z} (1 + \mu) \left( \frac{S_3}{H} \right) + \frac{\partial^2 e_{xz}^p}{\partial \eta_x \partial \eta_z} + \frac{\partial^2 \Delta e_{xz}^p}{\partial \eta_x \partial \eta_z} \right) \quad (29)
\]

The equilibrium equations are

\[
\frac{\partial \sigma_{xz}}{\partial z} = -\frac{\partial \sigma_{xx}}{\partial x} \quad (30)
\]

\[
\frac{\partial \sigma_{xz}}{\partial x} = -\frac{\partial \sigma_{zz}}{\partial z} \quad (31)
\]
In non-dimensional form, Eqs. (30) and (31) can be rewritten as

\[
\frac{\partial}{\partial \eta_x} \left( \frac{S_3}{H} \right) = - \frac{\partial}{\partial \eta_z} \left( \frac{S_1}{H} \right) \quad (32)
\]

\[
\frac{\partial}{\partial \eta_z} \left( \frac{S_3}{H} \right) = - \frac{\partial}{\partial \eta_x} \left( \frac{S_2}{H} \right) \quad (33)
\]

respectively. Differentiating Eq. (32) with respect to \( \eta_x \) and Eq. (33) with respect to \( \eta_z \), and then adding, results in

\[
2 \frac{\partial^2}{\partial \eta_x \partial \eta_z} \left( \frac{S_3}{H} \right) = - \frac{\partial^2}{\partial \eta_x^2} \left( \frac{S_1}{H} \right) - \frac{\partial^2}{\partial \eta_z^2} \left( \frac{S_2}{H} \right) \quad (34)
\]

Substitution of Eq. (34) into Eq. (29) gives

\[
(1 + \mu) \left[ \nabla^2 \left( \frac{S_1 + S_2}{H} \right) \right] - \mu \nabla^2 \left( \frac{S}{H} \right) + \nabla^2 (\tau) = - \frac{\partial^2}{\partial \eta_z^2} \left( e_{xx}^p + \Delta e_{xx}^p \right) - \frac{\partial^2}{\partial \eta_x^2} \left( e_{zz}^p + \Delta e_{zz}^p \right) + 2 \frac{\partial^2}{\partial \eta_x \partial \eta_z} \left( e_{xz}^p + \Delta e_{xz}^p \right) \quad (35)
\]
where \( \nabla^2 = \left( \frac{\partial^2}{\partial \eta_x^2} + \frac{\partial^2}{\partial \eta_z^2} \right) \)

Simplifying Eq. (35) yields

\[
\nabla^2 \left( \frac{S_1 + S_2}{H} \right) - \nabla^2 \tau = -\frac{\partial^2}{\partial \eta_z^2} \left( e_{xx}^p + e_{xx}^c \right) - \frac{\partial^2}{\partial \eta_x^2} \]

\[
\left( e_{zz}^p + \Delta e_{zz}^p \right) + \frac{2}{\partial \eta_x \partial \eta_z} \left( e_{xz}^p + \Delta e_{xz}^p \right) \quad (36)
\]

Eq. (36) is the general equation governing plane elasto-plastic problems.

Assume now that the quantities \( 2b \) and \( 2h \) are small compared to the length \( L \) and that the temperature varies in the 'Z' direction only, that is, \( T = T(Z) \). Because of the thinness of the beam, the plane stress assumption that \( \sigma_{yy} = \sigma_{yz} = \sigma_{xy} = 0 \) is made (see page 279 of ref. 12). The resulting two dimensional problem can be reduced to a one dimensional problem for the example discussed in Appendix B by making the assumption that

\[
\sigma_{zz} = \sigma_{xz} = 0
\]

and

\[
\sigma_{xx} = \sigma_{xx}(Z) \quad (37)
\]

Now Eq. (36) can be written as,
\[ \nabla^2 \left( \frac{S_1}{H} \right) + \nabla^2 \tau = \frac{\partial^2}{\partial \eta_z^2} \left( p_{xx} + \Delta p_{xx} \right) \]  

(38)

Note that from the above assumptions, stress varies only in the 'Z' direction and hence Eq. (38) becomes

\[ \frac{\partial^2}{\partial \eta_z^2} \left( \frac{S_1}{H} + \tau + p_{xx} + \Delta p_{xx} \right) = 0 \]  

(39)

Integrating Eq. (39) twice gives

\[ \frac{S_1}{H} + \tau + p_{xx} + \Delta p_{xx} = C_1 \eta_z + C_2 \]  

(40)

or

\[ p_{xx} = C_1 \eta_z + C_2 \]  

(41)

From Eqs. (40) and (41)

\[ S_1 = H \left( e_{xx} - \tau - e_{xx}^p - \Delta e_{xx}^p \right) \]  

(42)

is obtained. The constants $C_1$ and $C_2$ can be evaluated from the boundary conditions

\[ \int_{-h_1}^{+h_1} \sigma_{xx} \, dA = p \]  

(43)

and
In nondimensional form, Eqs. (43) and (44) can be written as

\[
\begin{align*}
\int_{-h_1}^{+h_1} \sigma_{xx} Z \, dA &= M \\
- \int_{-h_1}^{+h_1} \sigma_{xx} Z \, dA &= M
\end{align*}
\] (44)

Substituting for \( S_1 \) into Eqs. (45) and (46) yields

\[
\begin{align*}
\int_{-1}^{1} S_1 \eta_z \, d\eta_z &= \frac{P}{h\sigma_o} = p^* \\
- \int_{-1}^{1} S_1 \eta_z \, d\eta_z &= \frac{M}{h^2\sigma_o} = M^*
\end{align*}
\] (45) (46)

respectively.

Substituting for \( S_1 \) into Eqs. (45) and (46) yields

\[
\begin{align*}
\int_{-1}^{1} H (C_1 \eta_z + C_2) \, d\eta_z - \int_{-1}^{1} (\tau + \epsilon_{xx} + \Delta e_{xx}) \, d\eta_z &= p^* \\
- \int_{-1}^{1} H (C_1 \eta_z + C_2) \, d\eta_z - \int_{-1}^{1} (\tau + \epsilon_{xx} + \Delta e_{xx}) \, d\eta_z &= M^*
\end{align*}
\] (47) (48)
Solving Eqs. (47) and (48), for constants $C_1$ and $C_2$ yields

$$C_1 = B_1 (M^* + D) - B_2 (P^* + F)$$

(49)

and

$$C_2 = B_3 (P^* + F) - B_2 (M^* + D)$$

(50)

where

$$B_1 = \frac{\left\{ \int_{-1}^{1} H \eta_z \right\}^2}{\int_{-1}^{1} H \eta_z \int_{-1}^{1} H \eta_z d\eta_z - \left( \int_{-1}^{1} H \eta_z d\eta_z \right)^2}$$

and

$$B_2 = \frac{\left\{ \int_{-1}^{1} H \eta_z d\eta_z \right\}^2}{\int_{-1}^{1} H \eta_z \int_{-1}^{1} H \eta_z d\eta_z - \left( \int_{-1}^{1} H \eta_z d\eta_z \right)^2}$$

and

$$B_3 = \frac{\left\{ \int_{-1}^{1} H \eta_z^2 \right\}^2}{\int_{-1}^{1} H \eta_z^2 \int_{-1}^{1} H \eta_z d\eta_z - \left( \int_{-1}^{1} H \eta_z d\eta_z \right)^2}$$
Equation (41) can now be written as

\[ e_{xx} = (B_1 \eta_z - B_2)(M^* + D) + (B_3 - B_2 \eta_z)(P^* + F) \]  

(51)

where

\[ D = \int_{-1}^{1} \mathcal{H}(\tau + e_{xx}^P + \Delta e_{xx}^P) \eta_z d\eta_z \]

and

\[ F = \int_{-1}^{1} \mathcal{H}(\tau + e_{xx}^P + \Delta e_{xx}^P) d\eta_z \]

Equations (23) and (24) can also be written as

\[ e_{yy} = -\mu (C_1 \eta_z + C_2) + (1 + \mu) \tau + (\mu - 1) (e_{xx}^P + \Delta e_{xx}^P) - (e_{zz}^P + \Delta e_{zz}^P) \]  

(52)

and

\[ e_{zz} = -\mu (C_1 \eta_z + C_2) + (1 + \mu) \tau + (e_{zz}^P + \Delta e_{zz}^P) + \mu (e_{xx}^P + \Delta e_{xx}^P) \]  

(53)

Note that the relation \( \Delta e_{yy}^P = -\Delta e_{xx}^P - \Delta e_{zz}^P \) has been used in deriving Eq. (52).
From Eq. (40)

\[ \sigma_{xx} = \sigma_0 \ H (C_1 \eta_z + C_2 - \tau - e_{xx}^P - \Delta e_{xx}^P) \]  (54)

is obtained.

The strains \( \varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz} \) and stress \( \sigma_{xx} \) can now be calculated by Eqs. (51), (52), (53) and (54), respectively.
VII. THEORIES OF PLASTICITY

A. Total Deformation Theory of Plasticity

This theory establishes a relation between the stresses and the total strains so that if the stresses are known, the strains can be directly calculated. The path by which a given stress distribution is reached, presumably, does not influence the strains. This cannot be generally correct. However, it is a useful concept in certain cases. If the deformation theory is used, all the previous equations with Δ's and primes removed are valid.\(^{(10)}\)

Consider now the equation

\[ \varepsilon_{ep} = \varepsilon_{et} - \frac{2}{3} \left( 1 + \mu \right) \frac{\sigma_e}{E} \]  

\((55)\)

The relation between \(\varepsilon_{et}\) and \(\sigma_e\) is contained in the uniaxial stress-strain curve. For any selected value of \(\varepsilon_{ep}'\), the value of \(\sigma_e\) can be determined. Thus, Eq. (55) represents a direct relationship between \(\varepsilon_{et}\) and \(\varepsilon_{ep}\) which can be determined from the stress-strain curve (see Fig. 7). This curve can then be used in place of the original stress-strain curve. It can be seen that a plot of \(\varepsilon_{et}\) versus \(\varepsilon_{ep}\) will have a slope of approximately unity. A small error in \(\varepsilon_{et}\) will, therefore, produce the same order-of-magnitude error in \(\varepsilon_{ep}\) without any magnification.
B. The Incremental Theory of Plasticity

In this theory, each step consists of obtaining the increment in the plastic strain as the material passes from one stress state to the next stress state. Consider the equation

\[ \varepsilon_{xx} = \frac{1}{E} \{ \sigma_{xx} - \mu (\sigma_{yy} + \sigma_{zz}) \} + \alpha T + \sum \Delta \varepsilon^P_{xx} + \Delta \varepsilon^P_{xx} \]  

(56)

If the summation term is transposed to the left hand side of the equation, it will be seen that this equation becomes

\[ \varepsilon'_{xx} = \frac{1}{E} \{ \sigma_{xx} - \mu (\sigma_{yy} + \sigma_{zz}) \} + \alpha T + \Delta \varepsilon^P_{xx} \]  

(57)

where \( \varepsilon'_{xx} = \varepsilon_{xx} - \sum \Delta \varepsilon^P_{xx} \).

The expression \( \varepsilon_{xx} - \sum \Delta \varepsilon^P_{xx} \) is the total strain component in the X direction due to the thermal and mechanical stresses. It takes into account the fact that the free length of the element has been changed by the plastic flow (see page 113 of reference 2). Denoting this term as the "modified total strain component" it is evident that an equivalent modified total strain can be computed from the components in a manner analogous to that of Eq. (13). This strain will be related to the increment in the plastic strain by Eq. (11). This observation makes it
possible to perform any one computation in the same manner as in a deformation theory computation.

The solutions of problems by the incremental theory is desirable whenever unloading is possible and whenever a nonproportional loading condition exists. It consists thus, of several individual computations each representing a stage actually encountered by the body during its mechanical or thermal load history.
VIII. DETERMINATION OF THE MAGNITUDE OF THE PLASTIC STRAIN INCREMENT $\Delta \varepsilon_{ep}$

During the first increment of loading, there are two possibilities regarding the state of stress which may exist at the end of computation, (1) the stress may lie in the elastic range, and (2) the stress may exceed the elastic limit.

In the first case, the equivalent total mechanical strain will not exceed the limiting equivalent mechanical strain. This suggests that a condition during the computation should be imposed which will set $\Delta \varepsilon_{ep}$ and, hence, $\Delta \varepsilon_{xx}^p$, $\Delta \varepsilon_{zz}^p$, etc., to zero. The elastic solution may thus be obtained.

In the second case, there are two possibilities, (a) the stress may lie near the yield or limit stress ($A_1'$) or (b) the stress may be far from the yield point ($A_1''$).

These two cases have been illustrated in Fig. (9-a). Consider Eq. (17), that is,

$$\Delta \varepsilon_{ep} = \frac{\varepsilon_e' - 2/3[(1 + \mu)/E] \sigma_{e,i-1}}{1 + 2/3[(1 + \mu)/E] \left(\frac{d\sigma_e}{d\varepsilon_{ep}}\right)_{i-1}}$$

Obviously, the stress $\sigma_{e,i-1}$ and the slope $\left(\frac{d\sigma_e}{d\varepsilon_{ep}}\right)_{i-1}$...
should be taken as the stress and slope of the stress-strain curve at the current yield point. When the loading is such that the actual stress state is at \( A_1 \), then the plastic strain increment may be satisfactory. Eq. (17) shows that the strain hardening characteristic is linear with the slope \( \frac{d \sigma}{d \varepsilon_{ep}^{i-1}} \). If the actual stress state is to be at \( A_1'' \), then the plastic strain increment will be inaccurate. The alternative in such a case will be to use the expression

\[ \varepsilon_{ep} = \left( \frac{\sigma e}{B} \right)^n \]

and to calculate corresponding plastic strains for selected stress values. The least square method can then be used to find a straight line relationship between the stress \( \sigma e \) and the plastic strain \( \varepsilon_{ep} \). The slope of this line can be substituted into Eq. (17) in place of the term \( \frac{d \sigma e}{d \varepsilon_{ep}^{i-1}} \). Equation (17) in a sense limits the size of the load increment. For better results, it is necessary that smaller load increments be chosen in the area of the yield point.

Now consider any other increment of loading. At the end of the current load increment, the stress state may be at point \( B \) (see Fig. 9-b) if unloading is experienced, or it may be at \( C \) if the load is increased. Because of strain hardening, the new stress-strain curve during the current increment of loading is considered to be \( O'AC \).
The equivalent limit stress and equivalent limit strain are \( \sigma_{A,i-1} \) and \( \epsilon_{A,i-1}/E' \), where \( E' = E/\sqrt{3}(1+\mu) \). The subscript 'i' indicates the current increment of load. The stress \( \sigma_{A,i-1} \) is known. When the equivalent modified total strain \( \epsilon_{et}' \) does not exceed \( \sigma_{A,i-1}/E' \), unloading or loading in the elastic region occurs and the stress state reaches a point B somewhere on the line O'A. Observation of this condition, therefore, automatically ensures that unloading has been done elastically. Further, if the equivalent modified total strain \( \epsilon_{et}' \) is greater than the current equivalent limit strain, Eq. (17) can be used to determine the magnitude of the associated plastic strain increment. Point C, which indicates the state of stress at the end of the current load increment in case of further loading, should not lie far from the current stress limit (i.e., a large difference between the stress due to the current load increment and the stress in the plastic region due to next increment of load, should not be permitted). Also, it should be noted that if the magnitude of the stress at the end of the previous load increment is less than that of the limit stress, the value of the actual stress should be considered to be equal to the limit stress, in order to compute the plastic strain increment, \( \Delta\epsilon_{ep} \).
IX. ONE DIMENSIONAL TRANSIENT HEAT CONDUCTION

Most of the one dimensional problems can be treated in a manner similar to the flat plate problem (see page 276 of ref. 2). In this problem, the plate is assumed to have its faces and ends insulated so that the temperature varies in the \( z \) direction.

Consider the model shown in Fig. 10. The conduction equation is

\[
\frac{ \partial^2 T }{ \partial z^2 } = \frac{1}{k} \frac{\partial T}{\partial t}
\]  

(58)

where

\[
k = \frac{K}{\rho c} = \text{specific conductivity, ft./hr.}
\]

\[
K = \text{thermal conductivity, btu/hr. ft. } ^{\circ}\text{F.}
\]

\[
C = \text{specific heat, btu/lb. } ^{\circ}\text{F.}
\]

\[
\rho = \text{density, lbs./ft.}^3.
\]

Let

\[
\phi = T - T_f
\]

where \( T_f \) denotes the fluid temperature. The solution of Eq. (58) is

\[
\phi = \sum_{n=1,3,5}^{\infty} \frac{4(T_0 - T_f)}{n\pi} e^{-\frac{n^2\pi^2kt}{L^2}} \sin \frac{n\pi z}{L}
\]

(59)

where \( T_0 \) is the initial temperature of the plate.
X. EXPLANATION OF PROCEDURE AND FLOW LIST

The beam section shown in Fig. 10 was divided into a number of stations. An odd number was preferred in order to carry out the numerical integration (see Fig. 11). The stress-strain curve was determined for each station from the maximum temperature and the temperature range as explained previously. Numerical integration may be used or the equations for average properties may be derived by referring to curves such as shown in Fig. 3 (flow list: steps 2 and 3).

The proportional limit stress and strain were determined from the arbitrary condition that at the proportional limit the value of plastic strain $\varepsilon_p$ is approximately 0.0001. The equation

$$\sigma = B \varepsilon_{ep} \frac{1}{n}$$

was used to determine the value of limit stress at a plastic strain $\varepsilon_{ep}$ of 0.0001 (flow list: step 4).

Before beginning the application of mechanical and/or thermal load, the summation terms $\sum \varepsilon_{xx}^P$, $\sum \varepsilon_{zz}^P$, $\sum \varepsilon_{yy}^P$, etc., were set to zero. Then the loading cycle was started. To calculate the stresses and strains for any particular increment of loading, the iterative procedure was then started. The plastic strain increments $\Delta \varepsilon_{xx}^P$, $\Delta \varepsilon_{zz}^P$, etc., were set to zero for the first iteration. The total
strains \( \varepsilon_{xx} \), \( \varepsilon_{yy} \), \( \varepsilon_{zz} \) etc., were then calculated in order to determine the equivalent total strain. The integrals appearing in the equations for \( \varepsilon_{xx} \), \( \varepsilon_{yy} \), \( \varepsilon_{zz} \) (Eqs. (51), (52), (53), respectively) were evaluated by a numerical method. It should be noted that though Simpson's rule was used, any of the Newton-Cote formulae can be used (flow list steps 5 to 7-b).

The equivalent modified total strain \( \varepsilon'_{et} \) was then calculated by using Eq. (13). If \( \varepsilon'_{et} \) is less than the equivalent limit strain, then the stress and strains are in the elastic range. But if the equivalent modified total strain \( \varepsilon_{et} \) exceeds the equivalent limit strain, then the stress and strains due to the current load increment must be in the plastic range and hence \( \Delta \varepsilon_{ep} \) should be calculated as explained earlier. Step 7-g solves for the plastic strain components, \( \Delta \varepsilon_{xx}^P \), \( \Delta \varepsilon_{zz}^P \) and \( \Delta \varepsilon_{yy}^P \) by using the set of Eq. (11). Thus, the plastic strain increments \( \Delta \varepsilon_{xx}^P \), \( \Delta \varepsilon_{zz}^P \) and \( \Delta \varepsilon_{yy}^P \) were calculated at each station. At this stage, a check for convergence was carried out as shown by step 7-i. When convergence was not achieved, the values of the plastic strain increments \( \Delta \varepsilon_{xx}^P \), \( \Delta \varepsilon_{zz}^P \), and \( \Delta \varepsilon_{yy}^P \), obtained during the current iteration were substituted into the integrals appearing in the equations for total strains.

The procedure described by steps 7-b to 7-g was repeated until convergence was obtained.

The stresses were then computed by using Eq. (54) (flow list: step 8).
The next step was to determine a new limit stress and a limit strain before considering the next loading increment. Setting of the limit stress and strain has been discussed in detail in the following article.

Flow List:

(1) Data:
   (a) Geometry of a section.
   (b) Values of 'n' following the selection of a suitable number of stations (preferably an odd number).
   (c) Maximum temperature and temperature range for each station.
   (d) Thermo-physical properties such as specific heat, thermal conductivity and coefficient of thermal expansion.
   (e) Mechanical properties such as Poisson's ratio, modulus of elasticity, stresses \( \sigma_{0.85}, \sigma_{0.87} \) suitable for describing a stress-strain curve.
   (f) Mechanical loads.
   (g) Thermal loads.
   (h) Physical property: density

(2) Calculate average mechanical properties such as \( E, \sigma_{0.7} \) and \( \sigma_{0.85} \) which are used to compute an average stress-strain curve.
(3) Compute the average stress-strain curve mentioned in step 2.

(4) Calculate the limit stress and strain at each station for $\varepsilon_{ep} = 0.0001$ using Eq. (2).

(5) Set summation terms for plastic strain increment to zero, that is, $\sum \Delta \varepsilon_P^{xx} = \sum \Delta \varepsilon_P^{zz} = \sum \Delta \varepsilon_P^{yy} = 0$.

(6) Apply mechanical and thermal loads. Thermal loads are calculated by varying the time intervals in Eq. (59).

(7) Start the iteration procedure:
   (a) For the first iteration, set the plastic strain increments $\Delta \varepsilon_P^{xx}$, $\Delta \varepsilon_P^{zz}$, and $\Delta \varepsilon_P^{yy}$ to zero.
   (b) Evaluate integrals appearing in Eqs. (51), (52) and (53) by Simpson's rule.
   (c) Compute total strains $\varepsilon_{xx}$, $\varepsilon_{zz}$, and $\varepsilon_{yy}$ using Eqs. (51), (52) and (53).
   (d) Calculate modified total strains and the equivalent modified total strain using Eqs. (11) and (13).
   (e) If equivalent modified total strain is less than or equal to equivalent limit strain, then the plastic strain increment is zero. If not, go to (7-f).
   (f) Compute the equivalent plastic strain increment $\Delta \varepsilon_{ep}$ using Eq. (17).
(g) Compute the components of plastic strain
\( \Delta \varepsilon_{ep} \), i.e., \( \Delta \varepsilon_{xx}^p \), \( \Delta \varepsilon_{zz}^p \) and \( \Delta \varepsilon_{yy}^p \) using
Eq. (11).

(h) Calculate these components at all stations.

(i) Check for convergence at all stations.

Conditions:
\[ \left| \frac{\Delta \varepsilon_{xx}^p, i, j - \Delta \varepsilon_{xx}^p, i, j-1}{\Delta \varepsilon_{xx}^p, i, j} \right| \leq .001, \]

etc.

(j) If convergence is not achieved at one or more stations, substitute the values of
\( \Delta \varepsilon_{xx}^p \), \( \Delta \varepsilon_{zz}^p \) and \( \Delta \varepsilon_{yy}^p \) in (7-b) and repeat
the procedure. If convergence is achieved
at all stations, go to step 8.

(8) Calculate the stresses using Eq. (54).

(9) Calculate the equivalent stress \( \sigma_e \) and the
summation of the plastic strain increments.

(10) Determine the equivalent limit stress and strain
for the next increment of load.*

(11) Go to step 6.

*See section XI.
XI. DETERMINATION OF THE LIMIT STRESS AND LIMIT STRAIN FOR THE NEXT INCREMENT OF LOADING

When cyclic loading is encountered, it becomes necessary to consider the Bauschinger effect. According to this effect, plastic flow in one direction (say tension) reduces the stress at which yielding will occur in the opposite direction (compression). The stress-strain path which may be, presumably, traveled due to this effect has been shown by a broken line in Fig. 12.\(^{(13)}\)

Before considering the Bauschinger effect, let us examine the effect of the usual assumption of isotropic strain hardening. Assume that during an early stage of the loading cycle, the stress-strain state has reached the point A as shown in Fig. 12. Because of the assumption under consideration, during unloading or reversed loading, the limit stress will be considered to be equal to \(\sigma_C\), where \(\sigma_C\) is equal to \(\sigma_A\). The stress-strain state, therefore, will follow the path ABCD and hence plastic flow \(\varepsilon_{ep2}\) which could have occurred, will not be computed. The stress-strain state which would have been at \(C'\) would be calculated as if it were at some point on the line AC. Further, assume that the stress-strain state is at \(D'\), somewhere on the path CD, due to reversed loading. Reversed loading is then discontinued and once again loading in the initial direction is resumed. In this case, the
limit stress will be considered to be $\sigma_F$ where $\sigma_F$ is equal in magnitude to $\sigma_D$. It can be seen that this assumption will lead to incorrect results. To be closer to reality, it is, therefore, necessary to reset the limit stress and strain as per the Bauschinger effect. However, in order to reduce the error introduced due to the assumption of isotropic strain hardening, the limit stress can be calculated from $\sigma = B \varepsilon_{p3}^{\frac{1}{n}}$, where $\varepsilon_{p3}$ is the total plastic strain at the end of a loading cycle (see Fig. 13).

Due to the interaction of the mechanical load and different temperature gradients at different stations in a structure, it becomes quite difficult to predict when plastic flow in the direction opposite to that of previous plastic flow takes place. Thus, setting of the limit stress and limit strain according to the Bauschinger effect becomes difficult. The following discussion assumes simple mathematical models for the Bauschinger effect and describes the method of determination of limit stress and strain accordingly.

Assume now that during cyclic loading the stress-strain state follows the path shown in Fig. 14. The stresses $\sigma_{y,t}$ and $\sigma_{y,c}$ are the initial yield stresses in tension and compression, respectively. It is assumed that during the cyclic loading plastic flow occurs and a new yield stress $\sigma_A$ is obtained due to strain hardening. The
yield stress $\sigma_{y,c}$ is assumed here to have a value equal to the stress $\sigma_{y,t}$. Also assume, for simplification, that the ratio of the stress $\sigma_A$ to the stress $\sigma_{y,t}$ remains constant during the cyclic loading. The last assumption has been made in order to simplify the discussion that follows. If a mathematical model for the Bauschinger effect which relates the number of changes in the direction of plastic flow and the magnitude of plastic flow associated with each change were available, it could be easily employed. From Fig. (15-a) it can be seen that during a typical loading cycle, the path traveled is $OYAB_1$. The point $B_i$ (tensile stress) indicates the actual stress-strain state at the end of the current load increment. The point A indicates the limit stress-strain state for the current load increment. The point C indicates the limit stress-strain state for reverse plastic flow.

It can now be expected that at the end of the next load increment, there may exist three possible states of stress and strain, namely,

(1) The state may be on the elastic line AC (for example, point $B'_{i+1}$), if unloading occurs or in case of reverse loading which does not cause plastic flow in the opposite direction.

(2) The state may be on the strain hardening portion $AH$ (e.g., point $B''_{i+1}$), if further loading is continued and plastic flow occurs in the same direction.
(3) The state may be on the strain hardening portion CJ (e.g., point \( B_{i+1}'' \)), if the loading condition is such that plastic flow may occur in the opposite direction.

Assume now that the load increment size is small enough so that if the stress at the end of the current load increment is in one direction (tension), the stress that will be produced at the end of the next load increment will not exceed the yield limit \( C \) in the opposite direction. The purpose of this assumption will become clear from the discussion that follows. This assumption can be utilized by dividing the loading cycle into a number of load increments. The size of the loading increments must be kept small from other considerations as well. For a particular case such as shown in Fig. (15-a), the third possibility will be ruled out due to the small increment assumption. The actual stress-strain state can, therefore, be somewhere on the path CAH. It is now clear that the limit stress-strain state is at \( A \) for the next load increment.

In Fig. (15-b), the point \( B_i \) corresponds to the compressive stress \( \sigma_{B_i} \). From the previous discussion and in view of the assumptions made, the new limit stress-strain state can be determined to be at \( C \) (\( \sigma_C = \sigma_y, t \) for the first loading cycle). In other words, the limit stress state should be selected such that it has the same direction as that of the actual stress state at the end of the current
load increment (for example, if the actual stress is positive, the limit stress should also be positive). Assume that the actual stress is \( \sigma_{B_i} \). It is compressive and very near to zero. The stress \( \sigma_c \) will be selected as a new limit stress. Also, unloading or loading may occur during the next load increment. If loading occurs and the state of the actual stress is as shown by the point \( B_{i+1}' \), the stress indicated by point \( B_{i+1}' \) should not exceed the stress of \( \sigma_F \) (the magnitude of stress \( \sigma_F \) is equal to the magnitude of stress \( \sigma_c \)). If care is not taken and loading occurs, then yielding will be shown to occur at \( \sigma_F \), which is obviously incorrect because in case of loading, the limit stress should be \( \sigma_A \). Therefore, the size of the load increment should be kept small enough so that the difference in actual stress due to two consecutive load increments does not exceed the stress \( \sigma_c \) (in general, the lower of the two possible limit stresses).

Consider now Fig. (15-c). It is shown in this figure that reverse plastic flow has occurred. It can also be seen that the two possible stress limits will be \( \sigma_K \) and \( \sigma_L \). The magnitude of \( \sigma_L \) is given by

\[
\sigma_L = B \varepsilon_{ep} \frac{1}{n}
\]

Further, the same figure shows five limit stress-strain states for a typical loading cycle. These states are as follows:
(1) State Y (at the beginning of the cycle).
(2) State A (plastic flow in one direction has occurred).
(3) State C (unloading or loading in the reverse direction has occurred but not sufficient to cause plastic flow in the reverse direction).
(4) State K (plastic flow in the reverse direction has occurred).
(5) State L (loading or unloading along the elastic line LK).

It should, however, be noted that if plastic flow in the reverse direction does not take place, the point K will coincide with the point C and the point L will coincide with the point A.

From the above discussion, it should now be noted that knowledge of four conditions is required in order to choose the proper limit stress-strain state. These conditions are as follows:

(1) The magnitude and direction of the limit stress for the current load increment.
(2) The magnitude and direction of the actual stress at the end of the current load increment.
(3) The direction of the first plastic flow which occurs during the cycle under consideration.
(4) Occurrence of plastic flow in the reverse direction.
The direction of the first plastic flow is useful in the determination of the occurrence of plastic flow in the opposite direction. In Fig. (16-a), assume that the limit stress during the computation for a particular load increment is $\sigma_A$. The actual stress is $\sigma_{B_1}$. During the next and subsequent load increments the actual stress changes. As long as the actual stress state remains positive and on the elastic line O'A, the limit stress state remains at A. But when the actual stress state becomes negative, the limit stress state will have to be set at C, that is, the yield point in the opposite direction. This limit is obtained by dividing $\sigma_A$ by the constant established during the first cycle (see Fig. 14: $K = \frac{\sigma_{\text{f},t}}{\sigma_{\text{y},c}}$).

It should be noted that if a more appropriate mathematical model for the Bauschinger effect were available, the relation between these two limit stresses would be known. Also, if the model incorporates the effect of the number of stress strain cycles (hysteresis loops), then the constant used above may vary with the number of cycles.

Now consider a case wherein the actual stress state does not go beyond point C but reaches a point $C_1$ on the elastic line O'C. Then it returns to a point $D_1$ on the positive side during the next load increment. It will be seen that as soon as the actual stress-state becomes positive, a need arises to establish the limit stress-state at A. In the absence of knowledge of the direction of plastic flow,
the only known factors are

1. The magnitude and direction of the yield stress for the current load increment and,
2. The magnitude and direction of the actual stress at the end of the current load increment.

Reconsider now Fig. (16-a). Assume that during the current load increment, the limit stress was $\sigma_C$. This is because the actual stress-state was at $C_1$ at the end of the previous load increment. It is now observed that at the end of the current load increment the actual stress-state is at $\sigma_A$. In the absence of the knowledge of the direction of prior plastic flow, the situation would be as described below:

1. The yield stress for the current load increment is negative.
2. The actual stress at the end of the current load increment is positive.

Consider now Fig. (16-b). Two possible situations that may arise have been described by the points $C_1D_1$ and $C_2D_2$ on either side of the stress axis. Fig. (16-c) shows the typical paths that are possible in passing through these points during the first cycle. When a loading path passes through the points $D_1$ and $C_1$, tensile plastic flow occurs, whereas in the case of the loading path passing through the points $C_2$ and $D_2$, compressive plastic flow is observed. In the first case, it is necessary to multiply
the limit stress $\sigma_C$ by the constant $K$, whereas in the second case, division is necessary. The decision regarding multiplication or division can be made only when the direction of plastic flow is known.

Now reconsider Fig. (15-c). When the actual stress-strain state proceeds along the elastic line KL, the possible limit stress-strain states are at L and K. The limit stresses $\sigma_L$ and $\sigma_K$ can be related by a factor (e.g., $K_f = \sigma_L / \sigma_K$). It should be noted that as long as the value of the actual stress lies along the elastic line LK, the value of $K_f$ will remain constant. It may, however, change during subsequent cycles. The direction of plastic flow again helps to decide whether multiplication or division is necessary. Illustration of the procedure has been given below. The particular case shown in Fig. (15-c) has been considered.

$S_i$ is the actual stress state at the end of the current load increment.

$\sigma_K$ is the limit stress for the current load increment since $B_i$ was the actual stress at the end of the previous load increment.

$\sigma_L$ is another possible limit stress. This stress is obtained by the equation,

$$K_f = \frac{\sigma_L}{\sigma_K}$$

Since stress $\sigma_{S_i}$ is positive, the positive limit
stress $\sigma_L$ is to be selected. The limit stress $\sigma_L$ can be obtained by either

$$\sigma_L = B \varepsilon_{ep}^{\frac{1}{n}}$$

or

$$\sigma_L = \sigma_K K_f$$

During the computation, before starting a particular cycle of load, the number indicating the plastic flow for that cycle must be set to zero. In the program, plastic flow is indicated as shown below:

1. No plastic flow : 0
2. Compressive plastic flow : 1
3. Tensile plastic flow : 2

These help to indicate how the loading path is being traced for that particular loading cycle. However, it is necessary that the direction of the plastic flow which occurs for the first time during the previous loading history be retained in the memory of the computer. In order to avoid confusion in the discussion which follows, this plastic flow is termed the 'very first plastic flow'. The reason for retaining the direction of the very first plastic flow has been explained in Fig. 17.

Assume that at a certain station in the section, compressive stress occurs in the early stage of the loading
cycle (see Fig. (17-a)). However, this stress does not reach the yield point. Due to further load increments the stress-strain state travels along the path OYAC. The point C indicates the actual stress-strain state at the end of the first loading cycle. Similarly, point C indicates the actual stress-strain state at the end of the second loading cycle. The very first plastic flow, in this particular case, is tensile. It is seen that first plastic flow during the second cycle is compressive. Assume now that $\sigma_L$ is the limit stress for the current load increment. Due to the loading conditions of the current load increment, unloading or reverse loading occurs and the actual stress becomes $\sigma_B$. The stress $\sigma_B$ is compressive, and hence, the new limit stress should also be compressive. The limit stress $\sigma_K$, therefore, should be chosen as a new limit stress. This can be obtained by the equation,

$$\sigma_K = \frac{\sigma_L}{K_f}$$  \hspace{1cm} (61)

Suppose the direction of the very first plastic flow is not considered. Then one may depend only on the knowledge of the first plastic flow which takes place during the cycle. Considering Fig. (17-b), and following
the earlier discussion, the stress $\sigma_K$ will be obtained by the equation,

$$\sigma_K = \sigma_L K$$  \hspace{1cm} (62)

which is incorrect for this case.

Further, consider Fig. (17-c). The directions of the very first plastic flow and the first plastic flow for the second cycle are identical. The limit stress $\sigma_K$ is obtained by the equation,

$$\sigma_K = \sigma_L / K$$  \hspace{1cm} (63)

Compare now Eqs. (61) and (63). It can be seen that a decision regarding multiplication or division solely depends upon the direction of the very first plastic flow. It should be noted that in the absence of reverse plastic flow, the constant $K$ and $K_f$ will have the same value. Note that Fig. (17-c) represents a typical case where the very first plastic flow and the first plastic flow during a particular loading cycle have the same direction.

If no plastic flow is indicated during the particular cycle, it is necessary to consider the direction of the plastic flow in the previous cycle.

The preliminary computation procedure has been illustrated in the flow diagram shown in Fig. 18. This method of resetting the stress limit gives freedom of selection
of the stress limit according to available data. Hence, it does not confine one to the limitation of the assumption that the material has the same stress-strain curve in both directions and that alteration of the stress-strain curve is the same in compression and tension. In fact, mathematical models of the Bauschinger effect could be studied.
XII. DISCUSSION AND CONCLUSIONS

A computer program has been developed (see Appendix B) which has the potential to investigate the following phenomena.

A. The Influence of the Bauschinger Effect on the Cyclic Life of the Material

Because the Bauschinger effect is more the rule than the exception in the behavior of engineering alloys, and has the tendency of changing the yield stress with each change in the direction of the plastic flow, it has a very important bearing on the cyclic stress-strain patterns. As the yield point lowers at one station in the section, more load is shared by the portions which have not yielded as well as by those which have yielded but still possess higher yield points than the yield point of the station under consideration. This may cause gradual yielding at more and more stations as the number of cycles increases. It is, therefore, possible that a situation may develop when the entire section may yield causing a sudden collapse by failing to balance the applied load. It should, however, be noted that the life of the structure may be terminated because of excessive deformation or the appearance of cracks before a condition of sudden collapse is reached. It should be of interest to predict analytically when the condition of collapse is reached, that
is, to investigate the mechanical and thermal loads and the number of loading cycles required to cause collapse in typical structures.

B. The Phenomena of Thermal Ratcheting and Alternate Plasticity

In case of alternate plasticity, as defined by Parkes (1) there is both positive and negative plastic flow in each cycle of net zero value. This can be true when linearly elastic perfectly plastic material is considered. If strain-hardening characteristics are considered, it is not possible to show steady behavior of alternate plasticity immediately after the formation of the first hysteresis loop due to cyclic loading. The reason for this is that at any station in the section, the magnitude of tensile and compressive yield stresses should become smaller with each change in the direction of plastic flow and this, in turn, affects the stress-strain pattern not only at that particular station, but at other stations also. It should be noted that during an early stage of cyclic loading, the rate of change of the magnitude of the yield stresses due to the Bauschinger effect may be high. This rate would finally become negligible resulting in stationary yield points for the cycles that follow. One may, therefore, expect alternate plasticity to occur at a station when the yield stresses become steady subject to the condition that the stress-strain patterns at other
stations do not affect the stress-strain pattern at the station under consideration.

For thermal ratcheting, Ayer\textsuperscript{(5)} observed that "The stress pattern repeats itself after the first cycle and a constant increment of strain is added on each cycle." He considered the linearly elastic perfectly plastic material. Considering the linear strain hardening characteristic, the author has proven analytically that an asymptotic stress-strain condition may exist after a certain number of loading cycles and further strain growth may not occur (see Appendix A). This phenomenon has been observed experimentally (see page 128 of ref. 2). Thus, it is possible that different behavior may be observed at the same station with an increase in the number of loading cycles. Further, different stations in the section may exhibit different patterns of behavior at the same time and these patterns are likely to change. Thermal ratcheting may start at one or more stations and then progressively extend to the neighboring stations.

For design, it is important to investigate the loading conditions which may cause thermal ratcheting or other structural behavior in the section and how rapidly the detrimental effect of thermal ratcheting may spread to other stations. The behavior patterns should also be examined when the applied mechanical load is also varying. For example, the bending moment on the wing of
a bomber aircraft will vary with the variation in the aerodynamic loads. Variation in the mechanical loads may occur with the release of the bombs and consumption of fuel during flight. All of these ideas could be studied with the program which has been developed.

The plots shown in Fig. 19 illustrate structural behavior for cyclic thermal loading at four stations in a rectangular section of aluminum alloy 2024-T3 (see Appendix B). Thermal ratcheting is observed at all four stations. At the end stations (21) and (11), both tensile and compressive plastic flow are observed, whereas at neighboring stations, plastic flow in the opposite direction is not observed. An increase in stress and strain is observed for thermal ratcheting. It should be noted that the rate of increase in stress and strain reduces with an increase in the number of cycles as predicted in Appendix A.

Further, Fig. 20 illustrates the stress distribution along the Z axis of the section under consideration at several temperature conditions as described in the figure. Figure 21 has been reproduced from reference 3 to compare the pattern of the stress distribution while passing from one temperature to another. It can be seen that the stress distribution shown in Fig. 20 has the same tendency as that exhibited in Fig. 21.

The average stress-strain curve which the material
(aluminum 2024-T3) is supposed to follow has been shown by the path OAB in Fig. 22. The curve OYC is the path plotted during the computation procedure. It can be seen that the path OYC does not coincide with the path OAB and hence a cumulative error is observed. Possible reasons for this error are:

(1) The size of the load increments in the region of the yield point may be important. Eq. (17) is exact for linear strain hardening. Convergence takes place on a straight line of slope \( \frac{d\sigma}{d\varepsilon_{p}} \). A large increment, therefore, will cause a large error (see article IX).

(2) The number of terms retained in the Taylor series used in the development of Eq. (17) can also have an effect.
XIII. APPENDICES
APPENDIX A

Thermal Ratcheting of a Two Bar Model
Under Steady External Load

The model shown in Fig. 23 is a two bar assembly having areas A and B, respectively. The ends of the bars are attached to rigid plates, P₁ and P₂. Bar B is assumed to be heated and cooled in succession. When heated, the bar tends to expand, but the expansion is partially restrained by bar A through attachments P₁ and P₂. These plates are rigid enough so that the net lengths of both bars are always equal (see page 184 of ref. 2).

In addition to thermal loading resulting from nonuniform heating, there is present the external load P, which must be supported by the bars A and B acting jointly. The fraction of the load carried by each bar shifts, however, during heating and cooling, as a result of plastic flow.

Loading Cycle: Initially, it is considered that the model is at uniform temperature and the load P is uniformly distributed over the entire area A and B. Assume that the temperature of the outer bar is uniformly increased to a temperature T, while bar A is maintained at the reference temperature. Then the temperature of B again returns to the reference value.

This temperature cycle is assumed to be repeated while the external mechanical load P remains constant.
Analysis: For simplification of the analysis, the following assumptions are made.

(1) Plastic flow in the reverse direction does not occur.

(2) Strain hardening is linear.

When the model is at a uniform temperature initially, and the external load $P$ is applied, the average stress is

$$\sigma_m = \frac{P}{(A + B)} \quad (1-A)$$

When the temperature of the bar $B$ is uniformly increased to a temperature $T$ while the bar $A$ is maintained at the reference temperature, the attempted expansion of bar $B$ will transfer some of the load to $A$. If the expansion $\alpha T$ is small, the action may be completely in the elastic range, but if $\alpha T$ is large enough, the condition indicated by the points $A_{\frac{\pi}{2}}$ and $B_{\frac{\pi}{2}}$ may develop at the end of the first half cycle of heating (see Fig. 24). There is plastic flow in the bar $A$, while the bar $B$ has unloaded elastically to the point $B_{\frac{\pi}{2}}$. The strain at $A_{\frac{\pi}{2}}$ is greater than the strain at $B_{\frac{\pi}{2}}$ by an amount $\alpha T$. At the end of the half cycle, for equilibrium,

$$P = B\sigma_{B_{\frac{\pi}{2}}} + A\sigma_{A_{\frac{\pi}{2}}} \quad (2-A)$$
Therefore, with the help of Eq. (1-A), we obtain,

\[ \sigma_{B_{1/2}} = (1 + \frac{A}{B}) \sigma_m - \frac{A}{B} n_1 \sigma_y \]  \hspace{1cm} (3-A)

where

\[ n_1 = \frac{\sigma_{A_{1/2}}}{\sigma_y} \]

When the temperature of bar B is returned to the reference value, bar A, which has been stretched, unloads along the line \( A_{1/2}A_1 \) parallel to the elastic line and bar B assumes a greater portion of the load, straining along the line \( B_{1/2}NB_1 \). At the end of the first cycle, therefore, from the condition of equilibrium, we obtain

\[ \sigma_{A_1} = (1 + \frac{B}{A}) \sigma_m - \frac{B}{A} n_2 \sigma_y \]  \hspace{1cm} (4-A)

where

\[ n_2 = \frac{\sigma_{B_1}}{\sigma_y} \]

The overall length of the model is permanently increased by plastic strain \( \varepsilon_g \).

From Fig. 24,

\[ \varepsilon_g = JK \]

\[ = LM - LJ - KM \]
\[ \varepsilon_g = \alpha T + (2 + \frac{A}{B} + \frac{B}{A}) \frac{\sigma_m}{E} - (1 + \frac{A}{B}) \frac{n_1 \sigma_y}{E} - (1 + \frac{B}{A}) \frac{n_2 \sigma_y}{E} \]  

(6-A)

Eq. (6-A) can be rewritten as

\[ \varepsilon_g = \alpha T + K_1 \sigma_m - K_2 n_1 - K_3 n_2 \]  

(7-A)

where \( K_1, K_2 \) and \( K_3 \) are constants.

A relation can be established between \( n_1 \) and \( n_2 \).

Consider the triangle CIJK in Fig. 25. From this triangle we obtain

\[ HK = \frac{1}{\tan \theta} \left[ \sigma_{A_{1\alpha}} - \{ \sigma_m (1 + \frac{B}{A}) - \frac{B}{A} \sigma_{B_1} \} \right] \]  

(8-A)

Also from triangle CPJ we obtain

\[ HK = \cot \phi (\sigma_{A_{1\alpha}} - \sigma_{B_1}) \]  

(9-A)
Equating Eqs. (8-A) and (9-A) gives

\[(\sigma_{A_{1}} - \sigma_{B_1}) \tan \theta \cdot \cot \phi = \sigma_{A_{2}} - \sigma_{m} (1 + \frac{B}{A}) + \frac{B}{A} \sigma_{B_1}\]

\[(10-A)\]

Dividing both sides of Eq. (10-A) by $\sigma_{Y}$ and substituting $E$ and $E_{S}$ for $\tan \theta$ and $\tan \phi$, respectively, yields,

\[n_2 = \frac{\sigma_{m}}{\sigma_{Y}} \frac{(1 + B)}{A} + n_1 \frac{(E_{E} - 1)}{(E_{E} + B/A)}\]

\[(11-A)\]

Eq. (11-A) can be written as

\[n_2 = K_4 n_1 + K_5 \frac{\sigma_{m}}{\sigma_{Y}}\]

\[(12-A)\]

Substituting Eq. (12-A) into Eq. (7-A) yields,

\[\varepsilon_{g} = \alpha T + \sigma_{m} (K_1 - \frac{K_3 K_5}{\sigma_{Y}}) - n_1 (K_2 + K_3 K_4)\]

\[(13-A)\]

Also from Fig. 24 we obtain

\[\varepsilon_{B_{1}} = \frac{\sigma_{m}}{E} - \frac{\sigma_{m} - \sigma_{B_1}}{E}\]

\[(14-A)\]

\[\varepsilon_{A_{2}} = \frac{\sigma_{A_{2}} - \sigma_{Y}}{E} + \frac{\sigma_{Y}}{E}\]

\[(15-A)\]

\[\varepsilon_{A_{2}} = \varepsilon_{B_{2}} + \alpha T\]

\[(16-A)\]
Substituting for $\epsilon_{a_{\lambda}}$ and $\epsilon_{B_{\sigma}}$ into Eq. (16-A) and then solving for $n_1$ yields

$$n_1 = \frac{(1 + \frac{A}{B})\sigma_Y}{(1 + \frac{A}{BE})} \cdot \frac{\sigma_m}{E} + \frac{\sigma_Y}{(1 + \frac{A}{BE})} \alpha T$$

$$+ \frac{(\frac{1}{E_s} - \frac{1}{E})\sigma_Y^2}{(1 + \frac{A}{BE})}$$

(17-A)

Eq. (17-A) can be rewritten as

$$n_1 = K_6 \sigma_m + K_7 \alpha T + K_8$$

(18-A)

where $K_6$, $K_7$ and $K_8$ are constants.

Now substituting Eq. (18-A) into Eq. (17-A) yields

$$\epsilon_g = (1 - K_2 K_7 - K_3 K_4 K_7) \alpha T + (K_1 - \frac{K_3 K_5}{\sigma_Y} - K_6 K_2$$

$$- K_3 K_4 K_6) \sigma_m - (K_2 K_8 + K_3 K_4 K_8)$$

(19-A)

Rewriting Eq. (19-A) yields

$$\epsilon_g = K_9 \sigma_m + K_{10} \alpha T - K_{11}$$

(20-A)

where $K_9$, $K_{10}$ and $K_{11}$ are constants.
It should be noted that \( \varepsilon_g \) is the plastic strain growth which may occur at the end of the first cycle. Also, it can be seen that the factors to choose for design are \( T, \sigma_m, \) the ratio \( A/B \) and the plastic strain growth \( \varepsilon_g \).

Consider now the state of stress and strain at the end of 1\( \frac{1}{2} \) and 2 cycles (see Fig. 26). At the end of 1\( \frac{1}{2} \) cycles, the stress in bar B is

\[
\sigma_{B_{3/2}} = (1 + \frac{A}{B}) \sigma_m - \frac{\dot{A}}{B} \sigma_{A_{3/2}} \tag{21-A}
\]

Also, the strains in bars B and A are given by

\[
\varepsilon_{B_{3/2}} = \varepsilon_1 - \frac{\sigma_{B_1} - \sigma_{B_{3/2}}}{E} \tag{22-A}
\]

and

\[
\varepsilon_{A_{3/2}} = \frac{\sigma_y}{E} + \frac{\sigma_{A_{3/2}} - \sigma_y}{E_s}, \tag{23-A}
\]

respectively. And

\[
\varepsilon_{A_{3/2}} = \varepsilon_{B_{3/2}} + \alpha T \tag{24-A}
\]
Therefore,

$$\varepsilon_{A3/2} = \varepsilon - \frac{\sigma_{B1} - \sigma_{B3/2}}{E} + \alpha T \quad (25-A)$$

Substituting Eqs. (21-A) and (23-A) into Eq. (25-A) yields

$$\sigma_{A3/2} \left( \frac{1}{E_s} + \frac{A}{B} \right) = \left( \frac{1}{E_s} - \frac{1}{E} \right) \sigma_{B1} + (1 + \frac{A}{B}) \sigma_m + \alpha T \quad (26-A)$$

Further, at the end of the second cycle, the stress and strain are given by

$$\sigma_{A2} = \sigma_m (1 + \frac{B}{A}) - \frac{B}{A} \cdot \sigma_{B2} \quad (27-A)$$

and

$$\varepsilon_2 = \frac{\sigma_{A3/2} - \sigma_{A2}}{E} \quad (28-A)$$

Also,

$$\varepsilon_2 = \frac{\sigma_y}{E} + \left( \frac{\sigma_{B2} - \sigma_y}{E_s} \right) \quad (29-A)$$

Equating Eqs. (28-A) and (29-A) and then substituting for $\sigma_{A2}$ from Eq. (27-A) yields
\[
\sigma_{B_2} = \left\{ \frac{1 + \frac{B}{A}}{\frac{B}{A} - \frac{E}{E_s}} \right\} \sigma_m - \left\{ \frac{1}{\frac{B}{A} - \frac{E}{E_s}} \right\} \sigma_{A^{3/2}} \\
+ \left\{ \frac{\frac{B}{A} - \frac{E}{E_s}}{(\frac{B}{A} - \frac{E}{E_s})} \right\} \sigma_Y
\]

Substituting the value of \(\sigma_{A^{3/2}}\) obtained by Eq. (26-A) into Eq. (30-A) gives

\[
\sigma_{B_2} = \left\{ \frac{1 + \frac{B}{A}}{\frac{B}{A} - \frac{E}{E_s}} \right\} - \left\{ \frac{1 + \frac{A}{B}}{\frac{1}{E_s} + \frac{A}{B}} \left( \frac{B}{A} - \frac{E}{E_s} \right) \right\} \sigma_m \\
- \left\{ \frac{\frac{1}{E_s} + \frac{A}{B}}{(\frac{B}{A} - \frac{E}{E_s})} \right\} \alpha T - \left\{ \frac{\frac{1}{E_s}}{(\frac{B}{A} - \frac{E}{E_s})} \right\} \sigma_{B_1} \\
+ \left\{ \frac{\frac{B}{A} - \frac{E}{E_s}}{(\frac{B}{A} - \frac{E}{E_s})} \right\} \sigma_Y
\]

\[
= K_{12} \sigma_m - K_{13} \sigma_{B_1} - K_{14} \alpha T + K_{15}
\]

where \(K_{12}, K_{13}, K_{14}, \) and \(K_{15}\) are constants.

Assume now that a temporary decision has been made about mean stress \(\sigma_m\) and \(\alpha T\) (i.e., the external load and temperature range \(T\) have been selected). Consider, therefore, that \(\sigma_m\) and \(\alpha T\) are constants. Eq. (31-A) can
now be rewritten as

\[ \sigma_{B_2} = K_{16} - K_{13} \sigma_{B_1} \]  

(32-A)

From the repetitive nature of the loading and following the same arguments, Eq. (32-A) can be generalized as

\[ \sigma_{B_K} = \sum_{n=2}^{n=K} (-1)^{n-2} K_{13}^{n-2} K_{16} + (-1)^{K-1} K_{13}^{K-1} \sigma_{B_1} \]  

(33-A)

where \( \sigma_{B_K} \) is the stress in bar B at the end of the \( K_{th} \) cycle. Note that \( K > 1 \). The total strain at the end of the \( K_{th} \) cycle is obtained from

\[ \varepsilon_{KT} = \frac{\sigma_Y}{E} + \frac{\sigma_{B_K} - \sigma_Y}{E} \]  

(34-A)

The strain growth is the plastic component of the total strain. This plastic component is given by

\[ \varepsilon_{gK} = \sigma_Y \left( \frac{1}{E} - \frac{1}{E_s} \right) + \sigma_{B_K} \left( \frac{1}{E_s} - \frac{1}{E} \right) \]

\[ = K_{17} + K_{18} \sigma_{B_K} \]  

(35-A)

where \( K_{17} \) and \( K_{18} \) are constants.
Substituting Eq. (33-A) into Eq. (35-A) yields

\[ \varepsilon_{gK} = K_{17} + K_{18} \left[ \sum_{n=2}^{n=K} (-1)^{n-2}K_{13}^{n-2}K_{16}^{n-2} \right] \]

\[ + (-1)^{K-1}K_{13}^{K-1}g_{B_1} \]  \hspace{1cm} (36-A)

Also, from Eq. (12-A), solving for \( g_{B_1} \) gives

\[ g_{B_1} = K_{5}g_{m} + K_{4}\sigma_{A_2} \]  \hspace{1cm} (37-A)

Substituting Eq. (37-A) into Eq. (36-A) yields

\[ \varepsilon_{gK} = K_{17} + \sum_{n=2}^{n=K} (-1)^{n-2}K_{13}^{n-2}K_{19}^{n-2} \]

\[ + (-1)^{K-1}K_{13}^{K-1}K_{20}\sigma_{A_2}^{K-1} + (-1)^{K-1}K_{13}^{K-1}K_{21}^{K-1} \]  \hspace{1cm} (38-A)

where

\[ K_{19} = K_{16}K_{18} \]

\[ K_{20} = K_{4}K_{18} \]

and

\[ K_{21} = K_{5}K_{18}g_{m} \]
Expanding $K_{19}$, $K_{20}$ and $K_{21}$ yields

$$K_{19} = \sigma_m \frac{\left( \frac{1}{E_s} - \frac{1}{E} \right) \left( 1 + \frac{B}{A} \right) - \left( 1 + \frac{A}{B} \right)}{\left( \frac{B}{A} - \frac{E}{E_s} \right) \left( \frac{1}{E_s} + \frac{A}{B} \right)}$$

$$+ \left( \frac{1}{E_s} - \frac{1}{E} \right) \left( \frac{1 - \frac{E_s}{E}}{\left( \frac{B}{A} - \frac{E}{E_s} \right)} - \frac{\alpha T}{\left( \frac{1}{E_s} + \frac{A}{B} \right)} \right)$$

$$K_{20} = \left( \frac{E}{E_s} - 1 \right) \left( \frac{1}{E_s} - \frac{1}{E} \right) / \left( \frac{E}{E_s} + \frac{B}{A} \right)$$

and

$$K_{21} = \frac{\left( 1 + \frac{B}{A} \right) \left( \frac{1}{E_s} - \frac{1}{E} \right)}{\left( \frac{E}{E_s} + \frac{B}{A} \right) \sigma_m}$$

Expansion of $K_{13}$ gives

$$K_{13} = \frac{\left( \frac{1}{E_s} - \frac{1}{E} \right)}{\left( \frac{1}{E_s} + \frac{A}{B} \right) \left( \frac{B}{A} - \frac{E}{E_s} \right)}$$

The exponential nature of Eq. (38-A) indicates that the total growth of plastic strain increases as the number of cycles increases. However, the rate of growth of plastic strain decreases. After a certain number of cycles,
this rate may become negligible and an asymptotic stress condition may be achieved.

Substituting \( \sigma_m = 0 \) into Eq. (20-A) gives

\[
\varepsilon_g = K_{10} \alpha_T - K_{11}
\]  
(39-A)

When \( \varepsilon_g = 0 \), Eq. (39-A) yields

\[
\alpha_T = \frac{K_{11}}{K_{10}}
\]

\[
\frac{\sigma_y^3}{E} \left( \frac{1}{E_s} - \frac{1}{E} \right) (1 + \frac{A}{B}) + (1 + \frac{B}{A}) \left( \frac{E}{E_s} - 1 \right) \\
= \frac{1 - \frac{\sigma_y^2}{E} \left( \frac{1}{E_s} + \frac{A}{BE} \right) (1 + \frac{A}{B}) (1 + \frac{B}{A}) \left( \frac{E}{E_s} - 1 \right)}{\left( \frac{E}{E_s} + \frac{B}{A} \right)}
\]

\(40-A\)

Eq. (40-A) shows that when the free thermal expansion \( \alpha_T \) of bar B exceeds the value expressed by the right hand side of the equation, strain growth may be observed even in the absence of external load.
APPENDIX B

Statement of an Example Problem and the Program

1. **Statement of an example problem.** Study the structural behavior of a rectangular section for the following conditions:

(a) Bending moment 15000.0 lb. in.
(b) Axial load 0.0 lbs.
(c) Maximum temperature 375.0 °F
(d) Minimum temperature 75.0 °F
(e) Cross section
   (1) Height 4.0 in.
   (2) Width 0.23 in.

The structure undergoes cyclic thermal loading. The material of the structure is Aluminum alloy 2024-T3. The mechanical and thermo-physical properties have been listed in Table I.

In the computer program, the equations to calculate average properties cover the range of 75-700 F. The minimum temperature has been assumed to remain constant in this analysis.

The cross section has been divided into 21 stations as shown in Fig. 11.

Program Language: Fortran IV
Computer : IBM-360-50
Plotter : Calcomp-750
Start

Read $E, \sigma_{0.7}, \sigma_{0.85}$

Calculate average
$\varepsilon_T = \frac{\sigma}{E} + \left(\frac{\sigma}{B}\right)^n$

Calculate load thermal and/or mechanical

$\Delta \varepsilon_P = \Delta \varepsilon_P = \Delta \varepsilon_P = 0$

Evaluate integrals
Use Simpson's Rule

$\sum \varepsilon_P = \sum \varepsilon_P = \sum \varepsilon_P = 0$

Calculate
$\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}$

Compute
$\varepsilon'_{et}$

If not converged?
If converged

Calculate Stresses

Determine limit stress and strain

Write stresses and strains

Flow Chart
2. The program. The list of symbols used in the program:

A

Time, seconds.

ACRV(I)

Shape parameter of a stress-strain curve at 'I' th station.

ALPHA

Coefficient of thermal expansion, in./(in.) (F).

AM

Poisson's ratio.

AMT

Bending moment, lb/inch.

ANITA(I)

Ratio h/Z at 'I' th station.

B1, B2 & B3

Constants appearing in equations for

\[ \varepsilon_{xx}, \varepsilon_{yy}, \text{ and } \varepsilon_{zz}. \]

BCRV(I)

Constant B in equation \((\sigma/B)^n\) at 'I' th station.

BP

Axial load, lbs.

C

Specific heat, btu/(lb.) (F).

Cl and C2

Constants appearing in equations for

\[ \varepsilon_{xx}, \varepsilon_{yy} \text{ and } \varepsilon_{zz}. \]

CICLE

Temperature cycle.

CK

Thermal conductivity, btu/(hr.)(ft.) (F).

CONCON

Condition for convergence.

CRT EX

Modified strain in 'X' direction, in./in.

CRT EZ

Modified strain in 'Z' direction, in./in.

DISC(I)

Distance of an 'I' th station from one end of the section, in.
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>DIST(I)</td>
<td>Distance from the reference point in the section, in.</td>
</tr>
<tr>
<td>DLTEXP(I,J)</td>
<td>Dimensionless increment of plastic strain in 'X' direction at 'I'th station and 'J'th iteration.</td>
</tr>
<tr>
<td>DLTEZP(I,J)</td>
<td>Dimensionless increment of plastic strain in 'Z' direction at 'I'th station and 'J'th iteration.</td>
</tr>
<tr>
<td>DMTEXP(I,J)</td>
<td>Increment of plastic strain in 'X' direction at 'I'th station and 'J'th iteration in./in.</td>
</tr>
<tr>
<td>EC(I)</td>
<td>Modulus of elasticity at 'I'th station.</td>
</tr>
<tr>
<td>ECR</td>
<td>Modulus of elasticity at reference temperature, lb./in.$^2$.</td>
</tr>
<tr>
<td>EEP(I)</td>
<td>Equivalent plastic strain increment.</td>
</tr>
<tr>
<td>EQLIM(I)</td>
<td>Equivalent limit strain at 'I'th station.</td>
</tr>
<tr>
<td>EQSGMA(I)</td>
<td>Equivalent limit stress at 'I'th station.</td>
</tr>
<tr>
<td>EXM(I)</td>
<td>Mechanical strain in 'X' direction, in./in.</td>
</tr>
<tr>
<td>EZM(I)</td>
<td>Mechanical strain in 'Z' direction, in./in.</td>
</tr>
<tr>
<td>PFLOW1(M,I)</td>
<td>First plastic flow during 'M'th cycle.</td>
</tr>
<tr>
<td>PFLOW2(M,I)</td>
<td>Plastic flow in the direction opposite to that of the first plastic flow at 'I'th station during 'M'th cycle.</td>
</tr>
<tr>
<td>PLSTRN(I)</td>
<td>Plastic strain at 'I'th station.</td>
</tr>
<tr>
<td>PREFLO(I)</td>
<td>Previous plastic flow at 'I'th station.</td>
</tr>
<tr>
<td>PRESTS(I)</td>
<td>Initial yield stress, psi.</td>
</tr>
<tr>
<td><strong>SECFL(I)</strong></td>
<td>Plastic flow having the direction opposite to that of the very first plastic flow.</td>
</tr>
<tr>
<td>---------------</td>
<td>---------------------------------------------------------------------------------</td>
</tr>
<tr>
<td><strong>SECLIM(I)</strong></td>
<td>Limit Stress having the direction opposite to that of the stress which causes very first plastic flow.</td>
</tr>
<tr>
<td><strong>SDTEXP(I,2)</strong></td>
<td>Summation of plastic strain increment in 'X' direction (dimensionless).</td>
</tr>
<tr>
<td><strong>SUMEXP(I,2)</strong></td>
<td>Summation of plastic strain increment in 'X' direction, in./in.</td>
</tr>
<tr>
<td><strong>SIGMAR</strong></td>
<td>Yield stress at the reference temperature.</td>
</tr>
<tr>
<td><strong>SLOPE(I)</strong></td>
<td>Slope of stress-plastic strain curve at limit stress.</td>
</tr>
<tr>
<td><strong>TREF</strong></td>
<td>Reference temperature.</td>
</tr>
<tr>
<td><strong>TLOW</strong></td>
<td>Lower temperature.</td>
</tr>
<tr>
<td><strong>TMAX</strong></td>
<td>Maximum temperature.</td>
</tr>
<tr>
<td><strong>THKW</strong></td>
<td>Width of section.</td>
</tr>
<tr>
<td><strong>TM</strong></td>
<td>Time, hr.</td>
</tr>
<tr>
<td><strong>TNCRIS</strong></td>
<td>Increase in temperature.</td>
</tr>
<tr>
<td><strong>TDCRIS</strong></td>
<td>Decrease in temperature.</td>
</tr>
</tbody>
</table>
503 IF (TMX C(I) - 500.0) 504, 504, 505
504 EC(I) = (236550000.0 + 10585000.0*(TMX C(I) - 300.0) - 4750.0*
1/(TMX C(I)**2.0 - 90000.0))/(TMX C(I) - 75.0)
505 FOS(I) = (1860000.0 + 52050.0*(TMX C(I) - 300.0) - 27.25*
1/(TMX C(I)**2.0 - 90000.0))/(TMX C(I) - 75.0)
506 GO TO 507
507 STRESS(I) = 28000.0
508
509 EC(I) = (373350000.0 + 13400000.0*(TMX C(I) - 500.0) - 5000.0*0*
1/(TMX C(I)**2.0 - 250000.0))/(TMX C(I) - 75.0)
510 FOS(I) = (1466000.0 + 71300.0*(TMX C(I) - 500.0) - 46.5*
1/(TMX C(I)**2.0 - 250000.0))/(TMX C(I) - 75.0)
511 FOET(I) = (1226000.0 + 66050.0*(TMX C(I) - 500.0) - 43.25*
1/(TMX C(I)**2.0 - 250000.0))/(TMX C(I) - 75.0)
512 STRESS(I) = 22000.0
513 CCRV(I) = (FOS(I)/FOET(I))
514 ACRV(I) = 1.0*(((ALOG(I)**2.4285))/(ALOG(CCRV(I))))
515 ACRV(I) = (FOS(I)**(ACRV(I) - 1.0)/ACRV(I))**((EC(I)**(1.0/ACRV(I))))
516 STRES(I) = STRES(I)
517 PLSTRN(I) = (STRES(I)/ACRV(I))**ACRV(I)
518 IF (PLSTRN(I) - 0.00010) 512, 512, 513
519 STRES(I) = STRES(I) + 350.0
520 GO TO 514
521 IF (PLSTRN(I) - 0.00011) 515, 515, 516
522 STRES(I) = STRES(I) - 200.0
523 GO TO 514
524 ELLIM = (STRES(I)/EC(I))
525 ELLIM = (((2.0*(I.0 + AM))/3.0)*ELLIM
526 CONTINUE
527 WRITE(3,520)EC(I),FOS(I),FOET(I),STRES(I),PLSTRN(I)
528 DD 525 1=2,21
529 EC(I) = EC(I)
530 FOS(I) = FOS(I)
0059 F0ET(I)=F0ET(I)
0060 STRES(I)=STRES(I)
0061 FOLTM(I)=FOLTM(I)
0062 PLSTRN(I)=PLSTRN(I)
0063 ACRV(I)=ACRV(I)
0064 BCRV(I)=BCRV(I)
0065 CONTINUE
0066 C *** IN THIS PARTICULAR EXAMPLE, TMAXC(I) HAS BEEN ASSUMED TO BE
0067 C CONSTANT AT ALL POINTS. ALL PROPERTIES, THEREFORE, WILL BE SAME
0068 C AT ALL POINTS. DO LOOPS DEFINED BY STATEMENT NUMBERS 525 AND 501
0069 C HAVE BEEN INTRODUCED TO ELIMINATE REPEATITION OF COMPUTATIONS. ***
0070 SIGMAR=27500.0
0071 FCR=10300000.0
0072 STRNYP=SIGMAR/FCR
0073 ALPHA=0.000014
0074 P=172.8
0075 C=0.2375
0076 CK=75.0
0077 PK=(CK)/(10*C)
0078 TL=0.33333
0079 TKW=0.21
0080 HTCW=3.0
0081 155 DO 163 I=1,21
0082 SDTEXP(I,1)=0.0
0083 SDTEZP(I,1)=0.0
0084 PFLOW(I,1)=0.0
0085 FLOWNT(I)=0.0
0086 FIRSTF(I)=0.0
0087 CONST2(I)=1.0
0088 PREFLO(I)=0.0
0089 SECFL(I)=0.0
0090 CONST(I)=1.0
0091 PRESST(I)=STRES(I)
0092 EOSGM(I)=PRESST(I)
0093 EP(I)=PLSTRN(I)
0094 ESTIM(I)=PRESST(I)
0095 SECLM(I)=PRESST(I)
0096 163 SETSTS(I)=PRESST(I)
0097 STRESS(I)=STRES(I)+STRSIN-4000.0
0098 N=1
0099 2502 STRESS(I)=PRESST(I)+STRES(I)-STRESI-37500.0,12500.2500,2501
2500  N=N+1
2501  STRAIN=STRAIN+400.,0
   GO TO 2502
2502  CALL NEWPT(9.,0.,5.,0.,18.,0.)
   CALL ORIGIN(0.,0.,0.)
   CALL XSCALE(-0.0150,0.,0.,0.,0.015,15.,0.)
   CALL YSCALE(-50000.,0.,50000.,0.,0.,8.,0.)
   CALL XAXIS(5000.,0.)
   CALL YAXIS(5000.,0.)
   CALL XYPLT(STRAIN,STRESS,N,1.,-1)
   CALL ENDPRT
   NC=4
   KZ=4
   N=1
   DO 484  M=2,NC
   DO 2400  I=1,21
   FLOW1(I)=0.0
   FLOW2(I)=0.0
   PFLOW1(M,I)=0.0
   PFLOW2(M,I)=0.0
   2400  CICLE=0.25
   DO 550  I=1,21
   550  T(I)=TLOW
   GO TO 622
   551  T(I+1)=T(I+1)+TMCRT
   T(I+2)=T(I+2)
   560  CICLE=0.50
   A=0.2
   614  DO 11  J=1,11
   DO 120  K=1,4
   120  TF=TMAX
   121  T(J)=TLOW
   TM(J)=(A/3600.,0)
   123  R=(3.1428*(H(J)*((CL(J))/((T(I))
   124  AN=(4.0*(T(J)))/((H(J))*(3.1428)
   125  EXPT=((11./E*(H(J)**2*(3.1428)**2*(PK)*((TM(J)))))/((T(I)**2)))
   126  X(J)=R*AN*EXPT
   127  SUM=X(J)+X(21)*X(31)*X(4)
   128  TF(T(J)=TLOW)
   622  CONTINUE
   DO 178  I=12,21

82
174 T(J)=T(J-10)
145 GO TO 622
146 551 IF(A=2) B611,612,612
147 611 A=A+0.25
148 GO TO 614
149 612 IF(A=2) B617,618,618
150 617 A=A+1.0
151 GO TO 614
152 618 IF(A=4) B619,620,620
153 619 A=A+2.0
154 GO TO 614
155 620 IF(A=7) B2619,2620,621
156 2619 A=A+5.0
157 GO TO 614
158 2620 A=100.0
159 DD 31 J=1,21
160 T(J)=TF
161 31 CONTINUE
162 GO TO 622
163 621 CIRCLE=0.75
164 T(I)=T(I)-TOCRIS
165 T(2)=T(I)
166 IF(T(I)-TOLOW) 561,622,622
167 561 CIRCLE=1.0
168 A=0.2
169 DD 461 J=1,11
170 DD 401 K=1,4
171 TO(J)=TMAX
172 TF=TOLOW
173 TM(J)=(A/3600.0)
174 B=(3,1428*W(K)*(GL(J)))/(TL)
175 AN=(4.0*(TO(J)-TF))/W(K)*3,1428
176 EXP1=1/(1/(EXP(((H(K))**2+(3,1428)**2+(PK)**2)*T(M(J))))/(TL)**2))
177 401 X(K)=A*AN*EXP1
178 SUM=X(1)+X(2)+X(3)+X(4)
179 T(J)=SUM+TF
180 TF(T(J)-TOLOW) 402,403,403
181 402 T(J)=TO(J)
182 GO TO 461
183 403 T(J)=TO(J)
184 GO TO 461
185 461 CONTINUE
186 DD 405 J=12,21
405  T(J) = T(J-10)
406  GO TO 622
408  556  IF(A-2,2)429,430,430
409  429  A=A+0.25
410  GO TO 400
411  430  IF(A-24,2)431,432,432
412  431  A=A+1.0
413  GO TO 400
414  432  IF(A-44,2)433,434,434
415  433  A=A+2.0
416  GO TO 400
417  434  IF(A-74,2)2433,2434,436
418  2433  A=A+5.0
419  GO TO 400
420  2434  A=100.0
421  DO 435  J=1,21
422  435  TF(J)=TF
423  GO TO 622
424  436  WRITE(3,457)
425  GO TO 483
426  C  DOUBLE INTEGRATION BY SIMSON'S RULE IS REQUIRED FOR TWO
427  C  DIMENSIONAL PROBLEMS.
428  622  DO 179  J=1,21
429  179  CONTINUE
430  179  STARTH=AMT/(SIGMAR*HTCW*2.0*THKW)
431  180  STARPH=BP/(SIGMAR*HTCW*THKW)
432  1001  DO 175  I=1,21
433  175  CONTINUE
434  176  J=1,2
435  177  DMTEXP(T,J)=0.0
436  178  DLTEXP(T,J)=0.0
437  176  DMTEXP(I,J)=0.0
438  178  179  CONTINUE
440  179  32  DO 16  J=2,50
441  16  1144  DO 1 T=1,21
442  1144  WA(I)=EC(I)/ECR
443  1144  1  EN-1)=F(1)*ANITA(I)
444  1144  1  FNPRL=0.047320*F(21)+F(11)
445  1144  1+4.0*F(10)+F(10)+F(9)+F(16)+F(6)+F(13)+F(12)+F(12)+F(2)
445  1144  1+4.0*F(119)+F(119)+F(119)+F(119)+F(119)+F(119)+F(119)
DO 2 I=1,21
ATNGRL(I)=FINGRL
BINGRL(I)=TNGRL
C1(J)=B1(I)*STARMW+B1(I)*BINGRL(I)-B2(I)*STARPW-B2(I)*ATNGRL(I)
C2(J)=B3(I)*BINGRL(I)-B2(I)*BINGRL(I)+B3(I)*STARPW-B2(I)*STARMW
C3(J)=C1(J)*ANITA(I)+C2(J)
E1(I)=AM*(C1(J)+ANITA(I)+C2(J))+1.0*AM)*TOW(I)+(SDTEQP(I,2)+DLTEQP(I,1))*AM*(SDTEXP(I,2)+DLTEXP(I,1))
E2(I)=AM*(C1(J)+ANITA(I)+C2(J))+1.0*AM)*TOW(I)+1.0*(SDTEXP(I,2)+DLTEXP(I,1))-(SDTEQP(I,2)+DLTEQP(I,1))
EY(I)=EY(I)+STRNYR
fy(I)=fy(I)+STRNYR
fz(I)=fz(I)+STRNYR
fx(I)=fx(I)+STRNYR
CRIEX=FX(I)-SDTEXP(I,2)*STRNYR
CRIEY=FY(I)+SDTEXP(I,2)*STRNYR
CRIEZ=FZ(I)-SDTEXP(I,2)*STRNYR
CRTEY=FY(I)+SDTEXP(I,1)*STRNYR
CRTEX=FX(I)-SDTEXP(I,1)*STRNYR
CRTEZ=FZ(I)-SDTEXP(I,1)*STRNYR
EET(I)=((2.0**2.0+DIFYX**2.0+DIFYZ**2.0+DIFZX**2.0)**0.5)
EET(I)=ABS(EET(I))-EOLIM(I)165,165,144
K=J+1
DLTEXP(I,K)=0.0
DLTEQP(I,K)=0.0
GO TO 2
IF(J-2)1440,1440,147
1440 IF(EOSGMA(I)-STRES(I))146,146,147
146 EOSGMA(I)=STRES(I)
146 EOSGMA(I)=STRES(I)
E1(I)=(STRES(I)/ACRV(I)**A)**ACRV(I)
SLOPE(I)=(ACRV(I)**1.0/ACRV(I))/((ABS(E1(I))**((ACRV(I)-1.0)**ACRV(I))))
RAM=0.6667**((1.0*AM)/E1(I))
147 EET(I)=EET(I)-RAM*(EOSGMA(I)))/(1.0+RAM*SLOPE(I))
140.K=K+1
DLTEQP(I,K)=0.3333*(ABS(EET(I)/EET(I)))*(2.0*CRTEZ-CRTEY-CRTEX)
DLTEQP(I,K)=0.3333*(ABS(EET(I)/EET(I)))*(2.0*CRTEZ-CRTEY-CRTEX)
1/STRNYR
2 CONTINUE
522 DO 3 I=1,21
524 IFABS(DLTEXP(I,K))-0.0141,142,141
525 GO TO 3
526 DMTEXP(I,K)=DLTEXP(I,K)*STRNYR
527 CONCON=ABS(DMTEXP(I,K))-DMTEXP(I,J)
528 IFABS(CONCON)-0.00001),3,3,24
529 IFABS(CONCON)-0.0011,242,242,243
530 IFCONCON-0.00001),3,3,24
531 IFCONCON-0.001013,3,3,24
3 CONTINUE
533 DO 4 I=1,21
535 DICT(I)=G(I)*12.0
536 SIGMAX(I)=SIGMAR*H(A(I))*(C1(J)*ANITA(I)+C2(J)-TOW(I)-SDTEXP(I,2)
537 1-DLTEXP(I,K))
538 SDTEXP(I,2)=SDTEXP(I,2)+DLTEXP(I,K)
539 SDTEXP(I,2)=SDTEXP(I,2)+STRNYR
540 SUMEXP(I,2)=SUMEXP(I,2)+STRNYR
541 EP(I)=12.0/(3.0**0.5)*ABS((ABS(SUMEXP(I,2)))**2.0+
542 1*(ABS(SUMEXP(I,21))**2.0)*SUMEXP(I,21)*(SUMEXP(I,21))**0.5
4 CONTINUE
295 DO 2001 I=1,21
547 IFOSGMAT=ABS(SIGMAX(I))
549 2002 SETSST(I)=SIGMAX(I)
549 IF(SIGMAX(I))-0.012003,2003,2004
550 2003 IF(PFLOW1(M,I)=-1.0)2005,2005,2006
551 2005 PFLOW1(M,I)=1.0
552 PFLOW1(I)=1.0
553 IF(PFLOW1(I)=1.0)8001,8002,8001
554 8001 FLOWMAT(I)=FLOWMAT(I)+1
556 8002 PFLOW1=M,1)
557 GO TO 2200
558 2006 PFLOW2(M,I)=1.0
558 IF(PFLOW2(I)=1.0)8003,8004,8003
559 8003 FLOWMAT(I)=FLOWMAT(I)+1
560 8004 PFLOW2(M,I)
562 GO TO 2200
563 2004 IF(PFLOW4(M,I)=1.0)1.02007,2008,2007
564 2007 PFLOW4(M,I)=2.0
565
```plaintext
0304  PREFLOW(I)=2.0
0305  IF(FLOW1(I)=2)8005,8005,8006
0306  8005 FLOW(I)=FLOW(I)+1
0307  8006 FLOW1(I)=PFLOW1(M,I)
0308  GO TO 2200
0309  2008 PFLOW2(M,I)=2.0
0310  IF(FFLOW2(I)=2)8007,8007,8008
0311  8007 FLOW(I)=FLOW(I)+1
0312  8008 FLOW2(I)=PFLOW2(M,I)
0313  2200 IF(FIRST(I)-1.0)2201,2099,2099
0314  2201 IF(PFLOW1(M,I)-1.0)2203,2204,2205
0315  2203 FIRST(I)=0.0
0316  GO TO 2099
0317  2204 FIRST(I)=1.0
0318  SECFL(I)=2.0
0319  GO TO 2099
0320  2205 FIRST(I)=2.0
0321  SECFL(I)=2.0
0322  2099 IF(FIRST(I)-1.0)2001,2209,2210
0323  2209 IF(SIGMA(I)=0.)2401,2401,2212
0324  2212 SECLIM(I)=SIGMA(I)
0325  GO TO 2001
0326  2401 FSTLIM(I)=SIGMA(I)
0327  GO TO 2001
0328  2210 IF(SIGMA(I)-0.0)2214,2214,2402
0329  2402 FSTLIM(I)=SIGMA(I)
0330  GO TO 2001
0331  2214 SECLIM(I)=SIGMA(I)
0332  2001 CONTINUE
0333  WRITE(3,511)SIGMA(21),SETSTS(21),FSTLIM(21),SECLIM(21),STRES(21)
0334  1,CONST(21),CONST(21)
0335  WRITE(3,511)SIGMA(21),SETSTS(21),FSTLIM(21),SECLIM(21),STRES(21)
0336  1,CONST(21),CONST(21)
0337  511 FORMAT(17,5X,F10.2,5X,F10.2,5X,F10.2,5X,F10.2,5X,F10.2,5X,F5.2,5X,
0338  1F10.4)
0339  DO 2010 I=1,21
0340  2010 IF(SETSTS(I)-0.0)2011,2011,2012
0341  2011 IF(SIGMA(I)-0.0)2013,2013,2014
0342  2014 IF(PFLOW2(M,I)-1.0)2178,2278,2279
0343  2178 IF(SECLIM(I)-1.0)2178,2178,2041
0344  2278 IF(SECLIM(I)-1.0)2042,2042,2179
0345  2279 IF(SECLIM(I)-1.0)2142,2142,2142
```
CALL YAXIS(5000,0)
CALL XPLOT(FM10,XSIG10,N-1,1,-1)
CALL ENDPLT
CALL NEWPLT(9.0,5.0,18.0)
CALL ORIGIN(0.0,0.0)
CALL YSCALE(-0.0150,0.015,15.0)
CALL YSCALE(-50000.0,50000.0,8.0)
CALL XAXIS(0.0025)
CALL YAXIS(5000.0)
CALL XPLOT(FM20,XSIG20,N-1,1,-1)
CALL ENDPLT
CALL NEWPLT(9.0,5.0,18.0)
CALL ORIGIN(0.0,0.0)
CALL YSCALE(-0.0150,0.015,15.0)
CALL YSCALE(-50000.0,50000.0,8.0)
CALL XAXIS(0.0025)
CALL YAXIS(5000.0)
CALL XPLOT(FM21,XSIG21,N-1,1,-1)
CALL ENDPLT
2060 DO 2075 T=1,21
2075 CONSTR(T)=CONSTR(T)
2081 CONTINUE
C NEW LOADING CYCLE IS TO BE RESUMED.
IN=IN-1
WRITE(3,445)IN
445 IF(M-K)+484,2046,484
2046 M=1
KZ=KZ+3
484 CONTINUE
GO TO 231
489 CALL ENDPLT
25 WRITE(3,230)
201 FORMAT(7F10.6/7F10.6/7F10.6)
202 FORMAT(5F10.5)
203 FORMAT(7F10.2/7F10.2/7F10.2/7F10.2/7F10.2/7F10.2/7F10.2)
204 FORMAT(1X,'TOTAL FX',7X,'TOTAL FY',7X,'TOTAL EZ',8X,'SIGMAX',5X,
1 'SIMPLY',/)
226 FORMAT(4X,'FRX',4X,'FRY',4X,'FRZ',4X,'F12.8',4X,'F12.8',
14X,'F12.8',4X,'F12.8',4X,'F12.8',4X,'F12.8',
570 FORMAT(6X,'F10.2/6X','F10.2/6X','F10.2/6X','F10.2/6X','F10.2/6X','F10.2')
521 FORMAT(6X,'F12.7',6X,'F12.7','F12.7',6X,'F12.7')
624 FORMAT(15X,15X,15X,15X)**** COOLING BEGINS *****/)
457 FORMAT(15X,15X,15X)**** CYCLE ENDS *****/)
470 FORMAT(15X,15X,15X,15X)**** DOES NOT CONVERGE IN 40 ITERATIONS *****/)
471 FORMAT(15X,15X,15X,15X,15X)NO. OF INCREMENTS: 1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)
472 CALL ISTOP
474 END
### TABLE I

**LIST OF PROPERTIES OF ALUMINUM 2024-T3**

<table>
<thead>
<tr>
<th>Temp. °F</th>
<th>Time Hr.</th>
<th>E psi $\times 10^6$</th>
<th>$\sigma_{0.7}$ KSI</th>
<th>$\sigma_{0.85}$ KSI</th>
<th>n</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Room</td>
<td>2</td>
<td>10.7</td>
<td>39.0</td>
<td>36.0</td>
<td>11.5</td>
<td>0.3125</td>
</tr>
<tr>
<td>300</td>
<td>2</td>
<td>10.3</td>
<td>35.7</td>
<td>33.5</td>
<td>15.0</td>
<td>-</td>
</tr>
<tr>
<td>500</td>
<td>2</td>
<td>8.4</td>
<td>24.8</td>
<td>22.8</td>
<td>10.9</td>
<td>-</td>
</tr>
<tr>
<td>700</td>
<td>2</td>
<td>6.4</td>
<td>6.2</td>
<td>5.5</td>
<td>8.2</td>
<td>-</td>
</tr>
</tbody>
</table>

Material: Aluminum Alloy 2024-T3; sheet and plate, heat treated, thickness ≤ 0.25 inch.

\[ \rho = 0.10 \text{ lb./in.}^3 \]

\[ C = 0.23 \text{ BTU/(lb.)(F)} \]

\[ K = 75.0 \text{ BTU/[(hr)(Ft)(F)]} \]

\[ \alpha = 0.14 \text{ in./in.}(F) \]
XIV. FIGURES
Figure 1. Conventional stress-strain curve.

Figure 2. Stress-strain curve described by three parameters.
Figure 3. Behavior of materials under uniaxial stress.
Material: Aluminum alloy 2024-T3

Sheets and plates ≤ .25" (ref.9)

Figure 4. Variation of mechanical properties with changes in temperature.
Figure 5. Relation between $\varepsilon_{et}$, $\sigma_e$, and $\Delta \varepsilon_{ep}$.

Figure 6. Beam with rectangular section.
Figure 7. $\varepsilon_{et}$ versus $\varepsilon_{ep}$ curve.
Figure 8. Flow diagram for a rapidly convergent successive approximation scheme.
(a) Effect of size of load increment

(b) Limit stress $\sigma_{A,i-1}$ for the next load increment

Figure 9. Determination of magnitude of plastic strain increment $\Delta \varepsilon_{ep}$. 
Figure 10. Model for one dimensional transient temperature distribution (faces and ends are insulated).

\[ T = T(f) \]

\[ \eta_z = \frac{z}{h} \]

Figure 11. Model of the rectangular section with 21 stations.
— path which may, presumably be traveled if the Bauschinger effect is considered

Figure 12. Effect of isotropic strain hardening.
Figure 13. Limit stress for the next loading cycle;

\[ \sigma_L = B \quad \frac{1}{n} \quad \varepsilon_{p3} \]

\( \sigma_L \) = Limit stress
Figure 14. The simplified stress-strain curve for cyclic loading.

\[
\frac{\sigma_{1,t}}{\sigma_{1,c}} = K = \text{constant}
\]

\[
\frac{\sigma_{2,t}}{\sigma_{2,c}} = K
\]

\[
\sigma_B = B \varepsilon_p^{1/n}
\]
Figure 15. The size of the load increment and possible limit stresses.
Figure 16. Importance of direction of plastic flow.
Figure 17. Determination of a limit stress for the next load increment.
Figure 18. Preliminary flow diagram to determine a limit stress for the next increment of load.
Figure 18

* plastic flow in reversed direction
† First plastic flow
‡ Only to explain the basic idea. Division or multiplication by $K_f$ has not been included. Also, effect of change in the direction of the first plastic flow has not been included. See Article XI for further details.
\[ \sigma_{L,i} = \text{const} \]

\[ \sigma_{act,i} = \sigma_{L,i} + \frac{1}{n} B \cdot \varepsilon_{ep} \]

\[ \sigma_{L,i+1} = \sigma_{act,i} \]

\[ \sigma_{L,i+1} = \frac{1}{n} \sigma_{L,i} \]

\[ \sigma_{PFLOW1 \text{ tensile}} = \sigma_{L,i+1} = \sigma_{L,i} \cdot \text{const} \]

\[ \sigma_{PFLOW1 \text{ compressive}} = \sigma_{L,i+1} = \sigma_{L,i} / \text{const} \]

Figure 18
Figure 19. Structural behavior at different stations of the section due to cyclic thermal load.

Total cycles: 3
Figure 19

(a) Structural behavior at station (11).
Figure 19

(b) Structural behavior at station (10).
Figure 19

(c) Structural behavior at station (20).
Figure 19

(d) Structural behavior at station (21).
*Numbers in square brackets indicate the stages of the cyclic thermal loading described in Fig. 21.

Figure 20. Stress distribution in a rectangular section due to cyclic thermal loading.
Figure 21. Stresses in an I section due to cyclic thermal load.

\[\sigma_0 = \sigma_y \frac{(A_S - A_W)}{2} \quad \frac{1}{\left(A_S + A_W\right)} \]

\[\sigma' = \sigma_y \frac{(A_S - A_W)}{A_S}\]

\(A_S\) : Area of the flange of an I section

\(A_W\) : Area of the web of an I section

\(\sigma_y\) : Yield stress
[1] skin o, web o
[2] skin T, web o
[3] skin T, web T
[4] skin o, web T
[5] skin o, web o

Figure 21
Figure 22. Error in the computation procedure.
Figure 23. Two bar model.

Figure 24. State of stress and strain in bars A and B at the end of the first cycle.
Figure 25. Relation between stresses $\sigma_{A_2}$ and $\sigma_{B_1}$.

Figure 26. Growth of plastic strain in a two bar model.
XV . BIBLIOGRAPHY


(11) Lehnoff, Terry F., ME451, Thermal Stresses I, Class Notes, University of Missouri-Rolla, 1969.


XVI. VITA

Hazariwala Rameshchandra Chandulal was born on May 20, 1942, at Mandvi, District Surat, Gujarat State, India.

He graduated from Tapidas and Tulsidas Verajdas Sarvajanik High School, Surat, India in 1959. He received the B.S. degree in Mechanical Engineering, with distinction, from the Gujarat University, Ahmedabad, India, in 1964.

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