1970

Comparison of the Buckley-Leverett technique with a numerical simulation model containing heterogeneity and capillary pressure

Krishna Kumar Singh

Follow this and additional works at: http://scholarsmine.mst.edu/masters_theses

Department: Geosciences and Geological and Petroleum Engineering

Recommended Citation


This Thesis - Open Access is brought to you for free and open access by the Student Research & Creative Works at Scholars' Mine. It has been accepted for inclusion in Masters Theses by an authorized administrator of Scholars' Mine. For more information, please contact weaverjr@mst.edu.
COMPARISON OF THE BUCKLEY-LEVERETT TECHNIQUE
WITH A NUMERICAL SIMULATION MODEL CONTAINING HETEROGENEITY
AND CAPILLARY PRESSURE

BY
KRISHNA KUMAR SINGH, 1944 -

A
THESIS
submitted to the faculty of
THE UNIVERSITY OF MISSOURI-ROLLA

in partial fulfillment of the requirements for the
Degree of

MASTER OF SCIENCE IN PETROLEUM ENGINEERING
Rolla, Missouri
1970

Approved by
M.D. Arnold
(Advisor)
Abstract

A numerical model for a linear, two-phase fluid flow system has been developed with the capabilities of accounting for heterogeneity and capillarity. The model developed is general enough to analyze either water-oil or gas-oil systems. A hypothetical petroleum reservoir (containing only water and oil) under water flood was studied. The results for the homogeneous, no capillary pressure case were found to be similar to those obtained by the Buckley-Leverett technique. The study shows that the Buckley-Leverett is accurate even for a certain degree of heterogeneity. Also, capillarity does not substantially alter the results for the system studied. The results obtained using the mathematical model are considered more realistic where variations in a number of parameters or systems of variable geometry are involved.
ACKNOWLEDGEMENT

The author wishes to express his sincerest thanks and appreciation to his advisor, Dr. M. D. Arnold for his guidance and valuable suggestions throughout the development of this work. It is his pleasure also to thank Prof. J. P. Govier for his timely help in all the matters that came up during this work. His thanks are due also to Dr. R. E. Carlile.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>ii</td>
</tr>
<tr>
<td>ACKNOWLEDGEMENT</td>
<td>iii</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>v</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>vi</td>
</tr>
<tr>
<td>I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II. REVIEW OF LITERATURE</td>
<td>4</td>
</tr>
<tr>
<td>III. THEORY OF RESERVOIR SIMULATION</td>
<td>8</td>
</tr>
<tr>
<td>A. GENERAL</td>
<td>8</td>
</tr>
<tr>
<td>B. SOURCES OF ERRORS IN COMPUTED RESULTS</td>
<td>10</td>
</tr>
<tr>
<td>C. NUMERICAL TECHNIQUES TO SOLVE PARTIAL DIFFERENTIAL EQUATIONS</td>
<td>10</td>
</tr>
<tr>
<td>D. MATHEMATICAL DEVELOPMENT</td>
<td>11</td>
</tr>
<tr>
<td>E. MATERIAL BALANCE</td>
<td>28</td>
</tr>
<tr>
<td>IV. BUCKLEY-LEVERETT TECHNIQUE</td>
<td>33</td>
</tr>
<tr>
<td>V. RESULTS AND DISCUSSION</td>
<td>39</td>
</tr>
<tr>
<td>VI. CONCLUSIONS</td>
<td>50</td>
</tr>
<tr>
<td>VII. APPENDICES</td>
<td>52</td>
</tr>
<tr>
<td>A. NOMENCLATURE</td>
<td>53</td>
</tr>
<tr>
<td>B. DERIVATION OF MISCELLANEOUS RELATIONSHIPS</td>
<td>56</td>
</tr>
<tr>
<td>C. RESERVOIR DATA</td>
<td>57</td>
</tr>
<tr>
<td>D. COMPUTER FLOW DIAGRAMS</td>
<td>61</td>
</tr>
<tr>
<td>VIII. REFERENCES</td>
<td>68</td>
</tr>
<tr>
<td>IX. VITA</td>
<td>70</td>
</tr>
</tbody>
</table>
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure No.</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>A Reservoir Segmented into Blocks</td>
<td>12</td>
</tr>
<tr>
<td>2.</td>
<td>A Typical Elemental Block</td>
<td>12</td>
</tr>
<tr>
<td>3.</td>
<td>Central Grid Point Illustration</td>
<td>16</td>
</tr>
<tr>
<td>4.</td>
<td>Boundary Elements</td>
<td>25</td>
</tr>
<tr>
<td>5.</td>
<td>Grid System for Material Balance</td>
<td>29</td>
</tr>
<tr>
<td>6.</td>
<td>Saturation Distribution Curves</td>
<td>43</td>
</tr>
<tr>
<td>7.</td>
<td>Saturation Distribution Curves</td>
<td>44</td>
</tr>
<tr>
<td>8.</td>
<td>Oil Recovery Vs $\frac{\mu_o}{\mu_w}$</td>
<td>45</td>
</tr>
<tr>
<td>9.</td>
<td>Oil Recovery Vs $\frac{\mu_o}{\mu_w}$</td>
<td>46</td>
</tr>
</tbody>
</table>
## LIST OF TABLES

<table>
<thead>
<tr>
<th>Table No.</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>Summary of Results</td>
<td>41</td>
</tr>
<tr>
<td>II</td>
<td>Summary of Results</td>
<td>42</td>
</tr>
<tr>
<td>III</td>
<td>Heterogeneous Permeability (md) Distribution</td>
<td>58</td>
</tr>
</tbody>
</table>
I. INTRODUCTION

The injection of fluids in petroleum reservoirs to improve oil recovery over that obtainable by primary production has long been used in the oil industry. Water and gas are perhaps the most widely used fluids for this purpose. Waterflooding, over the years, has gained popularity over gas injection and is most widely used today because it is usually easily controlled and yields a favorable mobility ratio in many cases. Many techniques have been developed to explain flood behaviour (flow direction and distribution of fluids) when water is injected into the reservoir to displace oil, and to describe it qualitatively and quantitatively. A few of the techniques available for evaluating waterflood performance are those developed by Stiles (1), Dykstra-Parsons (2) and Buckley and Leverett (3). All these techniques involve some limiting assumptions and it is often not feasible to account for capillarity and gravity effects in practical calculations; thus, they deal with idealized systems.

Though gravity and capillarity are neglected in many practical calculations, nevertheless, as evidenced from practical cases in the field, these forces are often important when dealing with fluid flow in porous media. Under suitable circumstances, these may materially modify the behaviour of waterflood systems as compared with that predicted
by idealized behaviour. Capillary forces cannot always be
neglected in a waterflood system as they may exert a signifi-
cant influence on the fluid distribution, which in turn
is reflected in the macroscopic flow behaviour.

The objective in this work is to compare results of a
mathematical model with the results of Buckley-Leverett's
method for an idealized case and then to extend the model
to account for capillary pressure, thus checking the effect
of omitting this parameter. This should yield a double
check on using the Buckley-Leverett technique of calcula-
tions—the validity of Buckley-Leverett's method without
capillary pressure and on the validity of the discretized
model as compared to an analytical solution.

Recently developed reservoir simulation techniques also
permit the prediction of flood behaviour by numerical means
and these techniques make it possible to consider any num-
ber of parameters in an investigation without computational
difficulties. The present work deals with a one-dimensional,
two phase system containing compressible fluids and dealing
with the displacement of oil (non-wetting phase) by water
(wetting phase). Thus, it involves the solution of a non-
linear partial differential equation. The method is general
enough that either water-oil or gas-oil systems may be con-
sidered. The simulator accounts for capillarity, hetero-
geneity (with respect to permeability) and assumes the fluids
as compressible, all three features in contrast to Buckley-
Leverett assumptions.
Gravitational effects have not been considered and the treatment refers to immiscible displacing and displaced phases.
II. REVIEW OF LITERATURE

The problem of calculating the flow and distribution of fluids in an oil reservoir subjected to waterflood has long challenged the reservoir engineer. The special case of linear, incompressible, immiscible, two-phase flow has received much attention in the petroleum engineering literature. A basic paper on the subject is that of Buckley and Leverett (3) published in 1942. They derived the partial differential equation which describes the displacement of oil from a linear porous medium by an immiscible fluid. They solved this equation for the case when gravity and capillary forces are negligible, but their solution showed that with time, the saturation becomes a multi-valued function of distance. Since it is physically unrealistic for saturation to have more than one value at the same position, they interpreted the formation of multiple values as an indication that the \( S_w \) saturation-distance curve had become discontinuous. They surmounted this difficulty by introducing a saturation-discontinuity or "shock", the position of which they determined from material balance considerations.

Since then, a number of numerical solutions to the original Buckley-Leverett flow equation have been presented by different authors.
Fayers and Sheldon\textsuperscript{(4)} used a finite difference approximation for the partial differential equation including both gravity and capillarity for incompressible, two-phase flow. Their study shows that the inclusion of the capillary term does eliminate the multiple-valued saturation profile of Buckley and Leverett.

Sheldon, Zondek and Cardwell\textsuperscript{(5)} solved the Buckley-Leverett partial differential equation using the method of characteristics and the concept of shocks. They solved a one-dimensional, two-phase, incompressible fluid flow case, neglecting capillarity but accounting for gravity.

Douglas, et al.\textsuperscript{(6)} presented a method for calculating saturation profiles which includes the effect of $P_c$. The authors started with the one-dimensional displacement equation (non-linear) and by a change of variable, transform this equation to a semi-linear partial differential equation of parabolic type. This equation was then solved by a finite difference method on a high-speed digital computer. From this approximation (implicit form), a complete saturation distribution history can be calculated. They consider a linear, homogeneous, two-phase, incompressible fluid system, neglecting gravity. In their treatment, the initial conditions included a mobile initial water saturation, i.e., the permeability to water ahead of the front was not zero. This appeared to result in a region of slowly rising water saturation ahead of the front, at least for the particular
set of relative permeability and capillary pressure data used to illustrate the computation. The authors also postulated an "end effect" at the outflow face of the system, whereby no water could pass out of the system until the water saturation had built up to a high value. The saturation profiles computed under these assumptions show water accumulating at the downstream end of the system while the flood front proper is still travelling toward the outlet.

McEwen\(^{(7)}\) presented a numerical solution to the one-dimensional displacement equation including the effect of capillary forces but postulating no outlet end effect and no mobile water ahead of the flood front; i.e., zero water permeability ahead of the front (contrast with Douglas et al.). They used a finite-difference solution using a predictor-corrector method. They concluded, that at high rates, the computed behaviour approaches that predicted by the Buckley-Leverett method neglecting capillarity.

Hovanessian and Fayers\(^{(8)}\) extended the work of Douglas, et al. to include gravity and pressure profiles. They solved the same one-dimensional displacement equation for a homogeneous permeable medium, including the effects of capillary pressure and gravity forces. The method of solution resembles that by Douglas, et al. but it extends their work in that effect of gravity is included and in addition, pressure profiles may be calculated. Also the functions of saturation required are entered in tabular form rather than
as polynomials. Further, unlike the method of Fayers and Sheldon, the corresponding elapsed time required for the development of each saturation profile is calculated and also saturation profiles may be calculated after breakthrough. This study concluded that: the inclusion of capillarity eliminates the multiple-valued Buckley-Leverett saturation profiles; injection rates have a pronounced effect on saturation profiles and pressure distribution curves; the initial saturation distribution had little effect on the computed results when the initial water in place was small compared to the water in place at the time under consideration.

Byron S. Gottfried, Guilinger and Snyder's (9) is perhaps the only paper that presented numerical solutions of the equations for one-dimensional flow considering three-phase flow in porous media neglecting gravity and capillary pressure and thus extending capabilities beyond that of Buckley-Leverett. They considered flow equations with variable physical properties and used finite difference methods of solution which employed both implicit and explicit procedures. Results of their study of two-phase flow problem are in agreement with the prediction of Buckley-Leverett theory.
III. THEORY OF RESERVOIR SIMULATION

A. General

One of the latest techniques for solving a physical problem such as the prediction of reservoir performance, is to simulate the performance by a set of mathematical equations that express best the actual physical prototype. These mathematical equations usually include non-linear partial differential equations. The analytical solution of such equations is very difficult and often impossible. Thus, they must be solved by a numerical technique, most of which are suited only to high-speed digital computers.

In 1959, Douglas, et al.\(^{(10)}\) proposed the "leap frog" method for solving two-dimensional, two-phase flow problems. Since then, reservoir modeling as a science (or art) has grown considerably and many varied models have been described in the literature.

The simulation technique consists mainly in a) data preparation, b) model construction, c) history matching, and d) performance prediction. One important consideration is that all available data should be scrutinized carefully for consistency and accuracy. Care at this step may lead to fewer simulation runs since the answer is never better than the input data.\(^{(11)}\) Some authors\(^{(12)}\), however, believe that one need not demand accurate determination of all types of input data, rather the accuracy of the input data
should be proportional to the sensitivity of computed results to variations in those data. However, the application of models in cases where critical input data are poorly known constitutes its misuse and a model in such a case has limited usefulness.

Next, the entire reservoir is discretized, or broken up into a large number of cells, or grid points, consistent with the size of the reservoir. Then rock and fluid property values most representative to these particular locations are assigned to each of these cells. The grid sizes chosen should be large enough to be feasible and also justifiable by available information concerning reservoir properties. The rule should be to "select the least complicated model and grossst reservoir description that will permit the desired estimation of reservoir performance."(12)

The last step is to know how good the model is and it is important to do so because that is the justification for reservoir simulation. The accuracy of the model is checked by matching the results obtained with field recorded histories. If the agreement is not satisfactory, then pertinent input data are varied from one computer run to another until a match is achieved and the simulator is then used for the performance prediction. In general, the longer the matched history period, the more reliable the predicted performance will be. Actually, the best simulator would be the one which updates its data conforming to the physical picture of the reservoir at any time.
B. Sources of Errors in Computed Results

Computed results may be in error. Some of the reasons for this are as follows:

(1) The model itself is an approximation since it involves certain assumptions that are only partly valid.

(2) Numerical techniques do not yield exact solutions. Approximation of partial differential equations by difference equations introduces truncation error; i.e., the solution of the difference equation differs somewhat from the solution of the original differential equations. Further, the exact solution of the difference equations is not obtained due to round-off error.

(3) Reservoir description data are seldom accurately known. This, perhaps, is the most significant source of error.

The presence of appreciable truncation error can generally be determined by noting the sensitivity of calculated performance to changes in $\Delta x$ (spatial increment) or $\Delta t$ (time increment). A change in $\Delta x$ and $\Delta y$ values may require a major revision of the data and, therefore are not practical. It is a common practice, therefore, to study only the effect of the time step size. A significant truncation error warrants the need for small $\Delta t$ steps.

C. Numerical Techniques to Solve Partial Differential Equations

There are two general techniques available to solve a non-linear, partial differential equation using difference
equations - explicit and implicit. The decision to choose one or the other is an important step because it determines, to a large extent, the economic feasibility of using the model. Implicit procedures, however, are often preferred and, in the case of a two-dimensional simulator, ADIP (Alternating Direction Implicit Procedure), proposed by Peaceman and Rachford (13) is the most widely used technique.

The implicit technique is preferred because it has numerical stability for relatively large time steps without the necessity for excessive iteration at each step, is easily programmed, and appears to be the most economical from the standpoint of computing time.

D. Mathematical Development

The fluid flow in a porous medium is adequately described by using the law of mass conservation and Darcy's Law.

Considering a reservoir broken up into a number of blocks as shown in Fig. 1, and considering a typical elemental block such as (i) yields a basis for developing the theory. Assuming the flow from left to right as shown by Fig. 2, the diffusivity equations, for flow of any compressible phase, are developed as follows:

\[ \text{Mass rate in} - \text{mass rate out} + \text{injection} - \text{production} = \text{mass rate of accumulation} \]
Fig. 1  A Reservoir Segmented into Blocks

Fig. 2  A Typical Elemental Block
where,

\[ \text{Mass rate in} = (\nu_x A \rho_r)_{\text{in}} = (\nu_x \Delta y h \frac{\rho_{st}}{\beta})_{\text{in}} \]  

The equality \( \rho_r = \frac{\rho_{st}}{\beta} \) for a phase is assumed applicable and is shown in detail in the Appendix B.

\[ \text{Mass rate out} = (\nu_x \Delta y h \frac{\rho_{st}}{\beta})_{\text{in}} + \frac{\partial}{\partial x} (\nu_x \Delta y h \frac{\rho_{st}}{\beta}) \Delta x . \]  

(2)

Darcy's Law is given as follows:

\[ \nu_x = - \frac{k}{\mu} \frac{\partial \phi}{\partial x} . \]  

(3)

Subtracting Equation (2) from Equation (1) and applying Darcy's Law yields:

\[ \text{Mass rate in} - \text{mass rate out} = - \Delta x \frac{\partial}{\partial x} (\nu_x \Delta y h \frac{\rho_{st}}{\beta}) \]

\[ = \Delta x \Delta y \rho_{st} \frac{\partial}{\partial x} \left( \frac{k h}{\mu \beta} \frac{\partial \phi}{\partial x} \right) . \]

(4)

The rate of mass accumulation can be defined as;

\[ \text{Mass accumulation rate} = \frac{\partial M}{\partial t} \]

\[ = h \Delta x \Delta y \phi \rho_{st} \frac{\partial}{\partial t} \left( \frac{s}{\beta} \right) . \]  

(5)

Inserting other terms in the mass balance, the mass rate conservation relationship becomes:
Simplifying Equation (6) and expanding the time derivative term on the right hand side of the equation yields:

\[
\frac{\partial}{\partial x} \left( \frac{kh}{\mu B} \frac{\partial \phi}{\partial x} \right) + \frac{Q_{\text{inj}} - Q_{\text{PR}}}{\Delta x \Delta y} = \frac{h \Delta x \Delta y}{\rho_{st}} \frac{\partial}{\partial t} \left( \frac{s}{B} \right). \tag{6}
\]

Multiplying both sides of Equation (7) by \( \beta \) yields:

\[
\beta \frac{\partial}{\partial x} \left( \frac{kh}{\mu B} \frac{\partial \phi}{\partial x} \right) = \beta \frac{Q_{\text{inj}} - Q_{\text{PR}}}{\Delta x \Delta y} + \phi h \left[ \frac{1}{\beta} \frac{\partial s}{\partial t} - \frac{s}{\beta^2} \frac{\partial \phi}{\partial \beta} \frac{\partial P}{\partial t} \right]. \tag{8}
\]

Equation (9) is the partial differential equation describing the flow of any compressible phase in one dimension in a porous medium.

Now, considering the flow of oil displaced by water; i.e., a non-wetting phase being displaced by a wetting phase and keeping in mind that potential gradient rather than pressure gradient is considered, the following expressions are given.

The flow potential of wetting phase is defined as:

\[
\phi_w = P_w - \rho gh = P_w, \text{ neglecting gravity} \tag{10}
\]

Let us also arbitrarily designate \( P = P_w \), that is,
the dependent variable in the P.D.E. for the water phase will be \( P \), now defined as the pressure in the wetting phase. The capillary pressure between two immiscible phases is defined as:

\[
P_c = P_n - P_w
\]

Thus,

\[
P_n = P_w + P_c = P + P_c
\]  \hspace{1cm} (11)

Neglecting gravity terms and incorporating the preceding assumptions yields:

\[
\phi_n = P_n = P + P_c
\]

\[
\phi_w = P_w = P
\]  \hspace{1cm} (12)

To solve the non-homogeneous, second-order non-linear partial differential equation (Equation (9)), it must be discretized into finite difference form since no known analytical solution exists. This equation is non-linear because the coefficients are pressure and saturation dependent. The difference equations for the two phases are written as shown below:

**Wetting Phase:**

Expanding the term on the left side of the equality sign about some point \((i)\) in a grid system as shown in Fig. 1, assuming the location of pressure in each block to be at its midpoint (see Fig. 3) and applying Equation (11), yields the following relationships:
Fig. 3 Central Grid Point Illustration
Since there are two phases flowing, it is necessary to use effective permeabilities rather than absolute permeabilities. They are defined by the expressions:

\[ k_w = k k_{rw} \]
\[ k_o = k k_{ro} \]
\[ k_g = k k_{rg} \]  

Equation (13) may then be written as:

\[
\frac{\beta^{n+\frac{1}{2}}}{\Delta x_{i}} \left[ \frac{2(kh)_{i+\frac{1}{2}}}{\Delta x_{i} + \Delta x_{i+1}} \right. \\
\left. \frac{k_{rw}}{\mu_{w}^{\frac{1}{2}}} \right]^{n+\frac{1}{2}} \left( \frac{P_{i+1} - P_{i}}{\Delta x_{i} + \Delta x_{i+1}} \right)^{n+1} - \frac{2(kh)_{i-\frac{1}{2}}}{\Delta x_{i} + \Delta x_{i-1}} \\
\frac{2(kh)_{i-\frac{1}{2}}}{\Delta x_{i} + \Delta x_{i-1}} \left( \frac{P_{i} - P_{i-1}}{\Delta x_{i} + \Delta x_{i-1}} \right)^{n+1} \\
+ \text{terms involving } k_{rw} \left( \frac{1}{\mu_{w}^{\frac{1}{2}}} \right) \right]
\]  

Equation (13) may then be written as:

\[
\frac{\beta^{n+\frac{1}{2}}}{\Delta x_{i}} \left[ \frac{2(kh)_{i+\frac{1}{2}}}{\Delta x_{i} + \Delta x_{i+1}} \right. \\
\left. \frac{k_{rw}}{\mu_{w}^{\frac{1}{2}}} \right]^{n+\frac{1}{2}} \left( \frac{P_{i+1} - P_{i}}{\Delta x_{i} + \Delta x_{i+1}} \right)^{n+1} - \frac{2(kh)_{i-\frac{1}{2}}}{\Delta x_{i} + \Delta x_{i-1}} \\
\frac{2(kh)_{i-\frac{1}{2}}}{\Delta x_{i} + \Delta x_{i-1}} \left( \frac{P_{i} - P_{i-1}}{\Delta x_{i} + \Delta x_{i-1}} \right)^{n+1} \\
+ \text{terms involving } k_{rw} \left( \frac{1}{\mu_{w}^{\frac{1}{2}}} \right) \right]
\]  

Equation (15) may then be written as:

\[
\frac{\beta^{n+\frac{1}{2}}}{\Delta x_{i}} \left[ \frac{2(kh)_{i+\frac{1}{2}}}{\Delta x_{i} + \Delta x_{i+1}} \right. \\
\left. \frac{k_{rw}}{\mu_{w}^{\frac{1}{2}}} \right]^{n+\frac{1}{2}} \left( \frac{P_{i+1} - P_{i}}{\Delta x_{i} + \Delta x_{i+1}} \right)^{n+1} - \frac{2(kh)_{i-\frac{1}{2}}}{\Delta x_{i} + \Delta x_{i-1}} \\
\frac{2(kh)_{i-\frac{1}{2}}}{\Delta x_{i} + \Delta x_{i-1}} \left( \frac{P_{i} - P_{i-1}}{\Delta x_{i} + \Delta x_{i-1}} \right)^{n+1} \\
+ \text{terms involving } k_{rw} \left( \frac{1}{\mu_{w}^{\frac{1}{2}}} \right) \right]
\]
where,

$$HKX_i = \frac{2(kh)_{i+\frac{1}{2}}}{\Delta x_i + \Delta x_{i+1}}$$

$$HKX_{i-1} = \frac{2(kh)_{i-\frac{1}{2}}}{\Delta x_i + \Delta x_{i-1}}$$, by definition.

In this equation the groups superscripted \((n+\frac{1}{2})\) are evaluated at the half-time level; i.e., they are time-averaged values. The pressures are evaluated implicitly at the new time level \((n+1)\).

In the above Equations (13) and (15), it is noted that a time average of the pressure and saturation dependent variables has been used. Pressure dependent variables are evaluated at \((P_{n+1}+P_n)/2\) and relative permeabilities should be evaluated at \((s_{n+1}+s_n)/2\), where \(s\) represents the saturation of the phase being considered.

The parameters describing rock properties and model configurations are not time-dependent and hence are isolated and grouped to form a term \(HKX\) as defined above at \((i)\) spatial point. In this definition \(kh_{i+\frac{1}{2}}\) and \(kh_{i-\frac{1}{2}}\) are series-averaged values for rock permeability, evaluated at the interfaces bounding the \(i\)th block.

Non-Wetting Phase:

The left hand side of Equation (9), assuming flow of the non-wetting phase, is expanded in the same manner as that for the wetting phase. This expansion is shown as follows:
This equation differs from Equation (15) only in that it contains capillary pressure terms which are also calculated at the new \((n+1)\) time level, the evaluation of other terms being similar as in the case of the wetting phase.

If the accumulation term of Equation (9) is subscripted as a non-wetting phase term and equated to the result of Equation (17), a difference-differential equation for this phase results. The same application to the wetting phase added to the non-wetting phase equation yields:

\[
\frac{H_{KX_{i-1}}}{\Delta x_i} \left[ \beta_{wi} \left( \frac{k_{rw}}{\mu_w \theta_w} \right) + \beta_{n_i} \left( \frac{k_{rn}}{\mu_n \theta_n} \right) \right] + \frac{\beta_{n_i}}{\Delta x_i} \left[ H_{KX_i} \left( \frac{k_{rn}}{\mu_n \theta_n} \right) (P_{c_{i+1}} - P_{c_i}) \right]
\]

\[
\frac{H_{KX_{i-1}}}{\Delta x_i} \left[ \beta_{n_i} \left( \frac{k_{rn}}{\mu_n \theta_n} \right) \right] + \frac{\beta_{n_i}}{\Delta x_i} \left[ H_{KX_i} \left( \frac{k_{rn}}{\mu_n \theta_n} \right) (P_{c_{i+1}} - P_{c_i}) \right]
\]

\[
\frac{P_{n+1}}{\Delta x_i} \left[ \left( \frac{k_{rn}}{\mu_n \theta_n} \right) \right] + \frac{H_{KX_i}}{\Delta x_i} \left[ \left( \frac{k_{rn}}{\mu_n \theta_n} \right) \right]
\]
\[- \text{HKX}_{i-1} \left( \frac{k_{rn}}{\mu_n \beta_n} \right)_{i-\frac{1}{2}}^{n+1} + \beta_{n_i} \left( \frac{k_{rn}}{\mu_n \beta_n} \right)_{i-\frac{1}{2}}^{n+1} = - \beta_{w_i} \left( \frac{Q_{inj}}{\Delta x \Delta y} \right)_{w} \]

\[- \beta_{n_i} \left( \frac{Q_{inj}}{\Delta x \Delta y} \right)_{n} - \phi h \left( \frac{s_w}{\beta_w} \frac{d\beta_w}{dP} + \frac{s_n}{\beta_n} \frac{d\beta_n}{dP} \right) \frac{\partial P}{\partial t} \]  \hspace{1cm} (18)

This result is based on the fact that:

\[s_w + s_n = 1 \hspace{1cm} (19)\]

Taking the time derivative of Equation (19) yields:

\[\frac{\partial}{\partial t} s_w + \frac{\partial}{\partial t} s_n = 0 \hspace{1cm} (20)\]

\[\frac{\partial}{\partial t} (s_w + s_n) = 0\]

On the right hand side of Equation (18), the only term which must be written in difference form is \(\partial P/\partial t\) which, as a forward difference in time, may be expressed as:

\[\frac{\partial P}{\partial t} = \frac{P_{i+1} - P_{i}}{\Delta t} \hspace{1cm} (21)\]

Equation (18), after simplification, reduces to

\[\frac{\text{HKX}_{i-1}}{\Delta x_i} \left[ \beta_{w_i} \left( \frac{k_{rw}}{\mu_w \beta_w} \right)_{i-\frac{1}{2}}^{n+1} + \beta_{n_i} \left( \frac{k_{rn}}{\mu_n \beta_n} \right)_{i-\frac{1}{2}}^{n+1} \right] \hspace{1cm} P_{i-1}^{n+1} - (A_i + C_i) P_{i}^{n+1}\]

\[+ \frac{\text{HKX}_i}{\Delta x_i} \left[ \beta_{w_i} \left( \frac{k_{rw}}{\mu_w \beta_w} \right)_{i+\frac{1}{2}}^{n+1} + \beta_{n_i} \left( \frac{k_{rn}}{\mu_n \beta_n} \right)_{i+\frac{1}{2}}^{n+1} \right] \hspace{1cm} P_{i+1}^{n+1} = \text{QTERM}_i\]
\[
\begin{align*}
- \phi h \left( \frac{s_w}{\beta_w} \frac{d\beta_w}{dP} + \frac{s_n}{\beta_n} \frac{d\beta_n}{dP} \right) (P_{i+1}^n - P_i^n) - \frac{\beta n_i}{\Delta x_i} \left[ \begin{array}{c}
HKX_i \left( \frac{k_{rn}}{\mu_n \beta_n} \right) \\
(P_{i+1}^n - P_i^n) - HKX_i \left( \frac{k_{rn}}{\mu_n \beta_n} \right) (P_{i+1}^n - P_{i-1}^n) \end{array} \right] \end{align*}
\]

Equation (22) is a complete discretization of the partial differential Equation (8), which is ready to be solved numerically. Equation (22) can be written in a general form as:

\[
A_i P_i^{n+1} + B_i P_i^n + C_i P_i^{n+1} = D_i \quad (23)
\]

where,

\[
A_i = \frac{HKX_i}{\Delta x_i} \left[ \begin{array}{c}
\beta_{w_i} \left( \frac{k_{rw}}{\mu_w \beta_w} \right) + \beta_{n_i} \left( \frac{k_{rn}}{\mu_n \beta_n} \right) \\
\end{array} \right] \quad (24)
\]

\[
C_i = \frac{HKX_i}{\Delta x_i} \left[ \begin{array}{c}
\beta_{w_i} \left( \frac{k_{rw}}{\mu_w \beta_w} \right) + \beta_{n_i} \left( \frac{k_{rn}}{\mu_n \beta_n} \right) \\
\end{array} \right] \quad (25)
\]

\[
B_i = -(A_i + C_i + G) \quad (26)
\]

\[
G = -\frac{\phi h}{\Delta t} \left( \frac{s_w}{\beta_w} \frac{d\beta_w}{dP} + \frac{s_n}{\beta_n} \frac{d\beta_n}{dP} \right) \quad (27)
\]

\[
C_w = -\frac{1}{\beta_w} \frac{d\beta_w}{dP} \quad (28)
\]

\[
C_n = -\frac{1}{\beta_n} \frac{d\beta_n}{dP} \quad (29)
\]
\[
Q_{\text{TERM}}_i = \frac{1}{\Delta x \Delta y} \left[ \beta w_i \left( Q_{\text{inj}} - Q_{\text{PR}} \right)_{w} + \beta \frac{n_i}{n} \left( Q_{\text{inj}} - Q_{\text{PR}} \right)_{n} \right] (30)
\]

\[
D_i = Q_{\text{TERM}}_i - \frac{\beta n_i}{\Delta x} \left[ \frac{k_{\text{rn}}}{\mu n_i \beta n_{i+\frac{1}{2}}} \left( P_{c, i+1} - P_{c, i} \right) \right] - \frac{k_{\text{rn}}}{\mu n_i \beta n_{i-\frac{1}{2}}} \left( P_{c, i} - P_{c, i-1} \right) (31)
\]

Equation (23), written about each point in the grid system will result in a set of simultaneous linear equations whose coefficient matrix is tridiagonal. This matrix is shown as follows:

\[
\begin{align*}
B_1 P_1 + C_1 P_2 & = D_1 \\
A_2 P_1 + B_2 P_2 + C_2 P_3 & = D_2 \\
A_3 P_2 + B_3 P_3 + C_3 P_4 & = D_3 \quad (32)
\end{align*}
\]

\[
A_{M-1} P_{M-1} + B_{M-1} P_{M-1} + C_{M-1} P_M = D_{M-1}
\]

\[
A_M P_{M-1} + B_M P_M = D_M
\]

This system is readily solved by a Gaussian Elimination method. The results of such a solution are shown as follows:

\[
\begin{align*}
P_M & = \gamma_M \\
P_i & = \gamma_i - \frac{C_i P_{i+1}}{\beta_i} \quad i = M-1, M-2, \ldots 1
\end{align*}
\]
where $\gamma$'s and $\beta$'s are determined from the recursion formulae

$$\beta_1 = B_1,$$

$$\beta_i = B_i - \frac{A_i C_{i-1}}{\beta_{i-1}} \quad i = 2, 3, 4, \ldots M$$

$$\gamma_1 = \frac{D_1}{\beta_1}$$

$$\gamma_i = \frac{D_i - A_i \gamma_{i-1}}{\beta_i} \quad i = 2, 3, 4, \ldots M$$

It is one of the disadvantages of a Gaussian Elimination method that roundoff error may accumulate seriously. Douglas (14) has however, conducted an error analysis applicable to equations such as the set (32) and concluded that roundoff error will be small in comparison with discretization error for usual choices of $\Delta x$ and $\Delta t$.

The coefficients of Equation (23) are dependent upon both pressure and saturation at unknown time levels. Thus, the method is an iterative procedure. An estimate of future pressures ($P^{n+1}$) and saturations ($s^{n+1}$) are made by linear extrapolation. Then the pressure dependent properties are evaluated at $P^{n+\frac{1}{2}} = (P^{n+1} + P^{n})/2$ while the relative permeabilities are evaluated at $s^{n+\frac{1}{2}} = (s^{n+1} + s^{n})/2$. The unknown pressures are then calculated using these estimated values. Then, an explicit determination of future saturations are made as outlined in the Material Balance Section. These calculated pressures and saturations are compared with the previously assumed values. If the agreement is not reached
within a pre-specified tolerance, then an iteration is made; otherwise the calculations may proceed to the next time step. After the first time loop through, a linear extrapolation is made to predict pressures and saturations for the next time step.

**Boundary Conditions:**

The physical system simulated by this model is assumed to be closed at all boundaries except for those blocks containing wells, which for this model are those assumed to be at either end. No fluid is permitted to flow across these boundaries except that being produced or injected at the wells. The closed boundaries can be effected by either assigning zero permeability to an imaginary block outside the model or by assigning a pressure gradient of zero at these boundaries. Both are mathematically the same.

Assigning a pressure gradient of zero to Equation (23) and writing an equation about point \(i=1\), with no flow across the lefthand boundary gives (referring to Fig. 4):

\[
A_1 P_0^{n+1} + B_1 P_1^{n+1} + C_1 P_2^{n+1} = D_1
\]  

(36)

The lefthand boundary is closed by setting the pressure gradient to zero \(\left(\frac{\partial P}{\partial x}\right)_{1-\frac{1}{2}} = 0\), which is effected by setting \(P_0 = P_1\).

Substituting this result into Equation (36) for \(i = 1\), yields:

\[
(A_1 + B_1) P_1^{n+1} + C_1 P_2^{n+1} = D_1
\]  

(37)
Fig. 4 Boundary Elements
but, from Equation (26),

\[ B_1 = -A_1 - C_1 - G_1 \]

then the coefficient of \( P_{n+1} \) becomes

\[ (A_1 + B_1) = -C_1 - G_1 \]  (38)

This result is equivalent to setting the coefficient \( A_1 = 0 \) in Equation (36), which is obtained by closing the lefthand boundary by assigning a zero permeability at the boundary \( (k_o = 0) \). Thus, the equation for the first block may be written as

\[ B_1 P_{n+1} + C_1 P_{2} = D_1 \]  (39)

Using the same logic about point M (no flow across the right hand boundary), similarly results in \( C_M = 0 \), and the equation about point M, may be written as,

\[ A_M P_{n+1} + B_M P_{n+1} = D_M \]  (40)

where,

\[ B_M = -A_M - G_M \]

Injection End (left boundary):

For this study, a constant water injection rate is maintained in the injection well located in the first block at the left side of the model.
Production End (right boundary):

A production well is located in the last block to the right of the model. The sum of oil and water production rates (evaluated at reservoir conditions) at the production well is equal to the water injection rate (at reservoir conditions) into the system at the injection well. In this model the injection has been considered positive and production negative. Thus, the production-injection balance stated above may be written as follows:

\[
(Q_{w\text{PR}} \beta_w)_M = - (Q_{o\text{PR}} \beta_o)_M - (\frac{Q_{w\text{PR}}}{Q_{o\text{PR}}}) (\frac{k_{rw}}{k_{ro}}) \frac{\mu_o}{\mu_w}_M \tag{41}
\]

Darcy's Law is used to calculate water-oil ratio at reservoir conditions at the production well, and may be written as follows:

\[
\frac{(Q_{w\text{PR}} \beta_w)}{(Q_{o\text{PR}} \beta_o)_M} = \frac{k_{rw}}{k_{ro}} \frac{\mu_o}{\mu_w}_M \tag{42}
\]

Thus,

\[
(Q_{w\text{PR}} \beta_w)_M = (Q_{o\text{PR}} \beta_o)_M \frac{k_{rw}}{k_{ro}} \frac{\mu_o}{\mu_w}_M \tag{43}
\]

Substituting this result into the reservoir voidage balance given by Equation (41), then solving for oil production rate at reservoir conditions yields:
Having obtained reservoir oil production rate, the water production rate at reservoir conditions is calculated from Equation (43). Then both rates are converted to stock tank conditions by dividing the reservoir rates calculated by the appropriate formation volume factor for each phase.

E. Material Balance

Material balance calculations were made on each block in deriving the pressure equation and are also used in updating the saturations across each time step.

The updating of saturations for the two phases is discussed below.

Wetting Phase Saturation:

The saturation in block (i) (see Fig. 5) at the new time, \( t^{n+1} \), can be stated as the saturation at the beginning of the old time step \( t^n \), plus the change in saturation during the time increment \( \Delta t \) from \( t^n \) to \( t^{n+1} \). This procedure is shown by the following equation:

\[
\begin{align*}
\text{s}^{n+1}_{Wi} &= s^n_{Wi} \left( \frac{\beta_{w_{i}}^{n+1}}{\beta_{w_{i}}} \right) + (\Delta s_{Wi}) \\
&= s^n_{Wi} \left( \frac{\beta_{w_{i}}^{n+1}}{\beta_{w_{i}}} \right) + (\Delta s_{Wi})
\end{align*}
\]
Fig. 5 Grid System for Material Balance
\((\Delta s_w)\) is the change in volume of the wetting phase in block (i) during the time increment divided by that block's pore volume, \((PV)_i\).

However, to allow for the compressibilities of phases, the accuracy of the calculation will be enhanced by making a mass balance instead of a volumetric balance and then converting the change in mass in the block to an equivalent change in saturation. Thus, the following equation will be applicable:

\[
(\Delta s_w)_i = \frac{\text{(Mass in)}_i - \text{(Mass out)}_i - \text{(Mass produced)}_i}{(\rho_w)_{r_i} (PV)_i}
\]  

(46)

Darcy's Law can be used to implement the solution of the above equation, by evaluating the \((\text{mass in})_i\) and \((\text{mass out})_i\) terms at \((i-\frac{1}{2})\) and \((i+\frac{1}{2})\) interfaces respectively; thus,

\[
\text{Mass in} = v_{x_{\text{in}}} A \frac{\rho_{w_{\text{st}}}}{\beta_{w_{\text{in}}}} \Delta t
\]  

(47)

Recognizing that the mass entering block (i) came from block \((i-1)\) by the convention of assuming flow is left to right, the following equation can be written:

\[
\text{(Mass in)}_i = - \frac{\Delta y_i \rho_{w_{\text{st}}} \Delta t}{\beta_{w_{i-1}}} H K X_{i-1} \left( \frac{k_{rw}}{\mu_w} \right) (\phi_{i-\frac{1}{2}} - \phi_{i-1})
\]  

(48)
It follows, in like manner, that

\[
(Mass\ out)_i = - \frac{\Delta y_i \rho_w}{\beta_w} \frac{\Delta t}{\mu_w} HKX_{i-1} \left(\frac{k_{rw}}{\mu_w}\right) \left(\phi_{ri+1} - \phi_{ri}\right)
\]

(49)

where, \(HKX_{i-1}\) and \(HKX_i\) are as previously defined.

The production term is shown as follows:

\[
(Mass\ produced)_i = Q_w \rho_w \frac{\Delta t}{\rho_w}
\]

(50)

and, the pore volume term in the denominator of Equation (46) may be written as:

\[
(PV)_i = \Delta y_i h_i \Delta x_i \phi
\]

(51)

Combining the above results and substituting the relation for \(\phi\) as defined earlier, yields the following result:

\[
(\Delta s_w)_i = \frac{\Delta t}{\Delta x_i \phi} \left[ HKX_i \left(\frac{k_{rw}}{\mu_w}\right) (P_{i+1} - P_i)^{n+\frac{1}{2}} - \frac{\beta_w}{\beta_{wi-1}} HKX_{i-1} \right.
\]

\[
\left. \left(\frac{k_{rw}}{\mu_w}\right) (P_i - P_{i-1})^{n+\frac{1}{2}} - \left(\frac{Q_w \beta_w}{\Delta y_i}\right) \right]
\]

(52)

Thus, the updated wetting phase saturation is calculated by

\[
s_{wi}^{n+1} = s_{wi}^n \left(\frac{\beta_{wi}^{n+1}}{\beta_{wi}}\right) + (\Delta s_w)_i
\]

(53)
Non-wetting Phase Saturation:

The non-wetting phase saturations are updated in exactly the same manner as the wetting phase; the expression is:

\[
\frac{s_{n+1}^n}{s_n^n} = \frac{\beta_{n+1}^n}{\beta_n^n} + (\Delta s_n^n) \tag{54}
\]

\((\Delta s_n^n)\) is similarly defined and, using \(\phi_n\) as defined earlier in Equation (12), is given as:

\[
(\Delta s_n^n) = \frac{\Delta t}{h_i \Delta x_i \phi} \left[ \begin{array}{c}
HKX_i \left( \frac{k_{rn}}{\mu_n} \right) (P_{i+1} - P_i)^{n+\frac{1}{2}} - \frac{\beta_{n_i}}{\beta_{n_i-1}} HXX_{i-1} \\
\frac{k_{rn}}{\mu_n} (P_i - P_{i-1})^{n+\frac{1}{2}} + HXX_i \left( \frac{k_{rn}}{\mu_n} \right) \left( P_{c_i} - P_{c_{i-1}} \right)^{n+\frac{1}{2}} \\
- \frac{\beta_{n_i}}{\beta_{n_{i-1}}} HXX_{i-1} \left( \frac{k_{rn}}{\mu_n} \right) \left( P_{c_i} - P_{c_{i-1}} \right)^{n+\frac{1}{2}} - \frac{Q_{n_i} \beta_{n_i}}{\Delta y_i} \end{array} \right] \tag{55}
\]
IV. BUCKLEY-LEVERETT TECHNIQUE

In 1942 Buckley and Leverett proposed a technique which could quantitatively describe the characteristics of the displacement of oil by either gas or water "with an attempt to elucidate somewhat the mechanism by which such displacement is effected." This technique was the first developed which allows a quantitative evaluation to be made of waterflood performance in a linear reservoir. Their development was based on the following assumptions:

1) The reservoir can be represented as a linear system.

2) There are no permeability variations; i.e., permeability is uniform everywhere in the system.

3) Fluids are incompressible and immiscible.

4) Two and only two phases are flowing, there is no three-phase flow.

5) The initial saturations are constant throughout the reservoir and the initial water saturation need not be immobile.

6) The areal sweep efficiency is 100 per cent.

7) There is simultaneous flow of two phases. Thus, they do not assume a piston-like displacement as do Stiles and Dykstra-Parsons.

Stiles and Dykstra-Parsons obtain a continuously changing water-oil ratio after breakthrough by considering
a layered system, each layer having a different permeability. In the Buckley-Leverett method, in contrast, a changing water-oil ratio after breakthrough is obtained, not by introducing permeability variations, but because the two phases are allowed to flow simultaneously.

If permeability variations are not significant and if the system can reasonably well be represented as a linear system, the Buckley-Leverett method is the best method to use.

The Buckley-Leverett method is the only analytical method suitable for gas injection calculations. The reason for this is that all other methods assume piston-like displacement in any given layer—not a valid assumption for a gas drive. Calculations using the Buckley-Leverett method predict more realistic behavior than other methods; i.e., it shows a decrease in the breakthrough recovery as the viscosity ratio of the displacing to the displaced ($\mu_g/\mu_o$) phase is decreased. The Buckley-Leverett theory can be used for any wettability or viscosity ratio.

The calculation procedure of the Buckley-Leverett technique is based on conclusions derived from the use of two important relationships:

(1) Frontal Advance Formula, which is written as:

$$\left( \frac{\partial x}{\partial t} \right)_{SW} = \frac{q_t}{\phi A} \left( \frac{\partial f}{\partial S_w} \right)_t$$
\[
\Delta x = (\Delta t) \frac{q_t}{\phi A} \frac{\partial f_W}{\partial S_W} t
\]

Thus, this equation determines the linear advance of the various assumed saturations during the time that a volume equal to \((q_t \Delta t)\) is injected.

(2) Fractional Flow Formula - this formula derived from conservation of matter considerations with two incompressible fluids flowing in the system and taking capillarity and gravity into account, is given as follows:

\[
f_W = \frac{1 - \frac{k_o A}{q_t \mu_o} (\frac{dP_c}{dS_W} + g \Delta \rho \sin \alpha)}{1 + (\frac{k_o}{k_w})(\frac{\mu_w}{\mu_o})}
\]

The simplified formula, neglecting capillarity and gravity, is the one usually used for calculations. This is:

\[
f_W = \frac{1}{1 + \frac{k_o \mu_w}{k_w \mu_o}}
\]

Thus, the Buckley-Leverett method requires a knowledge of the ratio of effective oil permeability to the displacing phase permeability as a function of saturation and the viscosity ratio of the phases considered, at reservoir conditions.

A simplified computational procedure was proposed by Welge. This procedure was used in this study and is carried out as follows:
1) First step is to calculate

$$f_w = \left(1 + \frac{k_o}{k_w} \frac{\mu_w}{\mu_o}\right)^{-1}$$

from the known relative permeability-saturation relationships and the viscosity ratio, a plot of $f_w$ as a function of $s_w$ is obtained. This plot serves as the basis for all computations.

2) A straight line tangent to the curve is drawn from the point $(s_{w_1}, f_{w_1})$. The abscissa of the point of tangency gives the saturation $s_{wf}$ at the outlet end of the system when breakthrough occurs.

3) The outlet end saturation, $s_{ws}$, after breakthrough increases over the value $s_{wf}$ with time as the production is continued. Beginning with the breakthrough saturation $s_{wf}$, and up to residual oil saturation, a number of arbitrary values of outlet saturation, $s_{ws}$, are considered for calculations in the subordinate phase of oil production.

4) For each of the outlet saturation values chosen, the corresponding average saturation value $s_{w_{av}}$ is found by drawing the tangent at the outlet saturation value, extending it to $f_w = 1$, and reading the abscissa corresponding to this point.
5) The oil recovery corresponding to each $s_{w_s}$ is found by

$$N_p = \left( s_{w_{av}} - s_{w_i} \right) \frac{1}{\beta_o} (PV) \quad (60)$$

(STB)

6) Corresponding to each value of $s_{w_s}$ is noted the corresponding values of $f_w$. The producing water-oil ratio, then, corresponding to each $s_{w_s}$, is calculated from:

$$WOR = \frac{\beta_o}{f_w - 1} \quad (61)$$

7) Corresponding to each $s_{w_s}$, is found the value of $f'_w$ which is given by:

$$f'_w = \frac{s_{w_{av}}}{s_{w_s}} \quad (62)$$

The water injected corresponding to each $s_{w_s}$ is, then found by

$$w_i = \frac{(PV)}{f'_w} + w_f \quad (63)$$

8) For a constant injection rate, $Q_{w_{inj}}$, the time of production can be calculated from

$$T = \frac{w_i}{Q_{w_{inj}}} \quad (64)$$

9) If desired, the production rate can be obtained by
dividing the differences in production by the corresponding differences in time.
V. RESULTS AND DISCUSSION

The computational procedures for the mathematical model and Buckley-Leverett technique were developed and programmed in Fortran IV and run on an IBM 360/50 computer. Flow diagrams summarising the computer programs are shown in Appendix D.

The following basic mathematical models were run:
Model 1 - homogeneous (permeability), no capillary pressure
Model 2 - homogeneous (permeability), capillary pressure
Model 3 - heterogeneous (permeability), no capillary pressure
Model 4 - heterogeneous (permeability), capillary pressure
Model 5 - highly random heterogeneity, no capillary pressure
Model 6 - highly random heterogeneity, no capillary pressure
(series avg ≠ homogeneous)

The runs were made for two sets of water relative permeabilities. In Model 3, Model 4 and Model 5 the series averaged permeability was equal to the homogeneous permeability value used in the first two models. In each case the results were compared with Buckley-Leverett predictions. The results are tabulated in Table I. The results by the Buckley-Leverett technique are given at initial water breakthrough and those by the finite difference model are reported at the water-oil ratios equal to that at breakthrough shown by Buckley-Leverett method. In each case the finite difference model indicated a more gradual water-oil ratio
build-up than Buckley-Leverett, indicating that the sharp front assumed by Buckley and Leverett did not exist in the model. Reservoir configuration and computational data for the problem considered are presented in Appendix C.

Figures 6 and 7 show the saturation distribution given by both the Buckley-Leverett technique and by mathematical Model 1 at water breakthrough, for each value of water permeability used. The saturation distributions for Model 2, Model 3 and Model 4 match the saturation distribution of Model 1 very closely; they essentially overlap.

The effect of oil-water viscosity ratio on oil recovery is shown in Figures 8 and 9 where a comparison between the two recoveries—that by Buckley-Leverett and that by Model 1—for two values of water relative permeability, has been shown.

The mathematical model yielded higher recoveries than those obtained by the Buckley-Leverett technique for higher water relative permeability. But the recovery obtained in the model by material balance considerations matches Buckley-Leverett results very close. The material balance calculation is a part of the model based on the consideration that oil recovered is equal to the oil originally in place minus the oil remaining in place. This is calculated using the average oil saturation in the reservoir converted to standard conditions. This recovery compares favorably with Buckley-Leverett recovery as shown in Figures 8 and 9.
### Table I

Summary of Results  
$(\mu_o/\mu_w = 10.)$


\[
k_{rw} = \left( \frac{s_w - s_{wc}}{1-s_{wc}} \right)^4
\]

$WOR = 9.61$

\[
k_{rw} = \left( \frac{s_w - s_{wc}}{1-s_{wc}} \right)^3
\]

$WOR = 7.32$

<table>
<thead>
<tr>
<th></th>
<th>% Recovery by Numerical Model (Material Balance)</th>
<th>% Recovery by Buckley-Leverett at Breakthrough</th>
<th>% Recovery by Numerical Model (Material Balance)</th>
<th>% Recovery by Buckley-Leverett at Breakthrough</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model 1</td>
<td>47.877</td>
<td>47.566</td>
<td>42.524</td>
<td>43.350</td>
</tr>
<tr>
<td>Model 2</td>
<td>47.862</td>
<td>47.566</td>
<td>42.506</td>
<td>43.350</td>
</tr>
<tr>
<td>Model 3</td>
<td>47.827</td>
<td>47.566</td>
<td>42.425</td>
<td>43.350</td>
</tr>
<tr>
<td>Model 4</td>
<td>47.808</td>
<td>47.566</td>
<td>42.406</td>
<td>43.350</td>
</tr>
<tr>
<td>Model 5</td>
<td>47.873</td>
<td>47.566</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Model 6</td>
<td>46.171</td>
<td>47.566</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table II

Summary of Results

\( \mu_o / \mu_w = 10. \)

\[
k_{rw} = \left( \frac{s_w - s_{wc}}{1 - s_{wc}} \right)^4
\]

WOR = 9.61

\[
k_{rw} = \left( \frac{s_w - s_{wc}}{1 - s_{wc}} \right)^3
\]

WOR = 7.32

<table>
<thead>
<tr>
<th>% Recovery by Model</th>
<th>% Recovery by Buckley-Leverett at Breakthrough</th>
<th>% Recovery by Model</th>
<th>% Recovery by Buckley-Leverett at Breakthrough</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model 1</td>
<td>51.095</td>
<td>47.566</td>
<td>43.889</td>
</tr>
<tr>
<td>Model 2</td>
<td>51.051</td>
<td>47.566</td>
<td>43.844</td>
</tr>
<tr>
<td>Model 3</td>
<td>50.996</td>
<td>47.566</td>
<td>43.758</td>
</tr>
<tr>
<td>Model 4</td>
<td>50.947</td>
<td>47.566</td>
<td>43.710</td>
</tr>
<tr>
<td>Model 5</td>
<td>51.059</td>
<td>47.566</td>
<td></td>
</tr>
<tr>
<td>Model 6</td>
<td>49.489</td>
<td>47.566</td>
<td></td>
</tr>
</tbody>
</table>
$k_{rw} = \left(\frac{s_w - s_{wc}}{1 - s_{wc}}\right)^4$

1. Buckley-Leverett
2. Mathematical Model

Fig. 6 Saturation Distribution Curves
$$k_{rw} = \left( \frac{s_w - s_{wc}}{1 - s_{wc}} \right)^3$$

1. Buckley-Leverett
2. Mathematical Model

**Fig. 7 Saturation Distribution Curves**
\[ k_{rw} = \left( \frac{s_w - s_{wc}}{1 - s_{wc}} \right)^4 \]

1. Buckley-Leverett
2. Mathematical Model

Fig. 8 Oil Recovery Vs \( \left( \frac{\mu_o}{\mu_w} \right) \)
Fig. 9 Oil Recovery Vs \( \left( \frac{\mu_o}{\mu_w} \right) \)

\[
k_{rw} = \left( \frac{s_w - s_{wc}}{1 - s_{wc}} \right)^3
\]

1. Buckley-Leverett
2. Mathematical Model
For low water relative permeabilities, the model still gives higher oil recovery than Buckley-Leverett but matches the Buckley-Leverett recovery closer than the recovery obtained on material balance considerations (see Table II). However, the match shown in Figures 8 and 9 depicts the oil recoveries on material balance considerations. The match with material balance considerations indicates that the total fluid volumes are correct and also saturation distributions compare reasonably well.

The oil recovery, as indicated in Figures 8 and 9 declines with increasing oil-water viscosity ratio both in the case of Buckley-Leverett and using the mathematical model. This is an expected result. The curves show oil recovery for oil-water viscosity ratios of 2, 5, 10 and 20 and for two values of water relative permeability. In the mathematical model, the upstream saturations ($s_{wi}^{n+1}$) have been used and found good as a saturation-averaging technique for purposes of calculating oil recovery. This selection agrees with current wide usage in the field of mathematical modeling.

The Buckley-Leverett saturation distribution shows a double-valued curve. Truncation of this curve by a perpendicular plane at the efflux end results in the true saturation-distance curve, and the area under the curve (between the limits $s_w = 0.15, 0.80$ and $x = 0, 200$) does represent the oil recovered at breakthrough. Therefore, the Welge approach to the Buckley-Leverett technique appears correct for oil recovery calculations. But there is a difference,
as noted from Figures 6 and 7; the Buckley-Leverett assumption that oil saturation in the injection block immediately goes to residual value does not seem to hold well at least for finite models. Saturation distribution obtained by the numerical model shows it to be higher than the residual value, for both the values of water relative permeability used. However, the two saturation distributions do match reasonably well.

No significant differences in oil recovery were observed when accounting for heterogeneity as can be seen from the recovery value for Model 3 compared to that for the Model 1 (which simulates Buckley-Leverett flooding behaviour). Model 5 was run with another random permeability distribution in order to check the sensitivity of oil recovery to such changes, and as seen from Table I, there was yet no significant change in the oil recovery.

Realizing that Buckley-Leverett method does not consider the permeability distribution in the reservoir (a common occurrence in a natural reservoir) Model 6 was run using a highly random permeability distribution, for which the series averaged permeability was not equal to the homogeneous permeability. Recovery values obtained, as shown in Table I, indicate that the Buckley-Leverett method is not sensitive to areal heterogeneities. However, it may predict erroneous oil recoveries for highly heterogeneous reservoirs.
No significant variations in oil recovery were observed, even when accounting for capillarity, as can be seen from the values for Model 2 and Model 4, using either sets of values for water relative permeability. Thus the role of capillary forces appears to be subdued which is true if the viscous forces predominate the flooding behaviour.

The effect of heterogeneity and capillarity was noted in that they resulted in pressure values slightly higher than the pressure values obtained in the case of homogeneous systems with no capillary pressures. Also, the water saturation (wetting phase) values were slightly lower than those in the case of homogeneous, no capillary pressure systems.

For viscosity ratios higher than 10, the model shows for certain periods of production, negative pressures in a few blocks. This is mathematically correct (because we did not provide in the model a method to force the pressures to remain positive). Physically it means that pressure is building up in the reservoir. This situation cannot be anticipated in Buckley-Leverett technique. This feature gives a numerical model an edge over to Buckley-Leverett technique.
VI. CONCLUSIONS

Based on the results obtained and their discussion in the preceding chapter, the following conclusions can be made:

1. The numerical solution to the one-dimensional, two-phase flow problem matches the analytical solution of Buckley-Leverett satisfactorily both in the oil recovery obtained and the saturation distribution in the reservoir.

2. The performance prediction of a two-phase water flood can be accurately simulated by a one-dimensional, two-phase numerical model.

3. The Buckley-Leverett technique is accurate even for a certain degree of rock heterogeneity. Linear variations in permeabilities up to 800% were tested, as shown in Table III, Appendix C. The parameter which significantly governs the flood behavior is the relative permeabilities of the phases rather than the specific permeabilities variations as such. This conclusion is considered significant as reasonable heterogeneity does not invalidate the Buckley-Leverett approach, a much less expensive method than mathematical modeling.

4. Significant differences exist in oil recoveries for oil-water viscosity ratios of 2, 5, and 20.

5. The numerical model is more general than the Buckley-Leverett method, as it makes possible the study of changes in fluid properties as a function of pressure and geometry variations (such as variable thickness). Thus, it is concluded that the more expensive mathematical model
should always be used over the Buckley-Leverett technique for gas injection programs and systems of variable geometry.

6. The Buckley-Leverett technique should be used where applicable because of its relative simplicity which makes it inexpensive to run. The applicable cases include those in which the one-dimensional geometry is applicable, the system does not contain gas, and linear heterogeneity is reasonable. As shown by Table III, Appendix C, and as mentioned in Conclusion 3, linear variations in permeabilities up to 800% were tested and showed little effect on the results.
VII. APPENDICES
APPENDIX A

Nomenclature

\( A \) = cross-sectional area normal to the flow direction,
\( A = \Delta y h \) - sq-cm

\( \nu \) = Darcy's velocity potential - cm/sec

\( k \) = absolute permeability - Darcy

\( P \) = pressure - atm

\( h \) = reservoir thickness - cm

\( Q \) = production or injection rate - cc/sec

\( k_w, k_o, k_g \) = phase effective permeability

\( k_r \) = relative permeability

\( g \) = acceleration due to gravity - cm/sec/sec

\( P_c \) = capillary pressure - atm

\( C \) = compressibility of fluid - atm\(^{-1}\)

\( s \) = saturation of a phase - fraction

\( s_{w_c} \) = irreducible water saturation - fraction

\( SOR \) = residual oil saturation - fraction

\( SGC \) = equilibrium gas saturation - fraction

\( t \) = time - sec

\( PV \) = pore volume - res bbl

\( WOR \) = water-oil ratio

\( M \) = number of blocks

\( P_b \) = bubble point pressure - atm

\( q_t \) = total flow rate - cc/sec

\( \frac{ds_w}{dx} \) = saturation distribution in the system
\[ \frac{dP_c}{dx} = \frac{dP_c}{ds_w} \frac{ds_w}{dx}, \text{ capillary pressure gradient in the system} \]

\[ f_w \] = fraction of water in the produced fluids - fraction

\[ s_{wi} \] = initial water saturation - fraction

\[ f_{wi} \] = value of \( f_w \) corresponding to \( s_{wi} \) - fraction

\[ s_{ws} \] = average water saturation any time after breakthrough - fraction

\[ (f_w)_{s_{ws}} \] = value of \( f_w \) corresponding to any average saturation \( s_{ws} \) after the breakthrough - fraction

\[ (f'_w)_{s_{ws}} \] = value of \( f'_w \) corresponding to \( s_{ws} \)

\[ w_i \] = water injected - res bbl

\[ w_f \] = fill-up volume of water - res bbl

\[ v \] = volume - cc

Greek Letters and Derivatives:

\[ \alpha \] = angle with the horizontal, positive up-dip - radian

\[ \phi \] = flow potential - atm

\[ \phi \] = porosity - fraction

\[ \mu \] = viscosity - cp

\[ \rho \] = density - gm/cc

\[ \beta \] = formation volume factor - res cc/std cc

\[ \Delta \rho \] = density difference of the two phase involved - gm/cc

\[ \Delta x \] = length of elemental block - cm

\[ \Delta y \] = width of elemental block - cm

\[ \Delta t \] = time increment - sec

\[ \Delta s_i \] = change in saturation of a phase in the ith elemental block from time \( t^n \) to \( t^{n+1} \) - fraction

\[ \frac{\partial \beta}{\partial P} \] = change in formation volume factor with pressure
\( \frac{\partial M}{\partial t} \) = mass rate of accumulation - gm/sec

\( \left( \frac{\partial x}{\partial t} \right)_{s_w} \) = frontal advance rate for a constant saturation

\( s_w \) - cm/sec

**Subscripts:**

- \( x \) = indicating x-direction
- \( i \) = indicating ith elemental block
- \( i+1 \) = indicating \((i+1)\)th elemental block
- \( l \) = first elemental block
- \( M \) = the last block (producing block)
- \( w \) = wetting phase and water
- \( n \) = non-wetting phase
- \( o \) = oil phase
- \( r \) = reservoir conditions
- \( st \) = standard conditions of pressure and temperature
  
  (1 atm and 60°F)
- \( inj \) = injection
- \( PR \) = production

**Superscripts:**

- \( n \) = indicating nth time level
- \( n+1 \) = indicating \((n+1)\)th time level
APPENDIX B

Derivation of Miscellaneous Relationships

Derivation of the relationship $\rho_r = \frac{\rho_{st}}{\beta}$

$$\rho_r = \frac{\text{mass}}{V_r}, \quad \rho_{st} = \frac{\text{mass}}{V_{st}}$$

$$\frac{\rho_r}{\rho_{st}} = \frac{\text{mass}}{V_r} \times \frac{V_{st}}{\text{mass}} = \frac{V_{st}}{V_r} = \frac{1}{\beta}$$

Thus,

$$\rho_r = \frac{\rho_{st}}{\beta}.$$
APPENDIX C

Reservoir Data

The reservoir under water flood considered in this work has the following rock and fluid properties.

- Total reservoir length = 200 feet
- Reservoir thickness = 10 feet
- Reservoir width = 10 feet
- Reservoir porosity = 25%
- Reservoir homogeneous permeability = 218 md
- Irreducible water saturation = 15%
- Residual oil saturation = 20%
- Equilibrium gas saturation = 0.0
- Total oil in place = 548.817 STB

Numerical Model

Each model considered has 32 blocks. For each model,

- Length of each block, \( Ax = 6.25 \) feet
- Heterogeneous permeability (md) distribution is given in Table III.

- Initial pressure in each block = 200 atm
- Initial water saturation in each block = 15%
- Initial oil saturation in each block = 85%
- Initial gas saturation in each block = 0.0
- Bubble point pressure = 80 atm
- Oil compressibility = \( 7 \times 10^{-4} \) atm\(^{-1}\)
- Water compressibility = \( 5 \times 10^{-5} \) atm\(^{-1}\)
- Oil formation volume factor at bubble point = 1.5
Table III
Heterogeneous Permeability (md) Distribution

| Model 3 | 315 | 331 | 331 | 331 | 158 | 158 | 166 | 299 | 276 | 229 | 315 | 315 | 312 | 237 | 158 | 158 |
| Model 5 | 312 | 323 | 142 | 110 | 126 | 158 | 161 | 306 | 339 | 299 | 292 | 315 | 312 | 315 | 158 | 158 |
| Model 5 | 564 | 465 | 465 | 212 | 223 | 234 | 422 | 389 | 71  | 99  | 85  | 440 | 296 | 226 | 339 | 353 |
| Model 6 | 127 | 282 | 141 | 144 | 147 | 113 | 423 | 346 | 214 | 564 | 257 | 367 | 705 | 458 | 271 | 282 |
| Model 6 | 400 | 330 | 330 | 150 | 158 | 166 | 299 | 276 | 50  | 70  | 60  | 312 | 210 | 160 | 241 | 250 |
| Model 6 | 90  | 200 | 100 | 102 | 104 | 80  | 300 | 245 | 152 | 400 | 182 | 260 | 500 | 325 | 192 | 200 |
Water formation volume factor at bubble point = 1.0
Viscosity of water = 1.0 cp
Water injection rate (constant) = 60.6 bbl/day

Formation volume factors and viscosities as a function of pressure are calculated by the following relations:

\[ \beta_o = \beta_{ob} e^{-c_0(P-P_b)} \]

\[ \beta_w = \beta_{wb} e^{-c_w(P-1)} \quad P > P_b \]

\[ \mu_o = 0.2 + 0.0001(P\times14.7) \]

where,

\( \beta_{ob} \) = oil formation volume factor at bubble point pressure

\( \beta_{wb} \) = water formation volume factor at bubble point pressure

Relative Permeabilities \(^{(16)}\)

\[ k_{rg} = \frac{s^3(2-s_w-2s_w)}{g}\frac{s_w}{(1-s_w)^4} ; \quad s_w \leq s_{wc} \]

\[ k_{ro} = \frac{(1-s_g-s_w)^4}{1-s_{wc}} ; \quad s_w \leq s_{wc} \]

\[ k_{rw} = 0 ; \quad s_w \leq s_{wc} \]

\[ k_{rg} = \frac{s^3(2-s_g-2s_{wc})}{g}\frac{s_{wc}}{(1-s_{wc})^4} ; \quad s_w > s_{wc} \]
\[ k_{ro} = \frac{(1-s_g-s_w)^3(1-s_g+s_w-2s_w)}{(1-s_w)^4} ; \quad s_w > s_{wc} \]

\[ k_{rw} = \frac{s_w-s_{wc}}{1-s_{wc}} ; \quad s_w > s_{wc} \]

The capillary pressure-water saturation relationship used for this study is given as:

<table>
<thead>
<tr>
<th>( s_w )</th>
<th>( p_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>.05</td>
<td>.852</td>
</tr>
<tr>
<td>.10</td>
<td>.852</td>
</tr>
<tr>
<td>.15</td>
<td>.852</td>
</tr>
<tr>
<td>.20</td>
<td>.227</td>
</tr>
<tr>
<td>.25</td>
<td>.163</td>
</tr>
<tr>
<td>.30</td>
<td>.134</td>
</tr>
<tr>
<td>.35</td>
<td>.113</td>
</tr>
<tr>
<td>.40</td>
<td>.106</td>
</tr>
<tr>
<td>.45</td>
<td>.092</td>
</tr>
<tr>
<td>.50</td>
<td>.085</td>
</tr>
<tr>
<td>.55</td>
<td>.084</td>
</tr>
<tr>
<td>.60</td>
<td>.083</td>
</tr>
<tr>
<td>.65</td>
<td>.071</td>
</tr>
<tr>
<td>.70</td>
<td>.071</td>
</tr>
<tr>
<td>.75</td>
<td>.063</td>
</tr>
<tr>
<td>.80</td>
<td>.056</td>
</tr>
<tr>
<td>.85</td>
<td>.035</td>
</tr>
<tr>
<td>.90</td>
<td>.021</td>
</tr>
<tr>
<td>.95</td>
<td>.000</td>
</tr>
</tbody>
</table>

Note: The data is input in the computer program using field units. Conversion of field units to cgs units is made within the computer program.
APPENDIX D

Computer Flow Diagrams for Numerical Model

MAIN PROGRAM

START

Initialization of Model M, ϕ, PB, SWC, SOR, SGC, CW, CO, initial pressure and saturations, \( Q_{\text{inj}} \), PTOL, BTAOB

Data Input

\( \Delta x_i, \Delta y_i, h_i, k_i, p_c_i \)

CALL BETA

Calculate initial oil in place

\( i = 1, M-1 \)

Calculate \( HKX_i \)

Read ILOOP, \( \Delta t \)

ITER = 0
Estimate pressures and saturations for the next time level:
\[ P_{n+1}^{i}, S_{h+1}^{i}, S_{o+1}^{i}, S_{g+1}^{i} \]

CALL COEFF

CALL TRIDAG
Calculate pressures at new time level:
\[ P_{n+1}^{i} \]

CALL MBAL
Calculate saturations at new time level:
\[ S_{h+1}^{i}, S_{o+1}^{i}, S_{g+1}^{i} \]

ITER = ITER + 1

Pressure and Saturation in tolerance? (check key)

If YES, calculate cumulative oil production, oil recovery by material balance in blocks, percentage recoveries, WOR.

If NO and ITER > ITERLM, go back to A.

If YES, go back to C.
Output $p_{n+1}, s_{n+1}, s_{n+1}, s_{n+1}$

$\text{RECOIL, REC, ROILMB, RECF, TIME}$

$\text{DELT} = \text{DELT}$

Reset $\Delta t$

D
SUBROUTINE COEFF

I = 1
A(I) = 0.0
I = I+1

Calculate SWA, SOA, SGA, PAV, PX

CALL PERMR
CALL VISC
CALL BETA
CALL CAP
Calculate A, B, C, D

NO

Is I = M?

YES

C(M) = 0.0
Calculate A, B, D

Calculate reservoir oil and water production rates, calculate producing water-oil ratio.

Convert flow rates to standard conditions

RETURN TO MAIN PROGRAM
CALL CAP

Calculate DELPC2
QWTRM,QOTRM

CALL BETA

Calculate Updated saturations
$S_{n+1}, S_{n+1}, S_{n+1}$
$w_{i}, s_{oi}, s_{gi}$
Normalize saturations

Are saturations in tolerance?

NO

Key = 2

YES

Is $I = M-1$?

NO

B

YES

$I = M$

BBW = 0.0
BBC = 0.0

B
CALL CAP

Calculate DELPC2
QWTRM,QOTRM

CALL BETA

Calculate Updated saturations
\( s_{n+1}, s_{n+1}^w, s_{n+1}^o, s_{n+1}^g \)
Normalize saturations

Are saturations in tolerance?

\[ I = M - 1? \]

Key = 2

BBW = 0.0
BBO = 0.0
**SUBROUTINE TRIDAG**

\[
S(1) = R(1) \\
X(1) = D(1)/S(1)
\]

\[
I = 2,N
\]

\[
S(I) = R(I) - \frac{A(I) \times C(I-1)}{S(I-1)} \\
X(I) = \frac{D(I) - A(I) \times X(I-1)}{S(I)}
\]

Is \(I = N\)?

**YES**

\[
Y(N) = X(N)
\]

\[
I = 2,N \\
K = N+1-I
\]

\[
Y(K) = X(K) - \frac{C(K) \times Y(K+1)}{S(K)}
\]

Is \(I = N\)?

**YES**

RETURN TO MAIN PROGRAM

**NOTE:** \(Y\) is returned as pressure
VIII. REFERENCES


IX. VITA

The author, Krishna Kumar Singh, was born on March 15, 1944 in India. After his college and university education in Allahbad (U.P.), India, he entered the Indian School of Mines, Dhanbad, India, in 1964. He graduated from there in 1968 with a Bachelor's degree in Petroleum Engineering with a first class honours. He enrolled at the University of Missouri-Rolla in the Fall 1968 as a Master's candidate in Petroleum Engineering.

In India, he worked two summers with the Oil and Natural Gas Commission of India and Oil India Ltd., respectively.

The author is a member of Pi Epsilon Tau, The National Petroleum Honour Society of U.S.A. and a student member of A.I.M.E. (S.P.E.) since 1967 and was Vice-President of the I.S.M. student chapter of A.I.M.E. in Indian School of Mines, Dahnbad, India.