Magnetohydrodynamic channel flow with non-uniform inlet velocity profiles

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MAGNETOHYDRODYNAMIC CHANNEL FLOW WITH NON-UNIFORM INLET VELOCITY PROFILES

BY

GWOK-LIANG CHEN, 1942–

A

THESIS

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ABSTRACT

An analysis is made of laminar magnetohydrodynamic (MHD) flow development in the entrance region of a parallel-plate channel. The problem is formulated in general for any velocity distribution at the channel inlet by extending the linearization method of Sparrow, Lin, and Lundgren for non-MHD duct flows. This method involves linearization of the inertia terms in the equation of motion by introducing a stretched axial coordinate. A closed form solution is obtained for the velocity distributions which are continuous across the channel and along the length all the way from the entrance to the fully developed region. An expression for the pressure drop is also developed.

The general solutions are then specialized for two classes of non-uniform inlet velocity profile; linear and parabolic. Numerical results are obtained for the velocity distributions, pressure drop, and entrance lengths. In general, it is found that the effect of increasing the Hartmann number is to (1) flatten the velocity profile, (2) increase the pressure drop and (3) shorten the entrance length.

A comparison is made of the present results with those reported by other investigators using different
methods of solution. Excellent agreement between the two is obtained. This supports the validity of the present analysis for MHD flows in the entrance region of a parallel-plate channel. In addition, owing to the novel feature of the solution, it is felt that the present method of analysis is superior to previous methods of analysis.
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NOMENCLATURE

A \hspace{1cm} \text{channel cross sectional area} \\
B_0 \hspace{1cm} \text{magnetic field intensity applied in y direction} \\
b \hspace{1cm} \text{channel half width} \\
C_i \hspace{1cm} \text{coefficients of symmetric eigenfunctions} \\
D_n \hspace{1cm} \text{coefficients of asymmetric eigenfunctions} \\
E \hspace{1cm} \text{electric field vector} \\
E_0 \hspace{1cm} \text{electric field intensity in z direction} \\
e \hspace{1cm} \text{electric field magnitude factor, } E_0 = -e\bar{u}B_0 \\
H \hspace{1cm} \text{magnetic field intensity vector} \\
J \hspace{1cm} \text{electrical current density vector} \\
K \hspace{1cm} \text{pressure correction term, Equation (54)} \\
P \hspace{1cm} \text{pressure} \\
t \hspace{1cm} \text{time} \\
V \hspace{1cm} \text{velocity vector} \\
u, v \hspace{1cm} \text{velocity components in x, y directions} \\
\bar{u} \hspace{1cm} \text{mean velocity, Equation (9)} \\
W \hspace{1cm} \text{dimensionless x component of velocity, } W = u/\bar{u} \\
x, y, z \hspace{1cm} \text{space coordinates} \\
x^* \hspace{1cm} \text{stretched axial coordinate, Equation (15)} \\
X \hspace{1cm} \text{dimensionless axial coordinate, } X = (x/b)/(\bar{u}b/v) \\
X^* \hspace{1cm} \text{dimensionless stretched axial coordinate, } X^* = (x^*/b)/(\bar{u}b/v) \\
\alpha_i \hspace{1cm} \text{eigenvalues of } \alpha_i = \tan \alpha_i
NOMENCLATURE (Continued)

\( \alpha_n \)  
\text{eigenvalues of } \alpha_n = n\pi

\( \Delta \)  
\text{denominator terms of Equation (48)}

\( \varepsilon \)  
\text{scale factor, Equation (48)}

\( \eta \)  
\text{dimensionless transverse coordinate, } \eta = y/b

\( \Lambda \)  
\text{defined by Equation (13)}

\( \mu \)  
\text{dynamic viscosity}

\( \mu_e \)  
\text{permeability}

\( \nu \)  
\text{kinematic viscosity}

\( \rho \)  
\text{density}

\( \sigma_e \)  
\text{electrical conductivity}

Subscripts

\( o \)  
\text{condition at channel inlet}

\( \text{fd} \)  
\text{fully developed values}

\( x,y,z \)  
\text{refer to } x,y,z \text{ component}
I. INTRODUCTION

The studies of magnetohydrodynamic (MHD) channel flows in recent years have been stimulated by the applications of such flows to magnetohydrodynamic generators and accelerators. In the course of its flow through a MHD channel, the velocity distribution of flow in an electrically conducting fluid will, under the influence of an applied magnetic field, undergo a development from some initial profile at the entrance to a fully-developed Hartmann profile at large downstream distances. As a consequence, the pressure drop in the region of flow development will differ from that of a fully developed Hartmann flow. The length of channel in which such a flow development occurs is called the entrance length. A knowledge of the flow characteristics in the entrance region of a MHD channel is essential in many engineering applications, such as the operation of a magnetohydrodynamic accelerator.

The flow development of an electrically conducting fluid in the entrance region of a parallel-plate channel as described by the equations of motion does not permit an exact solution. The difficulties in analyzing the velocity distributions arise from the nonlinearity of the inertia terms which appear in the equations of motion.
Various approximate methods of solution have been devised and employed to provide information relating to the flow development and pressure drop in the entrance region.

One method of analysis is to apply the integral form of the equations of motion and continuity to the boundary layers which develop along the channel walls. The velocity profile is written in the form of a polynomial as in the Karman-Poulhausen method. This method was applied by Maciulaitis and Loeffler (1) in their study of flow development in a MHD channel.

In the second approach, the entrance region is divided into two zones. In the upstream zone near the duct entrance, an approximate solution based on a boundary layer model is obtained in terms of a perturbation series. In the downstream zone far from the duct entrance, a perturbation solution from the fully-developed Hartmann profile is obtained. The solution for the flow development throughout the entire entrance region is then found by patching together the upstream and downstream solutions at some intermediate location. The method just described was applied by Roidt and Cess (2) to a MHD channel.

The third approach involves the transformation of continuity and momentum equations into finite-difference form, with subsequent numerical solution on an electronic
digital computer. Hwang et al. (3,4) have employed this technique in their studies of MHD flows in a parallel-plate channel for uniform and parabolic inlet velocity profiles. The same numerical scheme was employed by Shohet et al. (5) in solving the MHD entrance region flow problem.

An altogether different method of solution to the problem of flow development is to linearize the inertia terms. In this method, a boundary-layer model is not necessary in the analysis, and a velocity solution is obtained which is continuous over the cross section and along the channel all the way from the entrance to the fully developed region. This method was devised by Sparrow et al. (6) for analyzing the non-MHD flow development in ducts. It was later applied to analyze the MHD channel flow development by Snyder (7) for the case of uniform inlet velocity profile.

Most of the published work on the flow development in a MHD channel deals with the case in which the velocity profile is cross-sectionally uniform at the inlet (see, for example, (2,3,7)). In practice, the inlet velocity profile may be asymmetric or cross-sectionally non-uniform, with the result that the flow development and pressure drop will differ from those induced by the uniform inlet velocity profile. The only studies which have treated the non-uniform velocity profile at the entrance are those of
Macinlaitis and Loeffler (1) and of Hwang et al. (4). These workers investigated the MHD flow development from a parabolic velocity profile (i.e., plane Poiseuille profile) at the channel inlet to the Hartmann profile in the fully developed region by employing, respectively, the momentum integral method and the finite-difference method of solution. The entrance lengths obtained from these two solution methods, however, do not agree well. In order to clarify the disagreement, this problem needs to be reinvestigated by a completely different method of analysis.

In the present study, the flow development in the entrance region of a MHD channel is analyzed for the case in which the inlet velocity profile is cross-sectionally non-uniform. The flow is assumed to be laminar and incompressible. The fluid properties are constant. A constant and uniformly distributed magnetic field is applied to the channel in a direction normal to the flow. The linearization technique of Sparrow et al. (6), devised for analyzing non-MHD channel flow with uniform inlet velocity profile, will be adapted and extended to solve the flow development in a MHD channel induced by non-uniform inlet velocity profiles. This technique was chosen because it not only represents a much simpler analysis, but also gives more accurate results than the other approximate methods of solution, as was amply verified by
Sparrow et al. (6) in their non-MHD channel flow analysis. In addition, the present method will provide expressions for velocity distribution, pressure drop, and other quantities of engineering interest which are continuous functions of the axial and transverse coordinates. This type of velocity solution is of advantage, for example, in the study of stability of laminar flow.

The problem will be first formulated in general for any type of inlet velocity profile. Solutions will be obtained for velocity and pressure fields. These solutions will be specialized for linear and parabolic inlet velocity profiles. Mathematical expressions and numerical results for the velocity profiles, pressure drops, and entrance lengths will be obtained. The present results for the case of parabolic inlet velocity profile will be compared with those reported by previous investigators (1,4). For the case of linear inlet velocity profiles, the effect of skewness parameter on the flow development will be analyzed.
II. ANALYSIS

A. Basic Equations

Consider the flow of an electrically conducting fluid in a parallel-plate channel whose walls are electrically non-conducting. The axial and transverse coordinates x and y are measured, respectively, from the inlet section and the center line of the channel. The height of the channel is 2b so that \(-b \leq y \leq b\).

The channel is assumed to be infinite in extent in the z direction. A constant magnetic field of intensity \(B_0\) is applied in the y direction normal to the walls. The fluid enters the channel with a non-uniform velocity profile. The velocity distribution will undergo a development from the initial profile at the inlet to a fully-developed Hartmann profile at large distances downstream.

The basic equations governing the laminar flow of an electrically conducting fluid under the influence of a magnetic field are:

Continuity equation:

\[
\frac{3\rho}{3t} + \nabla \cdot (\rho \mathbf{V}) = 0
\]  \hspace{1cm} (1)
Momentum equation:

\[ \frac{\partial V}{\partial t} + (V \cdot \nabla) V = -\frac{1}{\rho} \nabla p + \nabla^2 V + \frac{1}{\rho} (J \times \mu_e H) \]  

(2)

Maxwell's equations (in mks units):

\[ \text{curl } H = J, \text{ div } J = 0, \text{ curl } E = -\mu_e \frac{\partial H}{\partial t}, \text{ div } H = 0 \]  

(3)

Ohm's law for a moving fluid:

\[ J = \sigma_e (E + V \times \mu_e H) \]  

(4)

All the symbols are defined in the nomenclature.

For the problems under consideration, the following assumptions are made: (1) The flow is steady, laminar, and two dimensional; (2) All fluid properties are constant; (3) The Prandtl boundary-layer assumptions apply (that is, \( u \gg v, \frac{\partial u}{\partial y} \gg \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \gg \frac{\partial^2 u}{\partial x^2} \), and \( \frac{\partial p}{\partial y} = 0 \)); (4) Magnetic permeability \( \mu_e \) and electrical conductivity \( \sigma_e \) are constant; (5) No Hall currents are present; (6) The induced magnetic field in the x direction, \( B_x \), is negligible compared with the applied field \( B_o \); and (7) The electric field measured across the channel walls, \( E_y \), is zero.

With these assumptions, Equations (1) and (2) reduce to:

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]  

(5)
The maximum value of $E_0$ is obtained when the plates are open-circuited. This allows the maximum build-up of the electric field and is equivalent to no net current in the $z$ direction, namely,

$$
\int_{-b}^{b} J_z \, dy = 0
$$

(7)

Substituting $J_z = \sigma_e (E_0 + u B_0)$ into Equation (7), one obtains

$$
\sigma_e E_0 2b + \sigma_e B_0 \int_{-b}^{b} u \, dy = 0
$$

(8)

Next, upon introducing the average velocity $\bar{u}$ defined as

$$
\bar{u} = \frac{1}{2b} \int_{-b}^{b} u \, dy
$$

(9)

Equation (8) simplifies to $(E_0)_{\text{max}} = -\bar{u} B_0$. In the present study, $E_0$ is taken as

$$
E_0 = -e \bar{u} B_0
$$

(10)

where $e$ is the electric field factor which varies between zero and one with the external resistance varying from zero to infinity.
Substituting Equation (10) into Equation (6), the momentum equation becomes

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{dP}{dx} + v \frac{\partial^2 u}{\partial y^2} + \frac{\sigma e B_o^2}{\rho} (\bar{e}u - u) \]  

(11)

B. Velocity Solution

In the present study, the linearization method of analysis devised by Sparrow and coworkers (6) will be employed and extended to determine the flow development and the corresponding pressure drop in the entrance region of the MHD channel. To this end, Equation (11) is linearized. It is proposed here to obtain solutions of the following linearized momentum equation:

\[ \varepsilon(x) \bar{u} \frac{\partial u}{\partial x} = \Lambda(x) + v \frac{\partial^2 u}{\partial y^2} + \frac{\sigma e B_o^2}{\rho} (\bar{e}u - u) \]  

(12)

wherein \( \varepsilon(x) \) is a weighting function of \( x \) to be determined later. The function \( \Lambda(x) \) is another undetermined function which includes the pressure gradient as well as the residual of the inertia terms.

The function \( \Lambda(x) \) can be expressed in terms of other quantities by integrating Equation (12) over the cross sectional area \( A \). Upon noting that \( \frac{\partial}{\partial x} \left( \int_A \bar{u} dA \right) = 0 \) to satisfy mass conservation, the integration results in

\[ \Lambda(x) = -\frac{v}{A} \int_A \frac{\partial^2 u}{\partial y^2} dA - \frac{\sigma e B_o^2}{A \rho} \int_A (\bar{e}u - u) dA \]  

(13)
or

\[ \Lambda(x) = -\frac{\nu}{A} \oint_{c} \frac{\partial u}{\partial n} \, dl - \frac{\sigma e B_o^2}{\rho} (\bar{u} - \bar{u}) \]  

(14)

in which \( c \) is the circumference of the channel and \( \frac{\partial u}{\partial n} \) is the outward normal derivative of the velocity at the wall.

Next, a stretched axial coordinate \( x^* \) may be defined by

\[ dx = \varepsilon \, dx^* \]  

(15)

With Equations (14) and (15), the linearized momentum equation, Equation (12), becomes

\[ -u \frac{\partial u}{\partial x^*} = \frac{\nu}{2b} \left[ \frac{\partial u}{\partial y} \right]_b - \frac{\nu}{2b} \left[ \frac{\partial u}{\partial y} \right]_{-b} + \frac{\sigma e B_o^2}{\rho} (\bar{u} - u) \]  

(16)

This equation is to be solved subject to the boundary condition \( u = 0 \) on the channel walls. In addition, the velocity profile at the inlet, \( u_o \), will be assumed non-uniform across the section. The analysis is facilitated by introducing the following dimensionless parameters:

\[ \bar{W} = \frac{u}{u} \, , \, x^* = \frac{x^*/b}{\sqrt{b/\nu}} \, , \, \eta = \frac{y}{b} \, , \, M = \frac{B_o^2 \sigma e b^2}{4 \mu} \]  

(17)

Equation (16) then assumes the dimensionless form

\[ \frac{\partial W}{\partial \eta^2} = \frac{\partial^2 W}{\partial \eta^2} - \frac{1}{2} \left[ (\frac{\partial W}{\partial \eta})_{\eta=1} - (\frac{\partial W}{\partial \eta})_{\eta=-1} \right] + M^2 (1-W) \]  

(18)
with boundary conditions

\[ W = 0 \text{ at } \eta = \pm 1; \ W = W_0(\eta) \text{ at } X^* = 0 \]  

(19)

Since Equation (18) is linear, a solution for \( W \) can be sought by superposition method in the form

\[ W(X^*,\eta) = W_{fd}(\eta) + W^*(X^*,\eta) \]  

(20)

in which \( W_{fd} \) is the fully-developed velocity distribution and \( W^* \) is a difference velocity. Clearly, \( W^* \) approaches zero for sufficiently large values of \( X^* \). Upon substituting Equation (20) into Equation (18), it can be readily seen that \( W_{fd}(\eta) \) satisfied the fully-developed momentum equation

\[ \frac{\partial^2 W_{fd}}{\partial \eta^2} - \frac{1}{2}[ \left( \frac{\partial W_{fd}}{\partial \eta} \right)_{\eta=1} - \left( \frac{\partial W_{fd}}{\partial \eta} \right)_{\eta=-1}] + M^2 (1-W_{fd}) = 0 \]  

(21)

The solution of this equation, subject to the boundary conditions \( W_{fd} = 0 \text{ at } \eta = \pm 1 \) is easily found to be

\[ W_{fd}(\eta) = \frac{M(CoshM\eta-CoshM)}{SinhM-MCoshM} \]  

(22)

The difference velocity \( W^* \) obeys the equation

\[ \frac{\partial W^*}{\partial X^*} = \frac{\partial^2 W^*}{\partial \eta^2} - \frac{1}{2}[ \left( \frac{\partial W^*}{\partial \eta} \right)_{\eta=1} - \left( \frac{\partial W^*}{\partial \eta} \right)_{\eta=-1}] - M^2 W^* \]  

(23)

with the boundary conditions

\[ W^* = 0 \text{ at } \eta = \pm 1; \ W^* = W_0-W_{fd} \text{ at } X^* = 0 \]  

(24)
A separable solution for $W^*$ may be written in the form

$$W^*(X^*, \eta) = f(X^*) \ g(\eta)$$  \hspace{1cm} (25)

Substituting Equation (25) into Equation (23), one obtains

$$f' + (\alpha^2 + M^2) \ f = 0$$  \hspace{1cm} (26)

$$g'' + \alpha^2 g = \frac{1}{2}[g'(1) - g'(-1)]; \ g(\pm 1) = 0$$  \hspace{1cm} (27)

The solution of Equation (26) is

$$f(X^*) = \exp[-(\alpha^2 + M^2) X^*]$$  \hspace{1cm} (28)

The general solution of Equation (27) is

$$g(\eta) = A \ \sin \eta + B \ \cos \eta + \frac{1}{2\alpha^2}[g'(1) - g'(-1)]$$  \hspace{1cm} (29)

Upon noting that $g(\eta)$ satisfies the non-slip boundary conditions on the walls, it is clear that $g(\eta)$ and its boundary conditions $g(1) = g(-1) = 0$ constitute an eigenvalue problem.

Application of boundary conditions $g(\pm 1) = 0$ gives

$$B = - \frac{1}{2\alpha^2 \cos \alpha} \ [g'(1) - g'(-1)]$$  \hspace{1cm} (30)

and

$$A \ \sin \alpha = 0, \text{ that is } A = 0 \ \text{ or } \alpha = n\pi, \ n = 1, 2, \ldots$$  \hspace{1cm} (31)
Since non-uniform inlet velocity profiles of symmetric and asymmetric types are treated in the present study, consideration must be given to both symmetric and asymmetric eigenfunctions. It is evident from Equation (31) that $A=0$ leads to symmetric eigenfunction while $A\neq 0$ and $\alpha=n\pi$ leads to asymmetric eigenfunction.

The symmetric eigenfunctions and the corresponding eigenvalues are readily found to be

$$g_i = \frac{1}{\alpha_i} \left( 1 - \frac{\cos \alpha_i \eta}{\cos \alpha_i} \right)$$

and

$$\alpha_i = \tan \alpha_i$$

Likewise, the asymmetric eigenfunctions and eigenvalues are

$$G_n = \sin \alpha_n \eta$$

and

$$\alpha_n = n\pi, \ n = 1, 2, ...$$

Both sets of eigenfunctions are orthonormal so that

$$\int_{-1}^{1} g_i^2 \, d\eta = \int_{-1}^{1} G_n^2 \, d\eta = 1$$
Also, the eigenfunctions possess the following orthogonal properties

\[
\int_{-1}^{1} g_i \cdot G_n \, d\eta = 0 \quad \text{for all } i \text{ and } n \quad (37)
\]

and

\[
\int_{-1}^{1} g_i \cdot g_j \, d\eta = 0, \quad i \neq j; \quad \int_{-1}^{1} G_m \cdot G_n \, d\eta = 0, \quad m \neq n \quad (38)
\]

Owing to the linearity of Equation (23), the general solution for \( W^* \) can be obtained by adding the contributions of all the available eigenfunctions, that is

\[
W^* = \sum_{1 \leq i \leq 1} C_i g_i(\eta) \exp\left[-(\alpha_i^2 + M^2)X^*\right] + \sum_{n=1}^{\infty} D_n G_n(\eta) \exp\left[-(n^2 \pi^2 + M^2)X^*\right]
\]

(39)

The coefficients \( C_i \) and \( D_n \) are determined in the following manner. By applying the second condition from Equation (24), Equation (39) reduces, at \( X^* = 0 \), to

\[
W_0 - W_{\text{fd}} = \sum_{i \neq 1} C_i g_i(\eta) + \sum_{n=1}^{\infty} D_n G_n(\eta)
\]

(40)

Equation (40) is then multiplied by \( g_j(\eta) \) and integrated over the range \(-1 \leq \eta \leq 1\). Upon using the orthonormal properties (36) and the orthogonal properties, there is obtained
Next, Equation (40) is multiplied by $G_m(\eta)$ and integrated over $-1 \leq \eta \leq 1$. This gives

$$D_n = \int_{-1}^{1} W_0 \sin(n\pi \eta) \, d\eta$$

(42)

Both $C_i$ and $D_n$ can be determined once the velocity distribution at the inlet $W_0(\eta)$ is specified.

Finally, the complete solution for the velocity may be described by bringing together the results of $W_{fd}(\eta)$ and $W^*(X^*,\eta)$. This gives

$$W(X^*,\eta) = \frac{M(Cosh Mn - Cosh M)}{Sinh M - MCosh M}$$

(43)

$$+ \sum_{i=1}^{\infty} \frac{C_i}{\alpha_i} \left(1 - \frac{\cos \alpha_i \eta}{\cos alpha_i}\right) \exp[-(\alpha_i^2 + M^2)X^*] + \sum_{n=1}^{\infty} D_n \sin(n\pi \eta) \exp[-(n^2 \pi^2 + M^2)X^*].$$

The solution given by Equation (43) is still incomplete because the stretched axial coordinate $X^*$ appears in lieu of the physical axial coordinate $X$. It is, therefore, necessary to determine the relationship between $X$ and $X^*$. To this end, a knowledge of $\varepsilon(X^*)$ is required (See Equation (15)).
C. The Stretched Axial Coordinate

In the work of Sparrow et. al. (6), the weighting function \( \epsilon(x) \) is evaluated under the assumption that the local pressure gradient calculated from momentum consideration be equal to the local pressure gradient obtained from mechanical energy consideration. This approach will be taken in the present study.

First, an expression for the pressure gradient will be derived from the momentum Equation (11). For this purpose, it is convenient to rewrite, with the aid of Equation (5), the inertia terms in Equation (11) by

\[
\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial (u^2)}{\partial x} + \frac{\partial (uv)}{\partial y}
\] (44)

Integrating Equation (11) across the section with the inertia terms expressed by Equation (44), one obtains in dimensionless form

\[-\frac{1}{\rho u^2} \frac{dP}{dx} = \frac{1}{2} \frac{\partial}{\partial x} \left[ \int_1^1 W^2 d\eta - \frac{1}{2} \left( \frac{\partial W}{\partial \eta} \right)_1 - \left( \frac{\partial W}{\partial \eta} \right)_1 - M^2 (e-1) \right] \] (45)

Next, a representation for pressure gradient from mechanical energy consideration is obtained by multiplying Equation (11) by the velocity \( u \) and integrating across the channel. Upon noting that

\[
u^2 \frac{\partial u}{\partial x} + uv \frac{\partial u}{\partial y} = \frac{\partial}{\partial x} \left( \frac{u^3}{2} \right) + \frac{\partial}{\partial y} \left( \frac{u^2 v}{2} \right)
\] (46)
the integration gives

$$\frac{1}{\rho u^2} \frac{dp}{dx} = \frac{1}{4} \frac{\partial}{\partial x} \left[ \frac{1}{W^3} \frac{dr}{d\eta} + \frac{1}{2} \left( \frac{\partial W}{\partial \eta} \right)^2 d\eta - M^2 \left[ e^{-\frac{1}{2}} \right] \frac{1}{W^2} d\eta \right] \quad (47)$$

By equating the pressure gradients given by Equations (45) and (47), there follows, after some rearrangement,

$$\epsilon = \frac{\int_{-1}^{1} \left( 2W - \frac{3}{2} W^2 \right) \frac{\partial W}{\partial x^*} d\eta}{\left( \frac{\partial W}{\partial \eta} \right)_{-1} - \left( \frac{\partial W}{\partial \eta} \right)_{1} + M^2 \int_{-1}^{1} (W^2 - 1) d\eta + \int_{-1}^{1} \left( \frac{\partial W}{\partial \eta} \right)^2 d\eta} \quad (48)$$

Since \( W = W(x^*, \eta) \), it is clear that the right-hand side of Equation (48) is a function of \( x^* \) only. Consequently, by making use of Equation (15), \( x \) can be evaluated from the expression

$$x = \int_{0}^{x^*} \epsilon \ dx^* \quad (49)$$

The denominator terms of Equation (48) can be further simplified as described in Appendix A. The final expression for \( \epsilon \) reads

$$\epsilon = \frac{\int_{-1}^{1} \left( 2W - \frac{3}{2} W^2 \right) \frac{\partial W}{\partial x^*} d\eta}{\Delta(x^*, M)} \quad (50)$$
where

\[ \Delta(X^*, M) = 2 \sum_{i=1}^{\infty} C_i \alpha_i \exp\left[-(\alpha_i^2 + M^2)X^*\right] \]

\[ + \sum_{i=1}^{\infty} C_i^2 (\alpha_i^2 + M^2) \exp[-2(\alpha_i^2 + M^2)X^*] \]

\[ + \sum_{n=1}^{\infty} D_n^2 (n^2 \pi^2 + M^2) \exp[-2(n^2 \pi^2 + M^2)X^*] \]

(51)

With ε(X*) specified, the relationship between X and X* is determined and the velocity solution may now be considered as formally completed. Numerical evaluations of W must await the specification of the velocity profile at the inlet, W_0. The velocity solutions for the two classes of non-uniform inlet velocity profile will be presented later.

D. Pressure Drop

The pressure drop along the duct may be evaluated by integrating either Equation (45) or Equation (47). Since the momentum equation has a somewhat simpler form, it will be employed in the development. Upon integrating Equation (45), the pressure drop between the entrance section X=0 and any downstream section X may be determined. This gives after some arrangement
\[
\frac{P_o - P}{\frac{1}{2} \rho \bar{u}^2} = \frac{(P_o - P)_{fd}}{\frac{1}{2} \rho \bar{u}^2} + K(X) \quad (52)
\]

where

\[
\frac{(P_o - P)_{fd}}{\frac{1}{2} \rho \bar{u}^2} = - \int_0^X \left[ \frac{\partial W_{fd}}{\partial \eta} \right]_1 - \left[ \frac{\partial W_{fd}}{\partial \eta} \right]_{-1} dX + 2M^2 (1-e) X
\]

\[
= 2M^2 \left[ 1-e - \frac{\text{Sinh}M}{\text{Sinh}M - MCoshM} \right] X
\]

and

\[
K(X) = \int_{-1}^{1} W^2 d\eta - \int_{-1}^{1} W^2_0 d\eta - \int_0^{X*} \varepsilon \left[ \frac{\partial W^*_\eta}{\partial \eta} \right]_1 - \left[ \frac{\partial W^*_\eta}{\partial \eta} \right]_{-1} dX^*
\]

\[
(54)
\]

The term \((P_o - P)_{fd}/(\rho \bar{u}^2/2)\) represents the pressure drop that the flow would be sustained if it were fully developed right from the channel inlet. The expression \(K(X)\) represents the pressure drop due to flow development. Inspection of Equation (54) reveals that the pressure drop \(K(X)\) is due to two sources: (1) A change in momentum between the inlet section and any downstream section; and (2) An incremental wall shear stress exerted by the developing
flow relative to that of a fully developed flow. At sufficiently large downstream locations, $K(X)$ will approach a fully developed value $K_{fd}$. Therefore, in the hydrodynamic development region, $K(X)$ will vary from 0 at $X = 0$ to a constant value in the fully developed region. To facilitate the determination of pressure drop, the factor $K(X)$ is expressed in terms of the velocity solution developed in the preceding section. After substituting Equation (43) into Equation (54) and performing the integration, there is obtained (see Appendix B)

$$K(X) = \frac{M(2M\cosh^2 M - 3\sinh M \cosh M + M)}{(\sinh M - M \cosh M)^2}$$

$$+ 4 \sum_{i=1}^{\infty} \frac{C_i \alpha_i}{\alpha_i^2 + M^2} \exp[-\alpha_i^2 + M^2)X^*] + \sum_{i=1}^{\infty} C_i^2 \exp[-2(\alpha_i^2 + M^2)X^*]$$

$$+ \sum_{n=1}^{\infty} D_n \exp[-2(n^2 \pi^2 + M^2)X^*] - \left[ 1 + \frac{W_0^2 d\eta}{\int_{-1}^{1} X^*} \right]$$

$$- 2 \varepsilon \sum_{i=1}^{\infty} C_i \alpha_i \exp[-(\alpha_i^2 + M^2)X^*] dX^*$$

in which the coefficients $C_i$ and $D_n$ are expressed, respectively, by Equations (41) and (42). Once the velocity distribution at the channel inlet $W_0(\eta)$ is specified, the
final evaluation of $K(X)$ can be performed. Two classes of inlet velocity profile are treated in the present study. Their respective solutions and numerical results for the flow development characteristics and pressure drop are presented in the following section.
III. NUMERICAL RESULTS AND DISCUSSION

In the preceding sections, solutions for the velocity distribution and pressure drop in the entrance region were derived in general form so that they can be readily reduced to specific expressions corresponding to a specific inlet velocity profile. As examples, the flow development characteristics induced by two classes of inlet velocity profile were investigated. The first class of inlet profile treated is the linear type, that is, \( W_0 \) is a linear function of \( \eta \). The second class of inlet profile studied is that of a parabolic or plane Poiseuille flow type. This velocity profile corresponds to the situation in which the flow is already fully developed before it enters the MHD channel.

The analytical expressions and numerical results for the flow development and pressure drop corresponding to these two classes of inlet velocity profile will be presented in the following sections. These results will also be compared with those obtained by other methods of solution whenever possible. All calculations were performed with the aid of an IBM 360/50 electronic digital computer.

A. Linear Inlet Velocity Profile

The class of linear inlet velocity profile considered is represented as
\( W_0(\eta) = 1 + \Gamma\eta, \quad 0 \leq \Gamma \leq 1 \)  \hspace{1cm} (56)

The case of \( \Gamma = 0 \) corresponds to the uniform inlet velocity profile. With \( W_0 \) given by Equation (56), the expressions for \( C_i \) and \( D_n \) (see Equations (41) and (42)) now reduce, respectively, to

\[
C_i = -\frac{2\alpha_i}{\alpha_i^2 + M^2}, \quad D_n = -\frac{2\Gamma}{\pi} \left(\frac{-1}{n}\right)^n
\]  \hspace{1cm} (57)

The corresponding expressions for \( W, \varepsilon, \) and \( K(X) \) from Equations (43), (50), and (55) can then be expressed as follows:

\[
W(X^*, \eta) = \frac{M(CoshM\eta - CoshM)}{SinhM - MCoshM} + \sum_{i=1}^{\infty} \frac{-2\alpha_i}{\alpha_i^2 + M^2} (1 - \frac{\cos \alpha_i \eta}{\cos \alpha_i})
\]

\[
\cdot \exp [-\left(\alpha_i^2 + M^2\right)X^*] + \sum_{n=1}^{\infty} \left(\frac{-2\Gamma}{\pi}\right) \left(\frac{-1}{n}\right)^n \sin(n\pi \eta)
\]

\[
\cdot \exp [-\left(n^2 \pi^2 + M^2\right)X^*]
\]  \hspace{1cm} (58)

\[
\varepsilon = \frac{\int_{-1}^{1} (2W - \frac{3}{2} W^2) \frac{\partial W}{\partial X^*} d\eta}{\sum_{i=1}^{\infty} \frac{\alpha_i^2}{\alpha_i^2 + M^2} \exp [-2(\alpha_i^2 + M^2)X^*] - 4 \sum_{i=1}^{\infty} \frac{\alpha_i^2}{\alpha_i^2 + M^2} \exp [-\left(\alpha_i^2 + M^2\right)X^*]}
\]

\[
+ 4 \sum_{n=1}^{\infty} \frac{n^2 \pi^2 + M^2}{n^2 \pi^2} \exp [-2(n^2 \pi^2 + M^2)X^*]
\]  \hspace{1cm} (59)
and

\[ K(X) = \frac{M(2MCosh^2 M - 3Cosh MSinhM + M)}{(SinhM - MCoshM)^2} - 2 - \frac{2\pi^2}{3} \]

\[ + 4 \sum_{i=1}^{\infty} \frac{\alpha_i^2}{(\alpha_i^2 + M^2)^2} \{\exp[-(\alpha_i^2 + M^2)X^*] - 2\} \]

\[ \cdot \exp[-(\alpha_i^2 + M^2)X^*] + \frac{4\pi^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp[-2(n^2\pi^2 + M^2)X^*] \]

\[ + 4 \int_{0}^{X^*} \left[ \epsilon \left\{ \sum_{i=1}^{\infty} \frac{\alpha_i^2}{\alpha_i^2 + M^2} \exp[-(\alpha_i^2 + M^2)X^*] \right\} dX^* \right] \]

The eigenvalues \( \alpha_i \) appearing in these equations can be found from Equation (33). For convenience, the first fifty of the \( \alpha_i \) are listed in Table 1.
Table 1
The eigenvalues $\alpha_i$

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Numerical results were obtained for a total of six cases. They cover parametric values of the skewness parameter $\Gamma = 0, 0.1, \text{ and } 0.2$ for Hartmann numbers of 4 and 10.

1. Relationship Between Physical and Stretched Axial Coordinate

The relationship between $\varepsilon$ and $X^*$ as expressed by Equation (59) is plotted in Figure 1 for skewness parameters of $0$ (uniform inlet profile), $0.1$, and $0.2$ and Hartmann numbers of $M = 4$ and $10$. In order to preserve clarity, the $X^*$ for the case of $M = 10$ is referred to the upper abscissa. Inspection of the figure reveals that in the vicinity of the entrance (i.e., small $X^*$ values), $\varepsilon$ has a value well below unity. With increasing downstream distance, $\varepsilon$ increases and finally approaches a limiting value. For $M = 4$, the limiting value is $0.997$ whereas for $M = 10$, it is $0.830$. For a given Hartmann number $M$, the larger the skewness parameter $\Gamma$, the smaller is the $\varepsilon$ at a given $X^*$ in the neighborhood of the entrance. As $X^*$ increases, the effect of skewness on $\varepsilon$ diminishes, and all curves converge to a single one at large $X^*$ values.

When Equation (59) is substituted into Equation (49) and the integration is carried out numerically, the relationship between $X$ and $X^*$ can be obtained. This is
Figure 1: Relationship Between $\varepsilon$ and the Stretched Coordinate $X^*$, Linear Inlet Velocity Profile
described in Figure 2. As in Figure 1, the case of $M = 10$ is referred to the upper abscissa. It can be seen from the figure that $X$ is smaller than $X^*$ throughout the entire entrance region. In addition, $X$ is influenced by the skewness of the inlet velocity profile. At a given $X^*$ value, the smaller skewness parameter results in larger $X$ value. With the relationship between $X^*$ and $X$ established, the velocity distribution at any location $(X, \eta)$ in the channel can be evaluated from Equations (58) and (49).

2. Velocity Distribution

The development of the velocity profiles at various axial positions $X$ are illustrated in Figure 3 for the representative cases of $\Gamma = 0.2, 0$ and $M = 4, 10$. For $\Gamma = 0.2$, the profiles are skewed with respect to the channel centerline, the extent of the asymmetry decaying with increasing downstream distance. In addition, the asymmetry decays faster when $M$ is larger, as is illustrated in the figure. In particular, for $M = 10$ the two curves corresponding to $\Gamma = 0$ and 0.2 essentially coincide with each other for $X > 0.01$. Thus, the flow is essentially fully developed at $X = 0.01$ when $M = 10$. For $\Gamma = 0$, the velocity profiles are symmetric. When the flow is fully developed, i.e., at $X = \infty$, the dimensionless center line velocity attains maximum values of 1.284 and 1.1110, respectively, for $M = 4$ and 10.
Figure 2: Relationship Between the Physical Coordinate $X$ and the Stretched Coordinate $X^*$, Linear Inlet Velocity Profile.

$$X = (x/b)/(\bar{u}_b/\nu)$$

$$X^* = (x^*/b)/(\bar{u}_b/\nu)$$
Figure 3: Representative Developing Velocity Profiles, Linear Inlet Velocity Profile

\[ n = \frac{y}{b} \]
3. Pressure Drop

The incremental pressure drop $K$ as expressed by Equation (60) can be evaluated once the relationship between $\varepsilon$ and $X^*$ is known. Figure 4 is a plot of $K$ as a function of the physical axial coordinate. The insert of the figure shows, on an enlarged scale, the values of $K$ at small $X$ values. The general behavior of the $K$ is essentially similar for all cases. Starting with a value of zero at $X = 0$, $K$ increases monotonically with increasing downstream distance, approaching a constant, fully developed value $K_{fd}$ at sufficiently large distances. Inspection of the figure reveals that for a fixed Hartmann number $M$, larger skewness parameter $\Gamma$ gives smaller $K$ value at a given axial location in the entire entrance region. Also, a larger Hartmann number gives rise to a smaller value of $K$. Thus, larger $\Gamma$ (i.e., when the inlet profile is more skewed) and larger $M$ result in smaller incremental pressure drop in the entrance region of the channel. The values of $K_{fd}$ are listed in Table 2. Also included in the table are the $K_{fd}$ values reported by other investigators for the special case of $\Gamma = 0$ (uniform inlet velocity profile). It can be seen from the table that for $\Gamma = 0$, the present results agree well with those of Hwang and Fan (3) obtained by a finite difference scheme. On the other hand, the result of Snyder (7), for the case of $M = 10$
\[ r = \frac{0.0}{0.1} \]

\[ r = \frac{0.1}{0.2} \]

\[ K = 10 \quad 0.1 \]

\[ X = \frac{(x/b)}{(\bar{u}b/v)} \]

Figure 4: The Incremental Pressure Drop Due to Flow Development, Linear Inlet Velocity Profile
Table 2

\( K_{fd} \) Values for Linear Inlet Velocity Profile

<table>
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<th></th>
<th>( M = 4 )</th>
<th>( M = 10 )</th>
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<tr>
<td></td>
<td>( \Gamma = 0 )</td>
<td>( \Gamma = 0.1 )</td>
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<tr>
<td>Present Work</td>
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and $\Gamma = 0$, shows a discrepancy of about 15 percent on the low side when compared with the present result, and is believed to be inaccurate.

To calculate the pressure drop, one needs to specify the value of the electric field factor $e$. The pressure drops were calculated from Equation (52) with $K$ given by Equation (60) for $e$ values of 0, 0.5, and 1.0. The results are illustrated in Figure 5 for $M = 4$ and in Figure 6 for $M = 10$. To preserve clarity of the figure, the curves for $\Gamma = 0.1$ were omitted. From the figures, it can be seen that a smaller electric field factor $e$ results in a larger pressure drop and that for a given $e$, a smaller pressure drop is obtained when the skewness of the inlet profile increases (that is, $\Gamma$ deviates from zero). Also, for the same values of $e$ and $\Gamma$, larger Hartmann number causes a larger pressure drop in the entire entrance region. The effect of the velocity profile skewness causes only a small decrease in pressure drop when compared with the case of uniform inlet velocity profile ($\Gamma = 0$). For the purpose of comparison, curves for the case of $M=0$ (i.e., non-MHD flow) are also plotted in Figures 5 and 6. Thus, the effect of the magnetic field is to increase the pressure drop in the development region of a channel.

4. Entrance Length

One of the important aspects in the study of flow development in a channel is to determine the length of the
Figure 5: Pressure Drop Due to Flow Development, Linear Inlet Velocity Profile, $M = 4$
Figure 6: Pressure Drop Due to Flow Development, Linear Inlet Velocity Profile, $M = 10$
development region or the so-called entrance length. There are various possible criteria that can be used to define the entrance length. One such criterion is based on the approach of K value to some preassigned percentage of the fully developed value, K_{fd}. The other commonly used criterion is to define the entrance length as the distance from the channel inlet where the center line velocity reaches to within a certain percent of the center line velocity of the fully developed Hartmann profile. In the present study, both criteria were used to determine the entrance lengths. The entrance lengths based on K value were defined as the axial distances X from the channel inlet where K = 0.95 K_{fd} or 0.99 K_{fd}. The thus-defined entrance lengths are listed in Table 3. It can be seen from this table that the entrance length X_e decreases as the Hartmann number increases and that for a given Hartmann number, the effect of the skewness parameter \Gamma on the inlet velocity profile is to shorten the entrance length, except for the case of M = 10 when based on a 1 percent deviation.

The entrance lengths as determined on the basis of 1 percent or 5 percent deviation of the center line velocity from that of the Hartmann profile are listed in Table 4. Also included in this table are the results from other investigators for the special case of uniform inlet velocity
Table 3
Entrance Length $X_e$ Based on $K$ Value, Linear Inlet Velocity Profile

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<th>5%</th>
<th>1%</th>
<th>5%</th>
<th>1%</th>
<th>5%</th>
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Table 4
Entrance Length $X_e$ Based on Center Line Velocity, Linear Inlet Velocity Profile

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<td>1%</td>
<td>5%</td>
<td>1%</td>
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<tr>
<td>Present Work</td>
<td>0.06933</td>
<td>0.02639</td>
<td>0.06843</td>
</tr>
<tr>
<td>Hwang and Fan</td>
<td>0.0752</td>
<td>0.02929</td>
<td></td>
</tr>
<tr>
<td>Reidt and Cess</td>
<td>0.0668</td>
<td>0.0204</td>
<td></td>
</tr>
<tr>
<td>Snyder</td>
<td>0.077</td>
<td>0.035</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>M</th>
<th>$\Gamma = 0$</th>
<th>$\Gamma = 0.1$</th>
<th>$\Gamma = 0.2$</th>
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</thead>
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<tr>
<td></td>
<td>1%</td>
<td>5%</td>
<td>1%</td>
</tr>
<tr>
<td>Present Work</td>
<td>0.00820</td>
<td>0.00110</td>
<td>0.00757</td>
</tr>
<tr>
<td>Hwang and Fan</td>
<td>0.01216</td>
<td>0.000936</td>
<td></td>
</tr>
<tr>
<td>Reidt and Cess</td>
<td>--</td>
<td>--</td>
<td></td>
</tr>
<tr>
<td>Snyder</td>
<td>0.009</td>
<td>--</td>
<td></td>
</tr>
</tbody>
</table>
profile. The present analysis predicts a shorter entrance length than the one obtained from the finite difference solution of Hwang and Fan (3) for $\Gamma = 0$. The entrance lengths obtained by Snyder (7) for the case of $\Gamma = 0$ do not agree very well with those of the present work. This disagreement is probably attributed to the increased accuracy in the present numerical calculations.

B. Parabolic Inlet Velocity Profile

When the inlet velocity profile assumes the form of plane Poiseuille flow

$$W_0(\eta) = \frac{3}{2} (1 - \eta^2) \quad (61)$$

the series coefficients $C_i$ and $D_n$ are given, respectively, by

$$C_i = \frac{2M^2}{\alpha_i (\alpha_i^2 + M^2)} \quad \text{and} \quad D_n = 0 \quad (62)$$

$W$, $\varepsilon$, and $K(X)$ are then given by the following expressions:

$$W(X*,\eta) = \frac{M(Cosh\eta - CoshM)}{SinhM - MCoshM} + \sum_{i=1}^{\infty} \frac{2M^2}{\alpha_i^2 (\alpha_i^2 + M^2)} (1 - \frac{\cos\alpha_i \eta}{\cos\alpha_i}) \cdot \exp \left[ -(\alpha_i^2 + M^2)X* \right] \quad (63)$$

$$\varepsilon = \int_0^1 (2W - \frac{3}{2} W^2) \frac{2W}{\delta X*} \, d\eta$$

$$2 \sum_{i=1}^{\infty} \frac{M^4}{\alpha_i^4 (\alpha_i^2 + M^2)} \exp[-2(\alpha_i^2 + M^2)X*] + 2 \sum_{i=1}^{\infty} \frac{M^2}{\alpha_i^2 + M^2} \exp[-(\alpha_i^2 + M^2)X*] \quad (64)$$
and

\[ K(X) = \frac{M(2MCosh^2M-3CoshMSinhM+M)}{(SinhM-MCoshM)^2} - \frac{12}{5} \]

\[ + 4 \sum_{i=1}^{\infty} \frac{M^2}{\alpha_i^2+M^2} \left( \frac{M^2}{\alpha_i^2} \exp\left[-(\alpha_i^2+M^2)X^*\right] + 2 \right) \exp\left[-(\alpha_i^2 + M^2)X^*\right] \]

\[ - 4 \int_{0}^{X^*} \epsilon \sum_{i=1}^{\infty} \frac{M^2}{(\alpha_i^2+M^2)} \exp\left[-(\alpha_i^2+M^2)X^*\right] dX^* \quad (65) \]

For this class of inlet velocity profile, numerical calculations of the flow developmental characteristics were carried out for Hartmann numbers of 4 and 10. The results are presented in the following sections.

1. Relationship Between Physical and Stretched Axial Coordinate

The variation of \( \epsilon \) with \( X^* \), expressed by Equation (64), are plotted in Figure 7 for Hartmann numbers \( M=4 \) and 10. It can be seen that the curves increase sharply at small values of \( X^* \), attaining maxima, and then level off as \( X^* \) increases. For large \( X^* \) values, \( \epsilon \) approaches the limiting values of 0.997 for \( M=4 \) and 0.830 for \( M=10 \).
Figure 7: Relationship Between $\varepsilon$ and the Stretched Coordinate $X^*$, Parabolic Inlet Velocity Profile

$X^* = (x^*/b)/(\bar{u}b/\nu)$
With ε given by Equation (64), numerical integration of Equation (49) yields the relationship between X and X*. This is illustrated in Figure 8. It can be seen from the figure that X is smaller than X* throughout the entire entrance region. The relationship between X and X* is essentially linear in the region away from the channel inlet.

2. Velocity Distribution

The velocity profiles at three axial distances X=0.01, 0.05, and 0.10 are illustrated in Figure 9 for M=4 and 10. The flow enters the MHD channel with a plane Poiseuille profile. As the flow develops, the velocity profile undergoes a change and gradually transforms into the Hartmann profile characterized by a flat central core. The flow is virtually fully developed at X = 0.1.

Figure 10 shows the axial variation of the velocity at particular transverse locations \( \eta = 0, 0.2, 0.5, 0.7, \) and 0.9. Inspection of the figure reveals two distinctly different trends in evidence. In the region near the centerline, the flow is seen to decelerate with increasing axial distance. This deceleration is due to the retarding effect of the magnetic field. On the other hand, in the region near the wall, the flow accelerates with increasing axial distance. This is to make up for the deficit in mass flow resulting from the afore-mentioned retardation at the central core of the channel.
$X = \frac{(x/b)}{(\bar{u}b/\nu)}$

Figure 8: Relationship Between the Physical Coordinate X and the Stretched Coordinate $X^*$, Parabolic Inlet Velocity Profile
Figure 9: Representative Developing Velocity Profiles, Parabolic Inlet Velocity Profile
Figure 10: Axial Development of the Velocity at Various $\eta$ Values, Parabolic Inlet Velocity Profile
The velocity results of the present analysis are essentially identical to those obtained by Hwang et al. (4) from the finite-difference method of analysis. A comparison of the velocity results from these two different solution methods for two representative axial locations \(X = 0.01\) and \(0.05\) is made in Table 5. A study of the table reveals that the agreement between the two sets of velocity results is very excellent. Thus, the present method of analysis gives accurate results. Furthermore, in view of the closed form solutions which can be obtained, it is felt that the present analysis is superior to previous analyses.

3. Pressure Drop

The axial variation of \(K\), Equation (65), is plotted in Figure 11. Contrary to the case of linear inlet velocity profile, the \(K\) values due to parabolic inlet velocity profile are negative. Negative value of \(K\) implies that the pressure drop in the entrance region is smaller than the pressure drop in the fully developed region of equal distance. This effect is true because the centerline velocity decreases in the downstream direction, causing a decrease in momentum and an increase in pressure. This pressure increase compensates the pressure drop due to wall friction in the entrance region. As a consequence,
Table 5
Comparison of Velocity Solution at $X = 0.01$ and $0.05$, Parabolic Inlet Velocity Profile

| M | $\eta$ | $
\begin{array}{llll}
$X = 0.01$ & $X = 0.05$
\end{array}$
<table>
<thead>
<tr>
<th>Present work</th>
<th>Hwang et. al.</th>
<th>Present work</th>
<th>Hwang et. al.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.4281</td>
<td>1.4280</td>
<td>1.3204</td>
</tr>
<tr>
<td>0.1</td>
<td>1.4157</td>
<td>1.4153</td>
<td>1.3137</td>
</tr>
<tr>
<td>0.2</td>
<td>1.3785</td>
<td>1.3772</td>
<td>1.2934</td>
</tr>
<tr>
<td>0.3</td>
<td>1.3165</td>
<td>1.3140</td>
<td>1.2588</td>
</tr>
<tr>
<td>0.4</td>
<td>1.2298</td>
<td>1.2262</td>
<td>1.2081</td>
</tr>
<tr>
<td>0.5</td>
<td>1.1183</td>
<td>1.1146</td>
<td>1.1378</td>
</tr>
<tr>
<td>0.6</td>
<td>0.9817</td>
<td>0.9801</td>
<td>1.0411</td>
</tr>
<tr>
<td>0.7</td>
<td>0.8186</td>
<td>0.8214</td>
<td>0.9057</td>
</tr>
<tr>
<td>0.8</td>
<td>0.6217</td>
<td>0.6282</td>
<td>0.7115</td>
</tr>
<tr>
<td>0.9</td>
<td>0.3672</td>
<td>0.3713</td>
<td>0.4261</td>
</tr>
<tr>
<td>4</td>
<td>1.2080</td>
<td>1.2127</td>
<td>1.1114</td>
</tr>
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<td>0.1</td>
<td>1.2034</td>
<td>1.2114</td>
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<tr>
<td>0.2</td>
<td>1.1894</td>
<td>1.1915</td>
<td>1.1110</td>
</tr>
<tr>
<td>0.3</td>
<td>1.1662</td>
<td>1.1653</td>
<td>1.1102</td>
</tr>
<tr>
<td>0.4</td>
<td>1.1336</td>
<td>1.1296</td>
<td>1.1083</td>
</tr>
<tr>
<td>0.5</td>
<td>1.0917</td>
<td>1.0852</td>
<td>1.1035</td>
</tr>
<tr>
<td>0.6</td>
<td>1.0398</td>
<td>1.0332</td>
<td>1.0905</td>
</tr>
<tr>
<td>0.7</td>
<td>0.9739</td>
<td>0.9712</td>
<td>1.0555</td>
</tr>
<tr>
<td>0.8</td>
<td>0.8711</td>
<td>0.8762</td>
<td>0.9605</td>
</tr>
<tr>
<td>0.9</td>
<td>0.6414</td>
<td>0.6486</td>
<td>0.7022</td>
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</table>
Figure 11: The Incremental Pressure Drop due to Flow Development, Parabolic Inlet Velocity Profile
the overall pressure drop in the entrance region becomes smaller.

As \( X \) goes to infinity, \( K \) approaches a fully developed value \( K_{fd} \). The values of \( K_{fd} \) are listed in Table 6. Included also are the \( K_{fd} \) values reported by Hwang et al. (4). It is seen that the \( K_{fd} \) values obtained by the two different methods of analysis agree very well.

The pressure drops were calculated for three electric field factors \( e = 0, 0.5, \) and 1.0. The results are illustrated for Hartmann numbers \( M = 4 \) and 10 in Figure 12. It is seen from this figure that a smaller electric field factor gives rise to a larger pressure drop for a given Hartmann number. For a given electric field factor, a larger pressure drop occurs when the Hartmann number is larger. Thus, the effect of the magnetic field is to increase the pressure drop.

It is to be noted that, for both \( M = 4 \) and 10, the overall pressure drop has negative values at small \( X \) values when \( e = 1.0 \). This is possible when cognizance is made of the fact that the pressure drop corresponding to the fully developed flow \( (P_c - P)_{fd}/(\rho \overline{u}^2/2) \), Equation (53), decreases as \( e \) increases and that the incremental pressure drop \( K \), Equation (65), is negative for all \( M \) and \( X \) values. As explained earlier, negative \( K \) is due to the flow deceleration in the entrance region of the channel. Thus,
### Table 6

$K_f$ Values for Parabolic Inlet Velocity Profile

<table>
<thead>
<tr>
<th></th>
<th>$M = 4$</th>
<th>$M = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present Work</td>
<td>-0.2361</td>
<td>-0.3530</td>
</tr>
<tr>
<td>Hwang et al.</td>
<td>-0.232</td>
<td>-0.369</td>
</tr>
</tbody>
</table>
Figure 12: Pressure Drop Due to Flow Development, Parabolic Inlet Velocity Profile
the overall pressure drop (see Equation (52)) may become negative at some X values.

The pressure drop results for the case of e = 0.5 are compared with those obtained by Hwang et al. (4) in Table 7. As with the velocity results, the two sets of pressure drop data agree very well. This agreement lends further support to the accuracy of the present results and demonstrates again the superiority of the present analysis over others.

4. Entrance Length

The entrance lengths based on 1 per cent and 5 per cent deviations in Kfd value are listed in Table 8, whereas those based on the centerline velocity with 1 per cent and 5 per cent deviations are listed in Table 9. For the purpose of comparison, the results of Hwang et al. (4) are included in the latter. It can be seen from the tables that the entrance length decreases as the Hartmann number increases. For a fixed Hartmann number, the entrance length based on the K value is larger than that based on the centerline velocity. A 5 per cent deviation in the K gives entrance lengths which agree rather well with those of a 1 per cent deviation in the centerline velocity. The entrance lengths predicted by the present analysis agree well with those of Hwang et al. (4).
Table 7
Comparison of Pressure Drop for $e = 0.5$, Parabolic Inlet Velocity Profile

<table>
<thead>
<tr>
<th>X</th>
<th>M = 4 Present Work</th>
<th>M = 4 Hwang et al.</th>
<th>M = 10 Present Work</th>
<th>M = 10 Hwang et al.</th>
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<td>0.005</td>
<td>0.0644</td>
<td>0.0595</td>
<td>0.3576</td>
<td>0.3376</td>
</tr>
<tr>
<td>0.01</td>
<td>0.1578</td>
<td>0.1513</td>
<td>0.9045</td>
<td>0.8849</td>
</tr>
<tr>
<td>0.015</td>
<td>0.2639</td>
<td>0.2565</td>
<td>1.4959</td>
<td>1.4755</td>
</tr>
<tr>
<td>0.02</td>
<td>0.3770</td>
<td>0.3695</td>
<td>2.0976</td>
<td>2.0793</td>
</tr>
<tr>
<td>0.03</td>
<td>0.6158</td>
<td>0.6090</td>
<td>3.3147</td>
<td>3.2986</td>
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<tr>
<td>0.05</td>
<td>1.1206</td>
<td>1.1151</td>
<td>5.7961</td>
<td>5.7467</td>
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<tr>
<td>0.1</td>
<td>2.4345</td>
<td>2.4302</td>
<td>11.8655</td>
<td>11.8707</td>
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Table 8
Entrance Length $X_e$ Based on K Value, Parabolic Inlet Velocity Profile

<table>
<thead>
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<th>$M = 4$</th>
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<th>$M = 10$</th>
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<tr>
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<td>0.11235</td>
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<td>0.06856</td>
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<tr>
<td>1%</td>
<td>0.01536</td>
<td>5%</td>
<td>0.01170</td>
</tr>
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</table>

Table 9
Entrance Length $X_e$ Based on Center Line Velocity, Parabolic Inlet Velocity Profile

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<td>0.03346</td>
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<td>0.02643</td>
<td>5%</td>
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</tr>
<tr>
<td>1%</td>
<td>Hwang et al.: 0.0772</td>
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<td>0.03272</td>
</tr>
<tr>
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<td>0.02924</td>
<td>5%</td>
<td>0.01508</td>
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IV. CONCLUSION

The linearization method of Sparrow et al. for analyzing the hydrodynamic entrance region flows has been extended to the developing magnetohydrodynamic (MHD) flow in a parallel-plate channel. The case of non-uniform inlet velocity profiles is treated. In general, it is found that as the Hartmann number increases, the major effect on the flow development is to (1) flatten the velocity profile, (2) decrease the incremental pressure drop, (3) increase the overall pressure drop, and (4) reduce the entrance length.

A parabolic inlet velocity profile generally requires a much larger entrance length for the flow to develop than the uniform inlet velocity profile. In the case of the linear inlet velocity profile, a larger skewness parameter $\Gamma$ gives rise to a smaller pressure drop and shorter entrance length.

Comparison between the results of the present analysis with those of other investigators shows a good agreement. This comparison lends strong support to the validity of the present analysis. In addition, the present analysis is felt to be superior to other analyses because a closed form solution is obtained for the velocity distribution from which other physical quantities of interest can be evaluated with a minimum effort of numerical work.
V. REFERENCES


"Flow Development in the Hydrodynamic Entrance
Region of Tubes and Ducts," Physics of Fluids,
7, 338 (1964).

7. Snyder, W.T., "Magnetohydrodynamic Flow in the
Entrance Region of a Parallel-Plate Channel,"
To simplify the denominator terms of Equation (48), Equation (18) is first multiplied through by $W$ and then integrated across the channel cross section. This gives

$$
\int_{-1}^{1} W \frac{\partial W}{\partial x'} \, d\eta = \int_{-1}^{1} W \frac{\partial^2 W}{\partial \eta^2} \, d\eta - \frac{1}{2} \int_{-1}^{1} W \left( \frac{\partial W}{\partial \eta} \right)_1 - \left( \frac{\partial W}{\partial \eta} \right)_{-1} \, d\eta 
\hspace{2cm} (A-1)
$$

$$
+ M^2 \int_{-1}^{1} W(1-W) \, d\eta
$$

Upon facilitating the identity that

$$
\int_{A} W \nabla^2 W \, dA = - \int_{A} \nabla W \cdot \nabla W \, dA \hspace{2cm} (A-2)
$$

Equation (A-1) becomes

$$
\int_{-1}^{1} W \frac{\partial W}{\partial x'} \, d\eta = - \int_{-1}^{1} \left( \frac{\partial W}{\partial \eta} \right)^2 \, d\eta - \left[ \left( \frac{\partial W}{\partial \eta} \right)_1 - \left( \frac{\partial W}{\partial \eta} \right)_{-1} \right] + M^2 \int_{-1}^{1} (W^2 - 1) \, d\eta \hspace{2cm} (A-3)
$$
By comparing the right-hand side of Equation (A-3) with the denominator of Equation (48), it can be seen that the latter reduces to

$$
\Delta(X^*, M) = \int_{-1}^{1} W(\frac{\partial W}{\partial X^*}) d\eta \quad (A-4)
$$

To evaluate Equation (A-4), one needs the expressions for $W$ and $\partial W/\partial X^*$. The velocity solution is given by Equation (43), namely,

$$
W(X^*, \eta) = \frac{M (\text{Cosh } M \eta - \text{Cosh } M)}{\text{Sinh } M - M \text{ Cosh } M} \quad (A-5)
$$

$$
+ \sum_{i=1}^{\infty} \frac{C_i}{\alpha_i} (1 - \frac{\text{Cos} \alpha_i \eta}{\text{Cos} \alpha_i}) \exp[-(\alpha_i^2 + M^2)X^*] + \sum_{n=1}^{\infty} D_n \text{Sin}(n\pi \eta) \cdot \exp[-(n^2 \pi^2 + M^2)X^*] \quad (A-6)
$$

from which one finds

$$
\frac{\partial W}{\partial X^*} = - \sum_{i=1}^{\infty} \frac{C_i}{\alpha_i} (\alpha_i^2 + M^2) (1 - \frac{\text{Cos} \alpha_i \eta}{\text{Cos} \alpha_i}) \exp[-(\alpha_i^2 + M^2)X^*] \quad (A-7)
$$

$$
- \sum_{n=1}^{\infty} D_n (n^2 \pi^2 + M^2) \text{Sin}(n\pi \eta) \exp[-(n^2 \pi^2 + M^2)X^*] \quad (A-8)
$$

Substituting Equations (A-5) and (A-6) into Equation (A-4), one obtains
\[ \Delta(X^*,M) = \int_{-1}^{1} \left[ \frac{M(CoshMn-CoshM)}{SinhM-MCoshM} + \sum_{i \neq l}^{\infty} \frac{C_i}{\alpha_i^2} \frac{Cos\alpha_i}{(1-Cosh\alpha_i)} \exp\left\{ -(\alpha_i^2+M^2)X^* \right\} \right. \\
+ \sum_{n=1}^{\infty} D_n \sin(n\pi \eta) \exp\left\{ -(n^2 \pi^2+M^2)X^* \right\} \left\{ \sum_{i=1}^{c} \frac{C_i}{\alpha_i} (\alpha_i^2+M^2) \right\} \\
\left. \cos\alpha_i \sin(n\pi \eta) \right\} \] (A-7)

Equation (A-7) can be integrated term by term. The major integrals involved are expressed as follows:

\[ \int_{-1}^{1} \frac{M(CoshMn-CoshM)}{SinhM-MCoshM} \frac{1}{\alpha_i} \frac{1}{(1-Cosh\alpha_i)} d\eta = \frac{2 \alpha_i}{\alpha_i^2+M^2} \]

\[ \int_{-1}^{1} \frac{M(CoshMn-CoshM)}{SinhM-MCoshM} \sin(n\pi \eta) d\eta = 0 \]

\[ \int_{-1}^{1} \frac{1}{\alpha_i} \left( \frac{1}{1-Cosh\alpha_i} \right) \frac{1}{\alpha_j} \left( \frac{1}{1-Cosh\alpha_j} \right) d\eta = \begin{cases} 0 & , i \neq j \\ 1 & , i=j \end{cases} \]

\[ \int_{-1}^{1} \sin(n\pi \eta) \sin(m\pi \eta) d\eta = \begin{cases} 1 & , m=n \\ 0 & , m \neq n \end{cases} \]

and
\[ \int_{-1}^{1} \left( 1 - \frac{\cos \alpha_i \eta}{\cos \alpha_i} \right) \sin(n \pi \eta) \, d\eta = 0 \]

With these, Equation (A-7) becomes

\[
\Delta (X^*, M) = 2 \sum_{i=1}^{\infty} C_i \alpha_i \exp \left[ -(\alpha_i^2 + M^2) X^* \right] \\
+ \sum_{i=1}^{\infty} C_i^2 (\alpha_i^2 + M^2) \exp \left[ -2 (\alpha_i^2 + M^2) X^* \right] \\
+ \sum_{n=1}^{\infty} D_n^2 (n^2 \pi^2 + M^2) \exp \left[ -2 (n^2 \pi^2 + M^2) X^* \right]
\]

which is Equation (51) in the text.
Appendix B

Derivation of Equation (55)

The function $K(X)$ is expressed by Equation (54) in terms of the velocity solution described by Equation (43). Equation (55) can be obtained by substituting Equation (43) into Equation (54) and carrying out the integration. This is done as follows. One begins by evaluating $\partial W^*/\partial \eta$ at $\eta = 1$ and $-1$ from the velocity solution,

$$W(X^*, \eta) = \frac{M(CoshM\eta - CoshM)}{SinhM - MCoshM} + \frac{\partial}{\partial \eta} \sum_{i=1}^{\infty} \frac{C_i}{\cos \alpha_i} \left( \frac{\cos \alpha_i \eta}{\cos \alpha_i} \right) \exp \left[-(\alpha_i^2 + M^2)X^* \right]$$

$$+ \sum_{n=1}^{\infty} D_n \sin(n\pi \eta) \exp \left[-(n^2 \pi^2 + M^2)X^* \right]$$

This gives

$$\frac{\partial W^*}{\partial \eta} = \sum_{i=1}^{\infty} C_i \frac{\sin \alpha_i \eta}{\cos \alpha_i} \exp \left[-(\alpha_i^2 + M^2)X^* \right] + \sum_{n=1}^{\infty} D_n (n\pi) \cos(n\pi \eta) \exp \left[-(n^2 \pi^2 + M^2)X^* \right]$$

and

$$\left( \frac{\partial W^*}{\partial \eta} \right)_1 - \left( \frac{\partial W^*}{\partial \eta} \right)_{-1} = 2 \sum_{i=1}^{\infty} C_i \alpha_i \exp \left[-(\alpha_i^2 + M^2)X^* \right]$$
Substitution of Equations (B-1) and (B-3) into Equation (54), namely,

\[ K(X) = \int_{0}^{1} W^{2} \, d\eta - \int_{-1}^{1} W_{0}^{2} \, d\eta - \int_{-1}^{1} \epsilon \left( \frac{\partial W^{*}}{\partial \eta} \right)_{0}^{1} \left( \frac{\partial W^{*}}{\partial \eta} \right)_{-1}^{0} \, dx^{*} \]  \hspace{1cm} (B-4)

\[ K(X) = \left\{ \frac{M(CoshM\eta - CoshM)}{SinhM - MCoshM} + \sum_{i=1}^{n} \frac{C_{i}}{\alpha_{i}} \left(1 - \frac{\cos \alpha_{i} \eta}{\cos \alpha_{i}} \right) \exp \left[ -(\alpha_{i}^{2} + M^{2})X^{*} \right] \right\}^{2} \]  \hspace{1cm} (B-5)

After the integration is performed, Equation (B-5) can be reduced to a simpler form. The major integrations are

\[ \int_{-1}^{1} \frac{M^{2}(CoshM\eta - CoshM)^{2}}{(SinhM - MCoshM)^{2}} \, d\eta \]

\[ = \frac{M(2MCosh^{2}M - 3SinhMCoshM + M)}{(SinhM - MCoshM)^{2}} \]  \hspace{1cm} (B-6)
By making use of Equations (B-6) through (B-11), one obtains, after some arrangement, the final expression for $K$ in the form
\[ K(X) = \frac{M(2MCosh^2M - 3CoshMSinhM + M)}{(SinhM - MCoshM)^2} \]

\[ + 4 \sum_{i=1}^{\infty} \frac{C_i \alpha_i}{\alpha_i^2 + M^2} \exp[-(\alpha_i^2 + M^2)X^*] \]

\[ + \sum_{i=1}^{\infty} \frac{C_i^2}{\alpha_i^2 + M^2} \exp[-2(\alpha_i^2 + M^2)X^*] \]

\[ + \sum_{n=1}^{\infty} D_n \exp[-2(n^2 \pi^2 + M^2)X^*] \]

\[ - \int_{-1}^{1} W_0 \, d\eta \, -2 \int_{0}^{X^*} \varepsilon \{ \sum_{i=1}^{\infty} C_i \alpha_i \exp[-(\alpha_i^2 + M^2)X^*] \} \, dX^* \]

This is Equation (55) in the text.
VII. VITA

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He was graduated from Tainan First High School, Tainan, Formosa, in June 1962. He entered Cheng Kung University in September 1962 and received his Bachelor of Science degree in Mechanical Engineering in June 1966. After graduation from the university, he served one year under the ROTC program in the Nationalist Chinese 60th Arsenal in Kao Hsiung, Formosa. From August 1967 to September 1968 he worked in Seng Ta Industrial Company as a design engineer in Taipei, Formosa.

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