Application of approximate transmission matrices to describe transverse beam vibrations

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APPLICATION OF APPROXIMATE TRANSMISSION MATRICES
TO DESCRIBE TRANSVERSE BEAM VIBRATIONS

BY

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ABSTRACT

This paper is a report of an investigation of an approximate method for finding the principal mode frequency roots for beams in transverse vibration. The method utilizes approximate transmission matrices obtained by a power series expansion of the basic differential equation which governs the transmission matrix.

Investigation has been carried out to examine the efficiency of the method in producing the first several normal mode frequency roots. This has been achieved by applying the method to several uniform and non-uniform beams and comparing the results with the exact solutions. The technique has further been applied to non-uniform beams of tapered rectangular cross section and the results are given in non-dimensional form for use in practical application.

Frequency root errors obtained by this method when applied to uniform beams of Fixed-Fixed and Fixed-Free ends are seen to be proportional to $1/N^2$ (N, the number of segments), when two terms in the series are considered and N is large. With three and four series terms errors are proportional to $1/N^3$. The method utilizes only two variables of the beam to be analyzed, the cross sectional area and the area moment of inertia. Progressively better results are obtained by increasing the number of terms in the series.
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LIST OF SYMBOLS

A  =  Area of cross section
[A] = Matrix which defines state vectors
D  =  Ratio of the depth at output end to that at input end
E  =  Young's Modulus of Elasticity
e  =  Percentage frequency root error
g  =  Acceleration due to gravity
H  =  Ratio of the width at output end to that at input end
h  =  Width of the beam
I  =  Area moment of inertia
M  =  Bending Moment
N  =  Number of Segments
T_{ij} = Element of transmission matrix
[T] = Transmission matrix
V  =  Shear force
w  =  Vertical displacement
\phi  =  Slope of the beam
x  =  Distance along the length of the beam
TM = Numbers of terms in the series
\rho  =  Mass density per unit volume
L  =  Length of the beam
{\psi} = State vector (column matrix)
INTRODUCTION

The mathematical theory for vibration problems involving uniform and non-uniform continuous beams has been in existence for some time. Systems which are repeatedly confronted in practice involve beams of non-uniform and non-continuous cross sections, mass per unit length, loading, etc., and are very difficult to solve exactly. People have, therefore, tried to solve problems of this sort by various techniques. One common method used is the lumped parameter approximation whereby the beam is replaced by a finite N degree of freedom system consisting of lumped elements, i.e., massless springs, point masses, etc. Another approach is to obtain an approximate transmission matrix for a beam segment in terms of the governing matrix which defines the state vector of the beam element.

This later technique has been suggested and investigated for one-dimensional systems by Rocke [1]. The objective of this work is to apply this technique in determining the normal mode frequency roots for different types of uniform and non-uniform beams in transverse vibration.

In Chapter I the concept of the transmission matrix has been explained. Taking a general transmission element, the governing matrix is evaluated and the series expansion of the transmission matrix has been discussed. The transmission matrix has been evaluated for steady state sinusoidal variation of forces and motions which are transmitted through a linear elastic element.
In Chapter II the accuracy of the method has been examined on the basis of the frequency root errors for uniform beams with Fixed-Fixed and Fixed-Free ends. To qualitatively evaluate the accuracy, the approximate principal mode frequency roots have been compared with the corresponding exact solutions. These comparisons are displayed as a plot of the percentage frequency root error against the number of segments in the beam.

Chapter III deals with the application of this method to non-uniform beams. First, the transmission matrix is evaluated for a class of non-uniform elements. Secondly, the frequency roots have been calculated for several different cases of non-uniform beams. These results have been compared to those obtained by other investigators. H. H. Mabie [4] has calculated dimensionless frequency roots for beams of tapered rectangular sections with fixed-simply supported ends. G. W. Housner [3] has worked on tapered, cantilever beams of rectangular and circular sections. The results obtained by this method have been compared with their exact solutions and frequency root errors calculated. Finally, dimensionless frequency roots, for beams of tapered rectangular sections with fixed-fixed and pinned-pinned ends, have been found and are given in table and plot form such that they can be used in practical vibration problems.
1.0 Concept of Transmission Matrices

A transmission matrix relates the state vector, composed of forces and displacements, at the output end of an element with the state vector at the input end. It describes the manner in which sinusoidal forces and motions are transmitted through a linear elastic element during steady state conditions.

A general transmission element is shown below. The state vector, \(\{\psi\}\), is a column vector consisting of elements which describe the forces (and/or moments) and displacements (translational or angular) at the point of interest.

\[
\begin{align*}
\{\psi\}_{\text{input}} &= [T] \{\psi\}_{\text{output}} \\
\text{Fig. 1 General transmission element}
\end{align*}
\]

The form of the transmission matrix commonly used is given by:

\[
\{\psi\}_{\text{input}} = [T] \{\psi\}_{\text{output}}
\]

where \([T]\) is commonly designated the "Forward Transmission Matrix". The arrows in Fig. (1) indicate the direction of positive forces and displacements. The forces in the respective state vectors are those applied to the input and those applied by the output ends.

When the elements considered are the segments of a continuous body like a beam, they are in an end-to-end or chain-like arrangement.
for which the transmission matrix approach is best suited. The transmission matrix for a system composed of N segments has the following characteristics:

![Diagram of a beam segmented into N elements](image)

Fig. 2 Beam segmented into N elements

Employing the definition for the transmission matrix eq. (1.1) to the segments in Fig. (2)

\[ \{\psi_1\} = [T_1] \{\psi_2\} \]

Since from Fig. (2)

\[ \{\psi_1\}_{\text{input}} = [T_1] \{\psi_1\}_{\text{output}} \]

and

\[ \{\psi_1\}_{\text{output}} = \{\psi_2\}_{\text{input}} \]

similarly:

\[ \{\psi_2\} = [T_2] \{\psi_3\} \]
\[ \{\psi_3\} = [T_3] \{\psi_4\} \]
\[ \hdots \]
\[ \{\psi_{n-1}\} = [T_{n-1}] \{\psi_n\} \]

where the state vector at any point is designated as:

\[ \{\psi_i\} \text{ for } i = 1, 2, 3, 4, \ldots, n \]

representing the state vector of the ith segment at the input end. Thus, for the total end-to-end arrangement:

\[ \{\psi_1\} = [T_1][T_2][T_3] \ldots [T_{n-1}]{\psi_n} \] (1.2)

which is obtained by successive substitution of state vectors. It is
worth noting that the state vector at the output end of one segment is the same as that at the input end of the next, as indicated above.

As indicated in the introduction, this work deals only with transverse vibrations of straight beams in which case the displacement at a point $i$ is $w_i$ and slope is $\phi_i$. The internal forces at this point are the bending moment $M_i$ and the transverse shear force $V_i$ which are associated with the slope $\phi_i$ and displacement $w_i$, respectively. Hence, the state vector at point $i$ for the problem of interest has four components and is given by:

$$\{\psi\}_i = \begin{bmatrix} V \\ M \\ w \\ \phi \end{bmatrix}$$

where $i$ refers to a particular point. (1.3)

To treat uniform and non-uniform beams via the transmission matrix approach, we consider a straight beam as represented in Fig. (3) of length $\lambda$, divided into $N$ segments. In the following chapters an explanation as to the formation of the transmission matrix for different cases is given. The transmission matrices for the individual segments are different in the case of non-uniform beams and they are identically the same in the case of the uniform beam. In this work, the transmission matrix has been utilized to obtain the characteristic determinant for particular end conditions from which frequency roots
are calculated. Considering the beam shown in Fig. (3) and applying Eq. (1.2) for the total end-to-end arrangement, the state vector at the input end can be related to that at the output end as:

\[ \{\psi\}_{\text{input}} = [T_1][T_2][T_3] \ldots [T_{n-1}][T_n]\{\psi\}_{\text{output}} \]

or

\[ \{\psi\}_{\text{input}} = [T]\{\psi\}_{\text{output}} \] (1.4)

where

\[ [T] = [T_1][T_2] \ldots [T_{n-1}][T_n] \]

and is called the total transmission matrix, which is the product of the N matrices for the N segments of the beam. In the above \([T_i]\) represents the transmission matrices for the individual segments and are each of the same order. Let these matrices be of order \(m \times n\).

In order that we can multiply \([T_i]\) and \([T_{i+1}]\) of order \(m \times n\) and \(m \times n\), \(m\) has to be equal to \(n\). Therefore, \([T_i]\) are necessarily the matrices of order \(n \times n\) and the product matrix \([T]\) is also of the same order \(n \times n\). In the case of transverse vibration, \(\{\psi\}\) is a Column matrix with four elements and Eq. (1.4) becomes:

\[ \begin{bmatrix} 1 \\ 4 \end{bmatrix} \{\psi\}_{\text{input}} = n[T]\begin{bmatrix} 1 \\ 4 \end{bmatrix} \{\psi\}_{\text{output}} \]

Conformity with the matrix multiplication rule requires that the matrix \([T]\) in Eq. (1.4) has to be of order 4 by 4. Therefore, Eq. (1.4) may be written as:

\[ \begin{bmatrix} V \\ M \\ w \\ \phi_{\text{input}} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{bmatrix} \begin{bmatrix} V \\ M \\ w \\ \phi_{\text{output}} \end{bmatrix} \] (1.5)

The total transmission matrix is used to form the characteristic determinant for various boundary conditions, e.g., for fixed and free ends. The case of a Fixed-Free beam where the fixed end
is the input and the free end is the output is shown in Fig. (4).

```
M≠0  W=0  M=0  W≠0
V≠0  Φ=0  V=0  Φ≠0
```

Fig. 4 Beam with Fixed-Free ends

Characteristics of fixed and free boundary conditions are that bending moments and shearing forces at free ends are zero, whereas at fixed ends deflections and slopes are zero. Substituting these conditions into Eq. (1.5) for a fixed-free beam gives:

\[
\begin{bmatrix}
V_i \\
M_i \\
0 \\
0
\end{bmatrix} =
\begin{bmatrix}
T_{11} & T_{12} & T_{13} & T_{14} \\
T_{21} & T_{22} & T_{23} & T_{24} \\
T_{31} & T_{32} & T_{33} & T_{34} \\
T_{41} & T_{42} & T_{43} & T_{44}
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
w_o \\
ϕ_o
\end{bmatrix}
\]

(1.6)

where the subscripts \(i\) and \(o\) represent input and output quantities, respectively.

Writing the equations from expression (1.6) separately gives:

\[
\begin{align*}
V_i &= T_{13} w_o + T_{14} ϕ_o \\
M_i &= T_{23} w_o + T_{24} ϕ_o \\
0 &= T_{33} w_o + T_{34} ϕ_o \\
0 &= T_{43} w_o + T_{44} ϕ_o
\end{align*}
\]

(1.7)

The last two equations of (1.7) can be satisfied with nonzero values of \(w_o\) and \(ϕ_o\) only if the determinant formed by the coefficient of these two variables is equal to zero, that is:

\[
\begin{vmatrix}
T_{33} & T_{34} \\
T_{43} & T_{44}
\end{vmatrix} = 0
\]

(1.8a)
Therefore,

\[ T_{33} \cdot T_{44} - T_{34} \cdot T_{43} = 0 \]  

(1.8b)

which is the characteristic equation for Fixed-Free ends. The total transmission matrix is formed by multiplying in order the transmission matrices for N segments of the beam. The transmission matrix for each individual segment involves elements including \( \omega^2 \). Therefore, the characteristic Eq. (1.8b) becomes a polynomial of \( \omega^2 \) to some power. The roots of the characteristic equation are the normal mode frequencies for the system with the specific end conditions.

Other end conditions and the resulting characteristic determinants are given below.

### Fixed-Free

\[ \begin{align*}
    w &= 0 \\
    \phi &= 0 \\
    M &= 0 \\
    V &= 0
\end{align*} \]

Characteristic determinant

\[ \begin{vmatrix}
    T_{33} & T_{34} \\
    T_{43} & T_{44}
\end{vmatrix} = 0 \]  

(1.9a)

### Fixed-Fixed

\[ \begin{align*}
    w &= 0 \\
    \phi &= 0 \\
    \phi &= 0
\end{align*} \]

Characteristic determinant

\[ \begin{vmatrix}
    T_{31} & T_{32} \\
    T_{41} & T_{42}
\end{vmatrix} = 0 \]  

(1.9b)

### Fixed-Simply supported

\[ \begin{align*}
    w &= 0 \\
    \phi &= 0 \\
    M &= 0
\end{align*} \]
Characteristic determinant
\[
\begin{vmatrix}
T_{31} & T_{34} \\
T_{41} & T_{44}
\end{vmatrix} = 0 \quad (1.9c)
\]

Pinned-Pinned
\[w = 0 \quad w = 0\]
\[M = 0 \quad (d) \quad M = 0\]

Characteristic determinant
\[
\begin{vmatrix}
T_{21} & T_{24} \\
T_{31} & T_{34}
\end{vmatrix} = 0 \quad (1.9d)
\]

Free-Free
\[M = 0 \quad M = 0\]
\[V = 0 \quad (e) \quad V = 0\]

Characteristic determinant
\[
\begin{vmatrix}
T_{13} & T_{14} \\
T_{23} & T_{24}
\end{vmatrix} = 0 \quad (1.9e)
\]

Fig. 5 Beams with various end conditions and the corresponding characteristic determinants.

Several of the above characteristic determinants have been used in later sections.
1.1 Definition of State Vector

In the previous section the relationship between state vectors at two points through the transmission matrices has been shown. In this section the relationship between the different components of the state vector for a Bernoulli-Euler beam element is described. For a straight beam in transverse vibration, the displacements of interest at point \( i \) are the transverse deflection \( w_i \) and the slope \( \phi_i \). The internal forces are the moment \( M_i \) corresponding to slope \( \phi_i \) and the shear force \( V_i \) corresponding to the displacement \( w_i \). The state vector in this case has four components:

\[
\{ \psi(x) \} = \begin{bmatrix} V \\ M \\ w \\ \phi \end{bmatrix}
\]  

(1.11)

In order to derive expressions relating various components of the state vector, an elastic beam element of length \( dx \) has been considered (see Figure 6). The assumptions made consistent with the elementary Bernoulli-Euler beam theory are:

1. Shear deflection and rotary inertia effects are small and negligible.
2. \( \rho \) and \( E \) are assumed to be constants.
3. Area \( A(x) \), and stiffness \( EI(x) \) are variables.

The coordinate system and sign convention used are as follows. The righthanded cartesian coordinate system has been used, \( x \) axis coincides with the longitudinal centroidal axis of the element. Positive direction of the coordinate system has been indicated by the direction of the arrows. Positive displacements coincide with the positive direction of the coordinate system.
Fig. 6 Non-uniform elastic beam element.

Fig. (6) shows the elemental segment under consideration. For a negative displacement \( w \) of the element the forces and moments acting are as shown in Fig. (6). The inertia force is mass \( x \) acceleration acting opposite to displacement (downward). The acceleration is given by

\[
\frac{d^2w}{dt^2} = \ddot{w}
\]

Assuming sinusoidal displacements

\[ w = w_0 \sin \omega t \]

Therefore,

\[ \ddot{w} = -\omega^2 w \]

Summing forces on the \( dx \) element for dynamic equilibrium gives:

\[
V - V - \frac{\partial V}{\partial x} dx + \rho A(x)\omega^2 w dx = 0
\]

or

\[
\frac{dv}{dx} = \rho A(x) \omega^2 w
\]

Summing moments acting on the element about point A gives:

\[
M + \rho A(x)\omega^2 w dx \left( \frac{dx}{2} \right) - M - \frac{\partial M}{\partial x} dx - Vdx - \frac{\partial v}{\partial x} (dx)^2 = 0
\]

Neglecting terms involving \( (dx)^2 \) this reduces to:

\[
\frac{dM}{dx} = -V
\]

We also have from basic mechanics of materials:

\[
\frac{dw}{dx} = \phi, \quad \frac{d^2w}{dx^2} = \frac{d\phi}{dx}, \quad EI(x) \frac{d^2w}{dx^2} = -M
\]
or \[ \frac{d\phi}{dx} = -\frac{M}{EI(x)} \]

Collecting these terms and expressing them in matrix form gives:

\[
\begin{align*}
\frac{d}{dx} \begin{bmatrix} V \\ M \\ w \\ \phi \end{bmatrix} &= \begin{bmatrix} 0 & 0 & \rho A(x) \omega^2 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1/EI(x) & 0 & 0 \end{bmatrix} \begin{bmatrix} V \\ M \\ w \\ \phi \end{bmatrix} \\
\end{align*}
\]

Therefore, the differential equation for the state vector becomes:

\[
\frac{d}{dx} \{\psi(x)\} = [A(x)] \{\psi(x)\} \tag{1.14}
\]

where

\[
A(x) = \begin{bmatrix} 0 & 0 & Z(x) & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -Y(x) & 0 & 0 \end{bmatrix} \tag{1.15a}
\]

and

\[
Z(x) = \rho A(x) \omega^2 \tag{1.15b}
\]

\[
Y(x) = 1/EI(x)
\]
1.2 Direct Derivation of Transmission Matrices

The technique used herein to obtain the transmission matrix is relatively recent, used by Pipes [8] and Rocke [1] and is pertinent to the expansion methods to be employed to obtain approximate transmission matrices herein.

As derived in section (1.1), the differential equation which defines the state vector is given by:

\[
\frac{d}{dx} \{\psi(x)\} = [A(x)] \{\psi(x)\}
\] (1.20)

The matrix \([A(x)]\) has been shown to be entirely determined by the differential equations which govern a dx increment of the system being considered during steady state sinusoidal oscillation.

The forward transmission matrix, as it relates the input state vector of a continuous element to the state vector at any particular point \(x\) along the element, is given by:

\[
\{\psi(o)\} = [T(x)] \{\psi(x)\}
\] (1.21)

where \(\{\psi(o)\}\) represent the state vector at the input end, i.e., at \(x = 0\).

Differentiating the above with respect to \(x\) gives

\[
0 = [T(x)]' \{\psi(x)\} + [T(x)] \{\psi(x)\}'
\] (1.22)

where \('\) denotes \(\frac{d}{dx}\), differentiation with respect to \(x\).

Replacing \(\{\psi(x)\}'\) by Eq. (1.20), Eq. (1.22) reduces to:
\[ 0 = [T'(x)] \{\psi(x)\} + [T(x)][A(x)]\{\psi(x)\} \]

which upon elimination of \( \psi(x) \) gives:

\[
\frac{d}{dx} [T(x)] = - [T(x)][A(x)]
\]

(1.23)

Therefore, the transmission matrix is given directly by Eq. (1.23).

By shrinking \( \Delta x \to 0 \) in Eq. (1.21) the first initial condition becomes:

\[ \{\psi(0)\} = [T(0)]\{\psi(0)\} \]

or

\[ [T(0)] = [I] \]

(1.24)

Substituting this result into Eq. (1.23) gives:

\[ [T(0)'] = - [A(0)] \]

(1.25)

By differentiating Eq. (1.23), similar results are obtained for the second derivative:

\[
\frac{d}{dx} [T'(x)] = - [T'(x)] [A(x)] - [T(x)][A'(x)]
\]

but

\[ [T'(0)] = - [A(0)] \]

.: \[ [T''(0)] = [A(0)]^2 - [A'(0)] \]

(1.26)

Differentiating the basic equation again gives:

\[
\frac{d}{dx} [T''(x)] = - [T''(x)][A(x)] - [T'(x)][A'(x)] - [T'(x)][A'(x)] - [T(x)][A''(x)]
\]

\[ [T''(0)] = - [T''(0)][A(0)] - 2[T'(0)][A'(0)] - [T'(0)][A''(0)] \]

Substituting for \([T''(0)]\) and \([T(0)]\) gives:

\[ [T''(0)] = - \{[A(0)]^2 - [A'(0)]\} \ [A(0)] + [A(0)][A'(0)] + [A(0)][A'(0)] - [A''(0)] \]

\[ [T''(0)] = - [A(0)]^3 + [A'(0)][A(0)] + 2[A(0)][A'(0)] - [A''(0)] \]

(1.27)

This process can be continued to obtain as many initial conditions as required to evaluate the constants which arise in solving Eq. (1.23).
1.3 Power Series Expansion of Transmission Matrices

Pestel and Leckie [2] have briefly described how the Runge-Kutta and Picard iteration methods can be employed to numerically integrate the differential equations which define the state vector;

$$\frac{d}{dx} \{\psi\} = [A(x)]\{\psi\}$$  

for non-uniform continuous systems resulting in a transmission matrix. Another approach suggested by Rocke [1] is to use a Maclaurin series expansion which utilizes the following differential equation

$$\frac{d}{dx} [T(x)] = - [T(x)][A(x)]$$  

which is the basic definition of the transmission matrix. This approach appears attractive because of the fact that it utilizes the known variables $\rho A(x)$ and $EI(x)$ which appear as the only two variable terms in the governing matrix $[A(x)]$. This method, unlike common lumped parameter approximations, does not rely upon finding equivalent uniform cross sections or lumping masses at discrete points according to any established manner. However, this method is still an approximation whose accuracy depends upon the number of terms in the expansion series considered.

Considering the transmission matrix for one segment of a beam and expanding $[T(x)]$ in a Maclaurin series about the origin of the segment gives:

$$[T(x)] = [T(o)] + x[T'(o)] + \frac{x^2}{2!} [T''(o)] + \frac{x^3}{3!} [T'''(o)] + \ldots \ldots \ldots \ldots \ldots \text{higher order terms}$$  

(1.32)

The condition for the Maclaurin series that the function be piecewise analytic in this case requires through Eq. (1.31) that the variables in $[A(x)]$ be piecewise analytic. In particular they must be
analytic in the region between the points \( i \) and \( i+1 \) for which the transmission matrix is being approximated. By definition

\[
[T(o)] = [I], \text{ the identity matrix} \tag{1.33}
\]

Using the first three derivatives of \([T(x)]\) from Eqs. (1.25, 1.26, 1.27) and evaluating at the origin of the increment, Eq. (1.32) becomes:

\[
[T(x)] = [I] - x[A(o)] + \frac{x^2}{2!} \{[A(o)]^2 - [A'(o)]\} + \frac{x^3}{3!} \{-[A(o)]^3 + [A'(o)][A(o)] + 2[A(o)[A'(o)] - [A''(o)]\} + \ldots \ldots \tag{1.34}
\]

The notation \([\quad]\) designates a matrix. It is to be kept in mind that in the case of transverse beam vibrations this is a matrix of order four. Therefore, the algebraic sum of the right hand side results in a matrix consisting of four rows and four columns which becomes the desired transmission matrix for a particular segment. The value of \( x \) is to be replaced by the length of the segment in Eq. (1.34).

Since higher order terms are neglected this method becomes a low frequency approximation for the system, because of the fact that every \([A]\) or its derivative contains the square of the frequency, \( \omega^2 \).

It is conceivable at this point how the transmission matrix is constituted. As stated earlier, the variables in the governing matrix are area, \( A(x) \), and area moment of inertia of the section about an axis perpendicular to the neutral axis along the length, \( I(x) \). For beams of uniform section these terms are constant and therefore,

\[
[A'(x)] = [A''(x)] = [A'''(x)] = \ldots \ldots \ldots = 0 \tag{1.35}
\]

For the uniform beam, Eq. (1.34) then reduces to:

\[
[T(x)] = [I] - x[A(o)] + \frac{x^2}{2!} \{[A(o)]^2 - \frac{x^3}{3!} [A(o)]^3 + \ldots \ldots \tag{1.36}
\]

The above series has an infinite number of terms. For application to practical problems and to avoid cumbersome arithmetic, a finite
number of terms are considered. In general, results have confirmed that the accuracy of \([T(x)]\) is improved as more terms are included in the series. The primary objective of this work is to investigate this convergence behavior more thoroughly for uniform and non-uniform beams.
CHAPTER II

EVALUATION OF THE APPROXIMATE TRANSMISSION MATRIX

FOR UNIFORM BERNOULLI-EULER BEAM ELEMENTS

The derivation of the transmission matrix for a general beam element has been shown in the previous chapter. Equation (1.34) represents the transmission matrix in terms of an infinite series. In practice, to evaluate the transmission matrix only a finite number of terms in the series are used. Since only a fixed number of terms in the series are considered, the resulting matrix becomes an approximate transmission matrix. To examine the accuracy of this approximation work was carried out first with the uniform beam. Availability of the exact solutions for the uniform beam with various end conditions provides a basis of comparison for the results given by this method. To qualitatively evaluate the accuracy of a method for the solution of vibration problems, a common practice is to compare the frequency root obtained by the method with the corresponding exact solution. These comparisons are displayed as a plot of the percentage frequency root error against the number of segments in the beam. This procedure determines a gross level of error and illustrates the convergence to the exact solution.

2.0 The Approximate Transmission Matrix

The approximate transmission matrix evaluated herein consists of the first two, three, or four terms of the series solution. As derived previously, the expansion series for the uniform beam takes the form:
\[ [T(x)] = [I] - x[A(o)] + \frac{x^2}{2!} [A(o)]^2 - \frac{x^3}{3!} [A(o)]^3 + \ldots \]  \hspace{1cm} (2.1)

The governing matrix for the uniform beam element from Eq. (1.2) becomes:

\[
[A(x)] = \begin{bmatrix}
0 & 0 & \rho A(x) \omega^2 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & -1/\text{EI}(x) & 0 & 0 \\
\end{bmatrix}
\]  \hspace{1cm} (2.2)

where

\[ \rho = \text{density of the material of the beam.} \]

\[ A = \text{area of cross section which is constant.} \]

\[ I = \text{moment of inertia of the section area about an axis} \]
\[ \quad \text{perpendicular to plane of the figure.} \]

\[ E = \text{Young's modulus of elasticity of the material.} \]

Throughout this paper in the areas where numerical calculations have been made the value of \( E = 30 \times 10^6 \) psi and \( \rho = 0.339 \) lb/in\(^3\) has been taken for steel material. All results have been cast into non-dimensional form, however, to apply to general cases.

Since area \( A \), and moment of inertia \( I \) are constants, the governing matrix \([A]\) is constant, i.e., \([A]\) is independent of \( x \). Writing \( a = \rho A \omega^2 \) \( \quad b = 1/\text{EI} \). The governing matrix becomes:
\[
[A(o)] = \begin{bmatrix}
0 & 0 & a & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & -b & 0 & 0
\end{bmatrix}
\] (2.3)

and

\[
[A(o)]^2 = [A(o)][A(o)] = \begin{bmatrix}
0 & 0 & 0 & a \\
0 & 0 & -a & 0 \\
0 & -b & 0 & 0 \\
b & 0 & 0 & 0
\end{bmatrix}
\] (2.4)

\[
[A(o)]^3 = [A(o)]^2[A(o)] = \begin{bmatrix}
0 & ab & 0 & 0 \\
0 & 0 & 0 & -a \\
b & 0 & 0 & 0 \\
0 & 0 & ab & 0
\end{bmatrix}
\] (2.5)

In Eq. (2.1) \([T(x)]\) is the transmission matrix of 4 by 4 elements. Substituting the values of \([A(o)], [A(o)]^2\) etc., in this equation gives:

\[
[T(x)] = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} - x \begin{bmatrix}
0 & 0 & a & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & -b & 0 & 0
\end{bmatrix} + x^2 \begin{bmatrix}
0 & 0 & 0 & a \\
0 & 0 & -a & 0 \\
0 & -b & 0 & 0 \\
b & 0 & 0 & 0
\end{bmatrix} - \frac{x^3}{6} \begin{bmatrix}
0 & -ab & 0 & 0 \\
0 & 0 & 0 & -a \\
b & 0 & 0 & 0 \\
0 & 0 & ab & 0
\end{bmatrix} + \ldots \] (2.6)
Summation of right hand side gives:

\[
\begin{align*}
T_{11} &= 1 & T_{12} &= \frac{x^3}{6} ab & T_{13} &= -xa & T_{14} &= \frac{x^2 a}{2} \\
T_{21} &= x & T_{22} &= 1 & T_{23} &= \frac{x^2 a}{2} & T_{24} &= \frac{x^3 a}{6} \\
T_{31} &= \frac{x^3 b}{6} & T_{32} &= \frac{x^2 b}{2} & T_{33} &= 1 & T_{34} &= -x \\
T_{41} &= \frac{x^2 b}{2} & T_{42} &= xb & T_{43} &= -\frac{x^3 ab}{6} & T_{44} &= 1
\end{align*}
\]

Equations (2.7) represent the elements of the transmission matrix where \(x\) has to be replaced by the length of the segment of the beam.

For a beam of length \(L\), divided into \(N\) segments, \(x = L/N\). The transmission matrix so formed is for one particular segment. For the total beam \(N\) such transmission matrices (which are all the same for uniform cases) are to be multiplied to give a transmission matrix describing the whole system relating input quantities with output, or

\[
\{\psi\}_{\text{input}} = [T]\{\psi\}_{\text{output}}
\]

(2.8)

The transmission matrix given by Eq. (2.6) uses the first four terms in the power series. If two or three terms are taken, terms containing the appropriate power of \(x\) have to be included. When three terms are taken, terms which involve \(x^3\) would not be included. Hence, the transmission matrices formed with different numbers of terms are different. With three terms of the series Eq. (2.6) reduces to:

\[
[T(x)] = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
- x
\begin{bmatrix}
0 & 0 & a & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & -b & 0 & 0 \\
\end{bmatrix}
+ \frac{x^2}{2}
\begin{bmatrix}
0 & 0 & -a & 0 \\
0 & -b & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & b \\
\end{bmatrix}
\]

(2.9a)
And with two terms it reduces to:

\[
[T(x)] = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} - x \begin{bmatrix}
0 & 0 & a & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & -b & 0 & 0
\end{bmatrix}
\]  \hspace{1cm} (2.9b)

2.1 Calculation of the Frequency Roots

As mentioned in the previous section, the transmission matrices for each of the uniform beam segments are identical. With the transmission matrices as given by Eq. (2.7,2.9) at hand, the next step towards finding the frequency roots is to select the number of terms in the series and the corresponding transmission matrix. If \( N \) is the number of segments into which the beam is divided, then multiplication of \( N \) such transmission matrices will produce the total transmission matrix for the beam.

From the total transmission matrix, depending on the end condition of the beam considered, the characteristic determinants as given by Eq. (1.9) are formed. The values of \( \omega \) for which the characteristic determinant becomes equal to zero give the frequency roots of principal modes. An IBM System/360, Model 50 digital computer has been used to iteratively form the total transmission matrix and the characteristic determinant with changing values of \( \omega \) until the frequency root was obtained. Work has been carried out with beams of two different end conditions and the frequency roots obtained have been compared with the exact solutions.

The two cases of uniform beams chosen are of Fixed-Fixed ends and Fixed-Free ends. Frequency roots for these two cases have been
calculated by this method with the number of segments varying from four to twenty (in the increment of two) and for number of terms two, three and four in the series. For the computational work, the dimension and the material for the beams were chosen arbitrarily as follows:

Steel material for the beams with $E$, the Young's Modulus $= 30 \times 10^6$ psi, $\rho$, the weight density $= 488$ lb/ft$^3$. Cross section circular, with diameter of the beam $= 4\text{"}$, and the length $= 80\text{"}$

![Diagram of a beam with fixed-fixed and fixed-free ends.]

Fig. 8 Uniform beam with Fixed-Fixed and Fixed-Free ends

In the later part of this section, comparison of the frequency roots given in this method with exact solutions has been shown. The exact solutions for the frequency roots of the two cases of beams shown in Fig. (8) are found to be the following:

The characteristic equation for a beam of Fixed-Fixed ends is of the form:

$$\cosh (KL) \cdot \cos (KL) - 1 = 0$$

(2.10)

The characteristic equation for Fixed-Free ends is of the form:

$$\cosh (KL) \cdot \cos (KL) + 1 = 0$$

(2.11)

Where $L = $ length of the beam
and

\[ K = 4 \sqrt{\frac{\omega^2 \rho A}{EIg}} \]

The roots of the characteristic Equations (2.10,2.11) give the frequency roots for the respective cases. The frequency roots obtained from the above two characteristic equations for the first three principal modes are listed below:

<table>
<thead>
<tr>
<th>End Condition</th>
<th>1st Mode</th>
<th>2nd Mode</th>
<th>3rd Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed-Fixed</td>
<td>646.1064</td>
<td>1780.3271</td>
<td>3485.2165</td>
</tr>
<tr>
<td>Fixed-Free</td>
<td>101.5346</td>
<td>636.3220</td>
<td>1781.1257</td>
</tr>
</tbody>
</table>

By the approximate transmission matrix frequency roots are given by the characteristic determinants given by Eq. (1.9b) for the Fixed-Fixed ends and by Eq. (1.9a) for the Fixed-Free ends. The characteristic determinant in fact gives a polynomial in \( \omega^2 \), the solution of which gives the frequency roots.

Dimensions and all other parameters of the beam remaining constant the characteristic determinant becomes only dependent on the choice of initial selection of \( \omega \). Therefore, the characteristic determinant, \( D(\omega) \), becomes a function only of \( \omega \).

To obtain the frequency roots a digital iteration process was employed. An initial value was assumed for the frequency root and using this value, the characteristic determinant evaluated from Eq. (1.9). The value of the frequency roots was then successively changed so as to make the value of the determinant approach zero.

A general computer program was made to give the frequency root for various numbers of segments with 2, 3, and 4 series terms. With a small number of segments the root deviated nearly 100% from the exact
ones. This fact posed a difficult problem for the iteration process, because small increments in $\omega$ could not be used to cover the whole frequency range without consuming a large amount of computer time. For this iteration process the following technique was found adequate.

Suppose it is desired to find the third mode frequency root of the beam with Fixed-Free ends, as shown in Fig. (8), where the exact value is 1781.1257 RAD/SEC. (See table on page 24). Considering 10 segments of the beam and two series terms in the approximate matrix for each segment, the percentage frequency root error is found to be about 100%.

A starting value of 1500 for the frequency root was arbitrarily selected for this case and the characteristic determinant given by Eq. (1.9a) was evaluated. To reduce the number of iterations required a large increment $d\omega$ of 200 was first applied to the initial frequency root value of 1500 and $D(\omega)$ was again evaluated. This process was repeated with the same increment until the value of $D(\omega)$ changed sign from positive to negative or vice versa.

As soon as a sign change occurs $d\omega$ is made equal to $d\omega/2$ and the evaluation of $D(\omega)$ is repeated starting with the value of $\omega$ just before the sign change occurred. This process is continued each time reducing the value of $d\omega$ to half its previous value whenever a change in sign occurs for $D(\omega)$. After nine such reductions, with the initial incremental value of 200, $d\omega$ was reduced to about 0.375 which was considered to be a reasonable limit for further linear extrapolation as explained in Fig. (9).

Since this value of $d\omega$ is very small compared to the magnitude of the frequency root it is reasonable to assume that the function $D(\omega)$ varies nearly linearly with $\omega$ within this range of variation of $\omega$. 
Referring to Fig. (9), AB represents the assumed straight line for $D(\omega)$ for a variation $d\omega$ of the frequency root such that $\omega_2 = \omega_1 + d\omega$, where $d\omega$ is 0.375 and $D(\omega)$ and $D(\omega_2)$ are of opposite sign. The point of intersection of AB with the $\omega$ axis gives the desired frequency root which can be expressed by the relation:

$$\omega = \omega_1 + \frac{CD \cdot DB}{AC + DB}$$  \hspace{1cm} (2.11)

Considering the fact that the difference in the values of the frequency roots obtained by the exact method and the approximate transmission matrix method are in many cases large compared to the final value of the increment $d\omega$, the above assumption of a straight line variation for $D(\omega)$ within the range $d\omega$ does not indicate an appreciable error.

The statement above concerning the accuracy of the result, may be understood by taking a particular result. The frequency root for
Fixed-Free case with 20 segments in the beam and four terms in the series was found to be 1779.0513. Referring to Fig. (9) \( \omega_1 \) was found to be 1778.7430 and \( \omega_2 \) was \( \omega_1 + 0.375 \). The line segments AC and BD represent the values of the characteristic determinant given by Eq. (1.9a) for \( \omega_1 \) and \( \omega_2 \), respectively. Since it is clear that the solution by this method lies between \( \omega_1 \) and \( \omega_1 + 0.375 \), the root calculated by Eq. (2.11) can be assumed to be accurate to the second decimal place provided \( D(\omega) \) does not change abruptly between these two points.

Higher mode frequency roots have higher values and to reduce the number of iterations necessary to locate the first change in sign of the characteristic determinant, higher initial values of \( d\omega \) were used. For higher initial values of \( d\omega \), nine successive changes in \( d\omega \) resulted in correspondingly higher final values of \( d\omega \) for the iteration process shown in Fig. (9). Therefore, for higher modes the accuracy of the results are expected to be lower than those of lower modes.

The frequency roots obtained by this method have been compared with the respective exact solutions. The comparison has been interpreted as the percentage frequency root error, giving the extent by which the roots given by this method deviated from the exact solutions. The percentage frequency root error has been expressed in the following way:

\[
e = \frac{\omega_o - \omega_e}{\omega_e} \times 100
\]  

(2.12)

where

\( \omega_o \) = frequency root obtained by the method concerned.

\( \omega_e \) = frequency root obtained by the exact solution.

\( e \) = percentage frequency root error.
Frequency roots and percentage frequency root errors have been calculated for first, second, and third modes for Fixed-Fixed and Fixed-Free beams. Plots have been obtained for cases with two, three and four terms in the series constituting the transmission matrix for a beam element. The plot of percentage error against the number of segments in the beam has been shown in Figs. (10) for Fixed-Fixed case, and in Fig. (11) for Fixed-Free case.

2.2 Frequency Root Error Characteristics

The results for the frequency root evaluation are shown in Figs. (10) and (11) as plots of the percentage frequency root errors against the number of segments. A careful study of these plots indicates the behavior and characteristics of this approximate method.

It has been observed from the plots of the percentage frequency errors shown in Fig. (10,11) that the percentage frequency root error decreases as the number of segments increases. The more terms in the series, the better is the result. Percentage frequency error is always lower with four terms in the series than that with three and two terms in the series. This is, however, true when number of segments are equal for all the said cases. It is quite interesting to note that for the two kinds of beam considered, i.e., Fixed-Fixed and Fixed-Free ends, the frequency root obtained by taking two terms in the expansion series, given by the Eq. (1.18), was always higher than the exact solution, but with three and four terms in the series it was always lower than the exact solution. Percentage frequency root error as given by Eq. (2.12) was therefore positive with two terms in the series and negative with three and four terms in the series. For convenience in
representing the plot on semilog paper, absolute values of percentage frequency root errors are plotted against the number of segments.

Unlike the standard lumped parameter models, the number of normal modes obtainable by this method cannot be predicted from the number of segments into which the beam is divided alone. In this method, to get a frequency root in the third mode, the number of necessary segments is not necessarily three, but some other number which depends upon the number of terms in the series used to constitute the transmission matrix for the beam element. To illustrate this, let us take a beam with Fixed-Free ends. Consider the transmission matrix to have been formed with two terms in the series. The transmission matrix for a beam element in the above case as given by Eq. (2.9b) is:

\[
\begin{bmatrix}
T(x) \end{bmatrix} = \begin{bmatrix}
1 & 0 & -xa & 0 \\
x & 1 & 0 & 0 \\
0 & 0 & 1 & -x \\
0 & xb & 0 & 1 \\
\end{bmatrix}
\]  
(2.20)

where \( a = \rho A \omega^2 \)

\( b = 1/\text{EI} \)

\( x = \text{length of the beam element.} \)

The characteristic determinant for Fixed-Free ends beam as given by Eq.(1.9a) is:

\[
\begin{vmatrix}
T_{33} & T_{34} \\
T_{43} & T_{44} \\
\end{vmatrix} = 0
\]  
(2.21)

Now, to get a frequency root for the beam described above, if we select just one segment, then \( x \) becomes the length of the beam \( L \) and Eq. (2.20) represents the total transmission matrix for the beam.
Therefore, the characteristic determinant becomes:

\[
\begin{vmatrix}
1 & -x \\
0 & 1 \\
\end{vmatrix} = 0
\]  
(2.22)

Notice that this determinant has no element containing the constant 'a', therefore, it does not contain any $\omega^2$ term, showing that frequency root cannot be obtained in this case. However, when the number of segments is three, the total matrix is the product of three such matrices as given by Eq. (2.20) and becomes:

\[
[T(x)] = \begin{bmatrix}
1 & 0 & -xa & 0 \\
x & 1 & 0 & 0 \\
0 & 0 & 1 & -x \\
0 & xb & 0 & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & -xa & 0 \\
x & 1 & 0 & 0 \\
0 & 0 & 1 & -x \\
0 & xb & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & -xa & 0 \\
x & 1 & 0 & 0 \\
0 & 0 & 1 & -x \\
0 & xb & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & -xa & 0 \\
x & 1 & 0 & 0 \\
0 & 0 & 1 & -x \\
0 & xb & 0 & 1 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & x^3ab & -3xa & 3x^2a \\
3x & 1 & 13x^2a & x^3a \\
-x^3b & -3x^2b & 1 & -3x \\
3x^2b & 3xb & -x^3ab & 1 \\
\end{bmatrix}
\]  
(2.22)

In this case the characteristic determinant formed with the help of Eq. (2.21) gives:

\[
\begin{vmatrix}
1 & -3x \\
-x^3ab & 1 \\
\end{vmatrix} = 0, \text{ or } 3x^4ab = 1.
\]  
(2.23)

The above Eq. (2.23) when solved yields one frequency root. The similar process with two segments of the beam will not yield any frequency root. It is seen, therefore, that to get a first mode frequency root for Fixed-Free ends, with two terms in the series at least three segments in the beam are required.
Similarly, the minimum number of segments required for several other cases were found and are tabulated as follows:

<table>
<thead>
<tr>
<th>End Conditions</th>
<th>No. of series terms</th>
<th>1st Mode Frequency</th>
<th>2nd Mode Frequency</th>
<th>3rd Mode Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed-Free ends</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>Free fixed ends</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>Fixed fixed ends</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

Table I gives a quick means for selecting the number of segments for obtaining a particular normal mode. If for a beam of Fixed-Free ends, third mode frequency root is desired, then with the three terms in the series one has to take a minimum of four segments in the beam.

As mentioned earlier and as it is observed from the plot of the percentage frequency root error curves the frequency root error decreases as the number of segments increases. The nature of the decrease of the former with increase of the latter can be estimated by the slope of the curves. Comparison with the plots of $1/N$, $1/N^2$, $1/N^3$, $1/N^4$, etc., against $N$, the number of segments, in the same scale gave the approximate nature of variation of percentage error. As an example, consider the plot of percentage frequency error with four terms in the series, which is lowest of the three curves in Fig. (10a). This one was found very close to the plot of $1/N^3$ versus $N$ curve, showing,
therefore, that the frequency root errors for Fixed-Fixed ends beam in first mode with four terms in the series, are approximately proportional to $1/N^3$.

The errors in the natural frequencies using lumped parameter models (Rayleigh's Model and Duncan's Model) as mentioned by Rocke [1] are proportional to $1/N^4$ for large $N$, when neither end of the beam is free and to $1/N^2$ for large $N$, when one or both ends are free. In the method concerned the errors behaved differently with different number of terms. However, the dependence of errors on the number of terms shows a great deal of conformity with that of standard lumped parameter technique. From the curves shown in Fig. (10,11), the errors were found to be proportional to the number of segments $N$, in the way shown in Table (II) below:

### Table II: Variation of frequency root error with the number of segments in the beam

<table>
<thead>
<tr>
<th>End Conditions</th>
<th>No. of terms in the series</th>
<th>1st Mode Frequency</th>
<th>2nd Mode Frequency</th>
<th>3rd Mode Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed-Fixed</td>
<td>2</td>
<td>$1/N_2^2$</td>
<td>$1/N_2^2$</td>
<td>$1/N_2^2$</td>
</tr>
<tr>
<td>Fixed-Fixed</td>
<td>3</td>
<td>$1/N_3^2$</td>
<td>$1/N_3^2$</td>
<td>$1/N_3^2$</td>
</tr>
<tr>
<td>Fixed-Fixed</td>
<td>4</td>
<td>$1/N_4^2$</td>
<td>$1/N_4^2$</td>
<td>$1/N_4^2$</td>
</tr>
<tr>
<td>Fixed-Free</td>
<td>2</td>
<td>$1/N_2^2$</td>
<td>$1/N_2^2$</td>
<td>$1/N_2^2$</td>
</tr>
<tr>
<td>Fixed-Free</td>
<td>3</td>
<td>$1/N_3^2$</td>
<td>$1/N_3^2$</td>
<td>$1/N_3^2$</td>
</tr>
<tr>
<td>Fixed-Free</td>
<td>4</td>
<td>$1/N_4^2$</td>
<td>$1/N_4^2$</td>
<td>$1/N_4^2$</td>
</tr>
</tbody>
</table>
Fig. 10a First mode frequency root error for beams with Fixed-Fixed ends
Fig. 10b Second mode frequency root error for beams with Fixed-Fixed ends.
Fig. 10c Third mode frequency root error for beams with Fixed-Fixed ends
Fig. 11a First mode frequency root error for beams with Fixed-Free ends
Fig. 11b Second mode frequency root error for beams with Fixed-Free ends
Fig. 11c  Third mode frequency root error for beams with Fixed-Free ends
CHAPTER III
EVALUATION OF THE APPROXIMATE TRANSMISSION MATRIX
FOR NON-UNIFORM BERNOULLI-EULER BEAM ELEMENTS

In the previous chapter, the approximate transmission matrix with two, three, and four series terms for the uniform beam element has been examined. The objective of this chapter is similar to that of the previous except non-uniform beams are to be described. The term non-uniform is to imply strictly the non-uniformity of the cross-sectional area, rather than the material composition. There are several ways in which the cross-sectional area might vary. In this paper, two simple cases of tapered sections are considered, i.e., tapered rectangular sections and tapered circular sections.

The work in this chapter may be broadly classified into three parts. First, the transmission matrix has been formed for non-uniform cases in a manner similar to the previous cases. Secondly, the frequency roots have been calculated for several different cases of non-uniform beams and the frequency root errors have been plotted as a function of N, the number of segments in the beam. Finally, dimensionless frequency roots for different dimensions of non-uniform beams have been produced and given in curve form for use in practical vibration problems.

3.0 The Transmission Matrix for a Beam Element of Tapered Rectangular Section

The transmission matrix in Maclaurin's Series expansion form for the general case of a beam has been derived in Chapter I. Equation (1.34) shows the series expansion of the transmission matrix in terms of the
governing matrix $[A(x)]$, evaluated at input face of the element being considered, i.e., at $x = 0$, and its derivatives evaluated at the same point. In the case of non-uniform beams, $[A(x)]$ is not a constant and therefore derivatives of $[A(x)]$ exist and appear in the right hand side of the expansion series.

Taking up to four terms of the expansion series and rewriting Eq. (1.34) gives:

$$ [T(x)] = [I] - x[A(o)] + \frac{x^2}{2!} \{ [A(o)]^2 - [A'(o)] \} + \frac{x^3}{3!} \{ - [A(o)]^3 
+ [A'(o)] [A(o)] + 2[A(o)] [A'(o)] - [A''(o)] \} \quad (3.1) $$

In order to evaluate right hand side $x$ is replaced by the length of the segment and the values of $[A(o)]$, $[A'(o)]$, etc., are computed and combined according to the rules of matrix algebra. The transmission matrix has been evaluated taking a beam of tapered rectangular section as shown in Figure (12).

![Fig. 12 Non-uniform beam element of rectangular section](image)

The matrix $[A(x)]$ for such an element is given as:
\[
[A(x)] = \begin{bmatrix}
0 & 0 & \rho A(x) \omega^2 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & -1/EI(x) & 0 & 0
\end{bmatrix}
\] (3.2)

The only two variables appearing in \([A(x)]\), in Eq. (3.2) are \(A(x)\), the area of cross section of the element at any point some distance \(x\) from input face and \(I(x)\), the area moment of inertia of the same section about an axis perpendicular to the longitudinal axis of the beam.

The general expressions for the cross sectional area and area moment of inertia of the beams of tapered rectangular and circular sections have been evaluated in appendix A. These results are utilized to evaluate the quantities \([A(x)]\), \([A'(x)]\), etc. The series expansion of the transmission matrix derived in Chapter I refers to the transmission matrix for a particular beam segment. With non-uniform beams, unlike the case of uniform beams, the transmission matrices are different for different segments as \(A(x)\) and \(I(x)\) are different for each different section. If the beam in Fig. (21) is divided into \(N\) segments, then any segment taken would be similar to the original beam, with change in depth and width ratio and in length. Therefore, Eqs. (A.7) and (A.8) in appendix A are valid for any beam segment of rectangular section. For the beam segment shown in Fig. (12), Eqs. (A.7) and (A.8) reduce to:

\[
A(x) = A_1 \left(1 + C_1 x + C_2 x^2\right) \tag{3.3}
\]

\[
I(x) = I_1 \left(1 + B_1 x + B_2 x^2 + B_3 x^3 + B_4 x^4\right) \tag{3.4}
\]

where subscript \(i\) refers to the input face.
D = ratio of depth at output end to input end

H = ratio of width at output end to input end

and coefficients $C_1, C_2, B_1$, etc., are given by Eqs. (A.5) and (A.6).

Quantities $A_i$ and $I_i$ are the area of cross section and the area moment of inertia at the input face. Substituting the values of $A(x)$ and $I(x)$ from Eq. (3.3) and Eq. (3.4) in Eq. (3.2) and replacing $x$ by zero gives:

$$[A(o)] = \begin{bmatrix} 0 & 0 & \rho A_i \omega^2 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1/EI_i & 0 & 0 \end{bmatrix}$$

(3.5a)

Substituting

$$Z = A_i \rho \omega^2$$

(3.5b)

$$Y = 1/EI_i$$

(3.5c)

$$[A(o)] = \begin{bmatrix} 0 & 0 & Z & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -Y & 0 & 0 \end{bmatrix}$$

(3.5d)

Noting that differentiation of a matrix means differentiation of each of its elements with respect to the same variable, $\frac{d}{dx} [A(x)]$ is evaluated in the following manner.

$$\frac{d}{dx} [A(x)] = \frac{d}{dx} \begin{bmatrix} 0 & 0 & \rho \omega^2 A(x) & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1/EI(x) & 0 & 0 \end{bmatrix}$$

(3.6)
But
\[
\frac{d}{dx} [\rho A(x) \omega^2] = (\rho A_1) \frac{d}{dx} [1 + C_1 x + C_2 x^2]
\]
\[
= Z (C_1 + 2C_2 x)
\]  \hspace{1cm} (3.7)

and \[
\frac{d}{dx} \left[ \frac{1}{EI(x)} \right] = \frac{1}{EI_i} \left[ \frac{-1(B_1 + 2B_2 x + 3B_3 x^2 + 4B_4 x^3)}{(1 + B_1 x + B_2 x^2 + B_3 x^3 + B_4 x^4)^2} \right]
\]  \hspace{1cm} (3.8)

Substituting \(x = 0\) in Eq. (3.24) and Eq. (3.25), the Eq. (3.9) reduces to:
\[
[A(o)]' = \begin{bmatrix} 0 & C_1 Z & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & B_1 Y & 0 & 0 \end{bmatrix}
\]  \hspace{1cm} (3.10)

Since all of the elements of \([A(x)]\) except the two elements involving area \(A(x)\) and moment of inertia \(I(x)\) are constant, their derivatives are zero. It is understood, therefore, that in order to find the derivative of the matrix \([A(x)]\) we have to differentiate only the above mentioned two elements. Differentiating Eq. (3.7) and Eq. (3.8) gives:
\[
\frac{d^2}{dx^2} [\rho A(x) \omega^2] = \frac{d}{dx} [Z(C_1 + 2C_2 x)] = 2C_2 Z
\]  \hspace{1cm} (3.10a)

and
\[
\frac{d^2}{dx^2} \left[ \frac{1}{EI(x)} \right] = \frac{d}{dx} \left\{ \frac{1}{EI_i} \frac{-1(B_1 + 2B_2 x + 3B_3 x^2 + 4B_4 x^3)}{(1 + B_1 x + B_2 x^2 + B_3 x^3 + B_4 x^4)^2} \right\}
\]

or
\[
\frac{d^2}{dx^2} \left[ \frac{1}{EI(x)} \right] = \left\{ \frac{(-1)(-2)(B_1 + 2B_2 x + 3B_3 x^2 + 4B_4 x^3)}{(1 + B_1 x + B_2 x^2 + B_3 x^3 + B_4 x^4)^2} \right\}
\]

\[
- \frac{(-1)(-2)(B_1 + 2B_2 x + 3B_3 x^2 + 4B_4 x^3)^2}{(1 + B_1 x + B_2 x^2 + B_3 x^3 + B_4 x^4)^2}
\]  \hspace{1cm} (3.10b)
Substituting \( x = 0 \) in Eq. (3.10a) and Eq. (3.10b) gives:

\[
\frac{d^2}{dx^2} [\rho \omega^2 A(o)] = 2 C_2 Z
\]  
(3.11a)

\[
\frac{d^2}{dx^2} \left[ \frac{1}{EI(x)} \right] = Y (2 B_1 - 2 B_2)
\]  
(3.11b)

Differentiating Eq. (3.2) and evaluating the value at \( x = 0 \) using Eq. (3.11a, 3.11b) gives:

\[
[A(o)] = \begin{bmatrix}
0 & 0 & 2 C_2 Z & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -Y(2 B_1 - 2 B_2) & 0 & 0
\end{bmatrix}
\]  
(3.11)

It can be noted that Eq. (3.1) contains only up to second order derivatives of \( [A(x)] \). Therefore, when four terms in the series are selected, we need to evaluate only up to second order derivatives. The higher order derivatives of \( [A(x)] \) evaluated at \( x = 0 \) are, however, necessary when more than four terms in the series are considered.

From Eq. (3.1), the series expansion of the transmission matrices is written as follows:

With four terms in the series:

\[
[T(x)] = [I] - x[A(o)] + \frac{x^2}{2!} \{[A(o)]^2 - [A'(o)] \} + \frac{x^3}{3!} \{-[A(o)]^3 + [A(o)'][A(o)] \}
+ 2 [A(o)][A'(o)] - [A''(o)]
\]  
(3.12)

With three terms in the series:

\[
[T(x)] = [I] - x[A(o)] + \frac{x^2}{2!} \{[A(o)]^2 - [A(o)] \}
\]  
(3.13)

With two terms in the series:
\[ T(x) = [I] - x[A(o)]. \] 

(3.14)

With the help of Eq. (3.22) we get

\[
[A(o)]^2 = [A(o)] [A(o)] = \begin{bmatrix}
0 & 0 & 0 & Z \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & -Y & 0 & 0
\end{bmatrix} \begin{bmatrix}
0 & 0 & Z & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & -Y & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & Z \\
0 & 0 & -Z & 0 \\
0 & -Y & 0 & 0 \\
Y & 0 & 0 & 0
\end{bmatrix}
\]

(3.15)

Multiplication of the Eq. (3.5) and the Eq. (3.9) gives:

\[
[A'(o)] [A(o)] = \begin{bmatrix}
0 & 0 & C_1 Z & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & Y_B & 0 & 0
\end{bmatrix} \begin{bmatrix}
0 & 0 & Z & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & -Y & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & C_1 Z \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-YS_B & 0 & 0 & 0
\end{bmatrix}
\]

(3.16)

and

\[
[A(o)] [A(o)'] = \begin{bmatrix}
0 & 0 & Z & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & -Y & 0 & 0
\end{bmatrix} \begin{bmatrix}
0 & 0 & C_1 Z & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & Y_B & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -C_1 Z & 0 \\
0 & Y_B & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

(3.17)

Post multiplication of the Eq. (3.15) by the Eq. (3.5d) gives:

\[
[A(o)]^3 = [A(o)]^2 [A(o)] = \begin{bmatrix}
0 & 0 & 0 & Z \\
0 & 0 & -Z & 0 \\
0 & -Y & 0 & 0 \\
Y & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
0 & 0 & Z & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & -Y & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & -ZY & 0 & 0 \\
0 & 0 & -Z & 0 \\
Y & 0 & 0 & 0 \\
0 & 0 & ZY & 0
\end{bmatrix}
\]

(3.18)

In the right hand side of Eq. (3.12) substitution of the Equations (3.5d, 3.9, 3.11, 3.15, 3.16, 3.17, 3.18) gives:
\[
[T(x)] = \begin{bmatrix}
1 & \left(\frac{x^3 ZY}{6}\right) & (-xZ - \frac{x^2}{2} C_1 Z - \frac{x^3}{3} C_2 Z) & \left(\frac{x^2}{2} + \frac{x^3}{3} C_1 Z\right) \\
x & 1 & (-\frac{x^2}{2} Z - \frac{x^3}{3} C_1 Z) & \left(\frac{x^2}{2} Z\right) \\
-\frac{x^3}{6} & \left[-\frac{x^2 y}{2} + \frac{x^3}{3} Y B_1\right] & 1 & -x \\
\left(\frac{x^2}{2} Y - \frac{x^3}{6} Y B_1\right) & \left(xY - \frac{x^2}{2} Y B_1 + \frac{x^3}{6} Y (2B_1^2 - 2B_2)\right) & \left(-\frac{x^3}{6} Z Y\right) & 1
\end{bmatrix}
\]  

(3.19)

Which is the transmission matrix with four terms in the series.

Similar algebraic operations done with the Eq. (3.13), gives the transmission matrix with three terms in the series as:

\[
[T(x)] = \begin{bmatrix}
1 & 0 & (-xZ - \frac{x^2}{2} C_1 Z) & \left(\frac{x^2}{2} Z\right) \\
x & 1 & (-\frac{x^2}{2} Z) & 0 \\
0 & (-\frac{x^2}{2} Y) & 1 & -x \\
\frac{x^2}{2} Y & (xY - \frac{x^2}{2} Y B_1) & 0 & 1
\end{bmatrix}
\]  

(3.20a)

and with the Eq. (3.32) gives the transmission matrix with two terms in the series as:

\[
[T(x)] = \begin{bmatrix}
1 & 0 & -xZ & 0 \\
x & 1 & 0 & 0 \\
0 & 0 & 1 & -x \\
0 & xY & 0 & 1
\end{bmatrix}
\]  

(3.20b)

### 3.1 Transmission Matrices for the Beam Element of Tapered Circular Section

In the previous section we restricted ourselves in dealing with the non-uniform beams of rectangular section. As mentioned earlier,
the series expansion for the transmission matrix is true for the beam element of any shape. The factors which are dependent on the shape of the sections are the variables, area of cross section \(A(x)\) and the area moment of inertia \(I(x)\) which appear as the elements in the matrix \([A(x)]\) and its derivatives. This section has been devoted to examining the form of the transmission matrix for the beam element of tapered circular section.

The transmission matrix for a non-uniform beam element is given by Eq. (3.1) and the matrix \([A(x)]\) by Eq. (3.2) as used for rectangular sections. In order to evaluate Eq. (3.1) for circular sections we need to first evaluate the matrix \([A(x)]\) and its derivative at \(x = 0\), and hence, the expressions for \(A(x)\) and \(I(x)\) for the tapered circular section which are derived in section (A.2) of Appendix A.

We notice that Eq. (A.14) and Eq. (A.16) have the same form as Eq. (A.7) and Eq. (A.8), respectively. This is just a matter of coincidence that for the two cases of cross sections, the expressions for areas of cross section and moment of inertia are of the same order. The constant coefficients, e.g., \(C_1, C_2, B_1\), etc., however, have different values expressed in terms of the diameter ratio \(D\). As mentioned earlier, the series expression for the transmission matrix is given by Eq. (3.1) is true for any kind of sections. The expression for the matrix \([A(x)]\) as given by Eq. (3.2) is also true for all sections with various \(A(x)\) and \(I(x)\) for various sections. Since \(A(x)\) and \(I(x)\) for the tapered circular sections as given by Eq. (A.14, A.16) have the same form as given by Eq. (A.7, A.8) for the rectangular section, the derivation of the transmission matrix in this case has the same form as that for the rectangular section, but with different values of the
constants \((C_1, C_2, B_1, \text{etc.})\). Therefore, for the tapered circular section the transmission matrices are also given by Eq. (3.19, 3.20a, 3.20b) where the constants \(C_1, C_2, B_1, B_2, \text{etc.}\) are defined by Eq. (A.13, A.15).

In order to derive expressions for \(A(x)\) and \(I(x)\) a tapered circular beam as shown in Fig. (22), has been chosen. The transmission matrices are to be evaluated for a particular segment of the beam. Therefore, in the case of a beam segment the Eq. (A.14, A.16) are to be modified by replacing \(A_0, I_0\) by the same quantities at the input face and defining \(D\) as the ratio of the diameters at output end to input end.

3.2 Numerical Application and Programming Technique.

In the previous two sections the procedure for derivation of the transmission matrices for a beam element has been shown. The transmission matrices as given by Eq. (3.19, 3.20a, 3.20b) are derived, in particular for the beam element of rectangular section. The same equations are also true for tapered circular sections with different values of the constants \(C_1, C_2, B_1, \text{etc.}\) The procedure for deriving the transmission matrices as shown in section (3.0) and Section (3.1) is, in general, valid for any shape of non-uniform beam. References throughout the discussion in this section are made to the beam of rectangular section.

Equation (3.19) represents the transmission matrix for a beam segment when four terms in the series are taken. Notice that Eq. (3.20a) and Eq. (3.20b) are directly derived from Eq. (3.19) by dropping higher order \(x\) terms. For instance, when the number of terms taken are three, the coefficients of \(x^3\) terms in Eq. (3.37) are made equal to zero. In the computer program written for numerical evaluation the number of terms are changed by making the respective coefficients vanish.
With Eq. (3.19) representing the transmission matrix at hand, we are now in a position to apply the approximate matrix to each segment and obtain the total transmission matrix for the entire beam. Considering the beam as shown in Fig. (13), for a choice of the number of segments equal to \( N \), the segmental length becomes \( L/N \). The value of \( x \) in Eq. (3.19) is replaced by \( L/N \). Rewriting Eq. (1.2) as applied to the above beam gives:

\[
\{\psi_1\} = [T_1][T_2] \ldots [T_N] \{\psi_{N+1}\}
\]  

or

\[
\{\psi\}_{\text{input}} = [T] \{\psi\}_{\text{output}}
\]  

The immediate purpose is to get the transmission matrices for all the segments and multiply them together to get the total transmission matrix \([T]\) inter-relating the state vectors at the input end to that of the output end of the beam. Referring to Fig.(13) the left end of the beam is selected as the input end and the right end to be the output end.
Equation (3.19) gives the transmission matrix for a particular segment. All segments were chosen with the same length, $L/N$. The values of $Z(x)$ and $Y(x)$ are different for different segments as given by Eq. (3.5b, 3.5c). The values of the constants $B_1, B_2, C_1$ are also different for different segments as given by Eq. (3.15, 3.16). Looking at the expressions for $Z, Y, C_1, B_1, B_2$ it becomes clear that the only thing one has to know to have the transmission matrices for different segments is the depth and the width of the segment at the input and the output ends. For the two kinds of beams dealt with in this paper, i.e., circular section and rectangular section, the taper is linear. Therefore, the length of all the segments for a particular choice of $N$ being equal, the difference between the same dimension at the two ends of the segments is the same for all segments. Referring to Fig. (3.4) the difference of $d_0$ to $d_1$ is the same as that of $d_1$ to $d_2$. For the same beam divided into $N$ segments the differences in depth and width are given as:

\[
D_d = \frac{(d_0 - d_L)}{N} \quad \text{(3.23)}
\]
\[
D_h = \frac{(h_0 - h_L)}{N} \quad \text{(3.24)}
\]

Therefore, for the first segment, $D$ and $H$ in Eq. (A.5, A.6) are given as:

\[
D = \frac{d_0 - D_d}{d_0} \quad \text{(3.25a)}
\]
\[
H = \frac{h_0 - D_h}{h_0} \quad \text{(3.25b)}
\]

and for the second segment they are given as:
\[ D = \frac{d_0 - 2Dd}{d_0 - Dd} \quad (3.26a) \]
\[ H = \frac{h_0 - 2Dh}{h_0 - Dh} \quad (3.26b) \]

and quantities \( A_1 \), \( I_1 \) as given in Eq. (3.5b, 3.5c) for the second segments are:

\[ A_1 = (d_0 - Dd)(h_0 - Dh) \quad (3.27a) \]
\[ I_1 = \frac{1}{12} (h_0 - Dh)(d_0 - Dd)^3 \quad (3.27b) \]

Proceeding in a similar way the above quantities for other segments are readily obtained. The total transmission matrix \([T]\) as given by Eq. (3.22) is obtained by post multiplying transmission matrix for the segment 1 by the same for the segment 2, to give a product matrix. This product is then post multiplied by the transmission matrix for the third segment, and this process is continued to include all \( N \) segmental matrices.

After the formation of the total transmission matrix, the characteristic determinant as given by Eq. (1.9) is made to approach zero by changing the values of the frequency root \( \omega \). The iterative scheme used is the same as discussed for the uniform case in section (2.1).

### 3.3 Frequency Root Error

The procedure for obtaining the transmission matrix for each different segment has been explained in the previous section. A program was written in Fortran IV for the IBM System/360 computer to multiply the \( N \) transmission matrices to give a total transmission matrix, from which the characteristic determinant was found. The iteration procedure followed to find the frequency root was exactly the same as described in Section (2.1).
In order to investigate the application of the method for non-uniform beams, a beam of particular dimension was chosen. Tests were carried out with both rectangular and circular cross sections. Beams of tapered rectangular section were considered first. The frequency roots obtained for Fixed-Simply supported ends are compared with the exact solutions given by H. H. Mabie [3]. Work was also carried out for both rectangular and circular sections for Fixed-Free ends and the frequency roots obtained by this method were compared with the exact solutions as given by G. W. Housner [4]. The accuracy of the method has been shown in this section by the plot of the percentage frequency root errors for two cases of rectangular sections with Fixed-Simply supported end conditions.

Frequency roots have been calculated with two, three, and four terms in the series for first, second, and third modes and with the number of segments varying from 4 to 20 in increments of 2 segments. The frequency root error is expressed exactly the same way as Eq. (2.12) and is given as:

\[
e = \frac{(\text{Frequency obtained} - \text{Exact frequency}) \times 100}{\text{Exact frequency}} \tag{3.30}\]

The material for the beam was considered to be steel with \( \rho = 488 \text{ lb/ft}^3 \). \( E = 30 \times 10^6 \text{ psi} \).

H. H. Mabie [3] has worked with beams of rectangular section with Fixed-Simply supported ends and has expressed non-dimensional frequency roots for various depth and width ratios. Two such cases have been chosen in this section for the sake of comparison of the frequency roots.

Case 1. Constant width, variable depth: - The depth ratio (D) was selected as equal to 0.5. Keeping the same depth ratio, the
dimensions of the beam shown in Fig. (14) have been chosen arbitrarily for numerical manipulation.

Fig. 14 Rectangular beam of constant width but variable depth

With the dimensions of the beam shown in Fig. (14) the exact frequency roots calculated are:

First mode frequency root = 393.7497 rad/sec
Second mode frequency root = 1201.3360 rad/sec
Third mode frequency root = 2468.7790 rad/sec

Frequency roots obtained by the application of the method concerned are compared with the exact solution and plot of percentage frequency root error are shown in Fig. (16).

Case 2. Constant depth, variable width: - The width ratio of 0.5 was chosen and the dimensions shown for the beam in Fig. (15) below are chosen arbitrarily.

Fig. 15 Rectangular beam of constant depth but variable width
For the beam shown in the Fig. (15), the exact frequency roots calculated are:

First mode frequency root = 256.8017 rad/sec

Second mode frequency root = 809.1877 rad/sec

Third mode frequency root = 1678.3980 rad/sec

Plots of the percentage frequency root errors are shown in Fig. (17).

In either case the frequency roots as obtained with number of terms two in the series was found to be more than the exact root. With number of terms three and four in the series the frequency roots were always found to be less than the exact ones. That means that the percentage frequency roots were both positive and negative. In order to plot all curves on the same semi-log paper, the absolute values of the percentage frequency root errors were plotted against different numbers of segments of the beam.

The change of the frequency root errors with the change of the number of segments was determined by comparing the slope of each curve with the plot of \(1/N^n\) curve, where \(N\) is the number of segment and \(n = 1,2,3,4\) etc. The plot of \(1/N\), \(1/N^2\), \(1/N^3\), etc., are first accomplished in the semi-log paper with the same scale as done for the plot of percentage frequency root error. The frequency root error curve is then compared with these plots. The plot of frequency root error which has nearly the same slope as that of \(1/N^2\) curve is considered as the frequency root to be proportional to \(1/N^2\).

For both Case 1 and Case 2, the frequency root error with number of terms two and three in the series were found to be proportional to
Fig. 16a First mode frequency root error for beams of rectangular section (variable depth)
Fig. 16b  Second mode frequency root error for beams of rectangular section (variable depth)
Fig. 16c Third mode frequency root error for beams of rectangular section (variable depth)
Fig. 17a First mode frequency root error for beams of rectangular section (variable width)
Fig. 17b Second mode frequency root error for beams of rectangular section (variable width)
Fig. 17c Third mode frequency root error for beams of rectangular section (variable width)
With four terms in the series the frequency root error was found to be proportional to \(1/N^3\). This nature of the frequency root error relation with the number of segments was found to be true for all of the first three principal modes.

Leckie and Lindberg [6] have examined the effect of different choices of parameters on the beam frequencies for the Bernoulli-Euler beams. They have shown that the models which give the least frequency root errors display the following behavior:

(a) If neither end is free the errors are proportional to \(1/N^4\) for large \(N\).

(b) In cases where one or both ends are free the errors are proportional to \(1/N^2\) for large \(N\).

To eliminate the inconsistency shown by the different boundary conditions, Leckie and Lindberg [6] have applied a modified stiffness matrix to Bernoulli-Euler beam elements which they call the dynamic stiffness matrix. This matrix accounts for the stiffness properties of a massless elastic beam element and a first order distribution of the inertia forces.

The dynamic stiffness matrix, therefore, does not describe an actual lumped parameter model as neither mass nor stiffness are lumped into discrete point elements. The dynamic stiffness matrix brings into effect a distribution of inertia forces just as the consistent mass matrix which is a nondiagonal mass matrix.

Lindberg [5] has used the dynamic stiffness matrix for cantilever beams to get the frequency roots for the uniform beams \((d=H=1)\), the wedge \((D=0, H=1)\) and the cone \((D=H=0)\). The solutions for these three cases have been found to have errors proportional to \(1/N^4\). Applying
the approximate transmission matrix method, the errors have been found, herein, to be proportional to $1/N^2$ and $1/N^3$ for three (or two) and four terms in the series, respectively. Since inclusion of the more terms of the series in forming the transmission matrix, has been observed to yield better results, one can expect that when more than four terms in the series are taken the frequency root errors will have greater rate of convergence.

As mentioned earlier, the transmission matrix has been evaluated with a maximum of four terms in the series and has been found to produce better results than with two or three terms in the series. The results obtained with four terms in the series and with 20 segments of the beam are very close to the exact solution. For example, in Fig. (16a) for the beam described in Case 1 the percentage error for the first mode with four terms in the series is less than 0.1%. Assuming that similar results are obtainable with different boundary conditions and various shapes of the beam, one can proceed to apply the technique to obtain the frequency roots for any desired case.

Being convinced that the method is capable of producing frequency roots close to the exact solution, work has been carried out in determining the frequency roots for several cases of non-uniform beams.

3.4 Dimensionless Frequency Roots for Non-Uniform Beams of Rectangular Section.

Work described in the preceding sections was devoted to explaining the technique being followed in this paper and finding the efficiency of the method in producing the frequency roots of normal modes. Results were obtained by the method for uniform and non-uniform cases with several different boundary conditions and comparisons show them to be
equally good as other approximate methods. The method should, therefore, be capable of producing frequency parameters for any shape of beam with any end conditions. For practical use, the frequency parameter has been obtained in non-dimensional form.

Non-dimensional frequency roots have been calculated for wedge shaped beams with Fixed-Fixed and Pinned-Pinned ends as shown in the Fig. (18b) and Fig. (18c), respectively.

Fig. 18 End conditions for the rectangular beam
For the purpose of numerical manipulation the dimensions shown in Fig. (18a) were taken. Steel material with density $\rho = 0.339 \text{ lb/in}^3$ and $E = 30 \times 10^6 \text{ psi}$ was chosen. In order that the results of this work can be used in general for any length of the beam of any material a dimensionless parameter has been evaluated as expressed below.

Using the same definition as before:

$\tilde{\rho} = \text{weight density of the material}$

$E = \text{modulus of elasticity}$

$L = \text{length of the beam}$

$I_0 = \text{area moment of inertia of the section of maximum area}$

$A_0 = \text{maximum cross sectional area}$

$\omega = \text{frequency root in rad/sec}$

$g = \text{acceleration due to gravity}$

Then the quantity $\frac{\omega^2 A \tilde{\rho} L^4}{E I g}$ is seen to be dimensionless because

$$\frac{\omega^2 A \tilde{\rho} L^4}{E I g} = \frac{(\text{rad/sec})^2 \text{ in.}^2 \text{ lb/in.}^3 \text{ in.}^4}{\text{lb/in.}^2 \text{ in.}^4 \text{ in.}/\text{sec}^2} = 1 \quad (3.40)$$

To give a suitable dimensionless number, the above quantity has been slightly modified to give:

$$\beta = \frac{1}{10} \left( \frac{\omega^2 A \tilde{\rho} L^4}{E I g} \right)$$

The dimensionless parameter $\beta$ (Beta) has been calculated for beams with Fixed-Fixed and Fixed-Free ends. These calculations are done with four terms in the matrix series and 20 segments of the beam. The parameter $\beta$ is evaluated for first, second, and third modes in the above two cases. For each mode $\beta$ has been evaluated for various values of the depth and the width ratios. To obtain different depth and
width ratios the depth and the width of one end is kept fixed and the
dimensions of the other end are varied. The technique used to search
out the frequency roots is the same as described in Section (2.1).
The reason for producing output in the form of a dimensionless parameter
S is that the frequency root for a beam of any dimensions conforming to
the depth ratio D, width ratio H, and the end conditions (for which S
is selected) can be easily calculated.

The quantity D and H have the same definitions as before:

\[ D = \text{ratio of depth (variable) at right end of the beam to the depth at left end (constant)} \]

\[ H = \text{ratio of the width (variable) at right end of the beam to the depth at left end (constant)} \]

The values of dimensionless frequency root are calculated for first,
second, and third modes and are tabulated in the following tables.

The frequency roots have been calculated for various width and
depth ratios. Beta, the dimensionless parameter has been plotted against
the depth ratio D for different values of width ratio H. These plots
for different modes (1st, 2nd, and 3rd modes only) and for Fixed-Fixed
and Pinned-Pinned ends have been shown.
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TABLE V  Third mode dimensionless frequency parameter - Fixed-Fixed ends
TABLE VI  First mode dimensionless frequency parameter - Pinned-Pinned ends

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TABLE VIII  Third mode dimensionless frequency parameter - Pinned-Pinned ends

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<th>H=0.0</th>
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<th>H=0.4</th>
<th>H=0.6</th>
<th>H=0.8</th>
<th>H=1.0</th>
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<tr>
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Fig. 19a First mode dimensionless frequency root for beams with Fixed-Fixed ends
Fig. 19b Second mode dimensionless frequency root for beams with Fixed-Fixed ends
Fig. 19c Third mode dimensionless frequency root for beams with Fixed-Fixed ends
Fig. 20a First mode dimensionless frequency roots for beams with Pinned-Pinned ends
Fig. 20b Second mode dimensionless frequency roots for beams with Pinned-Pinned ends
Fig. 20c Third mode dimensionless frequency roots for beams with Pinned-Pinned ends
CONCLUSIONS

The objective of this work has been to investigate the efficiency of the approximate transmission matrix method in producing the normal mode frequency roots for different types of uniform and non-uniform beams in transverse vibration. To qualitatively evaluate the accuracy of the method, frequency roots obtained by applying this method have been compared with the corresponding exact solutions. For a comparison with uniform beam elements, boundary conditions of Fixed-Fixed and Fixed-Free ends were used. For the comparison with non-uniform beams, the boundary condition of Fixed-Pinned ends was used. Frequency root error characteristics shown by this method have been compared with that of the standard lumped parameter methods. On the basis of these investigations the following conclusions are drawn:

1. Progressively better results are obtained by increasing the number of terms used in the power series to form the transmission matrix. When two, three and four terms are used in the power series the frequency root errors behave proportional to $1/N^2$, $1/N^2$, $1/N^3$, respectively, for large $N$. These results appeared to be consistent for uniform or non-uniform beam elements and independent of the boundary condition used.

2. The better lumped parameter models produce frequency root errors which are proportional to $1/N^4$ and $1/N^2$, for large $N$, depending upon whether both ends are fixed or one end is free. The approximate transmission matrix produces results comparable to those of the lumped parameters models when four terms are used in the power series.
3. The approximate transmission matrix method is easily applied as it deals with only two unknowns, $A(x)$ and $I(x)$. When higher order terms in the series are considered, higher order derivatives of these two quantities must be determined, but no equivalent masses or spring elements need be determined as is necessary in any lumped parameter method.

4. Unlike the standard lumped parameter models, the number of normal modes obtainable by the approximate transmission matrix method cannot be easily predicted from the number of segments into which the beam is divided because this property depends also upon the number of terms used in the power series to form the transmission matrix.
APPENDIX A

DERIVATION OF A GENERAL FORM FOR THE AREA AND THE MOMENT OF INERTIA AT ANY SECTION OF NON-UNIFORM BEAMS

A.1 Beams of Wedge Shape

Consider a beam of tapered rectangular section as shown in Fig. (21).

Let the dimensions of the beam be:

\[ h_0 = \text{width at the left end } \text{at } x = 0 \]
\[ d_0 = \text{depth at the left end } \text{at } x = 0 \]
\[ h_L = \text{width at the opposite end of the beam, at } x = L \]
\[ d_L = \text{depth at the opposite end of the beam, at } x = L \]
\[ L = \text{length of the beam} \]

Let
\[ H = \frac{h_L}{h_0}, \text{ the ratio of the width} \]
\[ D = \frac{d_L}{d_0}, \text{ the ratio of the depth}. \]
Then the width, depth and area of cross section at a distance $x$ from left face are given as:

\[
\begin{align*}
  h &= h_0 \left\{1 - (1-H) \frac{x}{L}\right\} \quad \text{(A.1)} \\
  d &= d_0 \left\{1 - (1-D) \frac{x}{L}\right\} \quad \text{(A.2)}
\end{align*}
\]

and

\[
A = \left(\frac{d}{h}\right) = h_0 d_0 \left\{1 - (1-H) \frac{x}{L} - (1-b) \frac{x}{L} + (1-H)(1-D) \frac{x^2}{L^2}\right\} 
\]

or

\[
A = A_0 \left\{1 - (2-H-D) \frac{x}{L} + (1-H)(1-D) \frac{x^2}{L^2}\right\} \quad \text{(A.3)}
\]

Similarly, the moment of inertia of the section at that point is given as:

\[
I = \frac{1}{12} \frac{hd^3}{h_0^3} = \frac{1}{12} h_0^3 \left\{1 - (1-H) \frac{x}{L} \right\}^3 \left\{1 - (1-D) \frac{x}{L}\right\}^3 
\]

\[
= I_0 \left\{1 - 3(1-D) \frac{x}{L} + 3(1-D)^2 \frac{x^2}{L^2} - (1-D)^3 \frac{x^3}{L^3}\right\} 
\]

\[- I_0 \left(1-H \right) \frac{x}{L} \left\{1 - 3(1-D) \frac{x}{L} + 3(1-D)^2 \frac{x^2}{L^2} - (1-D)^3 \frac{x^3}{L^3}\right\} 
\]

or

\[
I = I_0 \left\{1 - \left\{3(1-D) + (1-H)\right\} \frac{x}{L} + 3 \left\{(1-D)^2 + (1-H)(1-D)\right\} \frac{x^2}{L^2} 
\]

\[- \left\{(1-D)^3 + 3(1-H)(1-D)^2\right\} \frac{x^3}{L^3} + (1-H)(1-D) \frac{x^4}{L^4}\right\} \quad \text{(A.4)}
\]

Where \( A_0 = d_0 h_0 \), \( I_0 = h_0^3 d_0^3 / 12 \).

Substituting \( C_1 = -(2-H)/L \) \quad \text{(A.5a)}

\( C_2 = (1-H)(1-D)/L^2 \) \quad \text{(A.5b)}

into Eq. (3.13) and

\[
\beta_1 = - \left\{3(1-D) + (1-H)\right\}/L \quad \text{(A.6a)}
\]

\[
\beta_2 = \left\{(1-D)^2 + (1-H)(1-D)\right\} \frac{3}{L^2} \quad \text{(A.6b)}
\]
\[ \beta_3 = \frac{(1-D)^3}{L^3} + 3(1-D)^2(1-H) \]  
\[ \beta_4 = \frac{(1-H)(1-D)^3}{L^4} \]  

into Eq. (3.14) gives:

\[ A = A_0(1 + c_1x + c_2x^2) \]  
\[ I = I_0(1 + \beta_1x + \beta_2x^2 + \beta_3x^3 + \beta_4x^4) \]

(A.6c)  
(A.6d)  
(A.7)  
(A.8)

A.2 Beams of Tapered Circular Section

Consider a beam of tapered circular section as shown in Fig. (22).

Fig. 22 Beam of tapered circular section

The beam shown has the length \( L \), the largest and the smallest diameters as \( d_0 \) and \( d_L \), respectively. Defining \( D = \frac{d_L}{d_0} \) as ratio of the smallest diameter to the largest diameter, i.e., \( d_L/d_0 \), the expressions for the area and the moment of inertia at a distance \( x \) from the left end of the beam in Fig. (A.2) are given as:

\[ d(x) = d_0 \left(1 - \frac{d_L}{d_0} \frac{x}{L} \right) \]  
\[ \text{where} \quad D = \frac{d_L}{d_0} \]  

(A.9)  
(A.10)
Therefore,
\[ A(x) = \frac{\pi}{4} d(x)^2 = \frac{\pi}{4} d_0^2 \left(1 - (1-D)\frac{x}{L}\right)^2 \]
or
\[ A(x) = A_0 \left(1 - 2(1-D)\frac{x}{L} + (1-D)^2 \frac{x^2}{L^2}\right) \quad (A.11) \]
where
\[ A_0 = \frac{\pi}{4} d_0^2, \text{ and} \]
\[ I(x) = \frac{\pi}{64} d(x)^4 \]
\[ = \frac{\pi}{64} d_0^4 \left(1 - 2(1-D)\frac{x}{L} + (1-D)^2 \frac{x^2}{L^2}\right)^2 \]
\[ = I_0 \left\{[1 - 2(1-D)\frac{x}{L}]^2 + (1-D)^4 \frac{x^4}{L^4}\right\} \]
or
\[ I(x) = I_0 \left[1 - 4(1-D)^3 \frac{x^3}{L^3} + 6(1-D)^2 \frac{x^2}{L^2} - 4(1-D)\frac{x}{L} + (1-D)^4 \frac{x^4}{L^4}\right] \quad (A.12) \]
where
\[ I_0 = \frac{\pi}{64} d_0^4. \]

In Eq. (3.42) substituting
\[ c_1 = -2(1-D)/L \quad (A.13a) \]
\[ c_2 = (1-D)^2 \quad (A.13b) \]
we get
\[ A(x) = A_0 [1 + c_1 x + c_2 x^2]. \quad (A.14) \]
And in Eq. (3.43) upon substitution of
\[ \beta_1 = -4(1-D)/L \quad (A.15a) \]
\[ \beta_2 = 6(1-D)^2/L^2 \quad (A.15b) \]
\[ \beta_3 = -4(1-D)^3/L^3 \quad (A.15c) \]
\[ \beta_4 = (1-D)^4/L^4 \quad (A.15d) \]
gives
\[ I(x) = I_0 (1 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 x^4). \quad (A.16) \]
REFERENCES


VITA

Ranjit Kumar Roy was born in the Barisal district of East Pakistan, January 1, 1947. He attended Model High School, Khulna and matriculated in the year 1961. In the year 1962 the author migrated to the State of West Bengal in India and completed Pre-University Course in Science at the University of Calcutta in the year 1963. The author then entered into five years degree course in Mechanical Engineering in Regional Engineering College, Durgapul, and earned the Bachelor of Engineering Degree in the year 1968. After graduation the author entered the University of Missouri - Rolla for graduate study in Mechanical Engineering. At the time this work was being completed the author has just started his work for doctoral degree. In the University of Missouri - Rolla, during the work for M.S. and Ph.D. the author was working as a teaching assistant.