An analysis of a synchronization scheme for 180° biphase modulated signals

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AN ANALYSIS OF A SYNCHRONIZATION SCHEME FOR
180° BIPHASE MODULATED SIGNALS

BY
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A
THESIS

submitted to the faculty of
UNIVERSITY OF MISSOURI - ROLLA
in partial fulfillment of the requirements for the
Degree of
MASTER OF SCIENCE IN ELECTRICAL ENGINEERING

Rolla, Missouri
1970

Approved by

R. E. Ziemer (advisor)
ABSTRACT

The problem of analyzing a synchronization system for demodulation of 180° biphase modulated signals is considered in this thesis. The system considered utilizes the so-called squaring loop technique.

Techniques derived from the theory of Markov processes are utilized in the analysis of the system. In particular the "fluctuation equation" approach for non-Markovian processes is used.

After the analysis of the system is performed, an expression for the bit error probability is obtained from knowledge of the phase error density function.
ACKNOWLEDGEMENTS

The author extends his thanks to Dr. Roger E. Ziemer of the University of Missouri - Rolla. Without his guidance and patience, the completion of this thesis could not have been realized.

Thanks are also extended to C. S. Williams, my supervisor at Sandia Laboratories in Albuquerque, New Mexico, for his helpful suggestions.

The author also wishes to thank Mrs. Ruby Cochrell and Mrs. Muriel Johnson for the typing of this manuscript.
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I. INTRODUCTION

A. Statement of Objectives

In order to accurately demodulate a $180^\circ$ biphase pulse code modulated (PCM) signal, information about the carrier frequency must be known, i.e., the carrier frequency must be continuously monitored. The demodulation system must be capable of tracking the carrier frequency and of adjusting to any slow fluctuation in its phase.

The demodulating system considered in this paper utilizes the phase-lock tracking loop. The behavior of the phase-lock loop is characterized by a nonlinear differential equation.

It is the object of this paper to analyze this tracking-demodulating system and to determine the statistics of the stochastic process which is a solution to the nonlinear stochastic partial differential equation describing the system behavior. Techniques taken from the theory of Markov processes are used to determine these statistics.

B. Reasons for Choosing Thesis Problem

The subject of synchronization of PCM modulated signals was chosen as a thesis topic because of its importance in efficient tracking and demodulating systems. If transmission of information occurs between a transmitter
and receiver which are moving relative to each other, it becomes necessary to account for the time varying frequency offset due to doppler shift. This frequency offset or other fluctuations in phase and frequency can be accounted for by the nonlinear feedback loop described in this paper. It, therefore, seems proper to present an analysis of this synchronization scheme.
II. REVIEW OF LITERATURE

Because of the nature of this thesis topic, there were three major areas in which literature research was conducted. These three areas were: 1) phase-lock loops; 2) the theory of Markov processes; and 3) tracking and synchronization systems.

A. Phase-Lock Loops

The literature surveyed on the subject of phase-lock loops (PLLs) overlapped to some extent that literature which was reviewed concerning tracking and synchronization systems - the reason being the fact that many tracking systems incorporate phase-lock loops.

A vast amount of research has been conducted during the past twenty years on phase-lock receivers. Consequently, there exists today a substantial number of publications on just this subject.

Some of the fundamental concepts basic to the PLL were presented by Appleton [1] in 1923-24 with his publication on the synchronization of oscillators. Later applications were advanced for frequency demodulation [2], and horizontal line sweep synchronization of televisions [3,4].

Gruen [5] was one of the first to analyze the phase-lock loop as we know it today. His analysis, however, was limited to signals without noise. A later analysis which
took into account additive noise was published by Jaffe and Rechtin [6]. Many other publications using different loop models and loop parameters or presenting techniques for analyzing the PLL have been written. The following includes those whose works have been consulted before the writing of this paper: Karshin [7]; Develet [8]; Van Trees [9,10]; Tausworthe [11,12]; Viterbi [13,14]; Tikhonov [15,16,17]; and Leon [18].

The publications by Tausworthe [11], and Leon present summaries of some of the important features of the PLL. An exhaustive coverage is presented by Viterbi [13].

B. The Theory of Markov Processes

On the theory of Markov processes, books consulted include those written by Cox and Miller [19]; Papoulis [20]; Middleton [21]; Stratonovich [22]; Bharucha-Reid [23]; and Blaquière [24].

Articles on the forward diffusion equation, or in the case of continuous Markov processes - the Fokker-Planck equation, include those on Brownian motion by: Uhlenbeck and Ornstein [25]; Wang and Uhlenbeck [26]; and Doob [27]. Another written by Chandrasekhar [28] looks at a macroscopic version of the Brownian movement - that of the motion of the heavenly bodies. A dissertation on the generalization and extension of the Fokker-Planck-Kolmogorov equations was also consulted [29].
In regard to the extensions of the Fokker-Planck equation to non-Markov processes, the author consulted Stratonovich [22].

C. **Tracking and Synchronization Systems**

As in the case of the PLL there exists a substantial number of publications on tracking and synchronization systems. However, those which were found helpful include the following: Tikhonov [15,16]; Van Trees [9]; Viterbi [13]; Stiffler [30,31]; Gilchriest [32]; Didday and Lindsey [33]; Lindsey and Simon [34]; Wintz and Hancock [35]; Layland [36]; and Feistel and Gregg [37].

Also found helpful were the proceedings of a panel discussion on synchronization which were published by the Institute of Electrical and Electronics Engineers [38].
III. PHASE-LOCK LOOPS

A. Preliminary Discussion on the Basic Loop

The phase-lock loop (PLL) is a fundamental component of many tracking systems. It is a narrow band nonlinear adaptive filter which is designed to acquire and maintain coherence with a received signal.

A block diagram of the basic PLL is illustrated in Figure 1 below. Its principle components are: the phase detector (multiplier); the loop filter; and the voltage controlled oscillator (VCO).

This PLL will lock onto a carrier frequency if that carrier is not suppressed and if it lies within the pull-in range of the loop.

Figure 1. The Basic Phase-Lock Loop
In many cases, however, the signal modulation is such that it completely suppresses the carrier, i.e., there is no power component at the carrier frequency. As an example consider the two signals below.

\[ s_1(t) = m_1(t) \sin(\omega_o t + \theta_1) \]
\[ s_2(t) = m_2(t) \sin(\omega_o t + \theta_2) \]

where \( m_1(t) = +1, 0 \) and \( m_2(t) = \pm 1 \) are the binary signals of bit period \( T \) modulating the carrier \( \omega_o \). \( \theta_1 \) and \( \theta_2 \) are statistically independent random variables uniformly distributed over \( 2\pi \) radians.

The autocorrelation function of \( s_1(t) \) is given as:

\[ R_{s_1}(\tau) \triangleq E[s_1(t) s_1(t + \tau)] \]
\[ = R_{m_1}(\tau) E[\sin(\omega_o t + \theta_1) \sin(\omega_o (t + \tau) + \theta_1)] \]
\[ = 1/2 R_{m_1}(\tau) \cos(\omega_o \tau) \]

and the autocorrelation function of \( s_2(t) \) is given by:

\[ R_{s_2}(\tau) \triangleq E[s_2(t) s_2(t + \tau)] \]
\[ = R_{m_2}(\tau) E[\sin(\omega_o t + \theta_2) \sin(\omega_o (t + \tau) + \theta_2)] \]
\[ = 1/2 R_{m_2}(\tau) \cos(\omega_o \tau) . \]

However, \( m_1(t) = 1/2 (1 + m_2(t)) \).

Then if +1 and -1 are equiprobable and the bit values are independent,
\[ R_{m1}(\tau) = E\{m_1(t) m_1(t + \tau)\} \]
\[ = 1/4 \ E\{(1 + m_2(t))(1 + m_2(t + \tau))\} \]
\[ = 1/4 + 1/4 \ R_{m2}(\tau). \]

\[ R_{s1}(\tau) \] then becomes
\[ R_{s1}(\tau) = (1/8 + 1/8 \ R_{m2}(\tau)) \cos(\omega_0 \tau) \]
\[ = 1/8 \cos(\omega_0 \tau) + 1/8 \ R_{m2}(\tau) \cos(\omega_0 \tau). \]

According to Papoulis [20], the spectral density of \( m_2(t) \), i.e., the Fourier transform of \( R_{m2}(\tau) \), is

\[ S_{m2}(\omega) \triangleq \mathcal{F}\{R_{m2}(\tau)\} = \frac{4 \sin^2(\omega \tau/2)}{\omega^2} \]

This is illustrated in Figure 2. \( S_{s1}(\omega) \) now becomes

\[ S_{s1}(\omega) = 1/16 \ \delta(\omega - \omega_0) + 1/16 \ \delta(\omega + \omega_0) \]
\[ + 1/4 \frac{\sin^2((\omega - \omega) \tau/2)}{T(\omega_0 - \omega)^2} + 1/4 \frac{\sin^2((\omega + \omega) \tau/2)}{T(\omega_0 + \omega)^2} \]

where \( \delta(\omega) \) is the Dirac delta, and \( S_{s2}(\omega) \) becomes

\[ S_{s2}(\omega) = \frac{\sin^2((\omega_0 - \omega) \tau/2)}{T(\omega_0 - \omega)^2} + \frac{\sin^2((\omega_0 + \omega) \tau/2)}{T(\omega_0 + \omega)^2} \]

These spectral densities are shown in Figure 3.

It can be seen from Figure 3 that \( s_1(t) \) contains power at the carrier frequency whereas \( s_2(t) \) does not.
Figure 2. Spectral Density of ±1 Random Binary Transmission
a) Spectral Density of $\pm 1, 0$ Random Binary Modulated Signal

b) Spectral Density of $\pm 1$ Random Binary Modulated Signal

Figure 3. Spectral Densities
Thus, $s_1(t)$, where the carrier is not totally suppressed, can be tracked by the basic PLL shown in Figure 1.

However, unless $s_2(t)$ is processed in some way so as to shift power to the carrier or one of its harmonics, the basic PLL will not be able to track any of these frequencies to establish a locally generated harmonically related signal, i.e., establish coherence.

If $s_2(t)$ is operated on by a nonlinear filter, power in the received signal can be shifted to either the carrier or one of its harmonics. The filter must be nonlinear because any operation by a linear filter cannot produce impulses in the spectral density.

The nonlinear filter to be considered in this paper will be one which behaves as a square-law device.

Consider the squaring filter below,

\[
S(t) \rightarrow \text{SQUARING FILTER} \rightarrow y(t)
\]

Figure 4. Squaring Filter

where

\[
y(t) = S^2(t),
\]

and let $S(t)$ be of the same form as $s_2(t)$, i.e.,

\[
S(t) = K m(t) \sin(\omega_0 t + \theta)
\]
where \( m(t) \) is \( \pm 1 \) binary modulation with zero mean and bit period \( T \), and \( \theta \) is a random variable which is uniformly distributed over \( 2\pi \) radians.

Then

\[
y(t) = K^2 \sin^2(\omega_0 t + \theta) = \frac{K^2}{2} \left( 1 - \cos(2\omega_0 t + 2\theta) \right)
\]

and its autocorrelation function is

\[
R_y(t) \triangleq E\{S^2(t) S^2(t + \tau)\} = \frac{K^4}{4} E\{[1 - \cos(2\omega_0 t + 2\theta)] [1 - \cos(2\omega_0(t + \tau) + 2\theta)]\}
= \frac{K^4}{4} + \frac{K^4}{8} \cos(2\omega_0 \tau).
\]

\( S_y(\omega) \) is then given by

\[
S_y(\omega) = \frac{K^4}{4} \delta(\omega) + \frac{K^4}{16} \delta(\omega - 2\omega_0) + \frac{K^4}{16} \delta(\omega + 2\omega_0).
\]

Thus, it can easily be seen that by squaring the signal \( S(t) \) whose carrier is completely suppressed (see Figure 5), power is shifted to the second harmonic of the carrier. This squared signal can then be tracked by the basic PLL of Figure 1. Because the PLL is tracking at twice the carrier frequency, the locally generated signal out of the voltage controlled oscillator (VCO) must be frequency divided by two to obtain a signal coherent with the carrier itself.
Figure 5. Spectral Density of the Square of a Random ±1 Binary Modulated Signal
The basic PLL preceded by the squaring filter is commonly referred to as the squaring-loop filter or squaring-loop phase-lock filter.

Analysis of this composite system when the received signal includes additive noise is presented in the next section.

B. The Squaring-Loop-Filter

In this section formulation of the behavior of the general order PLL is presented. The order of the PLL is defined as the number of poles in the PLL closed loop transfer function.

Consider the composite system as depicted in Figure 6 which is receiving a signal given by

\[ x(t) = S(t) + n(t) \]

where \( S(t) \) is, as before, a 180° biphase modulated signal, and \( n(t) \) is white zero mean Gaussian noise with spectral density \( N_0/2 \) watts/hertz.

The 180° biphase modulated signal is given by

\[ S(t) = K m(t) \sin(\omega_o t + \theta(t)) \]

where \( m(t) \) is binary ±1 modulation of zero mean and bit time \( T \). \( \theta(t) \) is the phase variation function which can be expressed as

\[ \theta(t) = \omega_m t + \theta_o \]
Figure 6. Squaring-Loop Phase-Lock Filter System
where $\omega_m$ is the frequency mismatch between $S(t)$ and the quiescent frequency of the squaring-loop system. It is assumed that the quiescent frequency of the squaring-loop system is designed to be set at the expected or mean frequency of the transmitter, $\omega_0$. The frequency mismatch can be attributed to either doppler shifts due to movement between transmitter and receiver or a frequency drift in the transmitter from its mean frequency.

Because the noise, $n(t)$, is white, it is necessary to bandlimit the received signal so as to increase the signal to noise ratio, i.e., bring the noise power down to a finite value. This accounts for the inclusion of a band-pass filter in the squaring-loop system as shown in Figure 6. This bandpass filter whose transfer function is given by $G(s)$ has $\omega_0$ as its center frequency and its bandwidth $B_g$ is wide enough so as to provide negligible distortion to $S(t)$. This bandpass filter is commonly referred to as the presquaring filter.

With $x(t)$ as the received signal, the output of the presquaring filter will be the convolution (*) of $g(t)$ with $x(t)$,

$$y(t) = g(t) * x(t) = S(t) + n_g(t),$$

where $g(t)$ is the impulse response function of $G(s)$ and $n_g(t)$ is a bandlimited zero-mean Gaussian process with power spectral density
\[
S_g(\omega) = \left| G(j\omega) \right|^2 N_o/2 .
\]

From Davenport and Root [39] it can be shown that if the bandwidth, \( B_g \), of \( G(s) \) is much less than \( f_o \triangleq \omega_o/2\pi \), then the bandlimited noise process can be represented by a narrow band process whose center frequency is the same as that of \( G(s) \), i.e., \( \omega_o \). So now \( n_g(t) \) can be given by

\[
n_g(t) = \sqrt{2}\{n_1'(t) \cos(\omega_o t) + n_2'(t) \sin(\omega_o t)\} ,
\]

where \( n_1'(t) \) and \( n_2'(t) \) are statistically independent Gaussian processes with spectral densities identical to that of \( n_g(t) \) at its baseband,

\[
S_{n_1'}(\omega) = S_{n_2'}(\omega) = \frac{N_o}{2} \left| G(j\omega) \right|^2 .
\]

If we define two new processes \( n_1(t) \) and \( n_2(t) \) such that

\[
n_1(t) \triangleq n_1'(t) \cos(\theta(t)) - n_2'(t) \sin(\theta(t))
\]

and

\[
n_2(t) \triangleq n_1'(t) \sin(\theta(t)) + n_2'(t) \cos(\theta(t))
\]

then it can be shown that

\[
n_g(t) = \sqrt{2}\{n_1(t) \cos(\psi(t)) + n_2(t) \sin(\psi(t))\}
\]

where \( \psi(t) \) is

\[
\psi(t) = \omega_o t + \theta(t)
\]

\[
= \omega_o t + \omega_m t + \theta_o .
\]
If \( \frac{f_m}{2\pi} \triangleq \frac{\omega_m}{2\pi} \) is much less than \( B_g \), then [13] \( n_1(t) \) and \( n_2(t) \) can be approximated by statistically independent zero mean Gaussian processes with spectra identical to those of \( n'_1(t) \) and \( n'_2(t) \) respectively.

The signal out of the presquaring filter can now be expressed as

\[
y(t) = (K m(t) + \sqrt{2} n_2(t)) \sin(\psi(t)) + \sqrt{2} n_1(t) \cos(\psi(t)).
\]  

(1)

Passing \( y(t) \) through the squaring filter, the output will be of the form

\[
z(t) = y^2(t).
\]  

(2)

Upon substituting Equation (1) into (2), we obtain

\[
z(t) = \left\{ \left( \frac{K^2}{2} + \sqrt{2} K m(t) n_2(t) + n_2^2(t) + n_1^2(t) \right) \\
+ \left( \frac{-K^2}{2} - \sqrt{2} K m(t) n_2(t) - n_2^2(t) + n_1^2(t) \right) \cos(2\psi(t)) \\
+ \left( \sqrt{2} K m(t) n_1(t) + 2n_1(t) n_2(t) \right) \sin(2\psi(t)) \right\}.
\]

If the quiescent frequency of the basic PLL were \( 2\omega_0 \), the loop would try to synchronize to the \( 2\psi(t) \) terms of \( z(t) \) and the output of the VCO would be of the form

\[
\tilde{z}(t) = K_v \sin(2\tilde{\psi}(t)) = K_v \sin[2(\omega_0 t + \tilde{\theta}(t))]
\]
where \( \tilde{\psi}(t) \) and \( \tilde{\theta}(t) \) would be the PLL estimates of \( \psi(t) \) and \( \theta(t) \) respectively.

The output of the phase detector (PD) would be given by

\[
\varepsilon(t) = K_{PD} z(t) \tilde{z}(t)
\]  

(3)

if the phase detector were a perfect multiplier, where \( K_{PD} \) is the gain constant of the phase detector. Substituting for \( z(t) \) and \( z(t) \) in Equation (3), we obtain

\[
\varepsilon(t) = K_{PD} K_{V} \left\{ \left( \frac{K^2}{2} + \sqrt{2} K m(t) n_2(t) + n_1^2(t) + n_1(t) \right) \right. \\
\times \sin(2\tilde{\psi}(t)) + \left( \frac{K}{\sqrt{2}} m(t) n_1(t) + n_1(t) n_2(t) \right) \\
\times \left( \cos 2(\theta(t) - \tilde{\theta}(t)) - \cos 2(\tilde{\psi}(t) + \tilde{\psi}(t)) \right) \\
+ \left( - \frac{K^2}{4} - \frac{K}{\sqrt{2}} m(t) n_2(t) + \frac{n_1^2(t)}{2} - \frac{n_1^2(t)}{2} \right) \\
\times \left( \sin 2(\tilde{\psi}(t) + \tilde{\psi}(t)) + \sin 2(\tilde{\theta}(t) - \tilde{\theta}(t)) \right) \}  
\]  

(4)

where \( \psi(t) - \tilde{\psi}(t) = \theta(t) - \tilde{\theta}(t) \).

The expression, \( \theta(t) - \tilde{\theta}(t) \), represents the phase error, \( \theta_{\varepsilon}(t) \), between the received signal and the output of the VCO frequency divided by two. However, the basic PLL is tracking with twice this error, so that the variable of interest in analyzing the loop behavior is

\[
\phi \triangleq 2\theta_{\varepsilon}(t) \\
= 2(\theta(t) - \tilde{\theta}(t)) .
\]
In Equation (4), the only terms of interest are those containing $\phi$. All other terms can be eliminated by either low-pass filtering of $\epsilon(t)$ or by choosing a PD which, instead of being a perfect multiplier, responds only to baseband frequencies. With the higher frequency terms eliminated $\epsilon(t)$ becomes

$$
\epsilon(t) = K_{PD} K_v \left\{ \left( \frac{K}{\sqrt{2}} m(t) n_1(t) + n_1(t) n_2(t) \right) \cos(\phi) 
+ \left( \frac{K^2}{4} + \frac{K}{\sqrt{2}} m(t) n_2(t) + \frac{n_2^2(t)}{2} - \frac{n_2^2(t)}{2} \right) \sin(\phi) \right\} 
$$

Proceeding on to the loop filter (LF), designated by its transfer function, $F(p)$, we see that its output is

$$
u(t) = F(p) \epsilon(t)
$$

where $p$ is the Heaviside differential operator, $d/dt$. This output, $u(t)$, then becomes the input to the VCO.

If $u(t)$ were zero, i.e., the dynamic error $\epsilon(t)$ is zero, this would imply either that there is no received signal or that there exists coherence between the input of the PLL and the output of the VCO. This in turn suggests that the PLL is operating at a certain quiescent frequency, $2\omega_o$, when the received signal is zero, i.e.,

$$
2 \frac{d\omega}{dt}_{\text{quies}} = 2\omega_o \text{ when } x(t) \equiv 0.
$$

However, when $\epsilon(t) \neq 0$, there will be a signal $F(p) \epsilon(t)$ fed into the VCO; and, therefore, the frequency
of the output of the VCO would be other than $2\omega_0$. Instead it will be of the form

$$2 \frac{d\tilde{\psi}}{dt} = 2 \frac{d\tilde{\psi}}{dt}_{\text{quies}} + K_{\text{VCO}} [F(p) \varepsilon(t) + n_{\text{VCO}}(t)] \quad (6)$$

where $K_{\text{VCO}}$ is the frequency gain (rad/(sec-volt)) of the VCO and $n_{\text{VCO}}(t)$ is the internal noise generated by the VCO.

Recalling that

$$\tilde{\psi}(t) = \omega_0 t + \theta(t)$$

and

$$\theta(t) = \omega_m t + \theta_0,$$

Equation (6) becomes

$$2 \frac{d\theta(t)}{dt} = K_{\text{VCO}} [F(p) \varepsilon(t) + n_{\text{VCO}}(t)] \quad (7)$$

and

$$2 \frac{d\theta(t)}{dt} = 2\omega_m. \quad (8)$$

Subtracting Equation (7) from (8), we get

$$2 \frac{d(\theta(t) - \tilde{\theta}(t))}{dt} = \frac{d\phi}{dt} = 2\omega_m - K_{\text{VCO}} [F(p) \varepsilon(t) + n_{\text{VCO}}(t)]. \quad (9)$$

From Equations (5) and (9) we get

$$\dot{\phi} = 2\omega_m - K_{\text{VCO}} K_{PD} K_v \left\{ \frac{K}{\sqrt{2}} m(t) m_1(t) + n_1(t) n_2(t) \right\} \cos(\phi)$$

$$+ \left( \frac{K^2}{2} + \frac{K}{\sqrt{2}} m(t) n_2(t) + \frac{n_2^2(t)}{2} - \frac{n_1^2(t)}{2} \right) \sin(\phi)$$

$$- K_{\text{VCO}} n_{\text{VCO}}(t), \quad (10)$$
where $\dot{\phi} = d\phi/dt$.

Defining $K_C \triangleq K_{VCO} K_{PD} K_V$ and

$$n_c(t) = \left\{ \frac{K}{\sqrt{2}} m(t) n_1(t) + n_1(t) n_2(t) \right\} \cos(\phi)$$

$$+ \left( -\frac{K}{\sqrt{2}} m(t) n_2(t) + \frac{n_2^2(t)}{2} - \frac{n_1^2(t)}{2} \right) \sin(\phi) \right\},$$

Equation (10) becomes

$$\dot{\phi}(t) = 2\omega_m - K_C F(p) \left[ \frac{K^2}{4} \sin\phi + n_c(t) \right] - K_{VCO} n_{VCO}(t). \quad (11)$$

Equation (11) represents the nonlinear stochastic differential equation which describes the behavior of the squaring-loop system.

In order to analyze Equation (11), it is necessary at this point to recall some techniques used in the theory of Markov processes.
IV. THE THEORY OF MARKOV PROCESSES

In this section some aspects of the theory of Markov processes are presented. However, before proceeding, it is essential that terminology be defined.

A. Terminology

1. Vector Random Process

\( \mathbf{x}(t) \) is a vector random process of m dimension if its m components \( x_1(t) \cdots x_m(t) \), defining the vector \( \mathbf{x}(t) \) are themselves random processes.

\[
\mathbf{x} \triangleq \left( \begin{array}{c}
x_1(t) \\
\vdots \\
\vdots \\
x_m(t)
\end{array} \right) = \{x_1(t), \cdots, x_m(t)\}.
\]

The probability density function of the vector random process \( \mathbf{x}(t) \) is given by the joint density of its components, i.e., \( p(\mathbf{x}(t)) = p(x_1(t) \cdots, x_m(t)) \).

Consider the joint probability density function of the random process at \( t = t \) and \( t = t + \tau \),

\[
p(\mathbf{x}(t), \mathbf{x}(t + \tau) \equiv p(\mathbf{x}, \mathbf{x}_{\tau})
\]

\[
= p(x_1(t) \cdots, x_m(t), x_1(t + \tau) \cdots x_m(t + \tau)).
\]

Then the conditional density of \( \mathbf{x}_{\tau} \), given \( \mathbf{x} = \mathbf{x}_0 \), is given by
\begin{align*}
\text{p}(\bar{x}_t | \bar{x} = \bar{x}_0) &= \frac{\text{p}(\bar{x}, \bar{x}_t)}{\text{p}(\bar{x})}.
\end{align*}

2. Characteristic Functions

The characteristic function of a random process $\bar{x}(t)$ at time $t$ is defined by

\begin{align*}
\theta(\bar{u}, \bar{x}) &= \text{E}\{\exp(j \bar{x}^T \bar{u})\} \\
&= \text{E}\{\exp(j \sum_{i=1}^{M} x_i u_i)\}
\end{align*}

where the expectation is with respect to the density function $p(\bar{x})$.

The conditional characteristic function of the random process $\bar{x}(t)$ at time $t = t + \tau$ given the previous value $\bar{x}(t)$ at time $t = t_0$, $\bar{x}(t_0) = \bar{x}_0$, is defined by

\begin{align*}
\theta(\bar{u}, \bar{x}_t | \bar{x} = \bar{x}_0) &= \text{E}\{\exp(j \sum_{i=1}^{M} x_{t,i} u_i)\}
\end{align*}

where now the expectation is with respect to $p(\bar{x}_t | \bar{x})$.

3. Correlation Time

The correlation time $\tau_{\text{cor}}$ of the random process $\bar{x}(t)$ is defined by the relationship
\[ \tau_{\text{cor}} \triangleq \int_0^\infty |R_x(\tau)| \, d\tau \]

where

\[ R_x(\tau) \triangleq \frac{E(\overline{x}x_\tau) - E(\overline{x})^2}{\sigma^2(\overline{x})} \]

is the correlation coefficient of \( \overline{x}(t) \),

(Stratonovich [22]).

The correlation time \( \tau_{\text{cor}} \) gives us a measure of how large the time interval over which correlation between values of the process extends.

B. Markov Processes

Define a random process \( \overline{x}(t) \), and let \( \overline{x}(t_1), \overline{x}(t_2) \ldots \). \( \overline{x}(t_n) \) be a set of values of \( \overline{x}(t) \) at different times

\( t_n > t_{n-1} > \cdots > t_1 \).

Then generalizing the joint probability function of the preceding section, we obtain

\[ p(\overline{x}(t_1) \cdots \overline{x}(t_n)) = p(x_1(t_1) \cdots x_1(t_n) \cdots x_m(t_1) \cdots x_m(t_n)). \]

Then extending the definition of conditional density function, we get

\[ p(\overline{x}(t_n) | \overline{x}(t_{n-1}) \cdots, \overline{x}(t_1)) \triangleq \frac{p(\overline{x}(t_1) \cdots, \overline{x}(t_n))}{p(\overline{x}(t_1) \cdots, \overline{x}(t_{n-1}))}. \quad (12) \]

The process which is described by the conditional density function, Equation (12), is said to be Markov if
p(\(\mathbf{x}(t_n)\mid \mathbf{x}(t_{n-1})\), \ldots, \(\mathbf{x}(t_1)\)) = p(\(\mathbf{x}(t_n)\mid \mathbf{x}(t_{n-1})\)).

In other words, a process is Markov if its value at some future time is dependent only upon its present value.

C. **Derivation of the Diffusion Equation**

In this section the generalized diffusion equation of \(m\)-dimension is derived. This derivation extends that which is presented by Stratonovich [22].

The derivation is as follows: Let \(\mathbf{x}(t)\) be an \(m\)-dimensional random process,

\[
\mathbf{x}(t) = \mathbf{x} \triangleq \{x_1(t), \ldots, x_M(t)\}
\]

and \(\mathbf{x}(t + \tau)\) represent the same process at \(t = t + \tau\),

\[
\mathbf{x}(t + \tau) = \mathbf{x}_\tau \triangleq \{x_1(t + \tau), \ldots, x_M(t + \tau)\}.
\]

Then \(p(\mathbf{x}_\tau, \mathbf{x})\) represents the joint probability density function of \(\mathbf{x}_\tau\) and \(\mathbf{x}\). And as before

\[
p(\mathbf{x}_\tau \mid \mathbf{x}) = \frac{p(\mathbf{x}_\tau, \mathbf{x})}{p(\mathbf{x})}.
\]

The marginal probability density function of \(\mathbf{x}_\tau\) is

\[
p_{\tau}(\mathbf{x}_\tau) \triangleq \int_{\mathbf{x}} p(\mathbf{x}_\tau, \mathbf{x}) \, d\mathbf{x} = \int_{\mathbf{x}} p(\mathbf{x}_\tau \mid \mathbf{x}) \, p(\mathbf{x}) \, d\mathbf{x}.
\] (13)
The conditional characteristic function of the random process $\bar{x}_\tau - \bar{x}$ is

$$\theta(\bar{u}; \bar{x}_\tau - \bar{x} | \bar{x}) = \mathbb{E}\{\exp(j \, \bar{u}^T (\bar{x}_\tau - \bar{x}))\}$$

$$= \mathbb{E}\{\exp(j \sum_{i=1}^{M} u_i (x_{\tau_i} - x_i))\}$$

where, as before

$$\theta(\bar{u}, \bar{x}_\tau - \bar{x} | \bar{x}) \triangleq \int \exp(j \sum_{i=1}^{M} u_i (x_{\tau_i} - x_i)) p(\bar{x}_\tau | \bar{x}) \, d\bar{x}_\tau. \quad (14)$$

Expanding $\exp(j \sum_{i=1}^{M} u_i (x_{\tau_i} - x_i))$ into its Taylor series we obtain

$$\exp(j \sum_{i=1}^{M} u_i (x_{\tau_i} - x_i)) = 1 + \sum_{K=1}^{\infty} \frac{j^K}{K!} \left( \sum_{i=1}^{M} u_i (x_{\tau_i} - x_i) \right)^K$$

$$= 1 + \sum_{K=1}^{\infty} \frac{j^K}{K!} \sum_{\alpha_1}^{M} \cdots \sum_{\alpha_K}^{M} (u_{\alpha_1} (x_{\tau_{\alpha_1}} - x_{\alpha_1}) \cdots u_{\alpha_K} (x_{\tau_{\alpha_K}} - x_{\alpha_K}))$$

$$= 1 + \sum_{K=1}^{\infty} \frac{j^K}{K!} \sum_{\alpha_1}^{M} \cdots \sum_{\alpha_K}^{M} (\prod_{\ell=1}^{K} u_{\alpha_\ell} \prod_{n=1}^{\tau} (x_{\alpha_n} - x_{\alpha_n})). \quad (15)$$

Taking the conditional expectation of Equation (15), we get
\[ \theta(\bar{u}, \bar{x}_\tau - \bar{x} | \bar{x}) = 1 + \sum_{K=1}^{\infty} \frac{1}{K!} \sum_{\alpha K = 1}^{M} \cdots \sum_{\alpha 1 = 1}^{M} \left( \Pi_{\alpha \ell = 1}^{K} u_{\alpha \ell} E\left\{ \Pi_{n=1}^{K} (x_{\tau \ell} - x_{\alpha n}) \right\} \right). \]  

(16)

By taking the inverse Fourier transform of Equation (14), we get \( p(\bar{x}_\tau | \bar{x}) \)

\[ p(\bar{x}_\tau | \bar{x}) = \left( \frac{1}{2\pi} \right)^M \int_{\bar{x}} \int_{\bar{u}} \exp(-j \sum_{i=1}^{M} u_i (x_{\tau i} - x_i)) \theta(\bar{u}, \bar{x}_\tau - \bar{x} | \bar{x}) d\bar{u} \ p(\bar{x}) d\bar{x}. \]  

(17)

Substituting Equation (17) into Equation (13), we arrive at an expression for \( p_\tau(\bar{x}_\tau) \),

\[ p_\tau(\bar{x}_\tau) = \left( \frac{1}{2\pi} \right)^M \int_{\bar{x}} \int_{\bar{u}} \exp(-j \sum_{i=1}^{M} u_i (x_{\tau i} - x_i)) \theta(\bar{u}, \bar{x}_\tau - \bar{x} | \bar{x}) d\bar{u} \ p(\bar{x}) d\bar{x} \]

\[ = \left( \frac{1}{2\pi} \right)^M \int_{\bar{x}} \int_{\bar{u}} \exp(-j \sum_{i=1}^{M} u_i (x_{\tau i} - x_i))(1 + \sum_{K=1}^{\infty} \frac{1}{K!} \sum_{\alpha K = 1}^{M} \cdots \sum_{\alpha 1 = 1}^{M} \left( \Pi_{\alpha \ell = 1}^{K} u_{\alpha \ell} E\left\{ \Pi_{n=1}^{K} (x_{\tau \ell} - x_{\alpha n}) \right\} \right) d\bar{u} \ p(\bar{x}) d\bar{x} \]

\[ = \prod_{i=1}^{M} \delta(x_{\tau i} - x_i) p(\bar{x}) \ d\bar{x} \]

\[ + \sum_{K=1}^{\infty} \frac{1}{K!} \sum_{\alpha K = 1}^{M} \cdots \sum_{\alpha 1 = 1}^{M} \left( \frac{1}{2\pi} \right)^M \int_{\bar{x}} \int_{\bar{u}} \prod_{\alpha \ell = 1}^{K} u_{\alpha \ell} \exp(-j \sum_{i=1}^{M} u_i (x_{\tau i} - x_i) d\bar{u} \ p(\bar{x}) d\bar{x} \]

\[ \times (x_{\tau i} - x_i) E\left\{ \Pi_{n=1}^{K} (x_{\tau n} - x_{\alpha n}) \right\} p(\bar{x}) d\bar{x}. \]  

(18)
Look at
\[ \left( \frac{1}{2\pi} \right)^M \int_{\mathbb{R}^M} j^K \prod_{\lambda=1}^K u_{\lambda l} \exp(-j \sum_{i=1}^M u_i (x_{\tau i} - x_i)) \, du \]

M integrals
\[ = \prod_{\lambda=1}^K (-\frac{\partial}{\partial x_{\tau \lambda}}) \left( \frac{1}{2\pi} \right)^M \int_{\mathbb{R}^M} \cdots \int_{\mathbb{R}^M} \exp(-j \sum_{i=1}^M u_i (x_{\tau i} - x_i)) \, du_1 \cdots du_M \]
\[ = \prod_{\lambda=1}^K (-\frac{\partial}{\partial x_{\tau \lambda}}) \prod_{i=1}^M \delta(x_{\tau i} - x_i) . \] (19)

But
\[ \prod_{i=1}^M \delta(x_{\tau i} - x_i) = \delta(\bar{x}_\tau - \bar{x}) . \] (20)

Using Equation (20) in Equation (19) and then substituting into Equation (18), we have
\[ p_\tau(x_\tau) = \int_{\mathbb{R}} \delta(\bar{x}_\tau - \bar{x}) \, p(\bar{x}) \, d\bar{x} \]
\[ + \sum_{K=1}^{\infty} \frac{1}{K!} \sum_{\alpha K} \sum_{\alpha 1} \prod_{\lambda=1}^K (-\frac{\partial}{\partial x_{\tau \lambda}}) \delta(\bar{x}_\tau - \bar{x}) E\{ \prod_{n=1}^K (x_{\tau n} - x_n) \}
\times p(\bar{x}) \, d\bar{x} \]
\[ = p(\bar{x}_\tau) + \sum_{K=1}^{\infty} \frac{1}{K!} \sum_{\alpha K} \sum_{\alpha 1} \prod_{\lambda=1}^K (-\frac{\partial}{\partial x_{\tau \lambda}}) E\{ \prod_{n=1}^K (x_{\tau n} - x_n) \} p(\bar{x}_\tau) . \] (21)
Transposing $p(\overline{x}_t)$ to the left side and dividing by $\tau$ where we then let $\tau \to 0$, we obtain

$$
\dot{p}(x_t) = \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{\alpha_k} \sum_{\alpha_1}^{M} \sum_{l=1}^{M} \sum_{l=1}^{K} \left( - \frac{\partial}{\partial x_{\tau \alpha_l}} \right) \lim_{\tau \to 0} \frac{E\{ \prod_{n=1}^{K} (x_{\tau \alpha_n} - x_{\alpha_n}) \}}{\tau} \frac{p(\overline{x}_t)}{\tau^{K}}
$$

$$
= \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{\alpha_k} \sum_{\alpha_1}^{M} \sum_{l=1}^{M} \sum_{l=1}^{K} \left( - \frac{\partial}{\partial x_{\tau \alpha_l}} \right) (K_{\alpha_1} \alpha_2, \ldots \alpha_k p(\overline{x}_t))
$$

(22)

where the intensity coefficients are defined by

$$
K_{\alpha_1} \ldots \alpha_k \triangleq \lim_{\tau \to 0} \frac{E\{ \prod_{n=1}^{K} (x_{\tau \alpha_n} - x_{\alpha_n}) \}}{\tau}
$$

and expectation is conditioned on $\overline{x} = \overline{x}_0$.

Equation (22) is the general equation describing the behavior of the probability density function of $p(\overline{x}_t)$. It is obvious that trying to solve Equation (22) would pose an insurmountable task. Fortunately, the processes which we shall be dealing with are approximated by continuous Markov processes which simplify (22).

Markov processes are said to be continuous if

$$
K_{\alpha_1} \ldots \alpha_k \triangleq \lim_{\tau \to 0} \frac{E\{ \prod_{n=1}^{K} (x_{\tau \alpha_n} - x_{\alpha_n}) \}}{\tau} \equiv 0
$$

for all $K > 2$. 
So that Equation (22) reduces to

$$\dot{p}(x_\tau) = - \sum_{i=1}^{M} \left( \frac{\partial}{\partial x_{\tau i}} (K_i \ p(x_\tau)) + \frac{1}{2} \sum_{i=1}^{M} \sum_{\lambda=1}^{M} \left( \frac{\partial^2}{\partial x_{\tau i} \partial x_{\tau \lambda}} (K_{i \lambda} \ p(x_\tau)) \right) \right)$$

for continuous Markov processes.

The intensity coefficients, $K_i$ and $K_{i \lambda}$, are given by

$$K_i \triangleq \lim_{\tau \to 0} \frac{E\{x_{\tau i} - x_i\}}{\tau}$$

$$K_{i \lambda} \triangleq \lim_{\tau \to 0} \frac{E\{(x_{\tau i} - x_i)(x_{\tau \lambda} - x_\lambda)\}}{\tau}.$$  

Equation (25) is analogous to the equation of fluid flow. Therefore, the Fokker-Planck equation can be interpreted as the equation of probability flow. Since probability can be neither created nor destroyed, the probability flowing into a region must, in the steady-state, equal the probability flowing out of that region.

Equation (23) is known as the forward diffusion equation or the Fokker-Planck equation.

If we rearrange Equation (23), we get

$$\dot{p}(x_\tau) = - \sum_{i=1}^{M} \frac{\partial}{\partial x_{\tau i}} \left( K_i - \frac{1}{2} \sum_{\lambda=1}^{M} \frac{\partial^2}{\partial x_{\tau i} \partial x_{\tau \lambda}} K_{i \lambda} \right) p(x_\tau).$$

(24)
If we define $A_i(\overline{x}_T) \triangleq \{K_{i} - \frac{1}{2} \sum_{\ell=1}^{M} \frac{\partial}{\partial x_T^\ell} K_{i \ell} \} p(\overline{x}_T)$, we get

$$\dot{p}(x_T) = - \nabla \cdot \overline{A}$$

or

$$\dot{p}(x_T) + \nabla \cdot \overline{A} = 0$$

where

$$\overline{A} \triangleq \{A_i(\overline{x}_T) \cdots A_M(\overline{x}_T)\}$$

and

$$\nabla = \{\frac{\partial}{\partial x_T^i} \cdots \frac{\partial}{\partial x_T^m}\}$$

Equation (25) can be interpreted as the "equation of conservation of probability" and $\overline{A}$ is referred to as the probability current (Stratonovich, [22]). Consider a hyperplane $H$ of dimension $M + 1$ contained "between two hyperplanes $\overline{x}(t)$ and $\overline{x}(t + \tau)$ of dimension $M$",

$$\overline{x}(t) \leq \overline{x} \leq \overline{x}(t + \tau)$$

Then $A(\overline{x})(\tau)$ is the amount of probability entering the hyperplane $H$ in a time $\tau$ across the hyperplane $\overline{x}(t)$ and $A(\overline{x}_T)(\tau)$ is the amount of probability leaving the hyperplane $H$ across boundary hyperplane $\overline{x}(t + \tau)$ in time $\tau$.

D. Discussion on the Fluctuation Equation

The derivation of the Fokker-Planck equation was based on the assumption that the process to be considered...
was continuous Markov. However, actual processes encountered are normally not Markov. Stratonovich [22] has presented a perturbation scheme for replacing actual processes by Markov processes.

If the process, $\bar{x}(t)$, is due to a forcing process $\xi(t)$ on a system whose behavior can be characterized by a first-order vector differential equation, i.e.,

$$\ddot{x} = \varepsilon \bar{F}(\bar{x}, \xi(t)),$$

then this equation is referred to as the "fluctuation equation" [22], and $\varepsilon$ is a small parameter. The parameter $\varepsilon$ does not have to be stated explicitly, but may be incorporated into the function $\bar{F}(\bar{x}, \xi(t))$.

Using expansion techniques [22], an equation corresponding to the Fokker-Planck equation may be obtained. This expansion involves a power series of the parameter $\varepsilon$. However, for small $\varepsilon$, the series may be truncated to include only the lower order terms in $\varepsilon$. Stratonovich shows that by including only terms through $\varepsilon^2$, the equation involving the power series in $\varepsilon$ reduces to an expression analogous to the Fokker-Planck equation.

For a more detailed presentation of the "fluctuation equation" approach, the reader should refer to Stratonovich [22].
V. NONLINEAR ANALYSIS OF SYSTEM

A. Phase Error Probability Density Function

Recalling the nonlinear stochastic differential equation which describes the behavior of the squaring loop system, Equation (11), we see from Stratonovich [22] that this process can be approximated as a Markov process by the "fluctuation equation" approach previously outlined.

If we assume that the loop filter transfer function can be expressed in terms of a Heaviside expansion, i.e.,

\[ F(p) = F_0 + \sum_{k=1}^{M} \frac{F_k}{1 + \tau_k p} \]

and substitute this expression into Equation (11), we obtain

\[ \dot{\phi} = 2\omega_m - K_C F_0 \left( \frac{K^2}{4} \sin \phi + n_C(t) \right) - K_C \sum_{k=1}^{M} \frac{1 - F_k}{1 + \tau_k p} \]
\[ \times \left( \frac{K^2}{4} \sin \phi + n_C(t) \right) - K_{VCO} n_{VCO}(t) \]  \hspace{1cm} (26)

Define a set of states as follows:

\[ x_0(t) = \phi(t) \]
\[ x_1(t) = -\left( \frac{1 - F_1}{1 + \tau_1 p} \right) K_C \left( \frac{K^2}{4} \sin \phi + n_C(t) \right) \]
\[ \cdots \]
\[ x_M(t) = -\left( \frac{1 - F_M}{1 + \tau_M p} \right) K_C \left( \frac{K^2}{4} \sin \phi + n_C(t) \right) . \]  \hspace{1cm} (27)
By addition of \( x_1(t), \cdots x_M(t) \) as given by (27), it can be seen that Equation (26) becomes

\[
\dot{\phi} = 2\omega_m - K_C F_0 \frac{K^2}{4} \sin \phi + n_0(t) - K_{VCO} n_{VCO}(t) + \sum_{\ell=1}^{M} x_{\ell}(t).
\]

Now if we look at each equation of (27), we see that

\[
(1 + \tau_\ell p) x_\ell(t) = -(1 - F_\ell) K_C \frac{K^2}{4} \sin \phi + n_0(t)
\]

(28)

\( \ell = 1, \cdots M \).

After manipulating (28) and remembering that \( p \) is the Heaviside differential operator, \( d/dt \), we arrive at

\[
\dot{x}_\ell(t) \equiv \frac{dx_\ell(t)}{dt} = -\frac{x_\ell(t)}{\tau_\ell} - \frac{(1 - F_\ell)}{\tau_\ell} K_C \frac{K^2}{4} \sin \phi + n_0(t);
\]

\( \ell = 1, \cdots M \)

Thus we have \( M + 1 \) first order differential equations, i.e.,

\[
\dot{x}_0 = \dot{\phi} = 2\omega_m - K_C F_0 \frac{K^2}{4} \sin \phi + n_0(t) + K_{VCO} n_{VCO}(t)
\]

\[
+ \sum_{\ell=1}^{M} x_\ell(t)
\]

\[
\dot{x}_1 = -\frac{x_1(t)}{\tau_1} - \frac{(1 - F_1)}{\tau_1} K_C \frac{K^2}{4} \sin \phi + n_0(t)
\]

\[
\cdots
\]

\[
\cdots
\]

\[
\dot{x}_M = -\frac{x_M(t)}{\tau_M} - \frac{(1 - F_M)}{\tau_M} K_C \frac{K^2}{4} \sin \phi + n_0(t).
\]
If the correlation times of \( \nu_{\text{vco}}(t) \) and \( \nu_{\text{c}}(t) \) are much less than the response time of the system then, according to Stratonovich, the vector random process defined by \( \overline{x}(t) \equiv \overline{x} = \{x_0(t), \cdots, x_M(t)\} \) can be approximated by a vector Markov process using the "fluctuation equation" approach. With this approximation, the Fokker-Planck equation may be used.

The processes which will be of interest to us are those which have been going on for such a time that the transients have dissipated, i.e., the steady state condition. For steady state the Fokker-Planck equation (Equation (23)) becomes,

\[
\sum_{i=1}^{M} \left( -\frac{\partial}{\partial x_{\tau_i}} \right) K_i p(\overline{x}_t) - \frac{1}{2} \sum_{\ell=1}^{M} \sum_{i=1}^{M} \left( \frac{\partial^2}{\partial x_{\tau_i} \partial x_{\tau_{\ell}}} \right) K_{i\ell} p(\overline{x}_t) = 0
\]

where \( K_i \equiv \lim_{\tau \to 0} \frac{E\{x_{\tau_i} - x_i\}}{\tau} = E\{\dot{x}_i\} \)

and

\[
K_{i\ell} \equiv \lim_{\tau \to 0} \frac{E\{(x_{\tau_i} - x_i)(x_{\tau_{\ell}} - x_{\ell})\}}{\tau}
\]

Using Equation (24), Equation (29) becomes

\[
\sum_{i=1}^{M} \frac{\partial}{\partial x_{\tau_i}} A_i(\overline{x}_t) = 0
\]
or alternately,
\[ \nabla \cdot \overline{A} = 0. \]

Using arguments similar to those used by Tikhonov [16], it follows that if the correlation times of \( n_{VCO}(t) \) and \( n_{C}(t) \) are much less than the response time of the system, then the \( K_{i\ell} \)'s can be expressed as

\[
K_{i\ell}(x) \triangleq \int_{-\infty}^{\infty} (E\{x_i(t) \dot{x}_\ell(t + \tau)\} - E\{x_i\} E\{\dot{x}_\ell\}) d\tau.
\]

Using Equation (30), and arguments similar to those used by Lindsey [40], the intensity coefficients become

\[
K_0(x) = 2\omega_m - F_0 \frac{K_C^2}{4} \sin \phi + \sum_{\ell=1}^{M} x_\ell
\]

\[
K_j(x) = -\frac{x_j}{\tau_j} - \frac{(1 - F_j)}{\tau_j} \frac{K_C^2}{4} \sin \phi
\]

\[
j = 1, \ldots, M
\]

\[
K_{oo}(x) = \frac{K_C^2}{4} F_0^2 \int_{-\infty}^{\infty} E\{n_C(t) n_C(t + \tau)\} d\tau
\]

\[
+ K_{VCO}^2 \int_{-\infty}^{\infty} E\{n_{VCO}(t) n_{VCO}(t + \tau)\} d\tau
\]

\[
K_{ij}(x) = \frac{K_C^2}{4} \frac{(1 - F_i)(1 - F_j)}{\tau_i \tau_j} \int_{-\infty}^{\infty} E\{n_{VCO}(t) n_{VCO}(t + \tau)\} d\tau
\]

\[
i, j = 1, \ldots, M
\]
Now if we apply these intensity coefficients to the general solution by Lindsey [40] and define new variables, $\beta$ and $\alpha$,

$$\beta \triangleq \frac{2}{K_{oo}(\bar{x})} \left[ 2\omega_m - \frac{K^2}{4} K_C E\{\sin \phi \} \sum_{\ell=1}^{M} (1 - F_{\ell}) \right.$$  

$$\times \left( 1 + \frac{8K_{oo}(\bar{x})}{K_C K^2 \tau K (E\{\sin^2 \phi \} - E^2\{\sin \phi \})} \right), \quad (31)$$

$$\alpha \triangleq \frac{2}{K_{oo}(\bar{x})} \left[ \frac{K^2 K_C F_0}{4} - \frac{2K_{oo}(\bar{x})}{K_C K^2 F_0} \sum_{\ell=1}^{M} \left( \frac{1 - F_{\ell}}{\tau_{\ell}} \right) \right], \quad (32)$$

we arrive at the following solution for the phase error probability density function, $p(\phi)$,

$$p(\phi) = \frac{\exp((\phi + \pi)\beta + \alpha \cos \phi)}{4\pi^2 \left| I_{j\beta}(\alpha) \right|^2} \int_{\phi}^{\phi+2\pi} \exp(-\beta x - \alpha \cos x) \, dx. \quad (33)$$

This probability density function will be used now to determine an expression for the bit error probability.

B. **Bit Error Probability**

The signal energy per bit in the $180^\circ$ biphase signal is

$$s \triangleq \int_0^T k^2 m^2(t)/2 \, dt = \frac{K^2}{2} \int_0^T dt \quad (34)$$

$$= \frac{K^2 T}{2},$$
and the one-sided power spectral density of the white zero-mean Gaussian noise, \( n(t) \), is \( N_0 \).

Viterbi [13] has shown that for optimum detection of antipodal signals in white Gaussian noise, the bit error probability, conditioned on \( \phi \), is

\[
P_E(\phi) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp(-y^2/2) \, dy
\]

\[
= \text{erfc} \left( \frac{2s}{N_0} \cos \phi \right).
\]

Combining this with the probability density of phase error (Equation (33)), we obtain the expression for the average error probability:

\[
P_E = E\{P_E(\phi)\} = \int_{-\pi}^\pi P_E(\phi) p(\phi) d\phi
\]

\[
= \frac{1}{4\pi^2} \int_{-\pi}^{\phi+2\pi} \text{erfc} \left( \frac{2s}{N_0} \cos \phi \right) \exp(\beta(\phi-x+\pi)+\alpha(\cos \phi - \cos x))
\]

\[
\times dx \, d\phi; \quad |\phi| < \pi.
\]

where \( \alpha \) and \( \beta \) are defined by Equations (31) and (32).

Equation (36) cannot be evaluated in closed form in the general case. This probability of bit error is a function of the signal energy-to-noise density ratio in the bit
rate bandwidth. Evaluation of Equation (36) can be performed by numerical integration. Thus, parametric curves of the probability of bit error, $P_E$, versus the signal energy-to-noise density ratio in the bit rate bandwidth can be obtained.
VI. CONCLUSIONS

This thesis has presented an analysis of the squaring loop synchronization scheme for 180° biphase modulated signals. The analysis uses extensions of the Fokker-Planck techniques to obtain the probability density function for the phase error of the squaring-loop.

In order to establish a foundation on which the analysis could be developed, some rudiments of the theory of Markov processes were presented. The derivation of the general M-dimensional diffusion equation given in this paper extends the derivation of the one-dimensional diffusion equation presented by Stratonovich [22].

The analysis of the tracking system follows, to a large extent, work performed previously by Lindsey [40] and Didday and Lindsey [33]. Their generalized methods have been applied to a squaring-loop tracking system which has a sinusoidal phase detector characteristic. A major premise on which this analysis was based is that the correlation times of the noise terms be much less than the response time of the system. This assumption enabled the approximation of the noise processes by delta correlated processes which, in turn, allowed the application of Fokker-Planck techniques to the analysis of the system.

After obtaining the probability density function of the phase error, it was straightforward to obtain the
probability of bit error. The phase error density function involves an integral which, in general, cannot be integrated in closed form. Consequently, numerical methods must be used to evaluate both the phase error density function and the bit error probability. Thus, parametric curves for the probability density function of the phase error as a function of \( \phi \) can be obtained, the parameters being \( \alpha \) and \( \beta \). Similarly, a set of curves with the same parameters can be obtained for the probability of bit error as a function of signal-to-noise ratio in the bit rate bandwidth. These results could then be used to evaluate the performance of receivers operating in changing doppler environments.

The parameter \( \alpha \) is directly related to the signal-to-noise ratio within the bandwidth of the linearized PLL; the parameter \( \beta \) is related to the frequency mismatch, \( \omega_m \). For a first order loop, i.e., \( F_0 = 1, F_1 = 0, \ldots, M \), with no frequency mismatch, \( \omega_m = 0 \), it can be seen that \( \beta \) is zero (Equation (31)).

This analysis is of practical importance because it provides performance capabilities of a squaring-loop synchronization scheme for 180° biphase modulated signals. As can be seen by Equations (31), (32), and (36), an expression is obtained for the bit error probability which depends on the signal-to-noise ratio in the bit rate bandwidth and parameters associated with the loop filter transfer function. Information about the signal-to-noise
ratio and the loop filter bandwidth are inherent in the parameters $\alpha$ and $\beta$. The functional forms for $\alpha$ and $\beta$ provide insight as to how the squaring-loop parameters associated with the presquaring and loop filter transfer functions should be chosen so as to minimize the bit error probability. By minimizing the probability of bit error, the amount of message redundancy normally included for accurate reception may be reduced. This, in turn, allows a savings in transmitter power.

Suggested areas for further study include: 1) the actual evaluation of Equation (36) for certain parameter values of $\alpha$ and $\beta$; 2) the extension of the "fluctuation equation" approach to include truncation of the power series in $\varepsilon$ to third order instead of second order; 3) the analysis of a tracking system which has variable loop bandwidth, initially wide-bandwidth for acquiring signals and then transcending to a narrow band loop for tracking after acquisition.
VII. BIBLIGRAPHY


VITA

James Joseph Spence was born on November 24, 1946, in Cairo, Illinois, where he received his primary and secondary education.

While attending the University of Missouri - Rolla, he was enrolled in the Co-operative Engineering Program through the Beloit Corporation, Beloit, Wisconsin. He also was employed for a summer by Shell Oil Company of Houston, Texas.

Upon receiving his Bachelor of Science degree in Electrical Engineering in June 1968, he went to work with Westinghouse Electric Company of Pittsburgh, Pennsylvania.

He returned to the University of Missouri - Rolla, September 1968, to enroll in Graduate School, where he held a National Science Foundation Fellowship.

Since June 1969, he has been employed at Sandia Laboratories in Albuquerque, New Mexico.

Mr. Spence is a member of Eta Kappa Nu, Tau Beta Pi, Phi Kappa Phi, and Kappa Mu Epsilon Honor Societies.