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Abel dynamic equations of the first and second kind

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ABEL DYNAMIC EQUATIONS
OF THE FIRST AND THE SECOND KIND

by

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ABSTRACT

In this work, we study Abel dynamic equations of the first and the second kind. After a brief introduction to time scales, we introduce the Abel differential equations of the first and the second kind, as well as the canonical Abel form in the continuous case. Using the existing information, we derive novel results for time scales. We provide formulas for the Abel dynamic equations of the second kind and present a solution method. We furthermore achieve a special class of Abel equations of the first kind and discuss the canonical Abel equation. We get relations between common dynamic equations by analyzing relations between common differential equations in $\mathbb{R}$. Examples for $\mathbb{T} = \mathbb{R}$ illustrate our results for the Abel dynamic equations.

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TABLE OF CONTENTS

1. INTRODUCTION

Niels Henrik Abel, one of the most active mathematicians of his time, was born in Norway in 1802 [14] and improved mainly the research field of functional analysis. He dedicated himself to integral equations, where he defined the Abel integral and worked on methods to solve special integral equations, which were later called *Abel integral equations* [14]. His research on integral equations and their solutions led him work on differential equations, where he verified the importance of the Wronskian determinant for a differential equation of order two [14]. It was studies of the theory of elliptic functions that got him involved in the analysis of special differential equations [12], which are a generalized Riccati differential equation. Due to his crucial research on and improvement of these differential equations, they are named after Abel. The Abel differential equation of the first and of the second kind are both nonhomogeneous differential equations of first order and are related by a substitution that is explained in more detail in Section 3.2.

Abel differential equations have various applications. For example, in physics to find solutions for equations describing the development of the universe [15] and in the theory of thin film condensation [18]. This illustrates the significance of the analysis of Abel equations and their solution. The purpose of this thesis is to introduce Abel dynamic equations of the first and of the second kind and illustrate the relation between both kinds, as well as to other common differential equations. It has been already mentioned that Abel differential equations generalize the Ricatti differential equation, which will be verified later on. The Abel differential equation of the first kind is furthermore a generalization of the common Bernoulli differential equation, which enables the establishment of a correlation to linear differential equations and to a special class of the logistic differential equation. These relations are discussed throughout the thesis, but mainly in Section 5.

Various classes of Abel differential equations can be solved by different methods, and to illustrate the idea of solving Abel differential equations in $\mathbb{R}$ and $\mathbb{T}$, a special method is described and proved, mainly based on a paper of Bougoffa [3].

Applied scientists are interested in modeling real-life situations mathematically, which often include differential equations such as Abel equations. The modeling, in general, enables the use of mathematical tools to analyze the modeled situations and optimize them. For the mathematical model, data is used, which is often based on observation and examination of experiments and is, therefore, not evaluated at each time step t. For the investigation of these models, differential equations are used, where the (variable) coefficients of these equations are constructed by the data. To apply methods to solve these differential equations, the variable coefficients have to be continuous, which is not satisfied by the data, since it is only evaluated at some time points. To transfer the data into a continuous function, approximation methods are used, such as the linear or exponential approximation method. This makes, on the one hand, the mathematical optimization by using differential equations possible but, on the other hand, more inaccurate, since approximations based on assumptions were made. This is one of the main reasons why the time scale in the models is valuable. The mathematical field of time scales generalizes the time set and helps improve the model of the real world. Thus the interest in translating differential equations and their solution (methods) into time scales.

In some cases, the strategy to solve differential equations in time scales, so-called dynamic equations, is more or less identical to the continuous case $\mathbb{R}$. In other cases, a novel idea has to be found to generate a general solution. That underlines the purpose of this thesis, namely to translate Abel differential equations, which are used in many important applications, into a more generalized time set. After an introduction to time scales $\mathbb{T}$, Abel equations of the second and of the first kind are expounded. In applied science, the solution of the differential equation is of main interest, which is the reason why a solution method of a class of the Abel differential equation of the second kind is explained in $\mathbb{R}$, and then converted into $\mathbb{T}$. This method refers to a strategy presented in $\mathbb{R}$ in a paper by Bougoffa [3].

Finally, relating the Abel dynamic equation with common differential equations in $\mathbb{T}$ is realized in Section 5. The connections are first derived in $\mathbb{R}$ and then analyzed in $\mathbb{T}$. In this context, further transformations between differential equations, such as between the linear and the logistic differential equation, are examined in time scales and the results are presented. This should help the readers to better understand mathematical behavior in time scales and to get familiar with the Abel differential equation in a generalized time set, $\mathbb{T}$.

2. TIME SCALES PRELIMINARIES

2.1. MAIN DEFINITIONS IN TIME SCALES

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$ [2, p. 1]. A subset of X is a closed subset if its complement set is an open subset. A closed subset of X has the following properties [8, p. 23]:

- X and $\emptyset$ are closed subsets of X,
- any union of finitely many closed subsets of X is a closed subset of X,
- any intersection of arbitrarily many closed subsets of X is a closed subset of X.

It is easy to see that the rational numbers do not satisfy the third property of a closed subset of $\mathbb{R}$ and therefore do not define a time scale. The complex numbers do not define a time scale either since they are not a subset of $\mathbb{R}$.

Definition 2.1. Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$, define [2, p. 1]:

- The *forward jump operator* $\sigma : \mathbb{T} \longrightarrow \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \text{ for all } t \in \mathbb{T}. \tag{2.1}$$

- The *backward jump operator* $\rho : \mathbb{T} \longrightarrow \mathbb{T}$ by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\} \text{ for all } t \in \mathbb{T}. \tag{2.2}$$

Define $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$.

Example 2.2. If $\mathbb{T} = \mathbb{R}$, then

$$\sigma(t) = \inf\{s \in \mathbb{R} : s > t\} = t = \rho(t) = \sup\{s \in \mathbb{R} : s < t\} \text{ for all } t \in \mathbb{R}.$$

Example 2.3. If $\mathbb{T} = \mathbb{Z}$, then

$$\sigma(t) = \inf\{s \in \mathbb{Z} : s > t\} = t + 1 \text{ for all } t \in \mathbb{Z},$$

$$\rho(t) = \sup\{s \in \mathbb{Z} : s < t\} = t - 1 \text{ for all } t \in \mathbb{Z}.$$

Definition 2.4. $t \in \mathbb{T}$ is called [2, p. 2]

- *right-scattered* if $\sigma(t) > t$,
- *left-scattered* if $\rho(t) < t$,
- *isolated* if t is left-scattered and right-scattered;
- *right-dense* if $\sigma(t) = t$,
- *left-dense* if $\rho(t) = t$, and
- *dense* if t is left-dense and right-dense.

It is trivial to realize that t is dense for all $t \in \mathbb{R}$ if $\mathbb{T} = \mathbb{R}$ and that t is isolated for all $t \in \mathbb{Z}$ if $\mathbb{T} = \mathbb{Z}$.

Definition 2.5. The *graininess function* $\mu : \mathbb{T} \longrightarrow [0, \infty)$ is defined by [2, p. 2]

$$\mu(t) := \sigma(t) - t \text{ for all } t \in \mathbb{T}. \tag{2.3}$$

If $t \in \mathbb{T}$ has a left-scattered maximum M, then we define $\mathbb{T}^\kappa = \mathbb{T} \setminus \{M\}$; otherwise, $\mathbb{T}^\kappa = \mathbb{T}$.

Example 2.6. For $\mathbb{T} = \mathbb{R}$,

$$\mu(t) = \sigma(t) - t = t - t = 0 \text{ for all } t \in \mathbb{R}.$$

Example 2.7. For $\mathbb{T} = \mathbb{Z}$,

$$\mu(t) = \sigma(t) - t = t + 1 - t = 1 \text{ for all } t \in \mathbb{Z}.$$

Remark 2.8. In the literature, $f(\sigma(t))$ is equivalently denoted as $f^{\sigma}(t)$.

2.2. DIFFERENTIATION ON TIME SCALES

Definition 2.9. Consider the function $f : \mathbb{T} \longrightarrow \mathbb{R}$. f is called *delta differentiable* at $t \in \mathbb{T}$, or short *differentiable*, if for all $\varepsilon > 0$, there exists $\delta > 0$ and a number $f^{\Delta}(t)$, such that [2, p. 2]

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s) \right| \leq \varepsilon \left| \sigma(t) - s \right| \text{ for all } s \in (t - \delta, t + \delta). \quad (2.4)$$

If f is differentiable in $\mathbb{T}^{\kappa}$, then f^{Δ} is called the *delta derivative* of f.

Theorem 2.10. *Assume $f : \mathbb{T} \longrightarrow \mathbb{R}$ is differentiable at $t \in \mathbb{T}^{\kappa}$. Then f is continuous at t [2, p. 2].*

Theorem 2.11. *Assume $f : \mathbb{T} \longrightarrow \mathbb{R}$ is continuous at $t \in \mathbb{T}^{\kappa}$ and t is right-scattered. Then f is differentiable at t and [2, p. 2]*

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}. \quad (2.5)$$

Theorem 2.12. *Assume $f : \mathbb{T} \longrightarrow \mathbb{R}$ is a function and $t \in \mathbb{T}^\kappa$ is right-dense, then f is differentiable at t if and only if the limit*

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s} \tag{2.6}$$

exists. In this case the limit is equal to the delta-derivative $f^\Delta(t)$ [2, p. 2].

Theorem 2.13. *If f is differentiable at $t \in \mathbb{T}^\kappa$, then*

$$f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t). \tag{2.7}$$

Proof. Let $f : \mathbb{T} \longrightarrow \mathbb{R}$ be differentiable at $t \in \mathbb{T}^\kappa$. If t is right-dense, then $\sigma(t) = t$ and therefore, by Definition 2.5, $\mu(t) = 0$, so

$$f(\sigma(t)) = f(t) = f(t) + \mu(t) f^\Delta(t).$$

If t is not right-dense, then $\sigma(t) \neq t$. So t is right-scattered and therefore, by Definition 2.5, $\mu(t) \neq 0$. Since f is differentiable at t, Theorem 2.11 yields

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)},$$

i.e.,

$$f^\Delta(t)\mu(t) = f(\sigma(t)) - f(t),$$

i.e.,

$$f^\Delta(t)\mu(t) + f(t) = f(\sigma(t)).$$

This completes the proof. □

Example 2.14. For $\mathbb{T} = \mathbb{R}$, any $t \in \mathbb{R}$ is right-dense. Theorem 2.12 states that f is differentiable if and only if

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists. In this case

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} = f'(t).$$

Example 2.15. For $\mathbb{T} = \mathbb{Z}$, any $t \in \mathbb{Z}$ is right-scattered. Assume f is continuous. By Theorem 2.11, f is differentiable at t with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(t+1) - f(t)}{1} = f(t+1) - f(t) = \Delta f(t).$$

The delta-operator is, as the derivative operator, a *linear operator*. Assuming $f, g : \mathbb{T} \longrightarrow \mathbb{R}$ are differentiable at $t \in \mathbb{T}^{\kappa}$ and $\alpha, \beta \in \mathbb{R}$, one has

$$(\alpha f + \beta g)^{\Delta}(t) = \alpha f^{\Delta}(t) + \beta g^{\Delta}(t). \tag{2.8}$$

Similar to the derivative of a product of two functions $f, g : \mathbb{R} \longrightarrow \mathbb{R}$, one can obtain the *product rule* for time scales in the following way [2, p. 3]. If $f, g : \mathbb{T} \longrightarrow \mathbb{R}$ are differentiable at $t \in \mathbb{T}^{\kappa}$, then

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f^{\sigma}(t)g^{\Delta}(t) = g^{\Delta}(t)f(t) + g^{\sigma}(t)f^{\Delta}(t). \tag{2.9}$$

Before deriving a formula for the *quotient rule* of two functions f, g in time scales, one has to differentiate $1/f$ for a function $f : \mathbb{T} \longrightarrow \mathbb{R}$.

Theorem 2.16. *If $f : \mathbb{T} \longrightarrow \mathbb{R}$ is delta-differentiable at $t \in \mathbb{T}^\kappa$ and $f(t)f(\sigma(t)) \neq 0$, then*

$$\left(\frac{1}{f}\right)^\Delta (t) = -\frac{f^\Delta(t)}{f(t)f(\sigma(t))}. \tag{2.10}$$

Proof. Assume f and $1/f$ are delta-differentiable at $t \in \mathbb{T}^\kappa$ and $f(s)$, $f(\sigma(t)) \neq 0$ for all s in surrounding of t. For $\varepsilon > 0$, define $\varepsilon^* = \varepsilon \left(\frac{\|f(\sigma(t))\| + \|f^\Delta(t)f(s)\|}{\|f(s)f(\sigma(t))\|}\right)^{-1}$. Then $\varepsilon^* > 0$. Since f is differentiable in $t \in \mathbb{T}^\kappa$, by Definition 2.4, there exists a neighborhood U_1 of t such that

$$\left|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)\right| \leq \varepsilon^* \left|\sigma(t) - s\right| \text{ for all } s \in U_1.$$

Since $1/f$ is differentiable at t, $1/f$ is continuous at t by Theorem 2.10. Therefore there exists a neighborhood U_2 of t such that

$$\left|\frac{1}{f(t)} - \frac{1}{f(s)}\right| \leq \varepsilon^* \text{ for all } s \in U_2.$$

Let $U = U_1 \cap U_2$ and $s \in U$. Then

$$\begin{aligned}
&\left|\frac{1}{f(\sigma(t))} - \frac{1}{f(s)} - \left(-\frac{f^\Delta(t)}{f(\sigma(t))f(t)}(\sigma(t) - s)\right)\right| \\
&= \left|\frac{f(s) - f(\sigma(t))}{f(\sigma(t))f(s)} + \frac{f^\Delta(t)}{f(\sigma(t))f(t)}(\sigma(t) - s) + \frac{f^\Delta(t)(\sigma(t) - s)}{f(s)f(\sigma(t))} - \frac{f^\Delta(t)(\sigma(t) - s)}{f(s)f(\sigma(t))}\right| \\
&\leq \left|[f(s) - f(\sigma(t)) + f^\Delta(t)(\sigma(t) - s)]\left(\frac{1}{f(s)f(\sigma(t))}\right)\right| \\
&\quad + \left|\frac{f^\Delta(t)(\sigma(t) - s)}{f(\sigma(t))}\left(\frac{1}{f(t)} - \frac{1}{f(s)}\right)\right|
\end{aligned}$$

$$\leq \frac{\varepsilon^*}{|f(s)f(\sigma(t))|}\,|\sigma(t)-s| + \frac{\varepsilon^*\left|f^\Delta(t)\right|}{|f(\sigma(t))|}\,|\sigma(t)-s|$$
$$= \varepsilon^*\frac{|f(\sigma(t))| + \left|f^\Delta(t)f(s)\right|}{|f(s)f(\sigma(t))|}\,|\sigma(t)-s| = \varepsilon\,|\sigma(t)-s|\,.$$

By Definition 2.9, $\left(\frac{1}{f}\right)^\Delta(t) = -\frac{f^\Delta(t)}{f(t)f(\sigma(t))}$ is the delta-derivative of $1/f$ at $t \in \mathbb{T}$. □

The *quotient rule* for delta-differentiable functions $f, g : \mathbb{T} \longrightarrow \mathbb{R}$ can be obtained by applying the product rule to $f\frac{1}{g}$. Therefore

$$\left(\frac{f}{g}\right)^\Delta(t) = \left(f\frac{1}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}. \tag{2.11}$$

Example 2.17. Let f be a delta-differentiable function at $t \in \mathbb{T}^\kappa$. The delta-derivative of f^2 at $t \in \mathbb{T}^\kappa$ can be found by applying the product rule to $f \cdot f$

$$(f^2)^\Delta(t) = (f\cdot f)^\Delta(t) = f^\Delta(t)f(t) + f^\Delta(t)f(\sigma(t)) = f^\Delta(t)(f(t) + f(\sigma(t)).$$

Example 2.18. Let $f : \mathbb{T} \longrightarrow \mathbb{R}$ be delta-differentiable at $t \in \mathbb{T}^\kappa$ with f and $f(\sigma) > 0$. The delta-derivative of $\sqrt{f}$ is then given by

$$(\sqrt{f})^\Delta(t) = \frac{f^\Delta(t)}{\sqrt{f(t)} + \sqrt{f(\sigma(t))}}.$$

Proof. Assume f and $\sqrt{f}$ are delta differentiable at $t \in \mathbb{T}^\kappa$. For $\varepsilon > 0$, define

$$\varepsilon^* = \varepsilon\left(\frac{m+1}{\sqrt{f(t)} + \sqrt{f(\sigma(t))}}\right)^{-1},$$

where $m = \max \sqrt{f(t_0)}^{\Delta}$ for $t_0 \in [\sigma(t), s]$ for all s in surrounding of t. Then $\varepsilon^* > 0$. Since f is differentiable in $t \in \mathbb{T}^\kappa$, there exists a neighborhood U_1 of t such that

$$\left|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)\right| \leq \varepsilon^* \left|\sigma(t) - s\right| \text{ for all } s \in U_1.$$

Since $\sqrt{f}$ is differentiable at t and $\sqrt{f}$ is continuous at t, by Theorem 2.10, there exists a neighborhood U_2 and U_3 of t such that

$$\left|\sqrt{f(t)} - \sqrt{f(s)}\right| \leq \varepsilon^* \text{ for all } s \in U_2$$

and

$$\left|\sqrt{f(\sigma(t))} - \sqrt{f(s)}\right| \leq m \left|\sigma(t) - s\right| \text{ for all } s \in U_3.$$

Let $U = U_1 \cap U_2 \cap U_3$ and $s \in U$. Then

$$\begin{aligned}
&\left|\sqrt{f(\sigma(t))} - \sqrt{f(s)} - \frac{f^{\Delta}(t)}{\sqrt{f(\sigma(t))} + \sqrt{f(t)}}(\sigma(t) - s)\right| \\
&= \left|\sqrt{f(\sigma(t))} - \sqrt{f(s)} - \frac{f^{\Delta}(t)}{\sqrt{f(\sigma(t))} + \sqrt{f(t)}}(\sigma(t) - s) \pm \frac{f(s)}{\sqrt{f(\sigma(t))} + \sqrt{f(t)}}\right| \\
&\leq \left|\frac{\sqrt{f(t)} - \sqrt{f(s)})(\sqrt{f(\sigma(t))} - \sqrt{f(s)})}{\sqrt{f(t)} + \sqrt{f(\sigma(t))}}\right| + \left|\frac{f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)}{\sqrt{f(t)} + \sqrt{f(\sigma(t))}}\right| \\
&\leq \frac{\varepsilon^*}{\sqrt{f(t)} + \sqrt{f(\sigma(t))}} \left|\sqrt{f(\sigma(t))} - \sqrt{f(s)}\right| + \frac{\varepsilon^*}{\sqrt{f(t)} + \sqrt{f(\sigma(t))}} \left|\sigma(t) - s\right| \\
&\leq \frac{\varepsilon^*}{\sqrt{f(t)} + \sqrt{f(\sigma(t))}} m \left|\sigma(t) - s\right| + \frac{\varepsilon^*}{\sqrt{f(t)} + \sqrt{f(\sigma(t))}} \left|\sigma(t) - s\right| \\
&= \frac{\varepsilon^*(m+1)}{\sqrt{f(t)} + \sqrt{f(\sigma(t))}} \left|\sigma(t) - s\right| = \varepsilon \left|\sigma(t) - s\right|.
\end{aligned}$$

By Definition 2.9, $\frac{f^{\Delta}(t)}{\sqrt{f(t)} + \sqrt{f(\sigma(t))}}$ is the derivative of $\sqrt{f}$ at $t \in \mathbb{T}$. □

Remark 2.19. In general, $(f^{\Delta})^{\sigma}(t) \neq (f^{\sigma})^{\Delta}(t)$, even if both exist. One can easily realize this by using Theorem 2.13 for $(f^{\sigma})^{\Delta}(t)$

$$(f^{\sigma})^{\Delta}(t) = (f + \mu f^{\Delta})^{\Delta}(t) = f^{\Delta}(t) + \mu^{\Delta}(t) f^{\Delta}(\sigma(t)) + \mu(t) f^{\Delta\Delta}(t).$$

Example 2.20. For $\mathbb{T} = \mathbb{R}$, consider $f : \mathbb{R} \longrightarrow \mathbb{R}$. If f is differentiable in t, then

$$(f^{\Delta})^{\sigma}(t) = f^{\Delta}(t) = (f^{\sigma})^{\Delta}(t).$$

Example 2.21. For $\mathbb{T} = \mathbb{Z}$, consider $f : \mathbb{Z} \longrightarrow \mathbb{R}$. If f^{σ} is differentiable and f is twice differentiable in t, then

$$(f^{\sigma})^{\Delta}(t) = f^{\Delta}(t+1) = \frac{f(\sigma(t+1)) - f(t+1)}{1} = \Delta f(t+1)$$

and

$$(f^{\Delta})^{\sigma}(t) = \frac{f(\sigma(\sigma(t))) - f(\sigma(t))}{1} = \Delta f(t+1).$$

Example 2.22. Consider a time scale $\mathbb{T}$ with $\mu(t) = 2^t$ for $t \in \mathbb{T}$ and $f : \mathbb{T} \longrightarrow \mathbb{R}$. Remember $\mu(t) = \sigma(t) - t$ and therefore $\sigma(t) = t + \mu(t)$. Then $(f^{\sigma})^{\Delta}(t) = (f^{\Delta})^{\sigma}(t)$ if and only if

$$\begin{aligned}(f^{\Delta})^{\sigma}(t) = (f^{\sigma})^{\Delta}(t) &= f^{\Delta}(t) + \mu^{\Delta}(t) f^{\Delta}(\sigma(t)) + \mu(t) f^{\Delta\Delta}(t) \\ &= f^{\Delta}(t) + t2^{t-1}(f^{\Delta})^{\sigma}(t) + 2^t f^{\Delta\Delta}(t),\end{aligned}$$

i.e., if and only if

$$(f^{\Delta})^{\sigma}(t) = \frac{f^{\Delta}(t) + 2^t f^{\Delta\Delta}(t)}{1 - t2^{t-1}}.$$

If $f^\sigma(t)$ is not delta-differentiable or f is not twice differentiable in t, then the equation is definitely not satisfied.

2.3. INTEGRATION ON TIME SCALES

To identify *delta-integrable functions*, it is critical to define some characteristics of delta-integrable functions.

Definition 2.23. A function $f : \mathbb{T} \longrightarrow \mathbb{R}$ is called *pre-differentiable* with region of differentiation D provided that the following conditions hold [2, p. 6]:

1. f is continuous on $\mathbb{T}$,
2. $D \subset \mathbb{T}^\kappa$,
3. $\mathbb{T}^\kappa \backslash D$ is countable and contains no right-scattered elements of $\mathbb{T}$,
4. f is differentiable at each $t \in D$.

Definition 2.24. A function $f : \mathbb{T} \longrightarrow \mathbb{R}$ is called *rd-continuous* at $t \in \mathbb{T}^\kappa$ if f is continuous at t for all right-dense points t and the left-sided limit exists for all left-dense points t [2, p. 7]. The set of rd-continuous functions is denoted by $C_{\mathrm{rd}} = C_{\mathrm{rd}}(\mathbb{T}) = C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$.

In order to define integrable functions, the characterization of *regulated functions* is necessary.

Definition 2.25. A function $f : \mathbb{T} \longrightarrow \mathbb{R}$ is called *regulated* provided its right-sided limits exist at all right-dense points in $\mathbb{T}$ and its left-sided limits exist at all left-dense points in $\mathbb{T}$ [2, p. 7].

Remark 2.26. All continuous functions $f : \mathbb{T} \longrightarrow \mathbb{R}$ are rd-continuous [1, p. 22].

Theorem 2.27. *For any time scale* $\mathbb{T}$*, we have the following.*

- *The jump-operator* σ *is rd-continuous.*
- *Every rd-continuous function* f *is regulated.*

Now the essential terms have been introduced and *pre-antiderivatives* can be defined.

Definition 2.28. Assume $f : \mathbb{T} \longrightarrow \mathbb{R}$ is regulated. Any pre-differentiable function F with region of differentiation D that satisfies $F^{\Delta}(t) = f(t)$ for all $t \in D$ is called a *pre-antiderivative* of f [2, p. 8].

The existence theorem for delta-integrable functions f is formulated as follows.

Theorem 2.29. *Let* $f : \mathbb{T} \longrightarrow \mathbb{R}$ *be regulated. There exists a function* F *which is* pre-differentiable *with region of differentiation* D *such that [2, p. 7]*

$F^{\Delta}(t) = f(t)$ *holds for all* $t \in D$.

The *indefinite integral* of a regulated function f is therefore defined by

$$\int f(t)\Delta t = F(t) + C,$$

where F is a pre-antiderivative of f and C an arbitrary constant in $\mathbb{R}$.

The *Cauchy-integral* of a regulated function f is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a),$$

where $a, b \in \mathbb{T}$ and F is a pre-antiderivative of f.

The properties of time scales integrals are similar to the properties of the integrals in $\mathbb{R}$. Let $f, g \in C_{\mathrm{rd}}$, $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$ [2, p. 8]. Then

1. $\int_a^b (\alpha f(t) + \beta g(t))\Delta t = \alpha \int_a^b f(t)\Delta t + \beta \int_a^b g(t)\Delta t$,
2. $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$,
3. $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$,
4. $\int_a^a f(t)\Delta t = 0$.

Similar to the *integration by parts formula* in $\mathbb{R}$, a formula for the integration of a product of two functions $f, g : \mathbb{T} \longrightarrow \mathbb{R}$ can be derived.

Theorem 2.30. *Let $f, g \in C_{\mathrm{rd}}$ and $a, b, c \in \mathbb{T}$. Then*

1. $\int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(t)\Delta t$,
2. $\int_a^b f(t)g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t$.

Proof. Assume $f, g : \mathbb{T} \longrightarrow \mathbb{R}$, $f, g \in C_{\text{rd}}$ and $a, b \in \mathbb{T}$. Then

$$\int_a^b f(\sigma(t))g^\Delta(t)\Delta t + \int_a^b g(t)f^\Delta(t)\Delta t = \int_a^b (f(\sigma(t))g^\Delta(t) + g(t)f^\Delta(t))\Delta t$$
$$= \int_a^b (fg)^\Delta(t)\Delta t = (fg)(b) - (fg)(a) = f(b)g(b) - f(a)g(a).$$

The second equation results similarly by using the fact that

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = g^\Delta(t)f(t) + g(\sigma(t))f^\Delta(t) = (gf)^\Delta(t).$$

This completes the proof. □

2.4. EXPONENTIAL FUNCTION ON TIME SCALES

To introduce the *exponential function* as a basic function in time scales, the definition of *regressive functions* is critical.

Definition 2.31. A function $p : \mathbb{T} \longrightarrow \mathbb{R}$ is called *regressive* if [2, p. 10]

$$1 + \mu(t)p(t) \neq 0 \quad \text{for all } t \in \mathbb{T}^\kappa. \tag{2.12}$$

The set of *regressive* and *rd-continuous functions* is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$. A function $p : \mathbb{T} \longrightarrow \mathbb{R}$ is called *positively regressive* if

$$1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}^\kappa.$$

Two often used operations that serve to simplify expressions and calculations in time scales are *circle plus* (denoted by $\oplus$) and *circle minus* (denoted by $\ominus$).

Definition 2.32. Assume $p, q \in \mathcal{R}$. The operations $\oplus$ and $\ominus$ are defined for $t \in \mathbb{T}^\kappa$ by [2, p. 10]

$$(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t), \tag{2.13}$$

$$(p \ominus q)(t) := (p \oplus (\ominus q))(t), \tag{2.14}$$

where (2.15)

$$(\ominus q)(t) := -\frac{q(t)}{1 + \mu(t)q(t)}. \tag{2.16}$$

Now we define the exponential function in time scales as the solution of a delta-differential equation problem.

Theorem 2.33. *Suppose $p \in \mathcal{R}$ and fix $t_0 \in \mathbb{T}$. Then the initial value problem*

$$y^\Delta(t) = p(t)y(t), \quad y(t_0) = 1 \tag{2.17}$$

has a unique solution on $\mathbb{T}$, denoted by $e_p(\cdot, t_0)$ [2, p. 10].

Remark 2.34. Another possibility to introduce the exponential function in time scales is by using the exponential function in $\mathbb{R}$. The exponential function in time scales can then be also defined by [1, p. 59]

$$e_p(t, s) = \exp\left\{\int_s^t \xi_{\mu(r)} p(\tau)\Delta\tau\right\} \text{ for } s, t \in \mathbb{T}, \tag{2.18}$$

where $\xi_{\mu(r)}$ is the so-called cylinder transformation. This definition implies that $e_p(\cdot, t_0)$ solves the initial value problem (2.17).

Using the definition of the exponential function as the solution of the initial value problem (2.17), some properties of the exponential function in time scales can be given. Consider $p, q \in \mathcal{R}$. The following properties of $e_p(t, t_0)$ hold for any $t, s, r \in \mathbb{T}$ [2, p. 10f]:

1. $e_0(t,s) = 1 \;$ and $\; e_p(t,t) = 1$,
2. $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t,s)$,
3. $(e_p(t,s))^{-1} = e_{\ominus p}(t,s) = e_p(s,t)$,
4. $e_p(t,s)e_p(s,r) = e_p(t,r)$,
5. $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$,
6. $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$.

The definition of the exponential function $e_p(t,s)$ yields furthermore the following results concerning the delta-derivative of the exponential function.

Theorem 2.35. *Let $p \in \mathcal{R}$ and $s \in \mathbb{T}$. Then*

$$\left(\frac{1}{e_p}\right)^{\Delta}(\cdot, s) = -\frac{p}{e_p^{\sigma}(\cdot, s)}$$

and

$$e_p^{\Delta}(s, \cdot) = -p e_p^{\sigma}(s, \cdot).$$

Proof. Let $p \in \mathcal{R}$ and $s \in \mathbb{T}$. Using the quotient rule and the properties of the exponential function in $\mathbb{T}$, it follows that

$$\left(\frac{1}{e_p}\right)^{\Delta}(\cdot, s) = \frac{-e_p^{\Delta}(\cdot, s)}{e_p(\cdot, s)e_p^{\sigma}(\cdot, s)} = -\frac{p e_p(\cdot, s)}{e_p(\cdot, s)e_p^{\sigma}(\cdot, s)} = -\frac{p}{e_p^{\sigma}(\cdot, s)}$$

and

$$e_p^{\Delta}(s,\cdot) = \left(\frac{1}{e_p}\right)^{\Delta}(\cdot,s) = -\frac{p}{e_p^{\sigma}(\cdot,s)} = -pe_p^{\sigma}(s,\cdot).$$

This completes the proof. □

To summarize the previous results and generalize them, one can see that the general solution of the delta-differential equation

$$y^{\Delta}(t) = p(t)y(t), \tag{2.19}$$

where $p \in \mathcal{R}$, is given by $y(t) = e_p(t,t_0)y(t_0)$, for an initial value $t_0 \in \mathbb{T}$ [2, p. 8]. The general solution of the delta-differential equation,

$$y^{\Delta}(t) = -p(t)y(\sigma(t)), \tag{2.20}$$

where $p \in \mathcal{R}$, is given by $y(t) = e_{\ominus p}(t,t_0)y(t_0)$ [2, p. 8].

3. ABEL DIFFERENTIAL EQUATIONS

3.1. SOLUTION OF THE ABEL DIFFERENTIAL EQUATION OF THE 2ND KIND

Definition 3.1. The general form of the dynamic Abel equation of the second kind is, for $f_i, g_k : \mathbb{R} \longrightarrow \mathbb{R}$, $i = 0, 1, 2$, $k = 0, 1$ [3]

$$[g_0(x) + g_1(x)u(x)]u'(x) = f_0(x) + f_1(x)u(x) + f_2(x)u^2(x). \tag{3.1}$$

The French mathematician Gaston Julia proved 1933 in [6, p. 82f] that the equation

$$\mathrm{d}y + \frac{Ay^2 + By + C}{Dy + E}\mathrm{d}x = 0, \tag{3.2}$$

for A, B, C, D, and E functions of x, has an implicit solution if the condition

$$E(2A - D') = D(B - E') \text{ with } D \neq 0 \tag{3.3}$$

is satisfied. Then the solution is implicitly given by

$$D\frac{y^2}{2}\exp\left\{\int\frac{2A - D'}{D}\mathrm{d}x\right\} + Ey\exp\left\{\int\frac{2A - D'}{D}\mathrm{d}x\right\} + \int C\exp\left\{\int\frac{2A - D'}{D}\mathrm{d}x\right\}\mathrm{d}x = \lambda, \tag{3.4}$$

where λ is any constant.

The result can be rewritten to match the equation (3.1), since Eq. (3.2) is equivalent to

$$\mathrm{d}y = -\frac{Ay^2 + By + C}{Dy + E}\mathrm{d}x,$$

i.e.,

$$\frac{\mathrm{d}y}{\mathrm{d}x}(Dy + E) = -Ay^2 - By - C,$$

i.e.,

$$(-Dy - E)y' = Ay^2 + By + C,$$

which has the form of Eq. (3.1) with

$$g_0(x) = -E, \quad g_1(x) = -D, \quad f_0(x) = C, \quad f_1(x) = B, \quad f_2(x) = A.$$

Using Julia's result, the Abel equation of the second kind has an implicit solution if the condition (3.3), namely

$$-g_0(x)(2f_2(x) + g_1'(x)) = E(2A - D') = D(B - E') = -g_1(x)(f_1(x) + g_0'(x)),$$

i.e.,

$$g_0(x)(2f_2(x) + g_1'(x)) = g_1(x)(f_1(x) + g_0'(x)), \quad g_1(x) \neq 0$$

is satisfied. The implicit solution is then given by (3.4) as

$$\begin{aligned} -D\frac{y^2}{2}\exp\left\{\int\frac{2A-D'}{D}\mathrm{d}x\right\} - Ey\exp\left\{\int\frac{2A-D'}{D}\mathrm{d}x\right\} \\ = \int C\exp\left\{\int\frac{2A-D'}{D}\mathrm{d}x\right\}\mathrm{d}x - \lambda, \end{aligned}$$

i.e.,

$$\begin{aligned} -Dy^2\exp\left\{\int\frac{2A-D'}{D}\mathrm{d}x\right\} - 2Ey\exp\left\{\int\frac{2A-D'}{D}\mathrm{d}x\right\} \\ = 2\int C\exp\left\{\int\frac{2A-D'}{D}dx\right\}dx + \Lambda, \end{aligned}$$

where $\Lambda := -2\lambda$. Using the expressions for A, C, D, and E, this yields

$$g_1y^2 + 2g_0y = 2\frac{\int f_0\exp\left\{\int\frac{2f_2-g_1'}{g_1}\mathrm{d}x\right\}\mathrm{d}x}{\exp\left\{\int\frac{2f_2-g_1'}{g_1}\mathrm{d}x\right\}} + \Lambda\frac{1}{\exp\left\{\int\frac{2f_2-g_1'}{g_1}\mathrm{d}x\right\}}. \tag{3.5}$$

Using furthermore the fact that

$$\begin{aligned} \exp\left\{\int\frac{2f_2+g_1'}{-g_1}\mathrm{d}x\right\} &= \exp\left\{\int-\frac{2f_2}{g_1}\mathrm{d}x\right\}\exp\left\{\int-\frac{g_1'}{g_1}\mathrm{d}x\right\} \\ &= \exp\left\{\int-\frac{2f_2}{g_1}\mathrm{d}x\right\}\frac{1}{g_1} \end{aligned}$$

and defining $J := \exp\left\{\int\frac{2f_2}{g_1}dx\right\}$, Eq. (3.5) turns into

$$\frac{y^2}{J} + \frac{2g_0y}{Jg_1} = 2\int\frac{f_0}{Jg_1}\mathrm{d}x + \Lambda.$$

This implicit solution is consistent with the solution given in [3].

Lazhar Bougoffa presented in [3] an additional method to solve a further class of Abel equations of the second kind. A different relation between the coefficients of the Abel equation of the second kind has to be satisfied in order to follow Bougoffa's idea, which is stated in the following theorem.

Theorem 3.2. *If there exists a constant* λ *such that*

$$2B_2(x)g_0(x) = \lambda B_1(x)g_1(x) \quad \textit{with} \quad g_0, g_1 \neq 0, \tag{3.6}$$

where

$$B_1(x) := \exp\left\{-\int_{x_0}^{x} \frac{f_1(t)}{g_0(t)} \mathrm{d}t\right\}, \quad B_2(x) := \exp\left\{-2\int_{x_0}^{x} \frac{f_2(t)}{g_1(t)} \mathrm{d}t\right\},$$

then Eq. (3.1) *admits the general solution* $u = u(x)$ *implicitly as*

$$B_2(x)u^2(x) + \lambda B_1(x)u(x) = 2\int_{x_0}^{x} \frac{f_0(t)}{g_1(t)} B_2(t)\mathrm{d}t + C, \tag{3.7}$$

where C *is an integration constant.*

Proof. Multiply B_1 on both sides of Eq. (3.1) to get

$$B_1 g_0 u' + B_1 g_1 u u' = B_1 f_0 + B_1 f_1 u + B_1 f_2 u^2.$$

Since $-B_1' g_0 = B_1 f_1$, we obtain

$$B_1 g_0 u' + B_1 g_1 u u' = B_1 f_0 - B_1' g_0 u + B_1 f_2 u^2,$$

i.e.,

$$B_1 g_0 u' + B_1' g_0 u + B_1 g_1 u u' = B_1 f_0 + B_1 f_2 u^2.$$

By using the product rule, we have

$$g_0(B_1u)' + B_1g_1uu' = B_1f_0 + B_1f_2u^2. \tag{3.8}$$

Similarly, we multiply B_2 on both sides of (3.8) to obtain

$$B_2g_0(B_1u)' + B_2B_1g_1uu' = B_2B_1f_0 + B_2B_1f_2u^2.$$

Since $-B_2'g_1 = 2B_2f_2$, we get

$$B_2g_0(B_1u)' + B_2B_1g_1uu' = B_2B_1f_0 - \frac{1}{2}B_2'B_1g_1u^2,$$

i.e.,

$$B_2g_0(B_1u)' + B_2B_1g_1uu' + \frac{1}{2}B_2'B_1g_1u^2 = B_2B_1f_0.$$

By using the product rule, we have

$$B_2g_0(B_1u)' + \frac{1}{2}B_1g_1(B_2u^2)' = B_2B_1f_0.$$

Dividing this by $\frac{B_1g_1}{2}$, we get

$$\frac{2B_2g_0}{B_1g_1}(B_1u)' + (B_2u^2)' = 2B_2\frac{f_0}{g_1}.$$

Since condition (3.6) is satisfied, we find

$$\lambda(B_1u)' + (B_2u^2)' = 2\frac{B_2}{g_1}f_0.$$

Integrating now both sides with respect to x gives the general solution $u = u(x)$ as

$$\lambda B_1(x)u + B_2(x)u^2 = 2\int_{x_0}^{x} \frac{f_0(t)}{g_1(t)} B_2(t)\mathrm{d}t + C,$$

where C is an integration constant, determined by an initial value x_0. □

Theorem 3.2 requires in particular $g_0(x) \neq 0$ and $g_1(x) \neq 0$. In case $g_1 = 0$, $g_0 \neq 0$, Eq. (3.1) becomes

$$g_0(x)u' = f_0(x) + f_1(x)u + f_2(x)u^2 \quad \text{with} \quad u = u(x),$$

i.e.,

$$u' = \frac{f_0(x)}{g_0(x)} + \frac{f_1(x)}{g_0(x)}u + \frac{f_2(x)}{g_0(x)}u^2 \quad \text{with} \quad u = u(x),$$

which is of the form [11, p. 1]

$$u' = h_1(x)u + h_2(x)u^2 + h_3(x) \quad \text{with} \quad u = u(x). \tag{3.9}$$

This is a (scalar) Ricatti differential equation, which enables the solution methods of the Ricatti differential equation to be used for the Abel differential equation. It is well-known [16, p. 73] that the Ricatti differential equation

$$y' = P(x) + Q(x)y + R(x)y^2$$

can be solved if a particular solution y_0 is known. The substitution $y = y_0 + u$ yields a Bernoulli differential equation in u and can afterwards be transformed into a linear differential equation in w by $u = \frac{1}{w}$ [16, p. 73].

In case $g_0 = 0$, $g_1 \neq 0$, Eq. (3.1) is of the form

$$g_1(x)uu' = f_0(x) + f_1(x)u + f_2u^2 \quad \text{with} \quad u = u(x).$$

By using the substitution $u(x) = y(x) + 1$, we obtain

$$g_1(x)(y+1)y' = f_0(x) + f_1(x)(y+1) + f_2(y^2 + 2y + 1),$$

i.e.,

$$(g_1(x)y + g_1(x))y' = (f_0(x) + f_1(x) + f_2(x)) + (f_1(x) + 2f_2(x))y + f_2y^2$$

which is of the form (3.1) with $g_0 = g_1 \neq 0$ and therefore satisfies the condition $g_0, g_1 \neq 0$. With this substitution, a form is obtained that is solvable with Bougoffa's method (assuming the condition (3.6) is additionally satisfied).

3.2. ABEL DIFFERENTIAL EQUATIONS OF THE 1ST KIND

3.2.1. Transform the Abel equation of the 2nd to the 1st kind. The Abel dynamic equation of the first kind appears especially in applications,such as physics. The Friedman equations, which describe a homogeneous, isotropic universe are given by [15]

$$\Theta'' + 3H\Theta' + \frac{dV}{d\Theta} = 0, \tag{3.10}$$

$$H^2 = \frac{1}{2}\Theta'^2 + V - \frac{k}{a^2}, \tag{3.11}$$

with a scalar, Θ the scalar field, V the self potential of the scalar field, and H the Hubble constant.

By introducing a functional of full energy, denoted by W, a relation between the self potential and the scalar field can be obtained. The assumption of a flat space time set ($k = 0$) allows the previous equations to become

$$\frac{dW}{d\Theta} = -3H\Theta', \tag{3.12}$$

$$H = \pm\sqrt{W}. \tag{3.13}$$

This enables the solving of the equation for a given self potential V of a scalar field, by using the differential equation for W. The function W is then given by an explicit formula, depending on V and y, where y is the solution of a particular Abel differential equation of the first kind [15].

Definition 3.3. The general Abel equation of the first kind, with $h_i : \mathbb{R} \longrightarrow \mathbb{R}$, $i = 0, 1, 2, 3$ is of the form [9]

$$y' = h_3(x)y^3 + h_2(x)y^2 + h_1(x)y + h_0(x) \quad \text{with} \quad y = y(x). \tag{3.14}$$

Starting with the Abel equation of the second kind, the Abel equation of the first kind can be derived by using a special substitution. Assume we are given a general Abel equation of the second kind (3.1) with $g_1(x)$, $g_0(x) \neq 0$, namely

$$[g_0(x) + g_1(x)u(x)]u'(x) = f_0(x) + f_1(x)u(x) + f_2(x)u^2(x),$$

i.e.,

$$[g(x) + u]u' = F_0(x) + F_1(x)u + F_2(x)u^2 \quad \text{with} \quad u = u(x),$$

where

$$g(x) = \frac{g_0(x)}{g_1(x)}, \quad F_i(x) = \frac{f_i(x)}{g_1(x)} \quad \text{for } i = 0, 1, 2.$$

Using two substitutions, the previous equation yields the general Abel equation of the first kind. First of all, one applies the substitution $u = \frac{w}{E} - g$ with $E = \exp\{-\int F_2 \mathrm{d}x\}$ to get

$$\left(g + \frac{w}{E} - g\right)\left(\frac{w'E - E'w}{E^2} - g'\right) = F_0 + F_1\left(\frac{w}{E} - g\right) + F_2\left(\frac{w^2}{E^2} - \frac{2gw}{E} + g^2\right),$$

i.e.,

$$\frac{w'w}{E^2} - \frac{E'w^2}{E^3} - \frac{g'w}{E} = F_0 + F_1\frac{w}{E} - F_1 g + F_2\frac{w^2}{E^2} - F_2\frac{2gw}{E} + F_2 g^2.$$

Using now that $E' = -F_2 E$, we obtain

$$\frac{w'w}{E^2} - \frac{-F_2 w^2}{E^2} - \frac{g'w}{E} = F_0 + F_1\frac{w}{E} - F_1 g + F_2\frac{w^2}{E^2} - F_2\frac{2gw}{E} + F_2 g^2,$$

i.e.,

$$\frac{w'w}{E^2} - \frac{g'w}{E} = F_0 + F_1\frac{w}{E} - F_1 g - F_2\frac{2gw}{E} + F_2 g^2.$$

Multiplying both sides with E^2 yields

$$ww' = F_0 E^2 - F_1 g E^2 + F_2 g^2 E^2 + g' E w + F_1 E w - 2F_2 g E w. \tag{3.15}$$

Eq. (3.15) is of the general form

$$ww' = G_0(x) + G_1(x) w, \tag{3.16}$$

where

$$G_0 = E^2(F_0 - F_1 g + F_2 g^2) \text{ and } G_1 = E(g' + F_1 - 2F_2 g).$$

Later on, Eq. (3.16) appears again, as the canonical Abel form. Apply the second substitution $w = \frac{1}{y+1}$, Eq. (3.15) results in

$$\left(\frac{1}{y+1}\right)\left(-\frac{y'}{(y+1)^2}\right) = E^2(F_0 - F_1 g + F_2 g^2) + E(g' + F_1 - 2F_2 g)\left(\frac{1}{y+1}\right).$$

Multiplying both sides with $-(y+1)^3$, this becomes

$$\begin{aligned} y' = -E^2[F_0 - F_1 g + F_2 g^2](y^3 + 3y^2 + 3y + 1) \\ - E[g' + F_1 - 2F_2 g](y^2 + 2y + 1). \end{aligned}$$

This is in the form of the Abel equation of the first kind, namely

$$y' = h_0(x) + h_1(x)y + h_2(x)y^2 + h_3(x)y^3,$$

where

$$\begin{aligned} h_0 &= -E[g' + F_1 - 2F_2 g] - E^2[F_0 - F_1 g + F_2 g^2], \\ h_1 &= -2E[g' + F_1 - 2F_2 g] - 3E^2[F_0 - F_1 g + F_2 g^2], \\ h_2 &= -E[g' + F_1 - 2F_2 g] - 3E^2[F_0 - F_1 g + F_2 g^2], \\ h_3 &= -E^2[F_0 - F_1 g + F_2 g^2]. \end{aligned}$$

In the following, a special class of the Abel differential equation of the first kind is presented whose time scales analogue is discussed in Section 4.2. Consider the general Abel equation of the second kind with $g_1(x) \neq 0$. Eq. (3.1) can then be written in the form

$$(u + g(x))u' = F_0(x) + F_1(x)u + F_2(x)u^2 \quad \text{with} \quad u = u(x).$$

By applying the substitution $u = \frac{1}{y} - g$, $y = y(x)$, a special class of the Abel equation of the first kind can be derived [7, p. 27], namely

$$\begin{aligned}\left(\frac{1}{y} - g + g\right)\left(-\frac{y'}{y^2} - g'\right) = -\frac{y'}{y^3} - \frac{g'}{y} & \\ &= F_0 + F_1\frac{1}{y} - F_1 g + F_2\frac{1}{y^2} - 2F_2\frac{g}{y} + F_2 g^2.\end{aligned}$$

Multiplying both sides with $-y^3$ yields

$$\begin{aligned}y' &= -g'y^2 - F_0y^3 - F_1y^2 + F_1gy^3 - F_2y + 2F_2gy^2 - F_2g^2y^3 \\ &= y^3(-F_0 + F_1g - F_2g^2) + y^2(-g' - F_1 + 2F_2g) - F_2y.\end{aligned}$$

This is of the special Abel equation of the first kind, namely

$$y' = h_1(x)y + h_2(x)y^2 + h_3(x)y^3 \quad \text{with} \quad y = y(x), \tag{3.17}$$

where

$$h_1 = -F_2, \tag{3.18}$$

$$h_2 = -g' - F_1 + 2F_2g, \tag{3.19}$$

$$h_3 = -F_0 + F_1g - F_2g^2. \tag{3.20}$$

This is the class of the Abel equation of the first kind where the variable coefficient h_0 of Eq. (3.14) satisfies $h_0(x) = 0$. This special class is later transferred into time scales, which helps to construct a special class of an Abel equation of the first kind.

3.2.2. Transform the Abel equation of the 1st to the 2nd kind. In particular, the transformation from an Abel equation of the first kind to the second kind is the subject under investigation, since various classes of Abel equations of the second kind can be already solved [10, p. 50–55]. By transferring the Abel equation of the first kind into the second kind, one can translate the conditions to solve the Abel equation of the second kind such as from Theorem 3.2 into required conditions for the Abel equation of the first kind. M. P. Markakis presented in [9] a transformation to reduce an Abel equation of the first kind to an Abel equation of the second kind. This is presented in more detail in the following.

Consider the general Abel equation of the first kind, Eq. (3.14). Note that in order to apply the substitution $y = y_0 + \frac{1}{u}$, where y_0 is a particular solution of Eq. (3.14), $f_0(x)$ has to be nonzero, otherwise $u = u(x)$ could be zero.

Apply to the general Abel equation of the first kind (3.14) the substitution $y = y_0 + \frac{1}{u}$ to obtain

$$y_0' - \frac{u'}{u^2} = h_3\left(y_0^3 + 3\frac{y_0^2}{u} + 3\frac{y_0}{u^2} + \frac{1}{u^3}\right) + h_2\left(y_0^2 + 2\frac{y_0}{u} + \frac{1}{u^2}\right) + h_1\left(y_0 + \frac{1}{u}\right) + h_0.$$

Since y_0 is a particular solution of Eq. (3.14), i.e.,

$$y_0' = h_3(x)y_0^3 + h_2(x)y_0^2 + h_1(x)y_0 + h_0(x),$$

where $y_0 = y_0(x)$, terms can be canceled. This results in

$$-\frac{u'}{u^2} = h_3\left(3\frac{y_0^2}{u} + 3\frac{y_0}{u^2} + \frac{1}{u^3}\right) + h_2\left(2\frac{y_0}{u} + \frac{1}{u^2}\right) + h_1\left(\frac{1}{u}\right).$$

To get $u'u$ on the left-hand side, both sides have to be multiplied by $-u^3$ to obtain

$$u'u = h_3(-3y_0^2u^2 - 3y_0u - 1) + h_2(-2y_0u^2 - u) - h_1u^2,$$

which is of the form

$$uu' = f_0(x) + f_1(x)u + f_2(x)u^2, \tag{3.21}$$

where

$$f_0 = -h_3\ , \quad f_1 = -3h_3y_0 - h_2\ , \quad f_2 = -3h_3y_0^2 - 2h_2y_0 - h_1.$$

This is the special form of the Abel equation of the second kind (3.1) with $g_0(x) = 0$, $g_1(x) = 1$, $h_i(x) = f_i(x)$ for $i = 0, 1, 2$.

The transformation of an Abel equation of the first kind to the second kind enables the classification of solvable Abel equations of the first kind. Various classes of Abel equations of the second kind are solvable under special conditions and can now be translated into conditions to solve Abel equations of the first kind.

3.3. CANONICAL ABEL DIFFERENTIAL EQUATIONS

Definition 3.4. The canonical form of the Abel equation, with $G_0, G_1 : \mathbb{R} \longrightarrow \mathbb{R}$, is defined by [9]

$$ww' - G_1(x)w = G_0(x) \quad \text{with} \quad w = w(x). \tag{3.22}$$

The interest in this special kind of the Abel equation is caused by the variety of solvable classes of this kind. In [10, p. 45–50], 37 types of solvable classes of the canonical Abel equation are presented, but 12 of them have to satisfy the special condition $G_0 = 1$. That is also one of the reasons of attempting to transfer an Abel equation of the first or of the second kind into the canonical form of an Abel equation. Some of the 44 solvable classes of the Abel differential equation of the first kind use the transformation into the canonical Abel differential equation and its solution methods [10, p. 55].

3.3.1. Transform the 2nd kind to the canonical Abel equation. A substitution that transfers an Abel equation of the second kind into the canonical form (3.22) is given by $u = \frac{w}{E} - g$ with $E = \exp\{-\int h_2(x)\mathrm{d}x\}$ [7, p. 27]. Consider an Abel equation of the form (3.1) and assume in particular $g_1 \neq 0$. Then

$$\left(\frac{g_0}{g_1} + u\right) u' = \frac{f_0}{g_1} + \frac{f_1}{g_1}u + \frac{f_2}{g_1}u^2. \tag{3.23}$$

Define

$$g(x) := \frac{g_0(x)}{g_1(x)}, \quad F_0(x) := \frac{f_0(x)}{g_1(x)}, \quad F_1(x) := \frac{f_1(x)}{g_1(x)}, \quad F_2(x) := \frac{f_2(x)}{g_1(x)}.$$

Eq. (3.23) is then of the form

$$(g(x) + u)u' = F_0(x) + F_1(x)u + F_2(x)u^2 \quad \text{with} \quad u = u(x).$$

By applying the substitution $u = \frac{w}{E} - g$ and $E = \exp\{-\int F_2(x)\mathrm{d}x\}$, the previous equation becomes

$$\left(g + \frac{w}{E} - g\right)\left(\frac{w'E - E'w}{E^2}\right) = F_0 + F_1\frac{w}{E} - F_1 g + F_2\frac{w^2}{E^2} - 2F_2\frac{wg}{E} + F_2 g^2.$$

Using furthermore the fact that $E' = -F_2E$, the equation results in

$$\frac{ww'}{E^2} + \frac{F_2w^2}{E^2} = F_0 + F_1\frac{w}{E} - F_1g + F_2\frac{w^2}{E^2} - 2F_2\frac{wg}{E} + F_2g^2,$$

i.e.,

$$\frac{ww'}{E^2} = F_0 + F_1\frac{w}{E} - F_1g - 2F_2\frac{wg}{E} + F_2g^2.$$

By multiplying E^2 on both sides, one can immediately recognize the canonical form (3.22), namely

$$\begin{aligned} ww' &= F_0E^2 + F_1wE - F_1gE^2 - 2F_2wgE + F_2g^2E^2 \\ &= F_0E^2 - F_1gE^2 + F_2g^2E^2 + w(F_1E - 2F_2gE), \end{aligned}$$

i.e.,

$$ww' - G_1w = G_0,$$

where

$$G_1 = E(F_1 - 2F_2g) \quad \text{and} \quad G_0 = E^2(F_0 - F_1g + F_2g^2).$$

3.3.2. Transform the 1st kind to the canonical Abel equation. The proof of the transformation of an Abel equation of the first kind into the second kind can be expanded to get the canonical form (3.22). We saw that the Abel equation of the first kind can be transformed into the form (3.21)

$$uu' = f_0(x) + f_1(x)u + f_2(x)u^2 \quad \text{with} \quad u = u(x),$$

where

$$f_0 = -h_3, \quad f_1 = -3h_3y_0 - h_2, \quad f_2 = -3h_3y_0^2 - 2h_2y_0 - h_1.$$

Assume $h_3 \neq 0$ (otherwise w could be zero) and suppose h_i are integrable for $i = 1, 2, 3$. By applying the further substitution $u = \frac{w}{E}$ with

$$E = \exp\left\{\int (3h_3(x)y_0^2 + 2h_2(x)y_0 + h_1(x))\mathrm{d}x\right\},$$

Eq. (3.21) becomes

$$\frac{w}{E}\left(\frac{w'E - E'w}{E^2}\right) = \frac{w^2}{E^2}(-3h_3y_0^2 - 2h_2y_0 - h_1) + \frac{w}{E}(-3h_3y_0 - h_2) - h_3.$$

Note that $E' = E(3h_3y_0^2 + 2h_2y_0 + h_1)$ and therefore

$$\frac{ww'}{E^2} - \frac{(3h_3y_0^2 + 2h_2y_0 + h_1)w^2}{E^2} = \frac{w^2}{E^2}(-3h_3y_0^2 - 2h_2y_0 - h_1) + \frac{w}{E}(-3h_3y_0 - h_2) - h_3,$$

i.e.,

$$\frac{ww'}{E^2} - \frac{(3h_3y_0^2 + 2h_2y_0 + h_1)w^2}{E^2} = \frac{w^2}{E^2}(-3h_3y_0^2 - 2h_2y_0 - h_1) + \frac{w}{E}(-3h_3y_0 - h_2) - h_3,$$

i.e.,

$$\frac{ww'}{E^2} = \frac{w}{E}(-3h_3y_0 - h_2) - h_3.$$

Multiplying both sides with E^2 results in

$$ww' = wE(-3h_3y_0 - h_2) - E^2h_3,$$

which is in canonical Abel form $ww' = wE[-3h_3y_0 - h_2] - E^2h_3$. In this case, $G_1 = -E(3h_3y_0 + h_2)$ and $G_0 = -E^2h_3$.

In the following chapter, the case of the Abel differential equations will be discussed in $\mathbb{T}$ and in particular the special case of the Abel equation of the first kind, which provides novel results in time scales. If the Abel equation of the first kind is given with $h_0 = 0$, then a particular solution of Eq. (3.14) is given by $y_0 = 0$, which reduces the previous substitution to

$$y = y_0 + \frac{E}{w} = \frac{E}{w}$$

with

$$E = \exp\left\{\int (3h_3y_0^2 + 2h_2y_0 + h_1)\mathrm{d}x\right\} = \exp\left\{\int h_1\mathrm{d}x\right\}.$$

4. ABEL DYNAMIC EQUATIONS

4.1. SOLUTION OF THE ABEL DYNAMIC EQUATION OF THE 2ND KIND

The Abel dynamic equation of the second kind has different expressions in $\mathbb{T}$, which are equivalent to the unique Abel equation of the second kind for $\mathbb{T} = \mathbb{R}$. In the following, the expressions of the Abel dynamic equations of the second kind are introduced and a method to solve a class of these equations is presented. For the general expression, the solution is derived explicitly, which is the foundation to extend the method to the other Abel dynamic versions of the second kind.

Definition 4.1. The general form of the Abel dynamic equation of the second kind, with $f_i, g_k : \mathbb{R} \longrightarrow \mathbb{R}$, $i = 0, 1, 2$, $k = 0, 1$ is

$$\left[g_0(x) + g_1(x)\left(\frac{u + u^\sigma}{2}\right)\right] u^\Delta = f_0(x) + f_1(x)u + f_2(x)u^2, \tag{4.1}$$

where $u = u(x)$.

Based on [3], one can transfer the idea of solving the Abel equation of the second kind under particular conditions from the continuous time set into time scales.

Theorem 4.2. *Consider the Abel equation of the second kind* (4.1) *with* $g_0(x) \neq 0$, $g_1(x) \neq 0$, $g_0, g_1, f_1, f_2 \in \mathcal{R}$, *and* $\frac{f_0}{g_1} \in C_{\mathrm{rd}}$. *If furthermore the condition*

$$2B_2^\sigma(x)g_0(x) = \lambda B_1^\sigma(x)g_1(x) \tag{4.2}$$

with

$$B_1^\Delta = -\frac{f_1}{g_0} B_1^\sigma, \quad B_2^\Delta = -2\frac{f_2}{g_1} B_2^\sigma$$

is satisfied, then the solution of the dynamic equation problem (4.1) *is implicitly given by*

$$B_2(x)u^2(x) + \lambda B_1(x)u(x) = 2\int_{x_0}^{x} B_2^\sigma(t)\frac{f_0(t)}{g_1(t)}\Delta t + C, \tag{4.3}$$

where C is an integration constant, determined by the initial value x_0.

Remark 4.3. The condition (4.2) is, by using Eq. (2.20), equivalent to

$$\begin{aligned} B_1(x) &= e_{\ominus p}(x, x_0)B_1(x_0) \quad \text{with} \quad p(x) = \frac{f_1(x)}{g_0(x)}, \\ B_2(x) &= e_{\ominus q}(x, x_0)B_2(x_0) \quad \text{with} \quad q(x) = \frac{2f_2(x)}{g_1(x)}, \end{aligned}$$

for an initial value x_0.

Proof of Theorem 4.2. Consider the Abel equation of the second kind in $\mathbb{T}$. Let furthermore the conditions from the theorem be satisfied. Multiply B_1^σ on both sides of equation (4.1) to get

$$B_1^\sigma g_0 u^\Delta + B_1^\sigma g_1 \left(\frac{u+u^\sigma}{2}\right) u^\Delta = B_1^\sigma f_0 + B_1^\sigma f_1 u + B_1^\sigma f_2 u^2.$$

By using the fact that $-g_0 B_1^\Delta = f_1 B_1^\sigma$, we have

$$B_1^\sigma g_0 u^\Delta + B_1^\sigma g_1 \left(\frac{u+u^\sigma}{2}\right) u^\Delta = B_1^\sigma f_0 - B_1^\Delta g_0 u + B_1^\sigma f_2 u^2,$$

i.e.,

$$B_1^\sigma g_0 u^\Delta + B_1^\Delta g_0 u + B_1^\sigma g_1 \left(\frac{u+u^\sigma}{2}\right) u^\Delta = B_1^\sigma f_0 + B_1^\sigma f_2 u^2.$$

Applying the product rule $(B_1 u)^\Delta = B_1^\Delta u + B_1^\sigma u^\Delta$, we get

$$g_0(B_1 u)^\Delta + B_1^\sigma g_1 \left(\frac{u+u^\sigma}{2}\right) u^\Delta = B_1^\sigma f_0 + B_1^\sigma f_2 u^2. \tag{4.4}$$

Similarly, we multiply B_2^σ on both sides of (4.4) to obtain

$$B_2^\sigma g_0(B_1 u)^\Delta + B_2^\sigma B_1^\sigma g_1 \left(\frac{u+u^\sigma}{2}\right) u^\Delta = B_2^\sigma B_1^\sigma f_0 + B_2^\sigma B_1^\sigma f_2 u^2.$$

Using the fact that $-g_1 B_2^\Delta = 2 f_2 B_2^\sigma$, the equation results in

$$B_2^\sigma g_0(B_1 u)^\Delta + B_2^\sigma B_1^\sigma g_1 \left(\frac{u+u^\sigma}{2}\right) u^\Delta = B_2^\sigma B_1^\sigma f_0 - \frac{1}{2} B_2^\Delta B_1^\sigma g_1 u^2,$$

i.e.,

$$B_2^\sigma g_0(B_1 u)^\Delta + B_2^\sigma B_1^\sigma g_1 \left(\frac{u+u^\sigma}{2}\right) u^\Delta + \frac{1}{2} B_2^\Delta B_1^\sigma g_1 u^2 = B_2^\sigma B_1^\sigma f_0.$$

Applying the product rule $(B_2 u^2)^\Delta = B_2^\Delta u^2 + B_2^\sigma u^\Delta (u + u^\sigma)$ yields

$$B_2^\sigma g_0(B_1 u)^\Delta + \frac{1}{2} B_1^\sigma g_1 (B_2 u^2)^\Delta = B_2^\sigma B_1^\sigma f_0.$$

Analogous to the case in $\mathbb{R}$, one continues by dividing both sides of the equation by $\frac{1}{2} B_1^\sigma g_1$ to get

$$\frac{2 B_2^\sigma g_0}{B_1^\sigma g_1} (B_1 u)^\Delta + (B_2 u^2)^\Delta = 2 B_2^\sigma \frac{f_0}{g_1}. \tag{4.5}$$

Using the λ condition (4.2) in (4.5) yields

$$\lambda(B_1u)^\Delta + (B_2u^2)^\Delta = 2B_2^\sigma \frac{f_0}{g_1},$$

and integrating this equation provides the general solution $u = u(x)$ of (4.1) implicitly as

$$\lambda B_1(x)u(x) + B_2(x)u^2(x) = \int_{x_0}^{x} 2B_2^\sigma(t)\frac{f_0(t)}{g_1(t)}\Delta t + C,$$

where C is an integration constant, determined by the initial value x_0. □

Remark 4.4. For $\mathbb{T} = \mathbb{R}$, Theorem 4.2 is the same as Theorem 3.2. Eq. (4.1) becomes

$$(g_0(x) + g_1(x)u)u' = f_0(x) + f_1(x)u + f_2(x)u^2 \quad \text{with} \quad u = u(x).$$

For $\mathbb{T} = \mathbb{R}$, $u = u^\sigma$ and $u^\Delta = u'$. Therefore condition (4.2) is

$$\begin{aligned}
&2B_2g_0 = \lambda B_1g_1,\\
&B_1(x) = e_{\ominus p}(x, x_0)B_1(x_0) = -B_1(x_0)\exp\left\{\int_{x_0}^{x} p(t)\mathrm{d}t\right\} \quad \text{with} \quad p = \frac{f_1}{g_0},\\
&B_2(x) = e_{\ominus q}(x, x_0)B_2(x_0) = -B_2(x_0)\exp\left\{\int_{x_0}^{x} q(t)\mathrm{d}t\right\} \quad \text{with} \quad q = \frac{2f_2}{g_1},
\end{aligned}$$

for an initial value x_0.

Example 4.5. For $\mathbb{T} = \mathbb{Z}$, Theorem 4.2 is formulated in the following way. The equation

$$\left(g_0 + g_1\left(\frac{u + Eu}{2}\right)\right)\Delta u = f_0 + f_1u + f_2u^2, \tag{4.6}$$

where E is the shift operator, $Eu(x) = u(x+1)$, has an implicit solution if $g_0(x) \neq 0$, $g_1(x) \neq 0$ and

$$2(EB_2)g_0 = \lambda(EB_1)g_1,$$

where B_1 and B_2 satisfy

$$(EB_1)f_1 = -g_0\Delta B_1, \quad (EB_2)f_2 = -\frac{g_1}{2}\Delta B_2.$$

To realize this, one should just apply Theorem 4.2 for $\mathbb{T} = \mathbb{Z}$ or otherwise follow the same steps as in proof of Theorem 4.2. First of all, we define B_1 as solution of the difference equation $(EB_1)f_1 = -g_0\Delta B_1$ and multiply EB_1 on both sides of Eq. (4.6). The product rule for $\Delta(B_1 u)$ can be used to connect the terms $g_0 u\Delta B_1 + g_0(EB_1)\Delta u$. Define furthermore B_2 as the solution of $2(EB_2)f_2 = -g_1\Delta B_2$ and multiply EB_2 on both sides of the equation. Further terms can be connected by applying the product rule for $\Delta(B_2 u^2)$.

To find the solution of the Abel equation

$$\left(g_0(x) + g_1(x)\left(\frac{u + u^\sigma}{2}\right)\right)u^\Delta = f_0(x) + f_1(x)u^\sigma + f_2(x)(u^\sigma)^2, \tag{4.7}$$

where $u = u(x)$, one uses the same procedure as in Theorem 4.2. This is formulated in the following theorem.

Theorem 4.6. *Consider the Abel equation of the second kind* (4.7) *with* $g_0(x) \neq 0$, $g_1(x) \neq 0$, $g_k, f_i \in \mathcal{R}$ *for* $k = 0, 1$, $i = 0, 1, 2$, *and* $\frac{f_0}{g_1} \in C_{\text{rd}}$. *If furthermore the condition*

$$2B_2(x)g_0(x) = \lambda B_1(x)g_1(x) \tag{4.8}$$

is satisfied, where

$$B_1(x) := e_p(x, x_0)B_1(x_0) \quad \textit{with} \quad p := \frac{-f_1}{g_0}$$

and

$$B_2(x) := e_q(x, x_0)B_2(x_0) \quad \textit{with} \quad q := \frac{-2f_2}{g_1},$$

for an initial value x_0, then the general solution $u = u(x)$ of Eq. (4.7) is given implicitly by

$$B_2(x)u^2(x) + \lambda B_1(x)u(x) = 2\int_{x_0}^{x} B_2(t)\frac{f_0(t)}{g_1(t)}\Delta t + C, \tag{4.9}$$

where C is an integration constant, determined by the initial value x_0.

Proof. Consider the Abel equation of the second kind (4.7) that satisfies the conditions in Theorem 4.6. Define $B_1(x) := e_p(x, x_0)B_1(x_0)$ with $p := \frac{-f_1}{g_0}$ and multiply B_1 on both sides of equation (4.7) to obtain

$$B_1 g_0 u^{\Delta} + B_1 g_1 \left(\frac{u + u^{\sigma}}{2}\right) u^{\Delta} = B_1 f_0 + B_1 f_1 u^{\sigma} + B_1 f_2 (u^{\sigma})^2.$$

Using the fact that $-g_0 B_1^{\Delta} = f_1 B_1$, the previous equation results in

$$B_1 g_0 u^{\Delta} + B_1 g_1 \left(\frac{u + u^{\sigma}}{2}\right) u^{\Delta} = B_1 f_0 - B_1^{\Delta} g_0 u^{\sigma} + B_1 f_2 (u^{\sigma})^2,$$

i.e.,

$$B_1 g_0 u^{\Delta} + B_1^{\Delta} g_0 u^{\sigma} + B_1 g_1 \left(\frac{u + u^{\sigma}}{2}\right) u^{\Delta} = B_1 f_0 + B_1 f_2 (u^{\sigma})^2.$$

The product rule gives the delta-derivative of $(B_1 u)$ as $(B_1 u)^\Delta = u^\Delta B_1 + u^\sigma B_1^\Delta$. Therefore the equation becomes

$$g_0(B_1u)^\Delta + B_1g_1\left(\frac{u+u^\sigma}{2}\right)u^\Delta = B_1f_0 + B_1f_2(u^\sigma)^2.$$

Define $B_2(x) := e_q(x,x_0)B_2(x_0)$ with $q := -2\frac{f_2}{g_1}$ and multiply B_2 on both sides. This yields

$$B_2g_0(B_1u)^\Delta + B_2B_1g_1\left(\frac{u+u^\sigma}{2}\right)u^\Delta = B_2B_1f_0 + B_2B_1f_2(u^\sigma)^2.$$

Using the fact that $-g_1B_2^\Delta = 2f_2B_2$, the equation results in

$$B_2g_0(B_1u)^\Delta + B_2B_1g_1\left(\frac{u+u^\sigma}{2}\right)u^\Delta = B_2B_1f_0 - \frac{1}{2}B_2^\Delta B_1g_1(u^\sigma)^2,$$

i.e.,

$$B_2g_0(B_1u)^\Delta + B_2B_1\frac{g_1}{2}(u+u^\sigma)u^\Delta + \frac{1}{2}B_2^\Delta B_1g_1(u^\sigma)^2 = B_2B_1f_0.$$

The product rule, $(B_2u^2)^\Delta = B_2u^\Delta(u+u^\sigma) + B_2^\Delta(u^\sigma)^2$ yields

$$B_2g_0(B_1u)^\Delta + \frac{1}{2}B_1g_1(B_2u^2)^\Delta = B_2B_1f_0.$$

Similar to the proof of Theorem 4.2, one divides both sides by $\frac{1}{2}B_1g_1$ to obtain

$$\frac{2B_2g_0}{B_1g_1}(B_1u)^\Delta + (B_2u^2)^\Delta = 2B_2\frac{f_0}{g_1}.$$

Using the λ condition (4.9) yields

$$\lambda(B_1u)^\Delta + (B_2u^2)^\Delta = 2B_2\frac{f_0}{g_1},$$

and by integrating both sides with respect to x the solution $u = u(x)$ of (4.7) is given implicitly by

$$\lambda B_1(x)u + B_2(x)u^2 = \int_{x_0}^{x} 2B_2(t)\frac{f_0(t)}{g_1(t)}\Delta t + C,$$

where C is an integration constant, determined by the initial value x_0. □

The other expressions of the Abel dynamic equation of the second kind in time scales are

$$\left(g_0(x) + g_1(x)\left(\frac{u+u^\sigma}{2}\right)\right)u^\Delta = f_0(x) + f_1(x)u^\sigma + f_2(x)u^2, \tag{4.10}$$

$$\left(g_0(x) + g_1(x)\left(\frac{u+u^\sigma}{2}\right)\right)u^\Delta = f_0(x) + f_1(x)u + f_2(x)(u^\sigma)^2, \tag{4.11}$$

$$\left(g_0(x) + g_1(x)\left(\frac{u+u^\sigma}{2}\right)\right)u^\Delta = f_0(x) + f_1(x)\left(\frac{u+u^\sigma}{2}\right) + f_2(x)u^2, \tag{4.12}$$

$$\left(g_0(x) + g_1(x)\left(\frac{u+u^\sigma}{2}\right)\right)u^\Delta = f_0(x) + f_1(x)\left(\frac{u+u^\sigma}{2}\right) + f_2(x)(u^\sigma)^2, \tag{4.13}$$

where $u = u(x)$. Especially the third and fourth expression are interesting, since they combine the u and u^σ also on the right-hand side of the Abel dynamic equation.

Remark 4.7. For $\mathbb{T} = \mathbb{R}$, any construction of the Abel dynamic equation of the second kind is the Abel equation of the second kind introduced in Section 3.1, i.e.,

$$(g_0(x) + g_1(x)u)u' = f_0(x) + f_1(x)u + f_2(x)u^2 \quad \text{with} \quad u = u(x).$$

Theorem 4.8. *There exists a solution of (4.12) provided $g_0 \neq 0$, $g_1 \neq 0$, g_k, $f_i \in \mathcal{R}$ for $k = 0, 1$, $i = 0, 1, 2$ and $\frac{f_0}{g_1} \in C_{\mathrm{rd}}$ and provided numbers λ and Λ exist such that*

$$B_2^\sigma(x)g_0(x) = \Lambda B_{11}(x)g_1(x) \quad \textit{and} \quad B_2^\sigma(x)g_0(x) = \lambda B_{12}^\sigma(x)g_1(x), \tag{4.14}$$

where

$$B_{11}(x) := e_{-p}(x,x_0)B_{11}(x_0) \quad \textit{with} \quad p := \frac{f_1}{g_0},$$

$$B_{12}(x) := e_{\ominus p}(x,x_0)B_{12}(x_0), \quad \textit{and}$$

$$B_2(x) := e_{\ominus q}(x,x_0)B_2(x_0) \quad \textit{with} \quad q := 2\frac{f_2}{g_1}.$$

Proof. One just has to separate equation (4.12) into

$$\frac{1}{2}g_0u^\Delta + \frac{1}{2}g_0u^\Delta + g_1\left(\frac{u+u^\sigma}{2}\right)u^\Delta = f_0 + \frac{f_1}{2}u + \frac{f_1}{2}u^\sigma + f_2u^2.$$

Define

$$B_{11}(x) := e_{-p}(x,x_0)B_{11}(x_0), \quad B_{12}(x) := e_{\ominus p}(x,x_0)B_{12}(x_0) \quad \text{with} \quad p := \frac{f_1}{g_0}$$

and multiply both sides of (4.12) by $B_{11}B_{12}^\sigma$ to get

$$B_{11}B_{12}^\sigma\frac{1}{2}g_0u^\Delta + B_{11}B_{12}^\sigma\frac{1}{2}g_0u^\Delta + B_{11}B_{12}^\sigma g_1\left(\frac{u+u^\sigma}{2}\right)u^\Delta$$
$$= B_{11}B_{12}^\sigma f_0 + B_{11}B_{12}^\sigma\frac{f_1}{2}u + B_{11}B_{12}^\sigma\frac{f_1}{2}u^\sigma + B_{11}B_{12}^\sigma f_2u^2.$$

By using the fact that

$$B_{11}f_1 = -g_0B_{11}^\Delta \quad \text{and} \quad B_{12}^\sigma f_1 = -g_0B_{12}^\Delta,$$

terms can be connected by the product rule, resulting in

$$B_{12}^\sigma\frac{g_0}{2}(B_{11}u)^\Delta + B_{11}\frac{g_0}{2}(B_{12}u)^\Delta + B_{12}^\sigma B_{11}g_1\left(\frac{u+u^\sigma}{2}\right)u^\Delta$$
$$= B_{12}^\sigma B_{11}f_0 + B_{12}^\sigma B_{11}f_2u^2.$$

Furthermore, define

$$B_2(x) := e_{\ominus q}(x, x_0)B_2(x_0) \quad \text{with} \quad q := 2\frac{f_2}{g_1}$$

and multiply B_2^σ on both sides to get

$$B_2^\sigma B_{12}^\sigma \frac{g_0}{2}(B_{11}u)^\Delta + B_2^\sigma B_{11}\frac{g_0}{2}(B_{12}u)^\Delta + B_2^\sigma B_{12}^\sigma B_{11} g_1 \left(\frac{u + u^\sigma}{2}\right) u^\Delta$$
$$= B_2^\sigma B_{12}^\sigma B_{11} f_0 + B_2^\sigma B_{12}^\sigma B_{11} f_2 u^2.$$

By using the fact that B_2 solves the dynamic equation

$$B_2^\sigma f_2 = -\frac{g_1}{2}B_2^\Delta.$$

Therefore we have

$$B_2^\sigma B_{12}^\sigma \frac{g_0}{2}(B_{11}u)^\Delta + B_2^\sigma B_{11}\frac{g_0}{2}(B_{12}u)^\Delta + B_{12}^\sigma B_{11}\frac{g_1}{2}(B_2 u^2)^\Delta = B_2^\sigma B_{12}^\sigma B_{11} f_0.$$

By applying the λ condition (4.14) and integrating both sides with respect to x, the solution $u = u(x)$ of Eq. (4.12) is given by

$$B_2(x)u^2 + \Lambda B_{11}(x)u + \lambda B_{12}(x)u = \int_{x_0}^{x} 2B_2^\sigma(t)\frac{f_0(t)}{g_1(t)}\Delta t + C,$$

where C is an integration constant, determined by the initial value x_0. □

Remark 4.9. The condition of Theorem 4.2, $g_0(x) \neq 0$, $g_1(x) \neq 0$, is critical to find an implicit solution to the different Abel dynamic equations of the second kind. Analogous to $\mathbb{R}$, one can apply a substitution to an Abel dynamic equation of the second kind, where $g_0 = 0$, to get a second kind with $g_0 \neq 0$. This enables, under the satisfaction of the additional conditions, the application of Theorem 4.2.

Consider the general Abel dynamic equation of the second kind in time scales with $g_0 = 0$, namely

$$g_1(x)\left(\frac{u+u^\sigma}{2}\right)u^\Delta = f_0(x) + f_1(x)u + f_2(x)u^2 \quad \text{with} \quad u = u(x). \tag{4.15}$$

The substitution $u = y + \frac{1}{2}$, $y = y(x)$ transfers Eq. (4.15) into

$$g_1\left(\frac{y+y^\sigma+\frac{1}{2}+\frac{1}{2}}{2}\right)y^\Delta = f_0 + f_1 y + \frac{f_1}{2} + f_2 y^2 + f_2 y + \frac{f_2}{4},$$

i.e.,

$$\left(g_1\left(\frac{y+y^\sigma}{2}\right) + \frac{g_1}{2}\right)y^\Delta = f_0 + \frac{f_1}{2} + \frac{f_2}{4} + f_1 y + f_2 y + f_2 y^2,$$

which is in the form (4.1)

$$\left(h_0 + h_1\left(\frac{y+y^\sigma}{2}\right)\right)y^\Delta = F_0 + F_1 y + F_2 y^2,$$

with

$$h_0 = \frac{g_1}{2}, \quad h_1 = g_1, \quad F_0 = f_0 + \frac{f_1}{2} + \frac{f_2}{4}, \quad F_1 = f_1 + f_2, \quad F_2 = f_2.$$

The methods that require $g_0 \neq 0$, such as Theorem 4.2, can now be also applied to Eq. (4.15).

The previous substitution transfers also the other expressions of the Abel dynamic equation of the second kind 4.10 to 4.13, where $g_0 = 0$, into the form of an Abel equation of the second kind, where $g_0 \neq 0$.

Theorem 4.10. *Consider the Abel equation*

$$\left(\frac{y+y^\sigma}{2}\right)y^\Delta = f_1(x)\left(\frac{y+y^\sigma}{2}\right) + f_2(x)y^2 \quad \textit{with} \quad y = y(x). \tag{4.16}$$

If a function F exists such that

$$\frac{F+F^\sigma}{2} = \lambda\frac{B_1}{B_3^\sigma}, \quad \textit{and} \quad \frac{F+F^\sigma}{2} = \Lambda\frac{B_2^\sigma}{B_3^\sigma},$$

with

$$\begin{aligned} B_1 &:= e_p(x,x_0)B_1(x_0), \quad p := \frac{-2(-F^\Delta+f_1)}{F^\sigma+F}, \\ B_2 &:= e_{\ominus q}(x,x_0)B_2(x_0), \quad q := \frac{2(-F^\Delta+f_1+4f_2F)}{F^\sigma+F}, \\ B_3 &:= e_{\ominus r}(x,x_0)B_3(x_0), \quad r := 2f_2, \end{aligned}$$

where x_0 is an initial value, then an implicit solution can be derived.

Proof. First of all, one uses the substitution $y = w + F$ on equation (4.16) to obtain

$$\begin{aligned} &\left(\left(\frac{F+F^\sigma}{2}\right) + \left(\frac{w+w^\sigma}{2}\right)\right)(w^\Delta + F^\Delta) \\ &\qquad = f_1\left(\frac{w+w^\sigma}{2}\right) + f_1\left(\frac{F+F^\sigma}{2}\right) + f_2w^2 + 2f_2Fw + f_2F^2, \end{aligned}$$

i.e.,

$$\begin{aligned} &\left(\left(\frac{F+F^\sigma}{2}\right) + \left(\frac{w+w^\sigma}{2}\right)\right)w^\Delta = (-F^\Delta+f_1)\left(\frac{F+F^\sigma}{2}\right) + f_2F^2 \\ &\qquad + \frac{w}{2}\left[-F^\Delta+f_1+4f_2F\right] + \frac{w^\sigma}{2}\left[-F^\Delta+f_1\right] + w^2f_2. \end{aligned}$$

Multiplying $B_1 B_2^\sigma$ on both sides yields

$$\begin{aligned}
&B_1 B_2^\sigma \left(\frac{F+F^\sigma}{4}\right) w^\Delta + B_1 B_2^\sigma \left(\frac{F+F^\sigma}{4}\right) w^\Delta + B_1 B_2^\sigma \left(\frac{w+w^\sigma}{2}\right) w^\Delta \\
=&B_1 B_2^\sigma \left[(-F^\Delta + f_1)\left(\frac{F+F^\sigma}{2}\right) + f_2 F^2\right] + B_1 B_2^\sigma \frac{w}{2}\left[-F^\Delta + f_1 + 4 f_2 F\right] \\
&+ B_1 B_2^\sigma \frac{w^\sigma}{2}\left[-F^\Delta + f_1\right] + B_1 B_2^\sigma w^2 f_2.
\end{aligned}$$

Using the conditions for B_1 and B_2, we have

$$\begin{aligned}
&B_1 B_2^\sigma \left(\frac{F+F^\sigma}{4}\right) w^\Delta + B_1 B_2^\sigma \left(\frac{F+F^\sigma}{4}\right) w^\Delta + B_1 B_2^\sigma \left(\frac{w+w^\sigma}{2}\right) w^\Delta \\
=&B_1 B_2^\sigma \left[(-F^\Delta + f_1)\left(\frac{F+F^\sigma}{2}\right) + f_2 F^2\right] - w B_1 B_2^\Delta \left(\frac{F+F^\sigma}{4}\right) \\
&- B_1^\Delta B_2^\sigma \left(\frac{F+F^\sigma}{4}\right) w^\sigma + B_1 B_2^\sigma w^2 f_2,
\end{aligned}$$

i.e.,

$$\begin{aligned}
B_2^\sigma \left(\frac{F+F^\sigma}{4}\right) (B_1 w)^\Delta + B_1 \left(\frac{F+F^\sigma}{4}\right) (B_2 w)^\Delta + B_1 B_2^\sigma \left(\frac{w+w^\sigma}{2}\right) w^\Delta& \\
= B_1 B_2^\sigma \left[(-F^\Delta + f_1)\left(\frac{F+F^\sigma}{2}\right) + f_2 F^2\right] + B_1 B_2^\sigma w^2 f_2.&
\end{aligned}$$

Define $B_3 := e_{\ominus r}(x, x_0) B_3(x_0)$, with $r := 2 f_2$. Multiplying B_3^σ on both sides, we get

$$\begin{aligned}
B_3^\sigma B_2^\sigma \left(\frac{F+F^\sigma}{4}\right) (B_1 w)^\Delta + B_3^\sigma B_1 \left(\frac{F+F^\sigma}{4}\right) (B_2 w)^\Delta + B_3^\sigma B_1 B_2^\sigma \left(\frac{w+w^\sigma}{2}\right) w^\Delta& \\
= B_3^\sigma B_1 B_2^\sigma \left[(-F^\Delta + f_1)\left(\frac{F+F^\sigma}{2}\right) + f_2 F^2\right] + B_3^\sigma B_1 B_2^\sigma w^2 f_2.&
\end{aligned}$$

Using the fact that $B_3^\Delta = -2f_2 B_3^\sigma$, we obtain

$$B_3^\sigma B_2^\sigma \left(\frac{F+F^\sigma}{4}\right)(B_1 w)^\Delta + B_3^\sigma B_1 \left(\frac{F+F^\sigma}{4}\right)(B_2 w)^\Delta + B_3^\sigma B_1 B_2^\sigma \left(\frac{w+w^\sigma}{2}\right) w^\Delta$$
$$= B_3^\sigma B_1 B_2^\sigma \left[(-F^\Delta + f_1)\left(\frac{F+F^\sigma}{2}\right) + f_2 F^2\right] - B_3^\Delta B_1 B_2^\sigma w^2 \frac{1}{2},$$

i.e.,

$$B_3^\sigma B_2^\sigma \left(\frac{F+F^\sigma}{4}\right)(B_1 w)^\Delta + B_3^\sigma B_1 \left(\frac{F+F^\sigma}{4}\right)(B_2 w)^\Delta + \frac{1}{2} B_1 B_2^\sigma (B_3 w^2)^\Delta$$
$$= B_3^\sigma B_1 B_2^\sigma \left[(-F^\Delta + f_1)\left(\frac{F+F^\sigma}{2}\right) + f_2 F^2\right].$$

Dividing both sides with $\frac{1}{2} B_1 B_2^\sigma$ and using the condition for F yields

$$\lambda (B_1 w)^\Delta + \Lambda (B_2 w)^\Delta + (B_3 w^2)^\Delta = 2B_3^\sigma \left[(-F^\Delta + f_1)\left(\frac{F+F^\sigma}{2}\right) + f_2 F^2\right].$$

Integrate both sides formulates the implicit solution w as

$$\lambda B_1 w + \Lambda B_2 w + B_3 w^2 = \int_{x_0}^{x} 2B_3^\sigma \left[(-F^\Delta + f_1)\left(\frac{F+F^\sigma}{2}\right) + f_2 F^2\right] \Delta t + C,$$

where C is an integration constant, determined by the initial value x_0. □

4.2. THE ABEL DYNAMIC EQUATION OF THE 1ST KIND

The special case of an Abel equation of the first kind in $\mathbb{R}$ with $f_0(x) = 0$ that has been discussed in Section 3.2 can also be derived in $\mathbb{T}$. Consider the Abel dynamic equation of the second kind, namely

$$\left(g(x) + \left(\frac{u+u^\sigma}{2}\right)\right) u^\Delta = F_0(x) + F_1(x)u + F_2(x)u^2, \tag{4.17}$$

with $u = u(x)$. As in $\mathbb{R}$, one gets a special class of the Abel equation of the first kind by applying the substitution $u = \frac{1}{w} - g$, $w \neq 0$.

Definition 4.11. A special class of the Abel equation of the first kind in $\mathbb{T}$ is

$$w^{\Delta}\left(\frac{w + w^{\sigma}}{2}\right) = h_1(x)(w^{\sigma})^2 + h_{21}(x)w^2w^{\sigma} + h_{22}(x)w(w^{\sigma})^2 + h_3(x)w^2(w^{\sigma})^2, \quad (4.18)$$

with $w = w(x)$.

Remark 4.12. Starting with the Abel differential equation of the second kind in $\mathbb{R}$, namely

$$(g(x) + u)u' = F_0(x) + F_1(x)u + F_2(x)u^2 \quad \text{with} \quad u = u(x), \quad (4.19)$$

the substitution $u = \frac{1}{w} - g$, $w \neq 0$ yields

$$\left(g + \frac{1}{w} - g\right)\left(-\frac{w'}{w^2} - g'\right) = F_0 - F_1g + F_2g^2 + \frac{F_1}{w} - \frac{2F_2g}{w} + \frac{F_2}{w^2},$$

i.e.,

$$-\frac{w'}{w^3} - \frac{g'}{w} = F_0 - F_1g + F_2g^2 + \frac{F_1}{w} - \frac{2F_2g}{w} + \frac{F_2}{w^2}.$$

Multiplying both sides with $-w^3$ yields

$$w' = -g'w^2 + w^3(-F_0 + F_1g - F_2g^2) + w^2(-F_1 + 2F_2g) - wF_2,$$

which is in the special form of an Abel equation of the first kind

$$w' = h_3w^3 + h_2w^2 + h_1w,$$

with

$$h_3 = -F_0 + F_1 g - F_2 g^2, \tag{4.20}$$

$$h_2 = -g' - F_1 + 2F_2 g, \tag{4.21}$$

$$h_1 = -F_2. \tag{4.22}$$

The coefficients match exactly the coefficients from Section 3.2.1, Equation (3.18).

Theorem 4.13. *The transformation $u = \frac{1}{w} - g$ transforms the Abel equation of the second kind into the special form of the Abel equation of the first kind* (4.18).

Proof. Consider the Abel equation of the second kind, namely

$$\left(\left(\frac{g+g^\sigma}{2}\right) + \left(\frac{u+u^\sigma}{2}\right)\right) u^\Delta = f_0 + f_1 u + f_2 u^2.$$

The substitution $u = \frac{1}{w} - g$, $w \neq 0$ yields

$$\left(\frac{g+g^\sigma}{2} + \frac{w+w^\sigma}{2ww^\sigma} - \frac{g+g^\sigma}{2}\right)\left(\frac{-w^\Delta}{ww^\sigma} - g^\Delta\right)$$
$$= f_0 + \frac{f_1}{w} - f_1 g + \frac{f_2}{w^2} - 2\frac{f_2 g}{w} + f_2 g^2,$$

i.e.,

$$-\frac{w^\Delta(w+w^\sigma)}{2(w^\sigma)^2 w^2} - \frac{g^\Delta}{ww^\sigma}\left(\frac{w+w^\sigma}{2}\right) = f_0 - f_1 g + \frac{f_1}{w} - 2\frac{f_2 g}{w} + \frac{f_2}{w^2} + f_2 g^2.$$

By multiplying both sides with $-w^2(w^\sigma)^2$, we have

$$w^\Delta\left(\frac{w+w^\sigma}{2}\right) = -\, g^\Delta\left(\frac{w+w^\sigma}{2}\right) ww^\sigma + w^2(w^\sigma)^2(-f_0 + f_1 g - f_2 g^2)$$
$$+ (w^\sigma)^2 w(-f_1 + 2f_2 g) + (w^\sigma)^2(-f_2),$$

which is of the form

$$w^{\Delta}\left(\frac{w+w^{\sigma}}{2}\right)=h_1(w^{\sigma})^2+h_{21}(w^{\sigma})^2w+h_{22}w^2w^{\sigma}+h_3(w^{\sigma})^2w^2,$$

with

$$h_1=-f_2,\quad h_{21}=-\frac{g^{\Delta}}{2}-f_1+2f_2g,\quad h_{22}=-\frac{g^{\Delta}}{2},\quad h_3=-f_0+f_1g-f_2g^2.$$

This completes the proof. □

Remark 4.14. For $\mathbb{T}=\mathbb{R}$, Eq. (4.18) becomes

$$w'w=h_1w^2+h_{21}w^2w+h_{22}ww^2+h_3w^2w^2,$$

i.e.,

$$w'w=h_1w^2+(h_{21}+h_{22})w^3+h_3w^4,$$

i.e.,

$$w'=h_1w+(h_{21}+h_{22})w^2+h_3w^3=h_1w+h_2w^2+h_3w^3,$$

where $h_2=h_{21}+h_{22}$. Theorem 4.13 provides

$$\begin{aligned}h_1&=-f_2,\\h_2&=-\frac{g'}{2}-f_1+2f_2g-\frac{g'}{2}=-g'-f_1+2f_2g,\\h_3&=-f_0+f_1g-f_2g^2.\end{aligned}$$

These variable coefficients h_i, $i = 1, 2, 3$ match the coefficients of equation (3.18) in Section 3.2.1.

Example 4.15. For $\mathbb{T} = \mathbb{Z}$, the Abel equation of the first kind is

$$\Delta w \left(\frac{w + Ew}{2} \right) = h_1(Ew)^2 + h_{21}w^2 Ew + h_{22}w(Ew)^2 + h_3 w^2 (Ew)^2,$$

i.e.,

$$\frac{(Ew)^2 - w^2}{2} = h_1(Ew)^2 + h_{21}w^2 Ew + h_{22}w(Ew)^2 + h_3 w^2 (Ew)^2,$$

i.e.,

$$(Ew)^2 \left[\frac{1}{2} - h_1 - h_{22}w - h_3 w^2 \right] + Ew[-h_{21}w^2] = \frac{w^2}{2}.$$

Theorem 4.16. *Consider a more general Abel equation of the second kind, where* $g_1(x) \neq 0$, *namely*

$$\left(g_0 + g_1 \left(\frac{u + u^\sigma}{2} \right) \right) u^\Delta = f_0 + f_1 u + f_2 u^2. \tag{4.23}$$

Assume there exists β *such that* $\frac{g_0}{g_1} = \frac{\beta + \beta^\sigma}{2}$. *Then there exists a substitution that transfers Eq.* (4.23) *into an Abel equation of the first kind.*

Proof. Consider Eq. (4.23). First of all, one divides the equation by g_1 to get

$$\left(g + \left(\frac{u + u^\sigma}{2} \right) \right) u^\Delta = F_0 + F_1 u + F_2 u^2,$$

with

$$F_i = \frac{f_i}{g_1}, \quad g = \frac{g_0}{g_1} \quad \text{for } i = 0, 1, 2.$$

Since there exists β such that $g = \frac{\beta+\beta^\sigma}{2}$, we have

$$\left(\left(\frac{\beta+\beta^\sigma}{2}\right) + \left(\frac{u+u^\sigma}{2}\right)\right) u^\Delta = F_0 + F_1 u + F_2 u^2.$$

Apply furthermore the substitution $u = \frac{1}{w} - \beta$ to obtain

$$\left[\left(\frac{\beta+\beta^\sigma}{2}\right) + \left(\frac{w+w^\sigma}{2ww^\sigma}\right) - \left(\frac{\beta+\beta^\sigma}{2}\right)\right]\left(-\frac{w^\Delta}{ww^\sigma} - \beta^\Delta\right)$$
$$= F_0 - F_1\beta + \frac{F_1}{w} + \frac{F_2}{w^2} - 2\frac{F_2\beta}{w} + F_2\beta^2,$$

i.e.,

$$\frac{-w^\Delta(w+w^\sigma)}{2w^2(w^\sigma)^2} - \beta^\Delta\left(\frac{w+w^\sigma}{2ww^\sigma}\right) = F_0 - F_1\beta + F_2\beta^2 + \frac{F_1}{w} - 2\frac{F_2\beta}{w} + \frac{F_2}{w^2}.$$

Multiplying both sides with $-w^2(w^\sigma)^2$ yields

$$w^\Delta\left(\frac{w+w^\sigma}{2}\right) = -\frac{\beta^\Delta}{2}w^2w^\sigma - \frac{\beta^\Delta}{2}(w^\sigma)^2 w - F_0 w^2(w^\sigma)^2 + F_1\beta w^2(w^\sigma)^2$$
$$- F_2\beta^2 w^2(w^\sigma)^2 - F_1(w^\sigma)^2 w + 2F_2\beta w(w^\sigma)^2 - F_2(w^\sigma)^2.$$

By changing the order of the terms, we get

$$w^\Delta\left(\frac{w+w^\sigma}{2}\right) = [-F_2](w^\sigma)^2 + w^2w^\sigma\left[-\frac{\beta^\Delta}{2}\right] + w(w^\sigma)^2\left[-\frac{\beta^\Delta}{2} - F_1 + 2F_2\beta\right]$$
$$+ w^2(w^\sigma)^2[-F_0 + F_1\beta - F_2\beta^2].$$

Putting

$$h_1 = -F_2, \quad h_{21} = -\frac{\beta^\Delta}{2}, \quad h_{22} = -\frac{\beta^\Delta}{2} - F_1 + 2F_2\beta, \quad h_3 = -F_0 + F_1\beta - F_2\beta^2,$$

we see that this is of the form of an Abel equation of the first kind (4.18). □

Theorem 4.16 also holds for $g_0 = 0$ ($g_1 \neq 0$). If $g_0 = 0$, then $\beta = 0$. The substitution to transfer this Abel equation of the second kind into the first kind is $u = \frac{1}{w}$. The resulting Abel dynamic equation is

$$w^\Delta \left(\frac{w + w^\sigma}{2}\right) = -F_0(x)w^2(w^\sigma)^2 - F_1(x)w(w^\sigma)^2 - F_2(x)(w^\sigma)^2, \quad w = w(x).$$

Example 4.17. Suppose $\frac{g_0}{g_1} = c =$ constant. Then the Abel equation (4.23) can be divided by g_1 to get

$$\left(\frac{g_0}{g_1} + \left(\frac{u + u^\sigma}{2}\right)\right) u^\Delta = F_0 + F_1 u + F_2 u^2,$$

i.e.,

$$\left(c + \left(\frac{u + u^\sigma}{2}\right)\right) u^\Delta = F_0 + F_1 u + F_2 u^2,$$

with $F_i = \frac{f_i}{g_1}$ for $i = 0, 1, 2$. Use the substitution $u = \frac{1}{y} - c$, $y \neq 0$. The left-hand side becomes

$$\begin{aligned}\left(c + \left(\frac{\frac{1}{y} + \frac{1}{y^\sigma}}{2}\right) - \frac{c + c}{2}\right)\left(-\frac{y^\Delta}{yy^\sigma} - 0\right) &= \left(\frac{\frac{1}{y} + \frac{1}{y^\sigma}}{2}\right)\left(-\frac{y^\Delta}{yy^\sigma}\right) \\ &= \left(\frac{y^\sigma + y}{2yy^\sigma}\right)\left(-\frac{y^\Delta}{yy^\sigma}\right) = -\frac{1}{y^2(y^\sigma)^2} y^\Delta \left(\frac{y^\sigma + y}{2}\right),\end{aligned}$$

while the right-hand side is

$$F_0 + \frac{F_1}{y} - cF_1 + \frac{F_2}{y^2} - 2\frac{cF_2}{y} + F_2c^2.$$

Combining the left and the right sides, we obtain

$$-\frac{1}{y^2(y^\sigma)^2}y^\Delta\left(\frac{y^\sigma + y}{2}\right) = F_0 + \frac{F_1}{y} - cF_1 + \frac{F_2}{y^2} - 2\frac{cF_2}{y} + F_2c^2.$$

Multiplying both sides with $-y^2(y^\sigma)^2$, the previous equation becomes

$$y^\Delta\left(\frac{y^\sigma + y}{2}\right) = y^2(y^\sigma)^2(-F_0 + cF_1 - c^2F_2) + (y^\sigma)^2y(-F_1 + 2cF_2) + (y^\sigma)^2(-F_2).$$

This is in the form of an Abel equation of the first kind

$$y^\Delta\left(\frac{y^\sigma + y}{2}\right) = y^2(y^\sigma)^2h_3 + (y^\sigma)^2yh_{22} + (y^\sigma)^2h_1$$

with

$$h_3 = -F_0 + cF_1 - c^2F_2, \quad h_2 = -F_1 + 2cF_2, \quad h_1 = -F_2.$$

4.3. CANONICAL ABEL DYNAMIC EQUATION

4.3.1. Transformation from the Abel dynamic equation of the 2nd kind to the canonical Abel dynamic equation.

First of all we give the definiton of the canonical Abel equation in $\mathbb{T}$.

Definition 4.18. The canonical form of an Abel equation in $\mathbb{T}$, $h_0, h_{11}, h_{12} : \mathbb{R} \longrightarrow \mathbb{R}$, is defined by

$$w^{\Delta}\left(\frac{w+w^{\sigma}}{2}\right) = h_0(x) + h_{11}(x)w + h_{12}(x)w^{\sigma} \quad \text{with} \quad w = w(x). \tag{4.24}$$

The previous section showed in Theorem 4.16 how to transfer the general Abel equation of the second kind

$$\left(g_0 + g_1\left(\frac{u+u^{\sigma}}{2}\right)\right)u^{\Delta} = f_0 + f_1 u + f_2 u^2$$

into

$$\left(\left(\frac{g+g^{\sigma}}{2}\right) + \left(\frac{u+u^{\sigma}}{2}\right)\right)u^{\Delta} = F_0 + F_1 u + F_2 u^2. \tag{4.25}$$

Using the same substitution from Section 3.3.1, Eq. (4.25) can be furthermore transformed into the canonical form of an Abel dynamic equation.

Theorem 4.19. *The substitution* $u = \frac{w}{E} - g$ *with* $E = e_{\alpha}(x, x_0)$ *transforms* (4.25) *into the canonical Abel dynamic equation* (4.24)*, where* α *is determined by*

$$\frac{\alpha}{2} + \frac{\alpha}{2}\frac{E^{\sigma}}{E} = -F_2\frac{(E^{\sigma})^2}{E^2} \quad \leftrightarrow \quad \alpha + \alpha^2\frac{\mu}{2} = -F_2(1+\mu\alpha)^2.$$

Remark 4.20. For $\mathbb{T} = \mathbb{R}$, Theorem 4.19 was discussed in Section 3.3.1. Eq. (4.25) is in $\mathbb{R}$

$$(g+u)u' = \left(\frac{g+g^{\sigma}}{2} + \frac{u+u^{\sigma}}{2}\right)u^{\Delta} = F_0 + F_1 u + F_2 u^2,$$

which is the Abel equation of the second kind in $\mathbb{R}$, introduced in Section 3.1. Theorem 4.19 is using exactly the same substitution in Section 3.2.1.

Proof. Note that $E = e_\alpha(x, x_0) = \exp\{\int \alpha(x)\mathrm{d}x\}$, where α satisfies

$$\alpha + \alpha^2 \frac{\mu}{2} = -F_2(1 + \mu\alpha)^2, \tag{4.26}$$

which is in $\mathbb{R}$ equal to

$$\alpha = \frac{\alpha}{2} + \frac{\alpha}{2} = -F_2 \cdot 1 = -F_2,$$

so E is defined in the same way as in the substitution in Section 3.2.1. □

Proof of Theorem 4.19. Consider the Abel equation of the second kind (4.25) and apply the substitution $u = \frac{w}{E} - g$ with $E = e_\alpha(x, x_0)$, where α satisfies

$$\alpha + \alpha^2 \frac{\mu}{2} = -F_2(1 + \mu\alpha)^2.$$

The left-hand side of Eq. (4.25) is then

$$\begin{aligned}
&\left(\left(\frac{g + g^\sigma}{2}\right) + \frac{1}{2}\left(\frac{w}{E} + \frac{w^\sigma}{E^\sigma} - g - g^\sigma\right)\right)\left(\frac{w^\Delta E - E^\Delta w}{EE^\sigma} - g^\Delta\right)\\
=&\frac{1}{2}\left(\frac{wE^\sigma + w^\sigma E}{EE^\sigma}\right)\left(\frac{w^\Delta E - E^\Delta w}{EE^\sigma} - g^\Delta\right)\\
=&\frac{1}{2}\left(\frac{w^\Delta E w E^\sigma + w^\Delta E w^\sigma E}{E^2(E^\sigma)^2}\right) + \frac{1}{2}\left(\frac{-E^\Delta w w E^\sigma - E^\Delta w w^\sigma E}{E^2(E^\sigma)^2}\right)\\
&- \frac{1}{2} g^\Delta \left(\frac{wE^\sigma + w^\sigma E}{EE^\sigma}\right)\\
=&\frac{1}{2}\left(\frac{w^\Delta E(wE^\sigma + w^\sigma E + Ew - Ew)}{E^2(E^\sigma)^2}\right) + \frac{1}{2}\left(\frac{-E^\Delta w w E^\sigma - E^\Delta w w^\sigma E}{E^2(E^\sigma)^2}\right)\\
&- \frac{1}{2} g^\Delta \left(\frac{wE^\sigma + w^\sigma E}{EE^\sigma}\right)
\end{aligned}$$

$$
\begin{aligned}
=&\frac{E^2w^\Delta}{E^2(E^\sigma)^2}\left(\frac{w+w^\sigma}{2}\right)+\frac{1}{2}\left(\frac{w^\Delta Ew(E^\sigma-E)}{E^2(E^\sigma)^2}\right)\\
&+\frac{1}{2}\left(\frac{-E^\Delta wwE^\sigma-E^\Delta ww^\sigma E}{E^2(E^\sigma)^2}\right)-\frac{1}{2}g^\Delta\left(\frac{wE^\sigma+Ew^\sigma}{EE^\sigma}\right)\\
=&\frac{w^\Delta}{(E^\sigma)^2}\left(\frac{w+w^\sigma}{2}\right)+\frac{1}{2}\left(\frac{w^\Delta w\mu E^\Delta}{E(E^\sigma)^2}\right)+\frac{1}{2}\left(\frac{-E^\Delta wwE^\sigma-E^\Delta ww^\sigma E}{E^2(E^\sigma)^2}\right)\\
&-\frac{1}{2}g^\Delta\left(\frac{wE^\sigma+Ew^\sigma}{EE^\sigma}\right)\\
=&\frac{w^\Delta}{(E^\sigma)^2}\left(\frac{w+w^\sigma}{2}\right)+\frac{1}{2}\left(\frac{w(w^\sigma-w)E^\Delta}{E(E^\sigma)^2}\right)+\frac{1}{2}\left(\frac{-E^\Delta wwE^\sigma-E^\Delta ww^\sigma E}{E^2(E^\sigma)^2}\right)\\
&-\frac{1}{2}g^\Delta\left(\frac{wE^\sigma+Ew^\sigma}{EE^\sigma}\right).
\end{aligned}
$$

After the substitution, the right-hand side of (4.25) is

$$
F_0+\frac{F_1w}{E}-F_1g+\frac{F_2w^2}{E^2}-\frac{2F_2gw}{E}+F_2g^2.
$$

Putting the left and the right sides together, Eq. (4.25) transfers into

$$
\begin{aligned}
&\frac{w^\Delta}{(E^\sigma)^2}\left(\frac{w+w^\sigma}{2}\right)+\frac{1}{2}\left(\frac{ww^\sigma E^\Delta}{E(E^\sigma)^2}\right)-\frac{1}{2}\left(\frac{w^2E^\Delta}{E(E^\sigma)^2}\right)+\frac{1}{2}\left(\frac{-E^\Delta wwE^\sigma-E^\Delta ww^\sigma E}{E^2(E^\sigma)^2}\right)\\
&\qquad-\frac{1}{2}g^\Delta\left(\frac{wE^\sigma+Ew^\sigma}{EE^\sigma}\right)=F_0+\frac{F_1w}{E}-F_1g+\frac{F_2w^2}{E^2}-\frac{2F_2gw}{E}+F_2g^2.
\end{aligned}
$$

Multiply both sides with $(E^\sigma)^2$ and use $E^\Delta=\alpha E$, as well as $E^\sigma=E(1+\mu\alpha)$ we have

$$
\begin{aligned}
&w^\Delta\left(\frac{w+w^\sigma}{2}\right)+\frac{1}{2}\alpha ww^\sigma-\frac{1}{2}\alpha w^2-\frac{1}{2}\alpha(1+\mu\alpha)w^2-\frac{1}{2}\alpha ww^\sigma-\frac{1}{2}g^\Delta E(1+\mu\alpha)^2w\\
&-\frac{1}{2}g^\Delta E(1+\mu\alpha)w^\sigma=F_0E^2(1+\mu\alpha)^2+F_1E(1+\mu\alpha)^2w-F_1gE^2(1+\mu\alpha)^2\\
&\qquad+F_2(1+\mu\alpha)^2w^2-2F_2gE(1+\mu\alpha)^2w+F_2g^2E^2(1+\mu\alpha)^2.
\end{aligned}
$$

Rearranging terms leads to

$$w^{\Delta}\left(\frac{w+w^{\sigma}}{2}\right) = -\frac{1}{2}\alpha w w^{\sigma} + \frac{1}{2}\alpha w^2 + \frac{1}{2}\alpha E(1+\mu\alpha)^2 w^2 + \frac{1}{2}\alpha w w^{\sigma}$$
$$+ \frac{1}{2} g^{\Delta} E(1+\mu\alpha)^2 w + \frac{1}{2} g^{\Delta} E(1+\mu\alpha) w^{\sigma} + F_0 E^2 (1+\mu\alpha)^2 + F_1 E(1+\mu\alpha)^2 w$$
$$- F_1 g E^2 (1+\mu\alpha)^2 + F_2 (1+\mu\alpha)^2 w^2 - 2F_2 g E(1+\mu\alpha)^2 w + F_2 g^2 E^2 (1+\mu\alpha)^2.$$

This is in the form

$$w^{\Delta}\left(\frac{w+w^{\sigma}}{2}\right) = h_0 + h_{11} w + h_{12} w^{\sigma} + h_{21} w w^{\sigma} + h_{22} w^2,$$

where $h_{21} = -\frac{\alpha}{2} + \frac{\alpha}{2} = 0$. By using condition (4.26), we get

$$h_{22} = \alpha + \alpha\frac{\mu}{2} + F_2(1+\mu\alpha)^2 = 0.$$

The form is therefore

$$w^{\Delta}\left(\frac{w+w^{\sigma}}{2}\right) = h_0 + h_{11} w + h_{12} w^{\sigma}, \tag{4.27}$$

with

$$h_0 = F_0 E^2 (1+\mu\alpha)^2 - F_1 g E^2 (1+\mu\alpha)^2 + F_2 g^2 E^2 (1+\mu\alpha)^2, \tag{4.28a}$$
$$h_{11} = \frac{g^{\Delta}}{2} E(1+\mu\alpha)^2 + F_1 E(1+\mu\alpha)^2 - 2F_2 g E(1+\mu\alpha)^2, \tag{4.28b}$$
$$h_{12} = \frac{g^{\Delta}}{2} E(1+\mu\alpha). \tag{4.28c}$$

This completes the proof. □

Remark 4.21. For $\mathbb{T} = \mathbb{R}$, the coefficients of (4.28) match exactly the variable coefficients from the substitution into the canonical form in $\mathbb{R}$, discussed in Section 3.2.1.

The canonical form (4.27) is for $\mathbb{T} = \mathbb{R}$

$$w'w = w^{\Delta}\left(\frac{w + w^{\sigma}}{2}\right) = h_0 + h_{11}w + h_{12}w^{\sigma} = H_0 + H_1 w, \tag{4.29}$$

with

$$H_0 = h_0 = F_0(E^{\sigma})^2 - F_1 g(E^{\sigma})^2 + F_2 g^2(E^{\sigma})^2 = F_0E^2 - F_1 gE^2 + F_2 g^2 E^2,$$

$$\begin{aligned} H_1 =& h_{11} + h_{12} = \frac{g^{\Delta}}{2}\frac{(E^{\sigma})^2}{E} + F_1\frac{(E^{\sigma})^2}{E} - 2F_2 g\frac{(E^{\sigma})^2}{E} + \frac{g^{\Delta}}{2}E^{\sigma} \\ =& \frac{g'}{2}E + F_1 E - 2F_2 gE + \frac{g'}{2}E. \end{aligned}$$

The coefficients (4.30) are identical to the coefficients from Section 3.2.1 and [7, p. 27f].

Example 4.22. For $\mathbb{T} = \mathbb{Z}$, the canonical form (4.24) is

$$\Delta w\left(\frac{w + Ew}{2}\right) = w^{\Delta}\left(\frac{w + w^{\sigma}}{2}\right) = h_0 + h_{11}w + h_{12}w^{\sigma} = h_0 + h_{11}w + h_{12}Ew.$$

Since $\Delta w = Ew - w$, we have

$$(Ew - w)\left(\frac{w + Ew}{2}\right) = h_0 + h_{11}w + h_{12}Ew,$$

i.e.,

$$(Ew)^2 - w^2 = h_0 + h_{11}w + h_{12}Ew,$$

i.e.,

$$Ew(Ew - h_{12}) = w^2 + h_{11}w + h_0.$$

4.3.2. Transformation from the Abel equation of the 1st kind to the canonical Abel dynamic equation.

Theorem 4.23. *Consider the special Abel equation of the first kind*

$$\left(\frac{y+y^\sigma}{2}\right)y^\Delta = h_1(x)(y^\sigma)^2 + h_{21}(x)y^2y^\sigma + h_{22}(x)y(y^\sigma)^2 + h_3(x)y^2(y^\sigma)^2, \quad (4.31)$$

with $y = y(x)$. *Use the substitution*

$$y = \frac{E}{w}, \qquad E = e_\alpha(x, x_0), \qquad \alpha = 2h_1\left(\frac{(1+\mu\alpha)^2}{2+\mu\alpha}\right),$$

to get the canonical Abel equation (4.24).

The canonical Abel form (4.24) can also be constructed by starting with the special Abel equation of the first kind that has been discussed in Section 4.2.

Proof. Let an Abel equation of the first kind (4.31) be given and apply the substitution $y = \frac{E}{w}$ with $E = e_\alpha(x, x_0)$ and $\alpha = 2h_1\left(\frac{(1+\mu\alpha)^2}{2+\mu\alpha}\right)$. The left-hand side of (4.31) is

$$\begin{aligned}
&\frac{1}{2}\left(\frac{E}{w}+\frac{E^\sigma}{w^\sigma}\right)\left(\frac{E^\Delta w - Ew^\Delta}{w^\sigma w}\right) = \frac{1}{2}\left(\frac{Ew^\sigma + wE^\sigma}{ww^\sigma}\right)\left(\frac{E^\Delta w - Ew^\Delta}{w^\sigma w}\right) \\
=&\frac{1}{2}\left(\frac{Ew^\sigma + wE^\sigma + Ew - Ew}{ww^\sigma}\right)\left(\frac{E^\Delta w - Ew^\Delta}{w^\sigma w}\right) \\
=&\left(E\left(\frac{w+w^\sigma}{2}\right)\frac{1}{ww^\sigma} + \frac{1}{2}(E^\sigma - E)\frac{w}{ww^\sigma}\right)\left(\frac{E^\Delta w - Ew^\Delta}{w^\sigma w}\right) \\
=&\left(\frac{EE^\Delta w}{w^2(w^\sigma)^2}\right)\left(\frac{w+w^\sigma}{2}\right) + \frac{(E^\sigma - E)wE^\Delta w}{2w^2(w^\sigma)^2} - \left(\frac{Ew^\Delta E}{w^2(w^\sigma)^2}\right)\left(\frac{w+w^\sigma}{2}\right) \\
&- \frac{Ew^\Delta \mu E^\Delta w}{2w^2(w^\sigma)^2}
\end{aligned}$$

$$
\begin{aligned}
=&\frac{EE^\Delta}{2(w^\sigma)^2}+\frac{EE^\Delta}{2ww^\sigma}+\frac{E^\sigma E^\Delta}{2(w^\sigma)^2}-\frac{EE^\Delta}{2(w^\sigma)^2}-\left(\frac{E^2}{w^2(w^\sigma)^2}\right)w^\Delta\left(\frac{w+w^\sigma}{2}\right)\\
&-\frac{EE^\Delta}{2ww^\sigma}+\frac{EE^\Delta}{2(w^\sigma)^2}\\
=&\frac{E^\sigma E^\Delta}{2(w^\sigma)^2}+\frac{EE^\Delta}{2(w^\sigma)^2}-\left(\frac{w^\Delta E^2}{w^2(w^\sigma)^2}\right)\left(\frac{w+w^\sigma}{2}\right)\\
=&\frac{E^\Delta(E^\sigma+E)}{2(w^\sigma)^2}-\left(\frac{w^\Delta E^2}{w^2(w^\sigma)^2}\right)\left(\frac{w+w^\sigma}{2}\right).
\end{aligned}
$$

By applying the substitution, the right-hand side of (4.31) is

$$
h_1\frac{(E^\sigma)^2}{(w^\sigma)^2}+h_{21}\frac{E^2}{w^2}\frac{E^\sigma}{w^\sigma}+h_{22}\frac{(E^\sigma)^2}{(w^\sigma)^2}\frac{E}{w}+h_3\frac{E^2}{w^2}\frac{(E^\sigma)^2}{(w^\sigma)^2}.
$$

By combining now the left and right sides, as well as dividing by $-\frac{E^2}{w^2(w^\sigma)^2}$, we get

$$
\begin{aligned}
&w^\Delta\left(\frac{w+w^\sigma}{2}\right)-\frac{E^\Delta(E^\sigma+E)w^2}{2E^2}\\
&\qquad=-h_1w^2\frac{(E^\sigma)^2}{E^2}-h_{21}E^\sigma w^\sigma-h_{22}w\frac{(E^\sigma)^2}{E}-h_3(E^\sigma)^2.
\end{aligned}
$$

Since $E^\Delta=\alpha E$, we have

$$
\begin{aligned}
w^\Delta\left(\frac{w+w^\sigma}{2}\right)=&\,w^2\left(\frac{\alpha(E^\sigma+E)}{2E}-h_1\frac{(E^\sigma)^2}{E^2}\right)+w\left(-h_{22}\frac{(E^\sigma)^2}{E}\right)\\
&+w^\sigma(-h_{21}E^\sigma)-h_3(E^\sigma)^2.
\end{aligned}
$$

Note that α satisfies $\alpha=2h_1\left(\frac{(E^\sigma)^2}{E^2+EE^\sigma}\right)$, which yields

$$
w^\Delta\left(\frac{w+w^\sigma}{2}\right)=w\left(-h_{22}\frac{(E^\sigma)^2}{E}\right)+w^\sigma(-h_{21}E^\sigma)-h_3(E^\sigma)^2.
$$

This complies with the common canonical Abel dynamic equation (4.24). □

Remark 4.24. For $\mathbb{T} = \mathbb{R}$, Theorem 4.23 is leading to the same result as in Section 3.3.2. We have

$$yy' = \left(\frac{y+y^\sigma}{2}\right) y^\Delta = h_1(y^\sigma)^2 + h_{21}y^2y^\sigma + h_{22}y(y^\sigma)^2 + h_3y^2(y^\sigma)^2$$
$$= h_1y^2 + (h_{21}+h_{22})y^3 + h_3y^4.$$

After dividing by y, the equation is equivalent to the Abel equation of the first kind in $\mathbb{R}$, Eq. (3.17) that has been discussed in Section 3.2, which is verified in the following. The substitution of Theorem 4.23 becomes for $\mathbb{R}$, $y = \frac{E}{w}$ with $E = \exp\{\int \alpha \mathrm{d}x\}$, where α satisfies $\alpha = 2h_1\frac{(E^\sigma)^2}{E^2+EE^\sigma}$. In $\mathbb{R}$ it is equal to $\alpha = 2h_1\frac{E^2}{E(E+E)} = h_1$. Applying now the substitution yields

$$\frac{E}{w}\left(\frac{E'w - w'E}{w^2}\right) = h_1\frac{E^2}{w^2} + (h_{21}+h_{22})\frac{E^3}{w^3} + h_3\frac{E^4}{w^4}.$$

Multiplying both sides with $-\frac{w^4}{E^2}$, we have

$$-\frac{E'}{E}w^2 + ww' = -h_1w^2 - (h_{21}+h_{22})Ew - h_3E^2.$$

Use furthermore $E' = \alpha E$ to obtain

$$ww' = (\alpha - h_1)w^2 - (h_{21}+h_{22})Ew - h_3E^2.$$

Since $\alpha = h_1$, the equation is

$$ww' = -(h_{21}+h_{22})Ew - h_3E^2.$$

This is consistent with the canonical Abel form that was derived in $\mathbb{R}$ in Section 3.2.

5. TRANSFORMATION BETWEEN COMMON DYNAMIC EQUATIONS

If Abel differential equations have a particular form, they already belong to a special class of common differential equations, such as for example the Bernoulli differential equation. In other cases, one needs a substitution to transfer a dynamic equation into another equation in $\mathbb{T}$. In the following, some of the relations between dynamic equations such as Abel equations, Bernoulli equations, logistic equations and linear equations are discussed.

5.1. TRANSFORMATION TO THE BERNOULLI DYNAMIC EQUATIONS

Whereas Abel differential equations are not solvable in every case, there exists a method to solve any Bernoulli differential equation in $\mathbb{R}$. Even in $\mathbb{T}$, there exists a general method to solve the Bernoulli dynamic equation [2, p. 38]. This is one of the reasons why the Bernoulli dynamic equation is so valuable. The Bernoulli differential equation has in $\mathbb{R}$ the general form [5, p. 73]

$$y' + a(x)y = r(x)y^{\alpha}, \tag{5.1}$$

where $\alpha \notin \{0, 1\}$ and $y = y(x)$.

Kamke explains in [7, p. 26] that an Abel differential equation of the first kind is of the Bernoulli form if $h_0 = h_2 = 0$, so

$$y' = h_3 y^3 + h_2 y^2 + h_1 y + h_0 = h_3 y^3 + h_1 y. \tag{5.2}$$

By comparing the equations (5.1) and (5.2), one can easily see that Eq. (5.2) is the same as

$$y' + (-h_1)y = h_3 y^3. \tag{5.3}$$

This is the Bernoulli form (5.1), with $a = -h_1$, $r = h_3$ and $\alpha = 3$. This differential equation is now solvable with the conventional methods of solving Bernoulli differential equations. The idea to solve the Bernoulli differential equation is to reduce it to a linear differential equation and use the simple methods to solve linear differential equations [4, p. 77]. This will be discussed in more detail in Section 5.2. There also exist methods to solve Bernoulli differential equations in $\mathbb{R}$ and $\mathbb{T}$ without reducing them to a simpler case. Bohner and Peterson introduced in [2] a method to solve a Bernoulli dynamic equation.

Theorem 5.1. *Suppose* $\alpha \in \mathbb{R}$, $y = y(x)$, $\alpha \neq 0$, $p \in \mathcal{R}$, *and* $f \in C_{\mathrm{rd}}$. *Let* $y_0 = y(x_0) \neq 0$ *for an initial value* x_0. *If*

$$\frac{1}{y_0^\alpha} + \int_{x_0}^{x} e_p^\alpha(t, x_0) f(t)\Delta t \quad > 0 \quad \textit{for all } x \in \mathbb{T}, \tag{5.4}$$

then the solution of the Bernoulli dynamic equation is given by [2, p. 38]

$$y(x) = \frac{e_p(x, x_0)}{\left[y_0^{-\alpha} + \int_{x_0}^{x} e_p^\alpha(t, x_0) f(t)\Delta t\right]^{\frac{1}{\alpha}}}. \tag{5.5}$$

The relation between the Abel equation of the first kind and the Bernoulli differential equation is also valid in time scales. Consider the special Abel dynamic equation of the first kind discussed earlier

$$\left(\frac{y + y^\sigma}{2}\right) y^\Delta = h_3(x)(y^\sigma)^2 y^2 + h_{21}(x) y^2 y^\sigma + h_{22}(x)(y^\sigma)^2 y + h_1(x)(y^\sigma)^2. \tag{5.6}$$

Compared to the Abel differential equation in $\mathbb{R}$, Eq. (5.6) is already an Abel dynamic equation of the first kind with $h_0 = 0$, since Eq. (5.6) is in $\mathbb{R}$

$$\begin{aligned} yy' = \left(\frac{y+y^\sigma}{2}\right) y^\Delta &= h_3(y^\sigma)^2y^2 + h_{21}y^2y^\sigma + h_{22}(y^\sigma)^2y + h_1(y^\sigma)^2 \\ &= h_3y^2y^2 + h_{21}y^2y + h_{22}y^2y + h_1y^2, \end{aligned}$$

i.e.,

$$y' = h_3y^3 + (h_{21} + h_{22})y^2 + h_1y.$$

Furthermore the h_2 term has to be zero which means in $\mathbb{T}$ $h_{21} = h_{22} = 0$. The Abel dynamic equation of the first kind is then

$$\left(\frac{y+y^\sigma}{2}\right) y^\Delta = h_3(y^\sigma)^2y^2 + h_1(y^\sigma)^2. \tag{5.7}$$

Equation (5.7) is a Bernoulli dynamic equation on time scales.

Remark 5.2. For $\mathbb{T} = \mathbb{R}$, Eq. (5.7) is equivalent to the Bernoulli differential equation (5.1) since

$$yy' = \left(\frac{y+y^\sigma}{2}\right) y^\Delta = h_3(y^\sigma)^2y^2 + h_1(y^\sigma)^2 = h_3y^4 + h_1y^2,$$

i.e.,

$$y' + (-h_1)y = h_3y^3.$$

The fact that the Bernoulli dynamic equation we achieved in $\mathbb{T}$ is for $\mathbb{T} = \mathbb{R}$ equal to the Bernoulli differential equation which is already known, proves that the definition of the Bernoulli dynamic equation in $\mathbb{T}$ is in the correct form. Note that in this case we examined, $\alpha = 3$.

5.2. TRANSFORMATION TO LINEAR DYNAMIC EQUATIONS

A Bernoulli differential equation can be transferred into a linear differential equation which enables the application of simple methods to solve a Bernoulli equation.

Theorem 5.3. *In $\mathbb{R}$, the Bernoulli differential equation [5, p. 73]*

$$y' + a(x)y = r(x)y^{\alpha}, \tag{5.8}$$

where $\alpha \notin \{0,1\}$ and $y = y(x)$, can be transferred into a linear inhomogeneous differential equation of the form

$$u' + (1-\alpha)a(x)u = (1-\alpha)r(x), \tag{5.9}$$

with $u = u(x)$, by using the substitution $u = y^{1-\alpha}$ [7, p. 19].

The same idea transfers a Bernoulli dynamic equation into a linear equation.

Theorem 5.4. *The special Bernoulli dynamic equation with $\alpha = 3$ which can be derived from the Abel dynamic equation of the first kind*

$$\left(\frac{y + y^{\sigma}}{2}\right) y^{\Delta} = h_3(x)(y^{\sigma})^2 y^2 + h_1(x)(y^{\sigma})^2 \quad \textit{with} \quad y = y(x), \tag{5.10}$$

can be transferred into a linear dynamic equation by applying the substitution

$$u = y^{1-3} = y^{-2}.$$

Remark 5.5. For $\mathbb{T} = \mathbb{R}$, Eq. (5.10) is a Bernoulli equation since

$$yy' = \left(\frac{y + y^\sigma}{2}\right) y^\Delta = h_3 (y^\sigma)^2 y^2 + h_1 (y^\sigma)^2 = h_3 y^4 + h_1 y^2,$$

i.e.,

$$y' = h_3 y^3 + h_1 y.$$

Theorem 5.3 states that this equation can be transferred into a linear equation by using the substitution $u = y^{1-\alpha} = \frac{1}{y^2}$. This leads by Theorem 5.3 to the linear differential equation

$$u' + 2h_1 u = -2h_3 \quad \text{with} \quad a = -h_1, \quad r = h_3. \tag{5.11}$$

Proof of Theorem 5.4. Consider the special Bernoulli dynamic equation (5.10) and use the substitution $u = y^{-2}$, $y(x) \neq 0$. That yields

$$\left(\frac{\frac{1}{\sqrt{u}} + \frac{1}{\sqrt{u^\sigma}}}{2}\right) \left(\frac{1}{\sqrt{u}}\right)^\Delta = h_3 \frac{1}{u^\sigma} \frac{1}{u} + h_1 \frac{1}{u^\sigma}.$$

In Section 2.2, the delta-derivative of $\frac{1}{\sqrt{u}}$ was shown to be $\left(\frac{1}{\sqrt{u}}\right)^\Delta = -\frac{u^\Delta}{(\sqrt{u}+\sqrt{u^\sigma})\sqrt{u^\sigma u}}$. Therefore we have

$$-\left(\frac{\sqrt{u} + \sqrt{u^\sigma}}{2\sqrt{uu^\sigma}}\right) \left(\frac{u^\Delta}{(\sqrt{u} + \sqrt{u^\sigma})\sqrt{u^\sigma u}}\right) = h_3 \frac{1}{u^\sigma} \frac{1}{u} + h_1 \frac{1}{u^\sigma},$$

i.e.,

$$-\frac{u^{\Delta}}{2uu^{\sigma}} = h_3 \frac{1}{u^{\sigma}} \frac{1}{u} + h_1 \frac{1}{u^{\sigma}}.$$

Multiply both sides with $-2uu^{\sigma}$ to obtain

$$u^{\Delta} = -2h_3 - 2h_1 u, \tag{5.12}$$

i.e.,

$$u^{\Delta} + 2h_1 u = -2h_3. \tag{5.13}$$

In [2, p. 17] the general linear dynamic equation is introduced as

$$y^{\Delta} = p(t)y + f(t). \tag{5.14}$$

It is trivial to see that for $y(t) = u(x)$, $p(t) = -2h_1(x)$ and $f(t) = -2h_3(x)$, Eq. (5.14) and Eq. (5.12) are identical. Eq. (5.12) is therefore a common linear dynamic equation in $\mathbb{T}$. □

For $\mathbb{T} = \mathbb{R}$, equation (5.13) is equivalent to Eq. (5.11).

That proves that a Bernoulli dynamic equation can be transferred into a linear dynamic equation, which simplifies the analysis and the solving method. A linear differential equation can be solved in $\mathbb{R}$, as well as in $\mathbb{T}$, by using the variation of constants method.

Theorem 5.6 (Variation of Constants)**.** *[2, p. 19] Suppose* $p \in \mathcal{R}$ *and* $f \in C_{\mathrm{rd}}$*. Let* $x_0 \in \mathbb{T}$ *and* $y_0 \in \mathbb{R}$*. The unique solution of the initial value problem*

$$y^{\Delta} = p(x)y + f(x) \text{ with } \quad y(x_0) = y_0,$$

is given by

$$y(x) = e_p(x, x_0)y_0 + \int_{x_0}^{x} e_p(t, \sigma(t))f(t)\Delta t.$$

5.3. TRANSFORMATION TO THE LOGISTIC DYNAMIC EQUATIONS

In the following, it is described how to transfer a linear dynamic equation into a logistic dynamic equation in $\mathbb{R}$ and $\mathbb{T}$. This will give the foundation to find a substitution to transform a Bernoulli dynamic equation into a logistic dynamic equation, since the transfer from Bernoulli to linear equations is already well known.

Theorem 5.7. *A general linear differential equation in* $\mathbb{R}$ *[4, p. 36], namely*

$$u' = p(x)u + g(x) \text{ with } \quad g(x) \neq 0 \tag{5.15}$$

can be transformed into a logistic differential equation

$$w' = G(x)w(H(x) + w), \tag{5.16}$$

by the substitution $u = \frac{1}{w}$ *and* $w(x) \neq 0$*.*

Proof. Consider the general linear differential equation (5.15) and apply the substitution $u = \frac{1}{w}$, $w(x) \neq 0$. Eq. (5.15) is then

$$-\frac{w'}{w^2} = \frac{p}{w} + g.$$

Multiplying both sides with $-w^2$ yields

$$w' = -pw - gw^2 = -gw\left(\frac{p}{g} + w\right),$$

which is of the form (5.16), with $H = \frac{p}{g}$ and $G = -g$. A logistic differential equation is obtained. □

The same method leads to the formulation of the relation between a linear dynamic equation and a logistic dynamic equation.

Theorem 5.8. *A general linear differential equation in* $\mathbb{T}$*,* $p, g \in \mathcal{R}$ *[2, p. 19]*

$$u^{\Delta} = p(t)u + g(x) \quad \textit{with} \quad g(x) \neq 0, \tag{5.17}$$

can be transformed into a logistic dynamic equation with variable coefficients [2, p. 30]

$$w^{\Delta} = w[\ominus(p(x) + g(x)w)], \tag{5.18}$$

by applying the substitution $u = w^{-1}$ *for* $w(x) \neq 0$.

Proof. Consider the linear dynamic equation (5.17) and apply the substitution $u = \frac{1}{w}$, $w(x) \neq 0$. Eq. (5.17) is then

$$-\frac{w^\Delta}{ww^\sigma} = \frac{p}{w} + g.$$

Multiplying both sides with $-ww^\sigma$ yields

$$w^\Delta = -pw^\sigma - gww^\sigma = -w^\sigma(p + gw).$$

To get the general expression of a logistic dynamic equation (5.18), one uses $w^\sigma = w + \mu w^\Delta$ to get

$$w^\Delta = -w^\sigma(p + gw) = -w(p + gw) - w^\Delta\mu(p + gw),$$

i.e.,

$$w^\Delta(1 + \mu(p + gw)) = -w(p + gw).$$

This is by definition of

$$\ominus(p(x) + g(x)w) = -\frac{(p(x) + g(x)w)}{1 + \mu(p(x) + g(x)w)}$$

equivalent to Eq. (5.18). □

The logistic differential equation is especially found in various mathematical biology applications, such as in population dynamics [17, p. 95]. To describe the development of the population mathematically, the logistic equation is used. Under assumptions in order to make the model simpler, the existence of a natural enemy is implemented by estimations for the birth-rate and death-rate. Furthermore it is assumed that a carrying capacity threshold exists, which is due to the natural fact of limited resources. If a special population is too big, then they restrict themselves. In the following, a population growth model is described in more detail.

Example 5.9. **The population model** [13, p. 702f]

Variables:

$$\begin{aligned} t = &\ \text{time; (usually) measured in years,} \\ P = P(t) = &\ \text{population at time } t, \\ L = &\ \text{carrying capacity,} \\ k = &\ \text{growth proportionality.} \end{aligned}$$

Assumptions:

- The population $P(t)$ is limited naturally by the carrying capacity L.
- The carrying capacity is constant.
- The growth proportionality is constant and includes any natural restrictions of a population, such as birth- and death-rate.
- The population is regarded in a non changing system.

The population is then modeled by the following logistic differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right). \tag{5.19}$$

The substitution to transfer a Bernoulli dynamic equation into a linear dynamic equation, has already been discussed in the previous part. To identify the required substitution from a Bernoulli dynamic equation into a logistic dynamic equation, one has to connect the transformations from Bernoulli to linear equations and then to a logistic equation.

Theorem 5.10. *A Bernoulli dynamic equation in y, given by*

$$y^{\Delta}\left(\frac{y+y^{\sigma}}{2}\right) = h_3(x)y^2(y^{\sigma})^2 + h_1(x)(y^{\sigma})^2 \quad \textit{with} \quad y = y(x), \tag{5.20}$$

can be transferred into a logistic dynamic equation (5.18) *by applying the substitution* $y(x) = \sqrt{w(x)}$.

Proof. In Section 5.2, it was proved that a Bernoulli dynamic equation (5.20) leads to a linear equation in u by applying $y = \frac{1}{\sqrt{u}}$. Moreover, in this section, a linear differential equation in u was transferred into a logistic dynamic equation in w by the substitution of $u = \frac{1}{w}$. To find a transformation from the Bernoulli dynamic equation into a logistic dynamic equation, one connects the two substitutions. That transfers a Bernoulli dynamic equation into a linear equation in $\mathbb{T}$ and afterwards into a logistic dynamic equation. The substitution that realizes the transformation is given by

$$y(x) = \frac{1}{\sqrt{u(x)}} = \frac{1}{\sqrt{\frac{1}{w(x)}}} = \sqrt{w(x)}. \tag{5.21}$$

This can now also be checked directly. □

The presented relations between differential equations in $\mathbb{R}$ remain therefore the same in time scales and can be used to ease the solving method for dynamic equations in $\mathbb{T}$.

6. CONCLUSION

In summary, one can conclude that the Abel dynamic equations of the second kind in time scales are defined similarly to the Abel differential equation of the second kind in the continuous case of $\mathbb{R}$, although they have different expressions, involving u and u^σ. The similarity of the Abel differential and dynamic equations of the second kind enables the translation of the solution methods from $\mathbb{R}$ to $\mathbb{T}$. The strategy of transferring solving methods to time scales is presented in an example in Section 4.1, where a solution of a special class of the Abel equation of the second kind is derived, based on a method used by Bougoffa in [3]. The purpose of this demonstration was to present the idea of how to use existing solution strategies in $\mathbb{R}$ to achieve methods and solutions in time scales. Various methods to solve Abel equations have been already generated [10, p. 45–62] and so form the base for possible solutions in time scales. The relation between the two kinds of an Abel equation has been verified by a substitution, applied to the Abel equation of the second kind. This led to a special class of the Abel equation of the first kind in $\mathbb{T}$. It was first derived in $\mathbb{R}$ and then in $\mathbb{T}$, to find the analogue of a special class of an Abel equation of the first kind in time scales. This relation has critical relevance because it enables the transfer of the solution methods for the Abel equation of the second kind to the first kind.

Analogous to the canonical Abel equation in the continuous time scale, a canonical form was defined in time scales. It was furthermore shown that both kinds of Abel equations, the first and the second kind, can be rewritten in canonical form. Some canonical Abel equations can be solved, depending on the satisfaction of special conditions, which let them refer to different classes of the canonical equation [10, p. 45–50].

Due to the possibility of transforming an Abel equation into canonical Abel form, the classes of solvable Abel equations extended, and additional solving methods can be established.

Finally the focus of Abel equations was extended to common differential equations and their relation to each other has been investigated. Based on the existing research on correlations and connections between the Abel, Bernoulli, logistic, and linear differential equations in $\mathbb{R}$, this was analyzed in the generalized time set $\mathbb{T}$. We realized that the investigated relation between these differential equations can be generalized into time scales. Nevertheless the definition for each of the common differential equations differ from the formula in $\mathbb{R}$ [2]. Throughout this thesis, examples were presented to illustrate the forms of an Abel equation in different time scales. They referred especially to the continuous and discrete cases, to compare the results from the derivation in time scales with the existing formulas for $\mathbb{R}$ and the discrete time scale $\mathbb{Z}$.

The purpose of this thesis was to introduce Abel dynamic equations of the first and the second kind in time scales. This builds the base for further analysis of Abel dynamic equations and the investigation of their general solution. Existing solution methods for Abel equations are useful in generating solutions for the different forms of an Abel dynamic equation and can be translated into time scales in a similar way as in Section 4.1.

Due to the fact that Abel equations are used to model real life situations and problems mathematically, the solution of these equations is critical. This is an additional advantage of the definition of Abel equations in time scales since the applications of the Abel equations in time scales offer a more precise reproduction of the reality.

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