Economical span length ratios for three-span continuous girder bridges

Donald Hindrik Timmer

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ECONOMICAL SPAN LENGTH RATIOS FOR
THREE-SPAN CONTINUOUS GIRDER
BRIDGES

BY
DONALD H. TIMMER

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A
THESIS
submitted to the faculty of the
SCHOOL OF MINES AND METALLURGY OF THE UNIVERSITY OF MISSOURI
in partial fulfillment of the work required for the
Degree of
MASTER OF SCIENCE IN CIVIL ENGINEERING
Rolla, Missouri
1950

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Approved by
E. W. Corsline
Professor of Civil Engineering
ACKNOWLEDGEMENT

The author wishes to express his appreciation to Professor E. W. Carlton, of the Civil Engineering Department, for his valuable advice and criticisms in the preparation of this thesis, and to Jonathan Jones, Chief Engineer of the Bethlehem Steel Company, for his kind suggestions that led to the selection of this thesis problem by the author.
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INTRODUCTION

The three-span continuous bridge is one of the most popular bridges used by highway departments at the present time. The low first cost and overall economy combined with the pleasing appearance of this type of bridge as compared to simple span structures has led most highway departments to use continuous spans whenever conditions permit.

The design of three-span continuous bridges, as well as the design of any of the other bridge types, is usually carried out as an analysis rather than a design. In the case of the three-span continuous girder, with rolled steel beams, an initial assumption as to the most economical span length ratio is made. If the analysis of the assumption shows that the arrangement is uneconomical, another assumption is made and so on until an economical arrangement is found.

The more experience the designer has, the better are his assumptions. In other words, unexperienced designers may have to analyze many situations before they obtain an economical design.

Although there has been much progress in the simplification of the analysis of stresses in indeterminate structures, there has been very little in their actual design.
The purpose of this paper is to present a method for determining the most economical span length ratios in a continuous girder directly and thus eliminate the need for the designer to obtain the same results by successive assumptions.
The Development of the Formulas Commonly used to Solve
the Stresses in Continuous Girders

The methods of determining stresses in indeterminate
structures as applied to continuous girders may be divided
into two groups, the classical methods and the modern
methods.¹

The classical methods are the older methods that
have been accepted and used for a relatively long period
of time. The more important of the classical methods
are the theorem of three moments, the method of least
work, Maxwell’s theorems, the use of influence lines,
and the method of virtual work.

The development of the classical methods of determining
stresses in indeterminate structures began in 1825 with the
theorem of three moments.² In that year Navier (1785-1836),
a French mathematician, published the first notes on
continuous beams. Actually the credit for the theorem of
three moments is given to Clapeyron, another Frenchman,
who in 1857, published the theorem in a usable form.

In a similar manner credit for the method of least
work is not given to Manabrea, an Italian General, who
first developed the method in 1858, but rather to

¹ L. E. Grinter, Theory of Modern Steel Structures,
² Ibid
Castigliano (1847-1884), an Italian Engineer, because of his more satisfactory statement of the least work theorem in 1879.\(^3\)

Another great advance in the development of methods of determining stresses in indeterminate structures was the publication of the theorem of reciprocal deflections developed by Maxwell, the renowned English Physicist, in 1864. Further developments of Maxwell's theorem were made by Mohr, the German Engineer, who applied the theorem to the development of the area moment principles in 1868, and later in 1885 when Müller-Breslau, another German Engineer, used the theorem of reciprocal deflections to develop influence lines.\(^4\)

The last of the important classical methods, the theorem of virtual work, was first conceived in 1833 by Poisson, a French Professor. For some reason Poisson's theorem wasn't published until later in the nineteenth century. In the meantime Fränkel published a portion of this theorem in 1875 in Germany. Nevertheless Poisson is given credit for the theorem of virtual work.\(^5\)

Although the classical methods of determining stresses in indeterminate structures were first developed in Europe

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\(^3\) H. M. Westergaard, One Hundred and Fifty Years Advance in Structural Analysis, ASCE Transactions, Vol. 94, pp 226-240 (1930).

\(^4\) Ibid.

\(^5\) Grinter, op. cit. p 61.
in the middle of the nineteenth century, it wasn't until 1930 that the majority of the engineers in the United States began to accept these theories of indeterminate structures.

The old engineering publications have many articles that show the ignorance and reluctance of the American engineers in using these methods.

Although there were many continuous girders in Europe as early as 1875, there were relatively few in America before 1900. The first continuous girder in the United States was the Scottsville Bridge which was designed by G. Lindenthal in 1880.

An article by F. H. Gilley, a prominent American Engineer, in the Transactions of the American Society of Civil Engineers, reflects the feeling American engineers had for indeterminate structure analysis. The article, written in 1900, states that, "Indeterminate forms such as trusses and continuous girders be abandoned entirely from practice in lieu of determinate structures because of the laboriousness and doubtful accuracy of computations."⁶

These were the dark ages of highway bridge design. Many bridges were designed and constructed in the early part of the twentieth century by incompetent men, and

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⁶ F. H. Gilley, Exact Design of Statically Indeterminate Frameworks, an Exposition of its Possibility, but Futility, ASCE Transactions, Vol. 43, pp 353-407, (1900)
in many cases the bridges were built without designs. Consequently there were many bridge failures and the engineering profession on the whole received a great deal of adverse publicity.

The first reinforced concrete continuous girder bridge of note in the United States, was constructed for an electric railway company in 1914. The bridge was located about six miles from Dallas, Texas, on White Rock Creek. It consisted of two continuous girders, each fifty feet long. The fourteen foot roadway was carried by two concrete girders.

The modern methods of determining stresses were developed for the most part in the United States after 1915. The modern methods most used by engineers in the United States are Slope Deflection, Moment Distribution, Column Analogy, and several of the graphical methods.

The method of Slope Deflection was first introduced in America in 1915 by Professor C. A. Maney, of the University of Minnesota. It was extended to a more usable form by the Danish Professor, A. Ostenfeld, in 1925.7

Undoubtedly the method of Moment Distribution first published by Professor Hardy Cross, of Illinois University in 1929,8 did the most to familiarize the American

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7 Westergaard, op. cit. pp 226-240
8 H. Cross, Analysis of Continuous Frames by Distributing Fixed-end Moments, ASCE Transactions, Vol. 96, pp 1-10, (1932)
Engineer with indeterminate structure analysis. This simple method of analysis consists of locking the joints of an indeterminate structure and releasing them individually and recording their effect on the other joints.

An exact method of moment distribution was introduced by J. A. Wise in 1939. His method gives more accurate results than Hardy Cross's method but the labor involved is considerably increased.

Hardy Cross's second contribution to the analysis of continuous structures, The Column Analogy Method, was published in 1930. In this method the structure is converted and analyzed as a short column with eccentric loadings.

The most commonly used graphical methods of indeterminate structure analysis are the Steinman-Nishkian method and Odd Albert's method. The Steinman-Nishkian method was published in 1927 jointly by the prominent bridge engineer D. B. Steinman and L. H. Nishkian. This method is a graphical analysis using the theorem of three moments and conjugate points. Odd Albert's method was

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published by Odd Albert in 1933 and is based on the graphical analysis of indeterminate structures, using fixed points.

There are many other modern methods of determining stresses in indeterminate structures, some of these will become well known in future years, but most of them will remain relatively unknown. The methods discussed above are the most important methods in use today.

Although most indeterminate structure problems could be solved by any one of the afore mentioned methods, one method isn't superior to the others for all problems. The person that can use three or four of these methods will be best able to tackle a variety of problems.

The most popular methods of determining stresses in continuous girders are shown in Table 1. This table was assembled from reports of thirty-four of the forty-eight state highway bridge departments.
<table>
<thead>
<tr>
<th>METHOD</th>
<th>USED EXCLUSIVELY</th>
<th>USED WITH OTHER METHODS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theorem of Three Moments</td>
<td>0</td>
<td>18%</td>
</tr>
<tr>
<td>Moment Distribution</td>
<td>30%</td>
<td>45%</td>
</tr>
<tr>
<td>Slope Deflection</td>
<td>0</td>
<td>21%</td>
</tr>
<tr>
<td>Influence Lines</td>
<td>6%</td>
<td>6%</td>
</tr>
<tr>
<td>Graphical Methods</td>
<td>3%</td>
<td>9%</td>
</tr>
<tr>
<td>Other Methods*</td>
<td>9%</td>
<td>30%</td>
</tr>
</tbody>
</table>

Table 1. Methods used by the state highway departments to analyze continuous girders. Percent using each method.

*Consists of fifteen individual methods.
Discussion

The development of the method for finding the most economical span length ratio for three-span continuous highway bridges, with rolled steel beams, was accomplished by the writer in the following manner.

1. A series of flexible influence lines were constructed for the three-span continuous girder with a varying span length ratio.

2. Using these flexible influence lines, formulas were evolved for the maximum moments at the critical points in the three-span continuous girder.

3. By varying the span length ratio of the three-span continuous girder and using the formulas found above, a method for finding the most economical span length ratio for any given set of conditions was developed.

In the development of this particular method the following conditions were assumed:

1. The spans had great enough length so that the live load was considered to be equivalent lane loadings rather than truck wheel loadings.

2. The overall length of the girder to be designed was known.

3. The dead load of the superstructure was known.
Influence Lines. In the construction of the flexible influence lines Hardy Cross’s method of moment distribution\(^\text{13}\) was used to determine the moments at the supports due to the unit load moving across the beam.

Figure 1 shows a method devised by the writer for finding the moments at the supports for the general symmetrical three-span continuous girder, due to a unit load in either of the end spans.

Figure 2 shows the same method as used to determine the moments at the supports due to the unit load in the center span.

The moments at the supports due to a unit load at any point "a" in the first span are then:

\[
\begin{align*}
M_1 &= 0 \\
M_2 &= -(F_2 + cF_1) \left( \frac{y - y^2 e^2}{1 - y^2 e^2} \right) \\
M_3 &= (F_2 + cF_1) \left( \frac{y e - y^2}{1 - y^2 e^2} \right) \\
M_4 &= 0
\end{align*}
\]

EQUATIONS I

When the unit load is in the center span, the moments at the supports are:

\[
\begin{align*}
M_1 &= 0 \\
M_2 &= -(F_3 + y e F_4) \left( \frac{1 - y}{1 - y^2 e^2} \right) \\
M_3 &= -(F_4 + y e F_3) \left( \frac{1 - y}{1 - y^2 e^2} \right) \\
M_4 &= 0
\end{align*}
\]

EQUATIONS II

### Carryover Factors

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$x$</td>
<td>$y$</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1st Distribution</td>
<td>$-F_1$</td>
<td>$-F_2$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1st Carryover</td>
<td>$-CF_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Sum</td>
<td>0</td>
<td>$-(F_2+CF_1)y$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

#### Fixed End Moments

**1st Distribution:**
- $-F_1$
- $-F_2$

**1st Carryover:**
- $-CF_1$

**Sum:**
- $0$
- $-(F_2+CF_1)y$

**2nd Distribution:**
- $(F_2+CF_1)y$

**2nd Carryover:**
- $+(F_2+CF_1)y^2e$

**3rd Distribution:**
- $-(F_2+CF_1)y^2e$

**3rd Carryover:**
- $+(F_2+CF_1)y^3e$

**4th Distribution:**
- $-(F_2+CF_1)y^3e$

**4th Carryover:**
- $+(F_2+CF_1)y^4e$

**5th Distribution:**
- $-(F_2+CF_1)y^4e$

**5th Carryover:**
- $+(F_2+CF_1)y^5e$

**nth Distribution:**
- $-(F_2+CF_1)y^ne$

**nth Carryover:**
- $+(F_2+CF_1)y^{n+1}e$

### Calculations

- $M_{R_2} = (F_2 + CF_1) [-y + y^2e^2 - y^3e^2 + y^4e^4 + \ldots + y^{n-1}e^{n-2}]$
- $M_{R_3} = (F_2 + CF_1) [(y + y^2e^2 - y^3e^3 + y^4e^3 + \ldots + y^{n-1}e^{n-2})]$

In both cases, the $y, e$ coefficients comprise two geometric series, since both $y$ and $e < 1$.

The sum of a geometric series, $\{a + ar + ar^2 + \ldots + ar^{n-1}\}$, as $n$ approaches infinity is $S_n = \frac{a(1-r^n)}{1-r}$.

The exact moments at the supports are then:

- $M_{R_1} = M_{R_4} = 0$
- $M_{R_2} = (F_2 + CF_1) \left[ \frac{y - y^2e^2}{1 - y^2e^2} \right]$
- $M_{R_3} = (F_2 + CF_1) \left[ \frac{y e - y^2e}{1 - y^2e} \right]$

**Fig. 1** - Moments at the supports due to a unit load in the end span.
\[ M_{k2} = F_3 \left[ -1 + y - y^e e^2 + y^2 e^3 \right] + F_4 \left[ -y e + y^e e - y^2 e^2 + y^3 e^3 \right] \]
\[ = F_3 \frac{(-1 + y)}{1 - y^e e^2} \quad + \quad F_4 \frac{(-y e + y^e e)}{1 - y^2 e^2} \]

\[ M_{k3} = F_4 \left[ -1 + y - y^e e^2 + y^2 e^3 \right] + F_3 \left[ -y e + y^e e - y^2 e^2 + y^3 e^3 \right] \]
\[ = F_4 \frac{(-1 + y)}{1 - y^e e^2} \quad + \quad F_3 \frac{(-y e + y^e e)}{1 - y^2 e^2} \]

\[ M_{k4} = 0 \]
\[ M_{k5} = - \left[ F_3 + y e F_4 \right] \frac{(1-y)}{1 - y^e e^2} \]
\[ M_{k6} = - \left[ F_4 + y e F_3 \right] \frac{(1-y)}{1 - y^2 e^2} \]

\[ M_{k7} = 0 \]

**Fig. 2** - Moments at the supports due to a unit load in the center span.
Where:

\[ F_1 \] - Fixed end moment on the left side of the beam of the first span due to a unit load in the first span "a" feet from \( R_1 \).

\[ F_2 \] - Fixed end moment on the right side of the beam.

\[ F_3 \] - Fixed end moment on the left side of the beam of the center span due to a unit load in the center span "b" feet from \( R_2 \).

\[ F_4 \] - Fixed end moment on the right side of the beam.

\( L \) - Length of either end span.

\( nL \) - Length of the center span.

(c) - Carry over factor of the outside edge of the end beams.

(d) - Carry over factor of the inside edge of the end beams.

(e) - Carry over factor of the edges of the center beam.

(x) - Distribution factor of the end beams.

(y) - Distribution factor of the center beam.

The moments at the supports due to a unit load in the last span are the reverse of those due to a load in the first span because of symmetry.

In the case of the three-span continuous girder with rolled steel beams, where the moment of inertia is constant throughout the entire beam, the coefficients of Equations I and II were found to have the following values.
Fixed-end Moments.

\[ F_1 = \frac{(1)a(L-a)^2}{L^2} \]
\[ F_2 = \frac{(1)a^2(L-a)}{L^2} \]
\[ F_3 = \frac{(1)(b)(nL-b)^2}{(nL)^2} \]
\[ F_4 = \frac{(1)b^2(nL-b)}{(nL)^2} \]

Carry-over Factors.

\[ c = a = e = 1/2 \]

Distribution Factors.

The distribution factors are proportional to the stiffness \((I/L)\) of the beams\(^\text{15}\) hence; \(I/L\), \(I/nL\), \(I/L\) for the three spans, if each of the beams was considered to be restrained at both ends. However it was more convenient to consider the free ends of the end spans as such, and so their stiffness factors had to be modified.

The relative stiffness factors became, \(\frac{2}{3} I/L\), \(I/nL\), and \(\frac{3}{4} I/L\). By multiplying through by \(4nL/I\) and letting \(3n + 4 = g\) the following relative stiffness factors and their corresponding distribution factors were found.

\[ \begin{array}{cccc}
3n & 4 & 3n \\
\frac{3n}{9} & \frac{4}{9} & \frac{3n}{9}
\end{array} \]

\[ R_1 \quad R_2 \quad R_3 \quad R_4 \]


\(^{15}\) *Ibid.* p 98.
therefore, \( x = \frac{3n}{g} \), \( y = \frac{4}{g} \).

Equations I and II with the coefficients found above have the following value.

\[
\begin{align*}
M_1 &= 0 \\
M_2 &= \frac{-(2(ax^2-a^3))}{L^2} \frac{(g-1)}{(g^2-4)} \\
M_3 &= \frac{(ax^2-a^3)}{L^2} \frac{(g-1)}{(g^2-4)} \\
M_4 &= 0
\end{align*}
\]

**EQUATIONS Ia**

\[
\begin{align*}
M_1 &= 0 \\
M_2 &= \frac{-(bg(nL-b)^2 + b^2(nL-b))}{(nL)^2} \frac{(g-4)}{(g^2-4)} \\
M_3 &= \frac{-(b^2g(nL-b) + 2b(nL-b)^2)}{(nL)^2} \frac{(g-4)}{(g^2-4)} \\
M_4 &= 0
\end{align*}
\]

**EQUATIONS IIa**

The moments at the supports were then calculated from Equations Ia and IIa as the unit load passed each tenth point of the spans as it moved across the girder. The values are shown in Table 2. From these influence lines all other influence lines for the three-splay girder can be calculated.

The moment diagrams in Figure 3 were made in order to help determine the critical points of the three-span continuous girder. These moment diagrams are of the girder loaded with a uniform load over its entire length.

From these figures it was observed that the maximum negative moment would occur at one of the supports and
<table>
<thead>
<tr>
<th>$R_1$</th>
<th>$M_{R_2}$</th>
<th>$R_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1L</td>
<td>$-0.198 \frac{L}{g^2} \frac{(g-1)}{(g^2-4)}$</td>
<td>0.1L</td>
</tr>
<tr>
<td>0.2L</td>
<td>$-0.384$</td>
<td>0.2L</td>
</tr>
<tr>
<td>0.3L</td>
<td>$-0.546$</td>
<td>0.3L</td>
</tr>
<tr>
<td>0.4L</td>
<td>$-0.672$</td>
<td>0.4L</td>
</tr>
<tr>
<td>0.5L</td>
<td>$-0.750$</td>
<td>0.5L</td>
</tr>
<tr>
<td>0.6L</td>
<td>$-0.768$</td>
<td>0.6L</td>
</tr>
<tr>
<td>0.7L</td>
<td>$-0.714$</td>
<td>0.7L</td>
</tr>
<tr>
<td>0.8L</td>
<td>$-0.575$</td>
<td>0.8L</td>
</tr>
<tr>
<td>0.9L</td>
<td>$-0.342$</td>
<td>0.9L</td>
</tr>
<tr>
<td>$R_2$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.1nL</td>
<td>$-(0.081g+0.018)nL\frac{(g-4)}{(g^2-4)}$</td>
<td>0.9nL</td>
</tr>
<tr>
<td>0.2nL</td>
<td>$-(0.126g+0.064)$</td>
<td>0.8nL</td>
</tr>
<tr>
<td>0.3nL</td>
<td>$-(0.147g+0.126)$</td>
<td>0.7nL</td>
</tr>
<tr>
<td>0.4nL</td>
<td>$-(0.144g+0.192)$</td>
<td>0.6nL</td>
</tr>
<tr>
<td>0.5nL</td>
<td>$-(0.125g+0.250)$</td>
<td>0.5nL</td>
</tr>
<tr>
<td>$M_{R_3}$</td>
<td>$M_{R_2}$</td>
<td></td>
</tr>
</tbody>
</table>

Table 2. Influence Lines for the Moments at the Supports

Note: For points on the left of the center line the table is read from the top and left side. For points on the right of the center line the table is read from the bottom and right side.
Figure 3: Moment Diagrams of Three-Span Continuous Girders with Various Span Length Ratios.

Note: uniform load $10^5$/ft on each girder. Overall length of each girder - 30 feet.
the maximum positive moment could occur either in the vicinity of the 4/10 point of the end span or at the center of the middle span. (The 4/10 point in the end span will be considered the point of maximum positive moment of that span in compliance with the AISC Steel Construction Manual. 16)

The influence lines for the maximum moments at the supports are the influence lines for the maximum negative moments. The influence lines for the maximum positive moments were constructed from these influence lines.

\[ M_{0.4L} \text{ with unit load at } 0.1L \]

\[ M_{0.4L} = R_2(0.6L) - M_{R_2} \]

\[ = ((0.1L)(1) + M_{R_2})(0.6L) - M_{R_2} \]

\[ = 0.06L - 0.4M_{R_2} \]

\[ = 0.06L - 0.0792L \frac{(g-1)}{(g^2-4)} \]

The other values of the critical point influence lines were found in a similar manner and recorded in Table 3, along with the influence lines of other important points of the three-span continuous girder.

<table>
<thead>
<tr>
<th>Table 3 - Flexible Influence Lines for 3-Span Continuous Girders</th>
</tr>
</thead>
<tbody>
<tr>
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</tbody>
</table>
Critical Moment Equations. The loads on the girder that produce maximum moments at the critical points consist of the dead load \((W)\) and the live loads \((AW)\) and \((PW)\).

\(W\) - Portion of the weight of the superstructure carried by the girder, and the weight of the girder.

\(AW\) - Portion of the AASHO uniform lane load carried by the beam.

\(PW\) - Portion of the AASHO concentrated load carried by the beam.

The AASHO loadings are placed so as to produce the maximum moment at the critical point in question.\(^{17}\)

Maximum Moment at 0.4L.

The loading arrangement to produce the positive maximum moment in the first span was found from the influence lines in table 3.

The sum of the influence line ordinates of each span are:

Span \#1 \(1.2L - 1.98L \frac{(g-1)}{(g^2-4)}\)

Span #2 \(-0.33 + 0.66\) nL \(\frac{(g-4)}{(g^2-4)}\)

Span #3 \(0.99L \frac{(g-4)}{(g^2-4)}\)

The influence line ordinates at the point of application of the concentrated loads are:

- at 0.4L \(0.24L - 0.2686L \frac{(g-1)}{(g^2-4)}\)
- at 0.6L' \(0.1536L \frac{(g-4)}{(g^2-4)}\)

These equations were then simplified.

Span #1 \((10.8n^2L + 22.86nL + 8.46L) x \frac{1}{(g^2-4)}\)

Span #2 \(-2.97n^3L + 5.94n^2L) x \frac{1}{(g^2-4)}\)

Span #3 \((2.97nL) x \frac{1}{(g^2-4)}\)

- at 0.4L \(-2.16n^2 + 4.9536n + 2.0736) L x \frac{1}{(g^2-4)}\)
- at 0.6L' \((0.4608n)L x \frac{1}{(g^2-4)}\)

Moment at 0.4L due to \((W)\):

\[M_{0.4L} = \frac{W(0.1)L}{(g^2-4)} (-2.97n^4L - 5.94n^3L + 10.8n^2L + 25.83nL + 8.46L)\]

Moment at 0.4L due to \((AW)\):

\[M_{0.4L} = \frac{W(0.1)L}{(g^2-4)} (10.8An^2L + 25.83AnL + 8.46AL)\]

Moment at 0.4L due to \((FW)\):

\[M_{0.4L} = \frac{W(0.1)L}{(g^2-4)} (21.6n^2P + 54.144nP + 80.736P)\]

The maximum moment at 0.4L is the sum of these three moment equations.
Max. \( M_{0.4L} = \frac{W(0.1)L}{(g^2-4)} \times \left(\frac{-2.97n^4L - 5.94n^5L}{g^2-4}\right) + n^2(10.8L + 21.6A) + n(25.83L(1+A) + 54.144A) + 8.46L(1+A) + 20.736P) \)

When this equation is used, the unknown factors are \( n \) and \( L \). The unknown \( L \) was eliminated by substituting the function of the overall length \( (S) \) for \( L \).

\[
S = 2L + nL
\]

\[
L = \frac{S}{2+n}
\]

With this substitution the maximum \( M_{0.4L} \) became:

Max. \( M_{0.4L} = \frac{W(0.1)S}{(g^2-4)[n+2]^2} \times \left(\frac{-2.97n^4S - 5.94n^5S}{g^2-4}\right) + n^2(10.83(1+A) + 97.344A) + n(25.83S(1+A) + 129.024A) + 8.46S(1+A) + 41.472P) \)

where the only unknown is \( n \).

**Maximum Moment at \( R_2 \)**

The loading arrangement to produce the maximum negative moment at the intermediate support \( R_2 \), was found from the influence lines in table 2.

The maximum moment equation at the intermediate support, \( (R_2) \) was developed from the influence line coefficients of
table 3, in a manner similar to the development of the maximum moment equation at 0.4L.

The maximum moment equation at R2 was found to be:

\[ \text{Max. } M_{R2} = \frac{-5W(0.1)3}{(g^2-4)(n+2)^2} \left( n^4(7.4253(1+A) + 12.96P) \right. \]

\[ + \left. n^3(14.853(1+A) + 48.96P) + n^2(69.12P) \right) \]

\[ + n(7.4253(1+2A) + 69.12P) + 14.853(1+A) \]

\[ + 46.08P \]

**Maximum moment at 0.5nL.**

The loading arrangement to produce the maximum moment at the center of the middle span, (0.5nL) was found from the influence lines in table 3.

![Diagram of moment equation](image)

The maximum moment equation at 0.5nL was then found in a manner similar to that of the development of the maximum moment equation at 0.4L.

\[ \text{Max. } M_{0.5nL} = \frac{W(0.1)3}{(g^2-4)(n+2)^2} \left( n^4(3.853(1+1) + 11.25P) \right. \]

\[ + \left. n^3(15.153(1+A) + 26.25P) + n^2(153(1+A) \right) \]

\[ + 10.5P) + n(-7.4253 + 6P) - 14.853) \]

The equations of the maximum moments at the critical points are reproduced in table 4 for the convenience of the reader.
\[ \text{Max. } M_{o.4L} = \frac{W(0.1)S}{(g^2-4)(n+2)} \left[ -2.975n^4 + (-5.945 + 21.6P) n^3 + (10.85S(1+A) + 97.344P)n^2 \\
+ (25.83S(1+A) + 129.024P)n + 8.465(1+A) + 41.472P \right] \]

\[ \text{Max. } M_{k_2} = -\frac{W(0.1)S}{(g^2-4)(n+2)^2} \left[ (7.425S(1+A) + 12.96P)n^3 + (14.85S(1+A) + 48.96P)n^2 \\
+ 69.12Pn^2 + (7.425S(1+A) + 69.12P)n + 14.85S(1+A) + 46.08P \right] \]

\[ \text{Max. } M_{o.5nL} = \frac{W(0.1)S}{(g^2-4)(n+2)^2} \left[ (3.85S(1+A) + 11.25P)n^4 + (15.15S(1+A) + 26.25P)n^3 \\
+ (15S(1+A) + 6.5P)n^2 + (-7.425S + 6P)n - 14.85S \right] \]

Table 4 - Equations of Maximum Moment at the Critical Points.
The maximum moment equations were then assigned logical coefficients in order to determine the method with which to apply these equations to find the most economical span length ratio of the three-span continuous girder bridge, with rolled steel beams.

The following design conditions were assumed to determine the logical length and loading coefficients.

**Overall Length.**

Overall lengths of 200 and 300 feet were used, as the range of the overall length for long three-span continuous girder bridges is usually between these values.

**Dead Load.**

The dead load coefficients were computed from typical cross sections of girder bridges with 28 foot roadways recommended by the Missouri State Highway Commission and shown in figure 4.

The center girder of the cross section in figure 4a using five carrying girders was selected to be analysed.

The dead load of the superstructure carried by the center girder is 690#/ft.

The girder is assumed to be 36" WF 170# for the overall length of 200 feet.

\[ w = 690#/ft. + 170#/ft. = 860#/ft. \]

**Live Load.**

The live load is assumed to be the AASHO H20-316-44 truck lane loading.
Figure 4 - Typical Cross sections of Girder Bridges with 28' Roadways.
The impact fraction is 30% (maximum).

The portion of the truck lane loading carried by the center girder was computed with AASHO bridge design specifications.

\[ \frac{G}{5.0} = \frac{6.75}{5.0} = 1.35 \]

where \( G \) is the average spacing of the interior girders, in feet.

Uniform live load.

\[ AW = \text{lane load per wheel} \times \text{impact} \times \frac{G}{5.0} \]

\[ 640 \times 1.3 \times 1.35 = 561.6\#/\text{ft}. \]

Concentrated live load.

\[ PW = \frac{18,000 \times 1.3 \times 1.35}{2} = 15,795\# \]

\[ A = \frac{AW}{W} = \frac{561.6}{860} = 0.654 \]

\[ P = \frac{PW}{W} = \frac{15,795}{860} = 18.35 \]

The coefficients used in the maximum moment equations when the overall length is 200 feet are:

\[ S = 200 \]

\[ P = 18.35 \]

\[ A = 0.654 \]

\[ W = 840\#/\text{ft} \]

These coefficients were then substituted into the maximum moment equations:

\[ \text{Max. } M_{0.4L} = \frac{860(0.1)S}{(g^2-4)(n+1)^2} \left( -594n^4 - 792n^3 + 5355n^2 ight) \]

\[ + 10,920n + 3560 \]
\[
\text{Max. } M_{R2} = -860(0.1)S \left( \frac{2698n^4 + 5817n^3 + 1270n^2}{(g^2-4)(n+2)^2} \right) + 4700n + 5765 \\
\text{Max. } M_{0.5NL} = \frac{860(0.1)S}{(g^2-4)(n+2)^2} \left( 1461n^4 + 5502n^3 + 5163n^2 \\
- 1375n - 2970 \right)
\]

By allowing \( n \) to vary from 0.8 to 2.0 in increments of 0.2, the values of the maximum moments at these values of \( n \) were found and recorded in table 5. These values of maximum moment were then plotted graphically in figure 5.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( M_{0.4L} )</th>
<th>( -M_{R2} )</th>
<th>( M_{0.5NL} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>928,000</td>
<td>860,000</td>
<td>331,000</td>
</tr>
<tr>
<td>1.0</td>
<td>785,000</td>
<td>669,000</td>
<td>481,000</td>
</tr>
<tr>
<td>1.2</td>
<td>680,000</td>
<td>502,000</td>
<td>491,000</td>
</tr>
<tr>
<td>1.4</td>
<td>585,000</td>
<td>966,000</td>
<td>611,000</td>
</tr>
<tr>
<td>1.6</td>
<td>489,000</td>
<td>1,028,000</td>
<td>688,000</td>
</tr>
<tr>
<td>1.8</td>
<td>420,000</td>
<td>1,140,000</td>
<td>832,000</td>
</tr>
<tr>
<td>2.0</td>
<td>346,500</td>
<td>1,230,000</td>
<td>926,000</td>
</tr>
</tbody>
</table>

When the overall length of the three-span continuous bridge with rolled steel beams was assumed to be 300 feet the following load coefficients were used.

**Dead Load.**

Use 36" WF 280'
Figure 5 - Maximum Moments at the Critical Points \( S = 200' \).
\[ W = 690\#/\text{ft.} + 280\#/\text{ft.} = 970\#/\text{ft.} \]

**Live Load.**

\[ PW = 15,795 \]

\[ AW = 561.6 \]

\[ F = \frac{PW}{W} = \frac{15,795}{970} = 16.28 \]

\[ A = \frac{AW}{W} = \frac{561.6}{970} = 0.58 \]

With these coefficients the maximum moment equations have the following values.

Max. \[ M_{0.4L} = \frac{970(0.1)3}{(g^2-4)(n+2)^2} \left( -89ln^4 + 1430n^3 + 5692n^2 + 14,350n + 4685 \right) \]

Max. \[ M_{R_2} = \frac{-970(0.1)3}{(g^2-4)(n+2)^2} \left( 373ln^4 + 7835n^3 + 1026n^2 + 5836n + 7889 \right) \]

Max. \[ M_{0.5nL} = \frac{970(0.1)3}{(g^2-4)(n+2)^2} \left( 2008n^4 + 7616n^3 + 7286n^2 + 2130n + 4455 \right) \]

The values of the maximum moments were recorded in table 6 as \( n \) increased from 0.8 to 2.0 in increments of 0.2. These values of the maximum moments were plotted graphically in figure 6.

The following information was obtained from the characteristics of the maximum moment curves in figures 5 and 6.

1. As \( n \) increases from 0.8 to 2.0 the values of the maximum moments at \( R_2 \) and 0.5nL increase and the value of Max. \( M_{0.4L} \) decreases.
Figure 6 - Maximum Moments at the Critical Points ... S = 300'}
### Table 6: Maximum Moment Values  \( S = 300 \)

<table>
<thead>
<tr>
<th>n</th>
<th>( M_{0.4L} )</th>
<th>(-M_{R2})</th>
<th>( M_{0.5nL} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>1,934,800(^{1/2})</td>
<td>1,881,100(^{1/2})</td>
<td>-----</td>
</tr>
<tr>
<td>1.0</td>
<td>1,685,000</td>
<td>1,890,000</td>
<td>742,000(^{1/2})</td>
</tr>
<tr>
<td>1.2</td>
<td>1,440,000</td>
<td>1,990,000</td>
<td>1,110,000</td>
</tr>
<tr>
<td>1.4</td>
<td>1,215,000</td>
<td>2,140,000</td>
<td>1,410,000</td>
</tr>
<tr>
<td>1.6</td>
<td>1,006,000</td>
<td>2,320,000</td>
<td>1,680,000</td>
</tr>
<tr>
<td>1.8</td>
<td>825,000</td>
<td>2,545,000</td>
<td>1,930,000</td>
</tr>
<tr>
<td>2.0</td>
<td>654,000</td>
<td>2,760,000</td>
<td>2,152,000</td>
</tr>
</tbody>
</table>

2. The value of \( n \) where the ordinate of the largest maximum moment is minimum, is the abscissa of \( n \) where the curves of the maximum moment at \( R_2 \) and 0.4L intersect. At this point the values of Max \( M_{0.4L} \) and Max. \( M_{R2} \) control the size of the carrying girder. For all values of \( n \) smaller than this the Max. \( M_{0.4L} \) alone controls the size of the carrying girder, and for all values of \( n \) larger than that at the point of intersection, Max. \( M_{R2} \) controls the size of the carrying girder.

3. The Max. \( M_{0.5nL} \) is never great enough to control the size of the carrying girder.

4. As the overall length increases the abscissa of \( n \) at the intersection of the maximum moment curves at \( R_2 \) and 0.4L decreases.
5. The characteristics of the maximum moment curves in figures 5 and 6 are similar to each other.

Using the maximum moment equations and the information obtained from figures 5 and 6, the following method was developed for finding the most economical span length ratio for three-span continuous girder bridges, with rolled steel beams.

The method consists of applying the values of (S), (W), (A), and (P) to the maximum moment equations at $R_2$ and 0.4$L$: setting these equations equal to each other, and solving the resulting equation for $n$.

The following example problem will serve to demonstrate this method.
Method for Finding the Most Economical Span Length Ratio for Long Three-span Continuous Girders, with Rolled Steel Beams.

Problem: To find the span length ratio that will enable the smallest size beam possible, be used as the interior carrying girder* for the three-span continuous girder bridge, with rolled steel beams, with the following conditions:

1. Dead Load - computed from cross section of girder bridge with 28' roadway, using four carrying girders, as recommended by the Missouri State Highway Commission. (Fig. 4b)

2. Overall length (S) = 240 feet.

3. Live Load = H2o = 316 = 44 truck lane loading.

4. Impact = 30% (Maximum)

*The interior girder carries the greatest combined live load and dead load and consequently is subjected to the greatest maximum bending moments.

Dead Load:

The weight of the superstructure carried by the interior girder, as shown in figure 4b is 716#/ft. A 36" WF 260#/ beam was assumed for the carrying girder.

\[ W = 716\text#/ft. + 260\text#/ft. = 976\text#/ft. \]

Live Load:

\[ AW = \text{Uniform lane load per wheel} \times \text{impact} \times \frac{g}{5.0} \]

\[ = \frac{640}{2} \times 1.3 \times \frac{8.75}{5.0} \]

\[ = 720\text#/ft. \]
\[ PW = \frac{18,000 \times 1.3 \times 8.75}{5.0} = 20,475 \]

\[ A = \frac{AW}{W} = \frac{728}{976} = 0.745 \]

\[ P = \frac{PW}{W} = \frac{20,475}{976} = 20.95 \]

The coefficients are:

\[ W = 976 \]
\[ L = 240 \]
\[ P = 20.95 \]
\[ A = 0.745 \]

These coefficients were then substituted with the maximum moment equations at 0.4L and R₂.

\[ \text{Max. } M_{0.4L} = 976(0.1)240 \left( \frac{-713n^4 - 974n^3 + 6560n^2}{(g^2-4)(n+2)^2} \right) + 13,525n + 4410 \]

\[ \text{Max. } M_{R₂} = -976(0.1)240 \left( \frac{3382n^4 + 7245n^3 + 1449n^2}{(g^2-4)(n+2)^2} \right) + 5889n + 7185 \]

By first setting \[ \text{Max. } M_{0.4L} = -\text{Max. } M_{R₂} \], then cancelling \[ \frac{976(0.1)240}{(g^2-4)(n+2)^2} \] from both sides of the equal sign, and subtracting the coefficients of the \( M_{0.4L} \) equation from the \( (-\text{Max. } M_{R₂}) \) equation the final equation in terms of \( n \) was obtained.

\[ 4095n^4 + 8219n^3 - 5111n^2 - 7636n + 2775 = 0 \]

Rather than employ one of the difficult solutions of power equations to find \( n \), the author chose to find \( n \) by the simple trial and error method.

From the study of the characteristics of the maximum moment equations in figures 5 and 6, the author expected to
find n to have a value close to 0.9

The first assumption of n was chosen as 0.9

when n = 0.9

\[ \frac{2688}{6000} - \frac{4140}{6860} + \frac{2775}{11,463} - \frac{11,000}{11,000} + 463 \]

From the above substitution the desired value of n is observed to be very nearly 0.9. The second assumption for n was taken as 0.87 in order for the valuation of the equation to more nearly approach zero.

then with n = 0.87, and again substituting we get:

\[ \frac{2344}{5400} - \frac{3860}{6640} + \frac{2775}{10,519} - \frac{10,500}{10,500} + 19 \]

The equation is now practically balanced, therefore for practical design analysis a value of n = 0.87 may be used.

when n = 0.87

\[ L = \frac{240}{2.57} = 83.6^\circ \]

\[ g = \frac{3n}{4} = 6.61 \]

\[ nL = 83.6(0.87) = 72.8^\circ \]

The maximum moments at the critical points are:

\[ \text{Max. } M_{Rg} = \frac{-976(0.1)240}{39.7} \left( \frac{3382(0.573)}{2.57} \right) + 7245(0.858) \]

\[ + 1449(0.756) + 5889(0.87) + 7185 \]

\[ = -71.7(20,098) = 1,440,000' \]
Max. $M_0 = 71.7 \left(-713(0.573) - 974(0.658) + 6560(0.756) \right) + 13,525(0.87) + 4410$

$= 71.7(20,100) = 1,440,000$\footnote{1,440,000}

Size of beam assumed for interior girder is a: 36" WF 260\# with $I = 17,233.8 \text{ in}^4$

and the size of beam needed to resist maximum moments is found by the flexure formula, $S = \frac{Mc}{I}$, solving for moment of inertia,

$I = \frac{Mc}{S} = \frac{1,440,000'}{17,233.8' \times 12/\text{ft} \times 18''} = 17,280 \text{ in}^4$

The value of the required moment of inertia is slightly greater than that supplied, according to the flexure formula. However, there will be no appreciable change in the safety factor, and the 36" WF 260\# beam may be used.
Summary and Conclusions

There is a need, at the present time, to simplify the methods of designing three-span continuous girder bridges, with rolled steel beams. The present methods require that a series of analyses of assumed span length ratios be made until, by trial and error, an economical ratio is found.

The method set forth in this thesis eliminates much of this trial and error analysis, the size of the carrying girder being the only assumed factor. The inexperienced designer may obtain the most economical span length ratio for a three-span continuous girder, with rolled steel beams, by successive solutions of this method, usually two substitutions will be sufficient to obtain the desired accuracy.

The method presented in this thesis assumes that the overall length of the bridge be known. With slight changes in the maximum moment equations in table 4, these equations could be altered so that the length of the middle span be known and the lengths of the end spans the unknown factors. The same procedure would then be followed to find the most economical span length ratio.

A common span length ratio used by highway departments, for long three-span continuous girders, with rolled steel beams is 1 : 1.25 : 1. With this ratio the maximum positive moments at 0.4L and 0.5nL are almost equal to
each other, but the maximum negative moment at $R_2$ is much larger. In this case the maximum moment that controls the size of the carrying girder is $M_{R_2}$.

The author suggests a ratio of $1 : 0.85 : 1$ be used as the common span length ratio for long three-span continuous girders. With this ratio the maximum moments at $R_2$ and $0.4L$ are almost equal to each other, the ratio of the maximum moment at $0.5L$ being much smaller. The maximum moment that controls the span length ratio of $1 : 0.85 : 1$ is considerably smaller than that for a span length ratio of $1 : 1.25 : 1$. Consequently the span length ratio of $1 : 0.85 : 1$ allows the smallest size carrying girder to be used.

This method was devised for long three-span continuous girders with rolled beams. Similar methods should be developed to obtain a greater degree of economy in three-span continuous girders. One low cost method would be to lower the elevation of the intermediate supports, to obtain equal maximum moments at $0.4L$, $R_2$, and $0.5nL$. 
BIBLIOGRAPHY


Vita

Donald H. Timmer was born on October 16, 1925 at New York City, New York. His parents are Geesje and Jan Timmer.

He received his primary education in the public schools of New York City.

Soon after graduation from high school in January of 1944, Mr. Timmer entered the United States Army. In 1945 while stationed at Fort Leonard Wood, Missouri he married Imogene, the daughter of Kirby and Georgia Hart of Houston, Missouri.

Mr. Timmer entered Indiana Technical College, at Ft. Wayne, Indiana, in June 1946, one month after his discharge from the army. He studied Civil Engineering there until June 1948, when he transferred to the Missouri School of Mines.

His daughter Linda Cheryl was born at Rolla in December 1948.

He graduated in August 1949, with a B. S. degree in Civil Engineering. He entered the graduate school in September 1949.