1950

Analytical investigation of bending effects on the stability of a tube subjected to external pressure

Charles Young

Follow this and additional works at: http://scholarsmine.mst.edu/masters_theses

Department: Mechanical and Aerospace Engineering

Recommended Citation

ANALYTICAL INVESTIGATION OF BENDING EFFECTS ON
THE STABILITY OF A TUBE SUBJECT TO EXTERNAL PRESSURE

BY
CHARLES YOUNG JR.

A
THESIS
submitted to the faculty of the
SCHOOL OF MINES AND METALLURGY OF THE UNIVERSITY OF MISSOURI
in partial fulfillment of the work required for the
Degree of
MASTER OF SCIENCE IN MECHANICAL ENGINEERING
Rolla, Missouri
1950

Approved by
Professor of Mechanical Engineering
ACKNOWLEDGEMENT

The author is grateful to Professor A. J. Miles under whose guidance this work was done. He also wishes to thank him for his reading of the manuscript.

The author is also thankful to Assistant Professor N. Costakos for his valuable assistance.
## CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acknowledgements</td>
<td>ii</td>
</tr>
<tr>
<td>List of illustrations</td>
<td>iii</td>
</tr>
<tr>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>Review of literature</td>
<td>3</td>
</tr>
<tr>
<td>Notations</td>
<td>5</td>
</tr>
<tr>
<td>Part I: The case of deflection of a tube in bending</td>
<td>7</td>
</tr>
<tr>
<td>Part II: The effect of the deflected middle surface on the critical external pressure of a bent tube</td>
<td>27</td>
</tr>
<tr>
<td>Conclusions</td>
<td>36</td>
</tr>
<tr>
<td>Summary</td>
<td>38</td>
</tr>
<tr>
<td>Bibliography</td>
<td>39</td>
</tr>
<tr>
<td>Vita</td>
<td>41</td>
</tr>
</tbody>
</table>
# LIST OF ILLUSTRATIONS

<table>
<thead>
<tr>
<th>Fig.</th>
<th>Illustration Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>View of tube length before deflection, showing moment direction</td>
<td>8</td>
</tr>
<tr>
<td>2.</td>
<td>Quarter section of tube, showing stresses on infinitesimal element</td>
<td>10</td>
</tr>
<tr>
<td>3.</td>
<td>Element of tube, showing oblique plane</td>
<td>12</td>
</tr>
<tr>
<td>4.</td>
<td>Cross section of tube, showing infinitesimal strip</td>
<td>14</td>
</tr>
<tr>
<td>5.</td>
<td>Deflected middle surface of tube subjected to pure bending</td>
<td>25</td>
</tr>
<tr>
<td>6.</td>
<td>Middle surface of bent tube split down the Z-axis, with end reactions and external pressure considered</td>
<td>28</td>
</tr>
<tr>
<td>7.</td>
<td>Tube subjected to a terminal transverse load</td>
<td>34</td>
</tr>
</tbody>
</table>
The behavior of thin and moderately thick-walled tubes under various types of loadings has been a major problem of several leading industries for many years. Predominant among these, and the one for which this thesis was primarily undertaken, is the oil industry. The problem attacked in this thesis, however, is not limited to that industry alone, having ample application in industries that are sub-marine in nature, or any industry in which the application of external pressures on equipment is likely to be responsible for buckling failures.

In the early days of the oil industry when operating pressures were moderate, failures were not common. The tubing invariably was much stronger than it had to be to withstand the low pressures, and failures were usually attributed to poor materials or, if failure occurred after a time, to fatigue. Combined stresses due to other types of loadings, such as axial tensile loads due to the weight of the casing, or bending, due to a non-homogeneous strata in the well hole, were not too important since the wells were not deep. However, as the depth of the wells became greater and the pressures became higher, the need for an exacting study of the conditions that exist became more and more apparent. In short, the cost of failure became great enough to warrant the expenditure of finances for research.
In this thesis, the problem of finding the effect of bending on the collapsing pressure of a thin-walled tube subjected to uniform external pressure will be attacked. The theory of elasticity will be used exclusively and failure will be assumed to have taken place as soon as the proportional limit has been exceeded. The problem will be divided into two parts. In the first part, the equation for the periphery of the middle surface of a tube subjected to pure bending in an axial plane will be derived. In the second part, the effect of the deflected middle surface of the bent tube on the external pressure required to cause failure will be found.
REVIEW OF LITERATURE

A survey of the work that has been done on thin and moderately thick-walled tubing reveals a remarkably large amount of literature on that subject. Most of the work has been done using the mathematical theory of elasticity. This can be readily justified by the fact that a study of the elasticity of material avails the investigator of a highly exacting and very reliable conclusion woven around Hooke's and Poisson's basic laws of mechanics.

Much of the work that has been completed in the laboratory has been undertaken on shells of comparatively short lengths. The results of many experiments appear in the writings of Timoshenko.

Prescott must be given much of the credit for his practical interpretation of the work of the master theorist, A. H. Love. It was through a study of these two men that this thesis was possible. Timoshenko furnished the approach used in this thesis, and it was his derivation of the critical pressure of thin-walled tubing subjected to external pressure alone, that was used as a comparison for the conclusions reached here.

Much of the work that has been done on tubing has taken into account the effects of one type of loading only, although an analysis of a tube subjected to external pressure and bi-axial loading may be found in Prescott. However, nowhere has the problem of bending plus external pressure been attacked. In fact, most
authors have hesitated to include in their writings any reference to a tube subjected only to bending in an axial plane. The problem of axial bending alone is so complex that the time necessary for a complete and exacting solution of its effects on a tube is not justifiable except as a curiosity. This is true, primarily, because there are ways of making comparative calculations between an axially bent tube and a tube bearing a uniform longitudinal axial load that give completely satisfactory working analyses. Axial bending produces stresses in the fibers of a tube that are very much like those produced by axial loading, the only exception being that the stresses are not uniform in the case of bending. Since the real problem involved is one of finding the effect of the bending on the critical external pressure, no attempt was made to analyze directly the effect of axial bending on a thin-walled tube.
NOTATIONS

D - outside diameter of tube
p - uniform external unit pressure
t - thickness of tube wall
$I_y$ - moment of inertia of tube section about $y$-axis
$x,y,z$ - distance along the respective axes
$\frac{1}{R}$ - curvature of strained centroidal axis of tube in bending
$u,v,w$ - deflections in the direction of $X$, $Y$, and $Z$ axes respectively
$E$ - Youngs' Modulus of Elasticity
$E' = \frac{E}{(1-\nu^2)}$
$\bar{Z}$ - distance, measured along $Z$-axis, between center of inertia of cross section and an elemental area
$l,m,n$ - direction cosines of $X$, $\Theta$, and $\Gamma$ respectively
$\sigma_x, \sigma_y, \sigma_z$ - tensile or compressive stresses parallel to respective axes
$\tau_{xy}$ - shear stress on plane perpendicular to $X$-axis acting parallel to $Y$-axis
$\tau_{yz}$ - shear stress on plane perpendicular to $Y$-axis acting parallel to $Z$-axis
$\tau_{xz}$ - shear stress on plane perpendicular to $X$-axis acting parallel to $Z$-axis
$\epsilon_x, \epsilon_y, \epsilon_z$ - strains in direction of $X$, $Y$, and $Z$ axes respectively
NOTATIONS
(cont.)

ω - deflection in a radial direction

\( \bar{X}, \bar{\Theta}, \bar{F} \) - components of distributed surface forces per unit area, acting on oblique plane parallel to \( X, \Theta, \) and \( R \) respectively

\( \lambda \) - Poisson's Ratio
PART I: THE CASE OF DEFLECTION OF A TUBE IN BENDING

The case of deflection in any cross section of a thin-walled tube subjected to pure bending will be considered first. This will be done by deriving the equation for the periphery of the deflected middle surface of the tube.

Consider a tube as shown in Figure 1 with a length much greater than its diameter. The tube is fastened in such a manner as to be a beam subjected to pure bending. The A-axis is taken as being the centroidal axis of the tube. The bending moment is applied in such a manner as to make the center line of the tube deflect in an upward direction and in the AZ-plane. The stress components of any section of the tube under bending become

\[ \sigma_x = \frac{E}{R} \frac{y^2}{J_x} \]  
\[ \sigma_y = \sigma_z = \tau_{xy} = \tau_{yz} = \tau_{xz} = 0. \]

It now becomes necessary to establish the equations of equilibrium for elemental sections of the tube. Since body forces are very small comparatively, they will be neglected. To avoid cumbersome calculations it is necessary to transform equations (1) and (2) to cylindrical coordinates. Thus, equations (1) and (2) become

\[ \sigma_x = \frac{E\rho \sin\theta}{R} \]  
\[ \sigma_\theta = \sigma_\rho = \tau_{\rho \phi} = \tau_{\rho x} = \tau_{x \phi} = 0. \]
Figure No. 1

View of Tube Length - Before Deflection
Showing Moment Direction

$M = \text{Bending Moment}$

$\theta = \text{Reference Angle}$

$r_m = \text{Middle Surface Radius}$

$r_o = \text{Inside Boundary Radius}$

$r_o = \text{Outside Boundary Radius}$
Referring to figure 2, note that the element considered is very small. The upper figure is drawn slightly out of projection to show the entire convex side of the section only. To handle three dimensions with clarity, it is necessary to have a third view. However, the legend used should eliminate all but the slightest of misunderstandings.

Looking at figure 2 and summing forces in a radial direction,

\[
\sigma_{rr} \rho \, d\theta \, dx - \left( \sigma_{rr} + \frac{\partial \sigma_{rr}}{\partial \rho} \rho \right) (\rho + d\rho) \, d\theta \, dx + \\
+ \sigma_{\theta \theta} \sin \frac{d\theta}{2} \, d\rho \, dx + \left( \sigma_{\theta \theta} + \frac{\partial \sigma_{\theta \theta}}{\partial \rho} \rho \right) \sin \frac{d\theta}{2} \, d\rho \, dx + \\
+ \mathcal{I}_{r\theta} \, d\rho \, dx - \left( \mathcal{I}_{r\theta} + \frac{\partial \mathcal{I}_{r\theta}}{\partial \rho} \rho \right) d\rho \, dx + \mathcal{I}_{r\rho} \left( \frac{d^2 \mathcal{I}_{r\rho}}{d\theta^2} \right) d\theta + \\
+ \mathcal{I}_{\theta x} \rho \, d\rho \, d\theta - \left( \mathcal{I}_{\theta x} + \frac{\partial \mathcal{I}_{\theta x}}{\partial x} \right) d\rho \, dx \left( \rho + \frac{d\rho}{2} \right) d\theta = 0.
\]

Neglecting small quantities of higher order, setting \( \sin \frac{d\theta}{2} \) equal to \( \frac{d\theta}{2} \), and dividing the result by \( \rho \, d\rho \, d\theta \, dx \),

\[
\frac{\partial \sigma_{rr}}{\partial \rho} + \frac{1}{\rho} \frac{\partial \mathcal{I}_{r\theta}}{\partial \theta} + \frac{\sigma_{\theta \theta} - \sigma_{\theta \theta}^0}{\rho} + \frac{\partial \mathcal{I}_{r\rho}}{\partial x} = 0. 
\]  

(5)

Summing forces in a tangential direction perpendicular to the radius,

\[
\sigma_{\theta \theta} \cos \frac{d\theta}{2} \, d\rho \, dx - \left( \sigma_{\theta \theta} + \frac{\partial \sigma_{\theta \theta}}{\partial \rho} \rho \right) \cos \frac{d\theta}{2} \, d\rho \, dx + \\
+ \mathcal{I}_{r\theta} \rho \, d\theta \, dx - \left( \mathcal{I}_{r\theta} + \frac{\partial \mathcal{I}_{r\theta}}{\partial \rho} \rho \right) (\rho + d\rho) \, d\theta \, dx - \\
- \mathcal{I}_{\theta \theta} \sin \frac{d\theta}{2} \, d\rho \, dx - \left( \mathcal{I}_{\theta \theta} + \frac{\partial \mathcal{I}_{\theta \theta}}{\partial \rho} \rho \right) d\rho \, dx \sin \frac{d\theta}{2} + \\
+ \mathcal{I}_{\theta x} \, d\rho \, (\rho + \frac{d\rho}{2}) \, d\theta - \left( \mathcal{I}_{\theta x} + \frac{\partial \mathcal{I}_{\theta x}}{\partial x} \right) d\rho \, dx \left( \rho + \frac{d\rho}{2} \right) d\theta = 0.
\]
Figure No. 2

Showing quarter section of tube and infinitesimal element. Legend applies to direction of stresses perpendicular to paper.
Setting \( \cos \frac{d\theta}{2} \) equal to unity, \( \sin \frac{d\theta}{2} \) equal to \( \frac{d\theta}{2} \), disregarding small quantities of higher order, and dividing the result by \( r \, d\varphi \, d\theta \, dx \),

\[
\frac{1}{r} \frac{\partial \sigma}{\partial \theta} + \frac{2 \tau_{x\theta}}{\mu} + \frac{\partial \tau_{x\theta}}{\partial \mu} + \frac{\partial \tau_{x\theta}}{\partial x} = 0.
\]  

(6)

Summing forces in the direction of the \( X \)-axis,

\[
\sigma_x (d\varphi (\rho + \frac{d\rho}{2}) d\theta - (\sigma_x + \frac{\partial \sigma_x}{\partial x} d\varphi (\rho + \frac{d\rho}{2}) d\theta +
+ \tau_{x\theta} d\varphi d\theta - (\tau_{x\theta} + \frac{\partial \tau_{x\theta}}{\partial \theta} d\varphi + \tau_{x\theta} d\varphi d\theta d\varphi +
- (\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial \varphi} d\varphi) (\rho + d\rho) d\theta d\varphi = 0.
\]

Neglecting higher orders of small quantities and dividing the result by \( r \, d\varphi \, d\theta \, dx \),

\[
\frac{\partial \sigma_x}{\partial x} + \frac{1}{\rho} \frac{\partial \tau_{x\theta}}{\partial \theta} + \frac{1}{\rho} \tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial \varphi} = 0.
\]  

(7)

Equations (5), (6), and (7) are the equations of equilibrium of any element on the ring. Equations (3) and (4) satisfy the equations of equilibrium and give

\[
\frac{\partial}{\partial x}(\sigma_x) = \frac{\partial}{\partial x} \left( \frac{E \mu \sin \theta}{\rho} \right) = 0.
\]  

(7a)

It is easily seen from the results just attained that \( \sigma_x \) is dependent upon \( r \) and \( \theta \), but not upon \( x \).

The stresses on an oblique plane, the direction cosines of which are \( l, m, \) and \( n \), are given as

\[
\begin{align*}
\sigma_x &= \sigma_x \cdot l + \tau_{x\theta} \cdot n + \tau_{x\theta} \cdot n \; ; \\
\tau_{x\theta} &= \sigma_{x\theta} \cdot m + \tau_{x\theta} \cdot n + \tau_{x\theta} \cdot l \; ; \\
\tau_{r\theta} &= \sigma_{r\theta} \cdot n + \tau_{r\theta} \cdot l + \tau_{r\theta} \cdot m.
\end{align*}
\]  

(8)
Figure No. 3

Showing element with an oblique plane - ABC. Oblique plane cuts tetrahedron - ABCO from element. Remainder of element is shown in phantom. Stress on sides of tetrahedron are also shown.

\[ \sigma_n = \bar{x}l + \bar{y}m + \bar{F}n \]
See Figure 3. Equations (8) are satisfied when combined with equations (3) and (4).

Choosing any cross section of the tube in a plane normal to the X-axis, it is easily seen that it is necessary to place the Y-axis in the neutral plane of the section in order to satisfy equation (3). The longitudinal stresses are then zero at \( \Theta \) equals zero and \( \Theta \) equals \( \pi \).

Taking a small strip \( F \), with area \( dA \), as shown in Figure 4, the total compression over the entire cross section of the tube is given as

\[ C_i = -\int_A \sigma_x \, dA, \]  
(9)

and from equation (1),

\[ C_i = -\frac{E}{R} \int_A \bar{z} \, dA. \]  
(10)

But, from the condition of symmetry,

\[ \int_A \bar{z} \, dA = 0 \]  
(11)

and \( \bar{z} = 0 \). Consequently,

\[ C_i = 0. \]

Taking moments about \( OY \) of the compressions on the elements of area, the total moment is found.

\[ M = \int_A \bar{z} \sigma_x \, dA = \frac{E}{R} \int_A \bar{z}^2 \, dA, \]

\[ M = \frac{E I_y}{R}. \]  
(12)

\( M \) will be considered positive if the moment bends the
Cross Section of Tube

Showing Infinitesimal Strip \( PQ \)
beam concave upward, or (as will be seen later) if it tends to straighten a bent strip.

If strips are taken parallel to the \( z \)-axis, the total moment of the compressions about it will be

\[
M' = -\int_A y \sigma_x \, dA \quad \text{or} \quad M' = -\frac{E}{R} \int_A y \gamma y \, dA.
\]

(13)

where \( I_{yz} \) denotes the product of the moment of inertia of the area relative to \( OY, OZ \). Since \( OY, OZ \) are the principal axes of inertia, the value of \( I_{yz} \) is zero, and \( M' \) equals zero.

Consideration will now be given to the case of deflection due to pure bending in a beam of any cross section. Using Hooke's Law and equation (1), and considering a small section on the compressive side of the neutral axis, the various axial strains are given as \(^{(1)}\)

\[
\varepsilon_x = \frac{\partial u}{\partial x} = \frac{R}{J_1} \quad \text{j}
\]

(14)

\[
\varepsilon_y = \frac{\partial v}{\partial y} = -\lambda \frac{R}{J_2} \quad \text{j}
\]

(15)

\[
\varepsilon_z = \frac{\partial w}{\partial z} = -\lambda \frac{R}{J_2} \quad \text{j}
\]

(16)

and the shear strain relations are

\[
\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0.
\]

(17)

From equation (14),
\[ u = \frac{2}{R} x + u_o, \] 
(18)
where \( u_o \) is a linear function of \( y \) and \( z \) only. Taking the partials of \( u \) with respect to \( y \) and \( z \),
\[ \frac{\partial u}{\partial y} = \frac{\partial u_o}{\partial y}, \] 
(19)
and
\[ \frac{\partial u}{\partial z} = \frac{x}{R} + \frac{\partial u_o}{\partial z}. \] 
(20)
Substituting equations (19) and (20) into the second and third of equations (17),
\[ \frac{\partial w}{\partial x} = -\frac{x}{R} - \frac{\partial u_o}{\partial y}, \] 
(21)
and
\[ \frac{\partial v}{\partial x} = -\frac{\partial u_o}{\partial y}. \] 
(22)
Integrating equations (21) and (22),
\[ w = -\frac{x^2}{2R} - x \frac{\partial u_o}{\partial z} + w_o, \] 
(23)
and
\[ v = -x \frac{\partial u_o}{\partial y} + v_o. \] 
(24)
Here, \( w_o \) and \( v_o \) are unknown functions of \( y \) and \( z \) which will be determined later. Taking the partial of \( w \) with respect to \( z \), and the partial of \( v \) with respect to \( y \),
\[ \frac{\partial w}{\partial z} = -x \frac{\partial^2 u_o}{\partial z^2} + \frac{\partial w_o}{\partial z}. \] 
(25)
and
\[ \frac{\partial \nu}{\partial y} = -\chi \frac{\partial^2 u_0}{\partial y^2} + \frac{\partial v_0}{\partial y} = -\lambda \frac{2}{\nu}, \tag{26} \]

From equations (15), (16), (25), and (26), it is easily seen that
\[ \frac{\partial \nu}{\partial y} = -\chi \frac{\partial^2 u_0}{\partial y^2} + \frac{\partial v_0}{\partial y} = -\lambda \frac{2}{\nu}, \tag{27} \]
and
\[ \frac{\partial \nu}{\partial \bar{z}} = -\chi \frac{\partial^2 u_0}{\partial \bar{z}^2} + \frac{\partial \rho_0}{\partial \bar{z}} = -\lambda \frac{2}{\nu}. \tag{28} \]

From formula (18), \( u_0 \) is a linear function of \( y \) and \( z \).
So,
\[ \frac{\partial^2 u_0}{\partial y^2} = 0, \tag{29} \]
and
\[ \frac{\partial^2 u_0}{\partial z^2} = 0. \tag{30} \]

from which, by equations (27) and (28),
\[ \frac{\partial \nu_0}{\partial y} = -\lambda \frac{2}{\nu}, \tag{31} \]
and
\[ \frac{\partial \nu_0}{\partial \bar{z}} = -\lambda \frac{2}{\nu}. \tag{32} \]

Integrating equations (31) and (32),
\[ v_0 = -\lambda \frac{2y}{\nu} + F_1(\bar{z}), \tag{33} \]
and
\[ \omega_0 = -\lambda \frac{\partial^2}{\partial y^2} + F(y), \] \hspace{1cm} (34)

Substituting the values of \( v_0 \) and \( w_0 \) into equations (23) and (24),

\[ \omega = -\frac{\chi}{2P} - \chi \frac{\partial u_0}{\partial y} - \frac{\lambda}{2P} \omega + F(y) \] \hspace{1cm} (35)

and

\[ v = -\chi \frac{\partial u_0}{\partial y} - \frac{\lambda}{2P} \omega + F_i(y). \] \hspace{1cm} (36)

Taking the partial of equation (35) with respect to \( y \), and the partial of equation (36) with respect to \( z \), the result is

\[ \frac{\partial \omega}{\partial y} = -\chi \frac{\partial^2 u_0}{\partial y \partial y} + \frac{\partial F(y)}{\partial y} \] \hspace{1cm} (37)

and

\[ \frac{\partial v}{\partial z} = -\chi \frac{\partial^2 u_0}{\partial y \partial z} - \frac{\lambda}{2P} \omega + \frac{\partial F_i(z)}{\partial z}. \] \hspace{1cm} (38)

From equations (37), (38), and the first of equations (17), it follows that

\[ -\chi \frac{\partial^2 u_0}{\partial y \partial y} + \frac{\partial F(y)}{\partial y} - \chi \frac{\partial^2 u_0}{\partial y \partial z} - \frac{\lambda}{2P} \omega + \frac{\partial F_i(z)}{\partial z} = 0 \]

or

\[ \frac{\partial F(y)}{\partial y} - 2\chi \frac{\partial u_0}{\partial y \partial z} - \frac{\lambda}{2P} \omega + \frac{\partial F_i(z)}{\partial z} = 0. \] \hspace{1cm} (39)

The second term of equation (39) is the only term dependent upon \( x \). Remembering that \( u_0 \) is a linear function of \( y \) and \( z \), the following may be concluded.
\[ \frac{\partial^2 u_o}{\partial y \partial z} = 0 \] (40)

and

\[ \frac{\partial F(y)}{\partial y} - \frac{\lambda y}{R} + \frac{\partial F(y)}{\partial z} = 0. \] (41)

From equations (29) and (30), by integration,

\[ \frac{\partial u_o}{\partial z} = K_1, \] (42)

and

\[ \frac{\partial u_o}{\partial y} = K_2, \] (43)

where \( K_1 \) and \( K_2 \) are constants not dependent on \( y \) or \( z \).

Integrating again,

\[ u_o = K_1 z + \overline{F_2(y) + K_3} \] (44)

and

\[ u_o = K_2 y + \overline{F_3(y) + K_4}. \] (45)

Now, from equations (44) and (45),

\[ F_2(y) = K_2 y + K_4 \] (46)

and

\[ F_3(y) = K_1 z + K_3. \] (47)

Therefore,

\[ u_o = K_1 z + K_2 y + K_3 + K_4 \]
or
\[ u_0 = K_1 z + K_2 y + K_5. \]  

Taking the second partial of \( r(z) \) with respect to \( z \), equation (41) becomes
\[ \frac{\partial^2 F_i(z)}{\partial z^2} = 0 \]  

from which,
\[ \frac{\partial F_i(z)}{\partial z} = K_7. \]  

Integrating equation (50),
\[ F_i(z) = K_7 z + K_8. \]  

Now, using equations (41) and (50),
\[ \frac{\partial F(y)}{\partial y} - \frac{\lambda y}{R} + K_7 = 0. \]  

Rearranging equation (52) and integrating,
\[ F(y) = \frac{\lambda y^2}{2R} - K_7 y + K_9. \]  

In the above, \( K_1, K_2, K_5, K_7, K_8, \) and \( K_9 \) are arbitrary constants. From equations (48), (51), and (53), and equations (18), (21), and (22), the deflections are
\[ u = -\frac{x^2}{2R} - K_1 x - \frac{\lambda y^2}{2R} + \frac{\lambda y^2}{2R} - K_7 y + K_9; \]  
\[ v = -\lambda K_2 - \frac{\lambda y^2}{R} + K_7 z + K_9; \]  
\[ w = \frac{\lambda x^2}{R} + K_7 z + K_2 y + K_5. \]
The arbitrary constants are determined from the condition of the fastening. Assuming that point 0, the centroid of the left end of the bar, together with an element of the \(\lambda\)-axis and an element of the \(\lambda\zeta\)-plane, are fixed, the values of \(x\), \(y\), and \(z\) are zero. Therefore, the deflections at the origin are

\[ u = v = w = 0, \]

and

\[ \frac{\partial w}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0. \]

The above conditions are satisfied if the arbitrary constants are taken equal to zero. Hence,

\[ w = -\frac{1}{2E} \sqrt{x^2 + \lambda(y^2 - \bar{y}^2)}; \quad (57) \]

\[ v = -\frac{\lambda \bar{y}}{R}; \quad (58) \]

\[ u = \frac{\bar{x}}{R}. \quad (59) \]

These are the general equations for a beam of any cross section subjected to axial bending. To obtain the deflection curve of the neutral axis of any beam, the values of \(z\) and \(y\) may be placed equal to zero, so long as the neutral axis and the \(\lambda\)-axis are the same. From equations (57), (58), and (59), and the conditions stated, it is obvious that

\[ w = -\frac{\bar{x}^2}{2E} = -\frac{M\bar{x}^2}{2EI_y} \]

\[ (60a) \]
Notice that deflection only occurs in the $xz$-plane.

Consider any cross section $x=c$, a distance $c$ from the left end of the beam. With $x$ equal to $c$, the section lies in a plane, and the equation of the plane is

$$x = C + u = C + \frac{c^2}{R}.$$  \hspace{1cm} (61)

The plane containing the section has rotated slightly, and lies on the radius $R$. The deflection of the section in this plane in any direction becomes

$$\begin{align*}
    w &= -\frac{1}{2R} \left[ c^2 + \lambda (y^2 - z^2) \right]; \\
    v &= -\frac{\lambda z y}{R}; \\
    u &= \frac{2c}{R}.
\end{align*}$$  \hspace{1cm} (62)

Equations (62) are the deflection equations of the symmetrical cross section of a beam when subjected to pure bending, where the cross section is taken at a distance $c$ from the origin of the beam.

If the beam is a tube with an inner radius $r_i$ and an outer radius $r_o$, a cross section of the tube may be chosen a distance $c$ from the origin. Before deflection, the equation of the tube will be
\[ r^2 = z^2 + y^2, \]  

(63)

where \( r \) is the distance from the \( X \)-axis of any point in the section of the tube. The value of \( r \) will depend upon the value of \( z \) and \( y \).

Now, consider the middle surface of the undeflected tube. All points on the middle surface will be a distance \( r_m \) from the \( X \)-axis and the values of \( z \) and \( y \) are given as

\[
\begin{align*}
    z &= r_m \sin \theta \\
    y &= r_m \cos \theta
\end{align*}
\]

(64)

and from equations (63) and (64), the value of \( r \) is easily found.

\[
r = (r_m^2 \sin^2 \theta + r_m^2 \cos^2 \theta).\]

(65)

After bending occurs, the cross section under consideration will assume a deformed shape and will lie in the plane given by formula (61). The values of \( z \) and \( y \), for the mean periphery of the cross section, will be given as

\[
    z = r_m \sin \theta + w
\]

(66)

and

\[
    y = r_m \cos \theta + v.
\]

(67)
From the general equation (63) and the particular equations (66) and (67), the value of \( r^2 \) will become

\[
r^2 = \left( r_m \sin \theta + \omega \right)^2 + \left( r_m \cos \theta + v \right)^2
\]  

(68)

and neglecting higher orders of \( w \) and \( v \), formula (68) is given as

\[
r^2 = r_m^2 + 2\omega r_m \sin \theta + 2 v r_m \cos \theta.
\]  

(69)

The deflections of the middle surface were given as \( w \) and \( v \), and were defined by equations (62). Expressed in polar coordinates, these deflections are given as

\[
w = -\frac{1}{2r} \left[ c^2 + \lambda r_m^2 \left( \sin^2 \theta - \cos^2 \theta \right) \right]
\]  

(70)

and

\[
v = -\frac{\lambda r_m^2 \sin \theta \cos \theta}{r}
\]  

(71)

From equations (69), (70), and (71), the equation of the middle surface of the deflected tube at a distance \( c \), along the \( X \)-axis, is

\[
r^2 = r_m^2 - \frac{r_m M \sin \theta}{E'I_y} \left[ c^2 + \lambda r_m^2 \right].
\]  

(72)

Observing equation (72), notice that \( E' \) has replaced \( E \) since there will be no deflection of the rectangular unit sections of the tube.

If \( \theta \) equals zero or \( \frac{\pi}{2} \), the value of \( r \) is \( r_m \).

The deflection \( v \), in the \( XY \)-plane, remains zero. Also, when \( \theta \) equals \( \frac{\pi}{2} \), the value of \( r^2 \) is a minimum and is
Figure No. 5

Showing $r = r_m$ as a dotted line and showing $r = \left\lfloor r_m^2 - \frac{r_m M \sin \theta}{E' I_y} \left( C^2 + \lambda r_m^2 \right) \right\rfloor^{\frac{1}{2}}$, as a solid line.
given as

\[
r_{\text{min}}^2 = r_m^2 - \frac{r_m M}{E'I_y} \left[ C^2 + \lambda r_m^2 \right],
\]

(72a)

and when \( \Theta \) equals \( \frac{3r}{2} \), the value of \( r^2 \) is a maximum and is given as

\[
r_{\text{max}}^2 = r_m^2 + \frac{r_m M}{E'I_y} \left[ C^2 + \lambda r_m^2 \right].
\]

(72b)

The curve, shown in Figure 5 as a solid line, is the exaggerated periphery of the deflected middle surface of any cross section of a thin-walled tube subjected to pure bending. The dotted line, shown as a circle, represents the original middle surface of the same section in its undeflected state.
PART II: THE EFFECT OF THE DEFLECTED MIDDLE SURFACE
ON THE CRITICAL EXTERNAL PRESSURE OF A BENT TUBE

Next, consideration will be given to the effect of the deflections, caused by pure bending of a thin or moderately thick-walled tube, on the stability of the tube when subjected to external pressure.

The deflected tube, shown in Figure 5, will now be assumed to be split down the Z-axis. This is the axis of symmetry of the deflected cross-section. Only half of the tube will be considered, together with the reactions at its ends. See Figure 6.

In Figure 6, the dotted line again represents the undeflected middle surface of the tube, where \( r \) equals \( r_m \). The solid line is an exaggerated view of the deflected middle surface. The values of \( r \), for the deflected middle surface, were found from equation (72) as \( \Theta \) varied from \(-\frac{\pi}{2}\) to \(+\frac{\pi}{2}\).

By the principle of inextensional deflection, the undeflected middle surface of the tube may be analyzed as though it is a ring of unit axial length. The unit bending moment \( M \), acting on the end of the ring, is given by the equation,(2)

\[
M = E' I \left( \frac{1}{r_i} - \frac{1}{r_m} \right) \tag{73}
\]

Figure No. 6

Showing middle surface of bent tube split down the Z-axis with end reactions and external pressure considered.
where \( l \) equals \( \frac{L^3}{Iz^2} \) and \( \tau \) is calculated from equation (72a). In a similar manner, \( M_2 \) may also be found.

Notice here, that \( M \) will be positive and \( M_2 \) will be negative. The unit bending moment of any point \( D \), on the deflected curve is consequently given by the equation,

\[
M = E'I\left(\frac{l}{r} - \frac{l}{r_m}\right).
\]

(74)

It is easy to see that \( \omega \) is given by the equation,

\[
\omega = r_m - r.
\]

(75)

The value of \( \omega \) at point \( A \), on Figure 6, will equal zero since, at that point,

\[
r = r_m.
\]

(75a)

Now, consider the effect of an external uniform pressure \( p \), acting on the deflected ring section, as shown in Figure 6. If the pressure is safe, the ring will assume its original circular periphery when the bending moment, acting on the tube, is released. However, if the bending moment is released and, under the influence of the pressure, the deflected cross-section remains deflected, the pressure will have assumed a major influence on the section and will be termed critical.

It may be seen from Figure 6 that

\[
F_i = p(r_m - \omega) = p \overline{OL}.
\]

(76)

The bending moment at any point \( D \), of the buckled ring is,
Considering the triangle, \( \triangle LDO \), it is easily seen that, by the law of cosines,
\[
OD^2 = DL^2 + \overline{OL}^2 - 2 \overline{OL} \cdot LT
\]
or
\[
\frac{1}{2} DL^2 - \overline{OL} \cdot LT = \frac{1}{2} (OD^2 - \overline{OL}^2).
\]
Substituting equation (78) into equation (77) gives
\[
M = M_i - \frac{P}{2} (OD^2 - \overline{OL}^2).
\] (79)

Now,
\[
\overline{OL} = r_m - \omega,
\]
and
\[
\overline{OD} = r_m - \omega.
\]

Therefore,
\[
M = M_i - \frac{P}{2} \left( r_m^2 - 2r_m \omega + \omega^2 - r_m^2 + 2r_m \omega - \omega^2 \right),
\] (80)

and neglecting higher orders of \( \omega \) and \( \omega' \),
\[
M = M_i - P r_m (\omega_i - \omega).
\] (81)

The differential equation of the deflected curve of a section of a unit ring is\(^{(3)}\)

\[
\frac{d^2 \omega}{d \Theta^2} + \omega = - \frac{M r_m^2}{E' I} \quad (82)
\]
and may be written for this case as,
\[
\frac{d^2 \omega}{d \Theta^2} + \omega = - \frac{r_m^2}{E' I} \left[ M, - \rho r_m (\omega_1 - \omega) \right]. \quad (83)
\]
Putting this equation in the form,
\[
\frac{d^2 \omega}{d \Theta^2} + \omega \left( 1 + \frac{\rho r_m^3}{E' I} \right) = \frac{-M r_m^2 + \rho r_m^3 \omega}{E' I} \quad (84)
\]
and using the notation,
\[
K^2 = 1 + \frac{\rho r_m^3}{E' I} \quad (85)
\]
the general solution of equation (84) becomes,
\[
\omega = A \sin K \Theta + B \cos K \Theta + \frac{-M r_m^2 + \rho r_m^3 \omega}{E' I + \rho r_m^3}. \quad (86)
\]
From Figure 6, it is easily seen that when \( \Theta \) equals zero, the value of \( \omega \) is also zero and formula (86) becomes
\[
0 = B + \frac{-M r_m^2 + \rho r_m^3 \omega_i}{E' I + \rho r_m^3}.
\]
or
\[
B = \frac{M r_m^2 - \rho r_m^3 \omega_i}{E' I + \rho r_m^3}. \quad (87)
\]
From equation (81) with the conditions of equation (87),
\[
M_i = \rho r_m \omega_i. \quad (88)
\]
Substituting equation (88) into equation (87), the value of \( B \) is zero.
Now, consider Figure 6. From symmetry, the value of $\frac{dw}{d\theta}$ is found.

$$\left(\frac{dw}{d\theta}\right)_{\theta=\frac{\pi}{2}} = 0,$$

from which it may be concluded that

$$\cos \frac{\pi r}{2} = 0.$$  \hspace{1cm} (90)

The smallest possible root of equation (90) is $k=1$.

However, it has been shown\(^{(4)}\) that this represents a translation of the ringed section as a rigid body and should not be considered in discussing elastic buckling of a ring.

Assuming, therefore, the smallest possible root for the elastic conditions present, $k=3$, and using this in equation (85), the critical pressure is found to be

$$p_{cr} = \frac{3E'I}{r_m^3}.$$  \hspace{1cm} (91)

The critical external pressure of a tube not subjected to any other stress is given by

$$p_{cr} = \frac{3E'I}{r_m^3}.$$  \hspace{1cm} (92)

so it is easily seen that the tube has actually been strengthened against a uniformly distributed pressure by the pure axial bending of the tube.

Now, remembering that

\[ I = \frac{t^3}{12} \]

and

\[ r_m = \frac{D - t}{2} \]

Equation (91) may be given as

\[ \kappa_{cr} = \frac{16E'}{3(\kappa_k - 1)^3} \quad (93) \]

and if the ratio \( \kappa_k \) is large with a linear distribution of circumferential stress assumed, the circumferential stress is given by

\[ \sigma = \frac{F D}{2t} \quad (93a) \]

Therefore, the stress at the critical pressure is

\[ \sigma_{cr} = \frac{8}{3} \frac{E'Dt^2}{(D-t)^3} \quad (94) \]

or

\[ \sigma_{cr} = \frac{8}{3} \frac{EDt^2}{(1-K)(D-t)^3} \quad (94a) \]

This is the critical compressive stress of a thin or moderately thick-walled tube subjected to simultaneous axial bending and uniform external pressure.

Now, consider the effect on the critical pressure of a tube, subjected to bending by a terminal transverse load instead of pure bending. The stresses in a longitudinal direction throughout the ring will be distributed around the periphery of the ring in the same manner as
Figure No. 7

Showing terminal transverse load, \( W \).

Moment at point \( J \) = \( W(x - x_i) \).

Moment at point \( O \) = \( WX \).
when the ring was subjected to pure bending. See Figure 7. However, the bending moment acting on the cross section at \( J \), which is a distance \( x \), from the origin, will be different from the moment acting on the cross section at the origin because the bending moment varies from zero at the point of application of the load, to a maximum at the origin. In addition to this, there will also be shearing stresses acting in a direction normal to the longitudinal axis. It has been shown, however, that any deflections caused by the addition of the shearing stresses will be so small as to be negligible. Therefore, the effect of bending of a long tube, under the influence of a terminal transverse load, on the critical external pressure of the tube will be the same as that already given for a thin or moderately thick-walled tube subjected to pure bending.\(^{(5)}\)

CONCLUSIONS

An important conclusion is that buckling of tubing under external pressure is much the same as buckling of columns under a compressive load. Therefore, much the same reasoning can apply in a mathematical analysis of the tube. Also, the principle of inextensional deflection has been shown to apply to a tube subjected to pure axial bending. Any cross section of the tube may be analyzed as though it were the cross section of a ring.

The higher the ratio of the diameter of the tube to the thickness of its walls, the more accurate the results of this thesis become.

The end conditions of the tube have been assumed rigid. This assumption is satisfactory for oil well casings but may not apply elsewhere. Care must be employed if the results of this thesis are used anywhere except on a tube that is rigidly fastened.

A tube bent by a terminal transverse load will have the same effect on the critical external pressure of the tube as will a pure axial bending load. This is true because the shearing load on the cross section of the tube is dependent on the ratio of the outside diameter of the tube to its length. For long tubes, where the length is much greater than the outside diameter, the effect of shear on deflection of the tube is negligible and the bending moment, at any point, may be considered constant.
The effect of pure axial bending on the external pressure of a thin or moderately thick-walled tube has been considered only within the elastic range. This suggests a wide field of endeavor for anyone who cares to investigate this problem from the standpoint of plasticity.
SUMMARY

The following are the important points brought out in this thesis:

1. An expression for the deflected middle surface of a thin or moderately thick-walled tube subjected to pure axial bending was derived.

2. An expression for the critical external pressure of a thin or moderately thick-walled tube, subjected to pure axial bending, was derived.

3. It was shown that the critical external pressure of a thin or moderately thick-walled tube is the same for pure axial bending and bending under a terminal transverse load, provided the ratio of the tube diameter to the tube length was sufficiently large.
1. Books:


2. Periodicals:


Clinedinst, W. O. Drill pipe yield strength lowered by severe drilling service. Oil and Gas Journal. Vol. 45, pp. 75-77 (Nov. 2, 1946)


O'Donnell, L. and Crake, W. S. Mechanical causes of casing failure and practices for their control. Oil and Gas Journal. Vol. 42, p. 46 (Dec. 18, 1943)


3. Publications of Learned Societies:

A.P.I. Information on collapsing pressures and setting depths for casing. Division of Production Bulletin No. 5-C-2. 2nd edition (March, 1950)


Main, W. C. Combining bending and hoop stresses to determine collapsing pressure of oil-country tubular goods. A.P.I. Drilling and Prod. Practice pp. 421-431 (1939)

4. Unpublished Material:

Cizek, F. J. Analytical investigation of thin and moderately thick-walled tubing under peripheral and biaxial loading. Thesis, Missouri School of Mines and Metallurgy, Rolla, Mo.
VITA

The author was born in Washington, Pennsylvania, on June 3, 1923. He received his primary and secondary education in Detroit, Michigan. Upon graduation from high school, he served a two-year tool and gauge apprenticeship with the Sterling Gauge Company and the Motor City Gauge Company, both of Detroit. In 1942 he entered the Lawrence Institute of Technology in Highland Park, Michigan. In 1943 he entered the service of the United States Navy where, in an officer training program, he was allowed to complete the work required for a Bachelor of Science degree in Mechanical Engineering. He received this degree from the Illinois Institute of Technology in 1945. From that time until the termination of his service with the United States Navy in 1946, he served in various engineering capacities on board ship in the Pacific Ocean.

In the fall of 1946, he became an instructor in mechanical engineering at Rhode Island State College. In June, 1947, he accepted a position as an instructor in mechanical engineering at the Missouri School of Mines and Metallurgy where he completed enough work for a Master's degree in Mechanical Engineering.