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Determining the transfer matrices of tapered multiwire transmission lines

Clyde A. Vandivort

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This new format contains a Prologue, an abbreviated report of thesis research, and expanded appendices.

The Prologue introduces the reader to the main research area. The abbreviated research report for this Electrical Engineering thesis appears in the form required by the IEEE Professional Society and thus is suitable for publication without rewriting. The appendices contain complete developments that were abbreviated in the research report.

The abbreviated research report will appear in the Proceedings of the 1967 IEEE Midwest Symposium on Circuit Theory. This symposium will be held at Purdue University on May 17, 18, and 19.

Sincerely yours,

C. A. Vandivort
C. A. Vandivort, Instructor
Electrical Engineering Department
DETERMINING THE TRANSFER MATRICES
OF TAPERED MULTIWIRE TRANSMISSION LINES

BY

CLYDE A. VANDIVORT - 1963 -

A
THESIS
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Approved by

E. C. Bertram (advisor)

[Signature]

[Signature]
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# TABLE OF CONTENTS

ABSTRACT ii  
ACKNOWLEDGEMENT iii  
LIST OF ILLUSTRATIONS v  
PROLOGUE TO THESIS vi  
DETERMINING THE TRANSFER MATRICES OF TAPERED MULTIWIRE TRANSMISSION LINES  
  I Introduction 1  
  II. The Network's Matric Differential Equation 2  
  III. The Exponential Network's Transfer Matrix 4  
  IV Example Calculation of Network Voltage Gain 9  
  V. Extensions and Conclusions 12  
  VI. References 13  
APPENDICES 16  
VITA 22
# LIST OF ILLUSTRATIONS

<table>
<thead>
<tr>
<th>Figures</th>
<th>Description</th>
<th>Pages(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>Incremental Section of Thin Film RC Microcircuit</td>
<td>viii</td>
</tr>
<tr>
<td>1.</td>
<td>2n Terminal Tapered RC Microcircuit</td>
<td>14</td>
</tr>
<tr>
<td>2.</td>
<td>Equivalent Circuit of Length $\Delta x$ for a 2n Terminal Tapered RC Network</td>
<td>14</td>
</tr>
<tr>
<td>3.</td>
<td>Six-Terminal Exponential RC Microcircuit</td>
<td>15</td>
</tr>
</tbody>
</table>
THESIS PROLOGUE

The advent of monolithic and integrated microelectronic circuits has aroused interest in passive multilayered thin film networks. The basic thin film network is shown in Figure 1. This structure consists of alternate resistive and dielectric layers sandwiched together. Such devices are referred to as distributed parameter RC networks. The reduction of volume and increased reliability offered by thin film devices has caused considerable effort to be extended in the analysis of their characteristics.

The distributed parameter approach to analysis subdivides the network lengthwise into elemental sections, each of length $\Delta x$. A representative section, located at position $x$, is shown in Figure i-1. The series resistances, $r_i$, of each network section are functions of the resistivity, $\rho_i$, the thicknesses, $h_i$, and the network’s width $w$.

$$\Delta x_i = \frac{\rho_i \Delta x}{w h_i}, \ (i=1,2,...,n). \quad (i-1)$$

In a similar fashion, the shunt admittances associated with the dielectric layers are composed of capacitance expressed by

$$\Delta c_i = \frac{\varepsilon_i w \Delta x}{s_i}, \ (i=1,2,...,n). \quad (i-2)$$

In equation (i-2), $\varepsilon_i$ is the permittivity of the $i$th dielectric layer, $s_i$ is the layer’s thickness and, as in equation (i-1), $w$ is the network’s width. Dividing equations (i-1) and (i-2) by $\Delta x$ and allowing $\Delta x$ to approach zero, the resistances and capacitances per unit length
\[ R_i(x) = \frac{\rho_i}{w(x)} h_i \]  
\[ C_i(x) = \frac{\varepsilon_i w(x)}{s_i} \]

where \( w(x) \) is the physical width of the RC network at position \( x \) and is defined as the network taper function.

The incremental equivalent circuit given in Figure 2 stems directly from the above physical parameters and is the circuit used as a basis for the analysis that follows. From this circuit the multilayered distributed parameter RC network's matrix partial differential equation may be written. The network analysis problem then becomes one of obtaining a solution to this equation.
Figure i-1 - Incremental Section of Thin Film RC Microcircuit
DETERMINING THE TRANSFER MATRICES
OF TAPERED MULTIWIRE TRANSMISSION LINES*

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University of Missouri at Rolla

I. INTRODUCTION

The development of multilayered, thin film, microelectronic circuits offers some important distributed parameter RC network analysis problems. The network analysis becomes difficult if a circuit is tapered and/or multilayered to provide certain desirable characteristics (3,5,7).

Figure 1 depicts a tapered multilayered distributed parameter RC network. The taper function $w(x)$ expresses how the network's width varies as a function of position $x$. The network's per unit length resistances and capacitances are $R_i(x) = R_i \frac{1}{w(x)}$ and $C_i(x) = C_i w(x)$.

For the four-terminal class of RC networks with certain tapers, exponential (3,7), trigonometric (2), etc., the exact sinusoidal steady state solutions have been found. Delay and rise time calculations for four-terminal RC networks were presented by Pronotatarious and Wing (5). Rice (4) found the exact steady state solution for a multiwire constant parameter transmission line. Bertnolli and Halijak (6,10) put forth numerical methods of solving the multiterminal variable parameter RC network problem. These references also mention that the matrizant solution to be discussed in what follows does not yield a closed form solution in terms of common functions.
Thus, the literature, although rich with analyses of distributed RC networks, is lacking any general method of solution in closed form for the 2n terminal arbitrarily tapered network problem. This paper presents an example of just such a method. A complete solution in the Laplace Transform domain is found for the exponentially tapered multilayered RC network and for its lossless transmission line counterpart. Also included is the outline of a demonstration that the previously published matrix matrifiant solution (10) can be summed to the closed form solution contained here.

II. THE NETWORK'S MATRIC DIFFERENTIAL EQUATION

The equivalent circuit for a section of tapered 2n terminal distributed RC network of length $\Delta x$ is shown in Figure 2. One formulation of the Laplace Transformed Kirchoff's Voltage and Current Law equations for the network is the matric difference equation.

\[
\begin{bmatrix}
V_1(x+\Delta x,s) \\
V_2(x+\Delta x,s) \\
\vdots \\
V_n(x+\Delta x,s) \\
I_1(x+\Delta x,s) \\
I_2(x+\Delta x,s) \\
\vdots \\
I_n(x+\Delta x,s)
\end{bmatrix}
- \begin{bmatrix}
V_1(x,s) \\
V_2(x,s) \\
\vdots \\
V_n(x,s) \\
I_1(x,s) \\
I_2(x,s) \\
\vdots \\
I_n(x,s)
\end{bmatrix}
= \begin{bmatrix}
o_n \\
sC(x) \\
o_n \\
sG(x)
\end{bmatrix}
\begin{bmatrix}
0 \\
\Delta x \\
0 \\
\Delta x
\end{bmatrix}
\begin{bmatrix}
R(x) \\
0 \\
V_n(x+\Delta x,s) \\
I_2(x,s)
\end{bmatrix}
\begin{bmatrix}
V_1(x+\Delta x,s) \\
V_2(x+\Delta x,s) \\
\vdots \\
V_n(x+\Delta x,s) \\
I_1(x+\Delta x,s) \\
I_2(x+\Delta x,s) \\
\vdots \\
I_n(x+\Delta x,s)
\end{bmatrix}
\]

(1)
where

\[
C(x) = \begin{bmatrix}
    c_1(x) & -c_1(x) & 0 & 0 \\
    -c_1(x) & c_1(x) + c_2(x) & -c_2(x) & 0 \\
    0 & -c_2(x) & \ddots & \ddots \\
    0 & 0 & \ddots & c_{n-1}(x) + c_n(x)
\end{bmatrix}
\]

\[\begin{align*}
  \begin{bmatrix}
    c_1 & -c_1 & 0 & 0 \\
    -c_1 & c_1 + c_2 & -c_2 & 0 \\
    0 & -c_2 & \ddots & \ddots \\
    0 & 0 & \ddots & c_{n-1} + c_n
  \end{bmatrix} \cdot w(x) &= C \cdot w(x)
\end{align*}\]

and

\[
R(x) = \text{diag.}[R_1(x), R_2(x), \ldots, R_n(x)] = \frac{1}{w(x)} \cdot [R_1, R_2, \ldots, R_n]
\]

In the previous equations \( R_1(x), R_2(x), \ldots, R_n(x) \) and \( c_1(x), c_2(x), \ldots, c_n(x) \) are the per unit length resistance and capacitance functions and \( w(x) \), a scalar quantity, is the network's taper function.

Dividing equation (1) by the scalar quantity \( \Delta x \) and taking the limit as \( \Delta x \) approaches zero produces the state space form of the 2n terminal network's equilibrium equations

\[
\begin{bmatrix}
    V_1(x, s) \\
    V_2(x, s) \\
    \vdots \\
    V_n(x, s) \\
    I_1(x, s) \\
    I_2(x, s) \\
    \vdots \\
    I_n(x, s)
\end{bmatrix}
\frac{\partial}{\partial x}
\begin{bmatrix}
    0_n \\
    R(x)
\end{bmatrix}
= \begin{bmatrix}
    V_1(x, s) \\
    V_2(x, s) \\
    \vdots \\
    V_n(x, s) \\
    I_1(x, s) \\
    I_2(x, s) \\
    \vdots \\
    I_n(x, s)
\end{bmatrix}.
\]
Defining column vectors

\[ V(x,s) = \text{col}[V_1(x,s), V_2(x,s), \ldots, V_n(x,s)], \text{ and} \]

\[ I(x,s) = \text{col}[I_1(x,s), I_2(x,s), \ldots, I_n(x,s)] \]

Equation (1) may be written as

\[
\frac{\partial}{\partial x} \begin{bmatrix} V(x,s) \\ I(x,s) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{\log(w(x))} \\ s C \log(w(x)) & 0 \end{bmatrix} \begin{bmatrix} V(x,s) \\ I(x,s) \end{bmatrix}
\]

or

\[
\frac{\partial}{\partial x} \begin{bmatrix} V(x,s) \\ I(x,s) \end{bmatrix} = K(x,s) \begin{bmatrix} V(x,s) \\ I(x,s) \end{bmatrix}
\]

This first order matrix space variable form of the partial differential equations for a 2n terminal, tapered, distributed RC network together with the values for \( V(0,s) \) and \( I(0,s) \), the network boundary values, constitute a mathematical formulation of the distributed RC network analysis problem. The solution of this mathematical problem produces the network's transfer matrix in a form that is useful for determining both steady-state and transient network behavior.

In what follows, an exponentially tapered distributed RC network will be employed as an example. The ideas and methods to be exemplified may be used as a general tool to generate results for the analysis of other multiport lines.

III. THE EXPONENTIAL NETWORK'S TRANSFER MATRIX

A. The Space Variable Taylor Series Method

The multilayered exponentially tapered distributed RC network's taper function is \( w(x) = e^{-2ax} \)

and its state space equation (7) is

\[
\frac{\partial}{\partial x} \begin{bmatrix} V(x,s) \\ I(x,s) \end{bmatrix} = \begin{bmatrix} 0 & Re^{-2ax} \\ s C_0 e^{-2ax} & 0 \end{bmatrix} \begin{bmatrix} V(x,s) \\ I(x,s) \end{bmatrix},
\]

\[
\frac{\partial}{\partial x} \begin{bmatrix} V(x,s) \\ I(x,s) \end{bmatrix} = K(x,s) \begin{bmatrix} V(x,s) \\ I(x,s) \end{bmatrix}.
\]
In the development that follows \( K(x,s) \) is abbreviated as \( K(x) \). One method to be proposed here for solving equation (10) is through the expansion of \( \text{col}[\tilde{V}(x,s), \tilde{I}(x,s)] \) in its Taylor Series in the space variable \( x \) about \( x = o \). The series is

\[
\begin{bmatrix}
\tilde{V}(x,s) \\
\tilde{I}(x,s)
\end{bmatrix} = \begin{bmatrix}
\tilde{V}(o,s) \\
\tilde{I}(o,s)
\end{bmatrix} + x \begin{bmatrix}
\tilde{V}(o,s) \\
\tilde{I}(o,s)
\end{bmatrix} + \frac{x^2}{2!} \begin{bmatrix}
\tilde{V}(o,s) \\
\tilde{I}(o,s)
\end{bmatrix} + \cdots + \frac{x^n}{n!} \begin{bmatrix}
\tilde{V}(o,s) \\
\tilde{I}(o,s)
\end{bmatrix} + \cdots
\]

(12)

where \( \begin{bmatrix}
\tilde{V}(o,s) \\
\tilde{I}(o,s)
\end{bmatrix} \) indicates \( \frac{\partial^n}{\partial x^n} \begin{bmatrix}
\tilde{V}(x,s) \\
\tilde{I}(x,s)
\end{bmatrix} \) evaluated at \( x = o \).

The terms of the series are generated by repeated differentiation of equation (11). When the differentiation is performed, equation (12) becomes

\[
\begin{bmatrix}
\tilde{V}(x,s) \\
\tilde{I}(x,s)
\end{bmatrix} = [\tilde{W} + \tilde{V}] \begin{bmatrix}
\tilde{V}(o,s) \\
\tilde{I}(o,s)
\end{bmatrix}
\]

(13)

where

\[
\tilde{W} = \left( \tilde{I}_{2n} x + \frac{x^2}{2!} 2aM + \frac{x^3}{3!} \left[ 4a^2 \tilde{I}_{2n} + K^2(o) \right] + \frac{x^4}{4!} \left[ 8a^3 \tilde{I}_{2n} + 4a K^2(o) \right] M + \frac{x^5}{5!} \left[ 16a^2 \tilde{I}_{2n} + 12a^2 K^2(o) + K^4(o) \right] + \cdots \right) K(o)
\]

(14)

and

\[
\tilde{V} = \left( \tilde{I}_{2n} + ox + \frac{x^2}{2!} K^2(o) + \frac{x^3}{3!} 2a K^2(o) M + \frac{x^4}{4!} \left[ 4a K^2(o) + K^4(o) \right] + \frac{x^5}{5!} \left[ 8a^3 K^2(o) + 4a K^4(o) \right] M + \cdots \right).
\]

(15)
In the previous equations for $W$ and $V$

$$M = \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix}$$

thus

$$M^{2n} = \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix} = I_{2n}$$

This relationship for $M$ is developed from

$$\frac{\partial}{\partial x} K(x) = 2a \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix} K(x)$$

or

$$\frac{\partial}{\partial x} K(x) = 2a M K(x) = -2a K(x) M.$$

Defining

$$\gamma = \left[ K^2(o) + a^2 I_{2n} \right]^{1/2} = \begin{bmatrix} (sCR + a^2 I_n) & 0 \\ 0 & (sCR + a^2 I_n) \end{bmatrix}^{1/2}$$

$$= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}^{1/2}$$

$W$ and $V$ may be written as

$$W = -K(o) \gamma^{-1} \begin{bmatrix} 0 & -2\gamma xM + 4a\gamma M^2 \frac{x^2}{2!} - [6a^2\gamma + 2\gamma^3] M^3 \frac{x^3}{3!} + \\
[8a^3\gamma + 8a\gamma^3] M^4 \frac{x^4}{4!} - [10a^4\gamma + 20a^2\gamma^3 + 2a\gamma^5] M^5 \frac{x^5}{5!} + \ldots \end{bmatrix} M$$

and

$$V = \gamma^{-1} \begin{bmatrix} 2\gamma I_n + aMx + 2\gamma K^2(o) M^2 \frac{x^2}{2!} + 2a\gamma K^2(o) M^3 \frac{x^3}{3!} + \\
2\gamma [4aK^2(o) + K^4(o)] M^4 \frac{x^4}{4!} + 2\gamma [8a^3 K^2(o) + 4aK^2(o)] M^5 \frac{x^5}{5!} + \ldots \end{bmatrix} M$$

or

$$W = -K(o) \gamma^{-1} \begin{bmatrix} (I_n - I_n) & \ldots \\ \ldots & \ldots \end{bmatrix} - [(aI_n + \gamma)^3 - (aI_n - \gamma)^3] M x + [(aI_n + \gamma)^4 - (aI_n - \gamma)^4] M^2 \frac{x^2}{2!} \\
- [(aI_n + \gamma)^5 - (aI_n - \gamma)^5] M^3 \frac{x^3}{3!} + [(aI_n + \gamma)^6 - (aI_n - \gamma)^6] M^4 \frac{x^4}{4!} \\
- [(aI_n + \gamma)^7 - (aI_n - \gamma)^7] M^5 \frac{x^5}{5!} + \ldots \end{bmatrix} M$$
\[
V = \frac{\gamma^{-1}}{2} \left( (\gamma-aI_n) + (\gamma+aI_n) \right) I_n + \left( (\gamma-aI_n) (\gamma+aI_n) - (\gamma+aI_n) (\gamma-aI_n) \right) M_x
\]
\[
+ \left( (\gamma-aI_n) (\gamma+aI_n)^2 + (\gamma+aI_n) (\gamma-aI_n)^2 \right) M_x^2 \frac{x^2}{2!}
\]
\[
+ \left( (\gamma-aI_n) (\gamma+aI_n)^3 - (\gamma+aI_n) (\gamma-aI_n)^3 \right) M_x^3 \frac{x^3}{3!} + \ldots
\]

These series sum to
\[
W = -K(o) \frac{\gamma^{-1}}{2} \left( \exp \left[ -(aI_n + \gamma) M_x \right] - \exp \left[ -(aI_n - \gamma) M_x \right] \right) M
\]
and
\[
V = \frac{\gamma^{-1}}{2} \left( (\gamma-aI_n) \exp \left[ (\gamma+aI_n) M_x \right] + (\gamma+aI_n) \exp \left[ (aI_n - \gamma) M_x \right] \right)
\]
or
\[
W = K(o) \frac{\gamma^{-1}}{2} \exp (-aMx) \left( - \exp (-\gamma Mx) + \exp (\gamma Mx) \right) M
\]
and
\[
V = \frac{\gamma^{-1}}{2} \exp (aMx) \left( \gamma \left[ \exp (\gamma Mx) + \exp (-\gamma Mx) \right] + a \left[ \exp (\gamma Mx) - \exp (-\gamma Mx) \right] \right)
\]

Employing the definition for \( \gamma \) gives
\[
W = \begin{bmatrix}
I_n e^{ax} & 0 \\
0 & I_n e^{ax}
\end{bmatrix}
\begin{bmatrix}
0 & R(T)^{-1} \sinh (T x) \\
sc \Gamma^{-1} \sinh (\Gamma x) & 0
\end{bmatrix}
\]
(22)

and
\[
V = \begin{bmatrix}
I_n e^{ax} & 0 \\
0 & I_n e^{ax}
\end{bmatrix}
\begin{bmatrix}
\cosh (\Gamma x) - a \Gamma^{-1} \sinh (\Gamma x) & 0 \\
0 & \cosh (\Gamma x) + a \Gamma^{-1} \sinh (\Gamma x)
\end{bmatrix}
\]
(23)

Thus the transfer matrix for a network of length \( x \) is
\[
\begin{bmatrix}
A(x) & B(x) \\
C(x) & D(x)
\end{bmatrix} = \begin{bmatrix}
I_n e^{ax} & 0 \\
0 & I_n e^{-ax}
\end{bmatrix}
\begin{bmatrix}
\cosh (\Gamma x) - a \Gamma^{-1} \sinh (\Gamma x) & R(T)^{-1} \sinh (T x) \\
sc \Gamma^{-1} \sinh (\Gamma x) & \cosh (\Gamma x) + a \Gamma^{-1} \sinh (\Gamma x)
\end{bmatrix}
\]
(24)

and for a network of length \( d \)
\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = \begin{bmatrix}
I_n e^{ad} & 0 \\
0 & I_n e^{-ad}
\end{bmatrix}
\begin{bmatrix}
\cosh (\Gamma d) - a \Gamma^{-1} \sinh (\Gamma d) & R(T)^{-1} \sinh (T d) \\
sc \Gamma^{-1} \sinh (\Gamma d) & \cosh (\Gamma d) + a \Gamma^{-1} \sinh (\Gamma d)
\end{bmatrix}
\]
(25)
B. The Laplace Transform Variable Taylor Series Method

Another form for expressing the transfer solution matrix of equation (10) is given by Bertnolli and Halijak (10) as

\[
\begin{bmatrix}
A & B \\
C & D \\
\end{bmatrix} = \sum_0^d \left[ K(x,s) \right] \\
\text{where} \sum_0^d \left[ K(x,s) \right], \text{the matrizant of } K(x,s), \text{is the network's transfer matrix. Reference (8) gives the matrizant of } K(x,s) \text{ as}
\]

\[
\sum_0^d \left[ K(x,s) \right] = I_{2n} + \int_0^d K(x_1,s) \, dx_1 + \int_0^d \int_0^{x_1} K(x_1,s) \, dx_2 \, dx_1 \\
+ \ldots + \int_0^d \int_0^{x_1} \ldots \int_0^{x_n} K(x_1,s) K(x_2,s) \ldots K(x_n,s) \, dx_n \ldots dx_2 \, dx_1 + \ldots
\]  

For the exponentially tapered network

\[
K(x,s) = \begin{bmatrix}
0 & R e^{2ax} \\
s C e^{-2ax} & 0
\end{bmatrix}
\]

Using this relationship and performing the indicated integration gives

\[
\sum_0^d \begin{bmatrix}
0 & R e^{2ax} \\
s C e^{-2ax} & 0
\end{bmatrix} = \left[ I_n \begin{bmatrix} R f_2(ad) \\ s C f_3(ad) \end{bmatrix} I_n \right] + \left[ s R C f_5(ad) \begin{bmatrix} R s C R f_6(ad) \\ (s C R)^2 f_7(ad) \end{bmatrix} \right] + \ldots
\]

Thus the matric elements of the network's transfer matrix are of the form

\[
A = I_n + (s R C) f_5(ad) + (s R C)^2 f_9(ad) + \ldots + (s R C)^n f_{4n+1}(ad) + \ldots
\]

\[
B = R \left[ f_2(ad) + (s C R)^2 f_6(ad) + \ldots + (s C R)^n f_{4n+2}(ad) + \ldots \right]
\]

\[
C = s C \left[ f_3(ad) + (s R C)^2 f_7(ad) + \ldots + (s R C)^n f_{4n+3}(ad) + \ldots \right]
\]

\[
D = I_n + (s C R) f_8(ad) + (s C R)^2 f_{12}(ad) + \ldots + (s C R)^n f_{4n+4}(ad) + \ldots
\]
The $f_n(\alpha d)$'s in the previous equations may be generated by the integral matrizing process and are

\begin{align*}
  f_1(\alpha d) &= \frac{e^{\alpha d}}{a} \sinh(\alpha d) \\
  f_2(\alpha d) &= \frac{e^{\alpha d}}{a} \sinh(\alpha d) \\
  f_3(\alpha d) &= \frac{e^{-\alpha d}}{a} \sinh(\alpha d) \\
  f_4(\alpha d) &= \frac{e^{-\alpha d}}{a} \sinh(\alpha d) \\
  f_5(\alpha d) &= \frac{e^{\alpha d}}{2a^2} \left[ \sinh(\alpha d) - ade^{-\alpha d} \right] \\
  f_6(\alpha d) &= \frac{e^{\alpha d}}{2a^3} \left[ \alpha d \cosh(\alpha d) - \sinh(\alpha d) \right] \\
  f_7(\alpha d) &= \frac{e^{-\alpha d}}{2a^3} \left[ \alpha d \cosh(\alpha d) - \sinh(\alpha d) \right] \\
  f_8(\alpha d) &= \frac{e^{-\alpha d}}{2a^2} \left[ -\sinh(\alpha d) + ade^{\alpha d} \right] \\
  \vdots
\end{align*}

or in general

\begin{equation}
  \begin{bmatrix}
    f_{n+4}(\alpha d) & f_{n+5}(\alpha d) \\
    f_{n+6}(\alpha d) & f_{n+7}(\alpha d)
  \end{bmatrix} = \int_0^x \begin{bmatrix}
    f_n(\alpha d) & f_{n+1}(\alpha d) \\
    f_{n+2}(\alpha d) & f_{n+3}(\alpha d)
  \end{bmatrix} K(x, s) \, dx .
\end{equation}

(31)

To show that the matrizing solution yields the identical transfer matrix given in equation (25), equation (25) is expanded in its Taylor Series in $s$ about $s = 0$. It is then apparent by comparison that the solution obtained, a series of functions, is the same as would be obtained from the integral matrizing process.

IV. EXAMPLE CALCULATION OF NETWORK VOLTAGE GAIN

The circuit analyzed is the 6 terminal, 3 input and 3 output specialization of Figure 1. $V_{11}(s)$ is treated as the input by applying an ideal voltage source. All other terminals are unloaded. The equation relating
the terminal variables in this circuit is

\[
\begin{bmatrix}
V_{11}(s) \\
V_{21}(s) \\
I_{11}(s) \\
I_{21}(s)
\end{bmatrix}
= \begin{bmatrix} A & B \\ C & D \end{bmatrix}
\begin{bmatrix}
V_{12}(s) \\
V_{22}(s) \\
I_{12}(s) \\
I_{22}(s)
\end{bmatrix}
\tag{32}
\]

where \(A, B, C\) and \(D\) for the exponentially tapered circuit are given by equation (25). The open circuit terminal conditions used in this example force \(I_{21}(s), I_{12}(s)\) and \(I_{22}(s)\) to equal zero. Therefore equation (32) may be written as

\[
\begin{bmatrix}
V_{11}(s) \\
V_{21}(s) \\
I_{11}(s) \\
0
\end{bmatrix}
= \begin{bmatrix} A \\ C \end{bmatrix}
\begin{bmatrix}
V_{12}(s) \\
V_{22}(s)
\end{bmatrix}
= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}
\begin{bmatrix}
V_{12}(s) \\
V_{22}(s)
\end{bmatrix}
\tag{33}
\]

where the lower case notation indicates scalar elements. From equation (33) one voltage transfer function of interest is

\[
\frac{V_{12}(s)}{V_{11}(s)} = \frac{c_{22}}{a_{11}c_{22} - a_{12}c_{21}} \frac{1}{a_{11} - \frac{a_{12}c_{21}}{c_{22}}}.
\tag{34}
\]

To evaluate this voltage transfer function, the necessary scalar transfer matrix elements are found by application of Sylvester's Theorem (8) to the matric elements \(A\) and \(C\). For the example calculation

\(R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\), \(C = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\) and \(a = d = 1\).

Thus

\[
\mathbf{I}^2 = (\mathbf{I}^T)^2 = \begin{bmatrix} s+1 & -s \\ -s & 2s+1 \end{bmatrix}.
\tag{36}
\]
The necessary scalar elements are

\[
\begin{align*}
 a_{11} &= \frac{e}{2 \sqrt{5}} \left[ (\cosh \sqrt{\mu_1} - \frac{\sinh \sqrt{\mu_1}}{\sqrt{\mu_1}})(\sqrt{5} - 1) + (\cosh \sqrt{\mu_2} - \frac{\sinh \sqrt{\mu_2}}{\sqrt{\mu_2}})(\sqrt{5} + 1) \right] \quad (37) \\
 a_{12} &= \frac{e}{2 \sqrt{5}} \left[ (\cosh \sqrt{\mu_1} - \frac{\sinh \sqrt{\mu_1}}{\sqrt{\mu_1}})(-2) + (\cosh \sqrt{\mu_2} - \frac{\sinh \sqrt{\mu_2}}{\sqrt{\mu_2}})(2) \right] \quad (38) \\
 c_{21} &= \frac{e^{-1} s}{2 \sqrt{5}} \left[ \frac{\sinh \sqrt{\mu_1}}{\sqrt{\mu_1}} (-\sqrt{5} - 3) + \frac{\sinh \sqrt{\mu_2}}{\sqrt{\mu_2}} (-\sqrt{5} + 3) \right] \\
 c_{22} &= \frac{e^{-1} s}{2 \sqrt{5}} \left[ \frac{\sinh \sqrt{\mu_1}}{\sqrt{\mu_1}} (2\sqrt{5} - 4) + \frac{\sinh \sqrt{\mu_2}}{\sqrt{\mu_2}} (2\sqrt{5} + 4) \right] \quad (40)
\end{align*}
\]

where \( \mu_1 = 1 + \frac{s}{2} (3 + \sqrt{5}) \), \( \mu_2 = 1 + \frac{s}{2} (3 - \sqrt{5}) \) are the eigenvalues of \((r)^2\).

Thus, the voltage gain \( \frac{V_{12}(s)}{V_{11}(s)} \) is

\[
\frac{V_{12}(s)}{V_{11}(s)} = \frac{e^{-1}}{2 \sqrt{5}} \left[ \frac{a + b}{20(c+d+e)} \right]
\]

where

\[
\begin{align*}
 a &= (4 + 2\sqrt{5}) \frac{\sinh \sqrt{\mu_1}}{\sqrt{\mu_1}} \\
 b &= (-4 + 2\sqrt{5}) \frac{\sinh \sqrt{\mu_2}}{\sqrt{\mu_2}} \\
 c &= (20 + 8\sqrt{5}) \frac{\sinh \sqrt{\mu_1} \cosh \sqrt{\mu_2}}{\sqrt{\mu_1}} \\
 d &= (20 - 8\sqrt{5}) \frac{\sinh \sqrt{\mu_2} \cosh \sqrt{\mu_1}}{\sqrt{\mu_2}}
\end{align*}
\]
This equation shows the type of transfer function obtained when dealing with tapered multiterminal RC networks.

V. EXTENSIONS AND CONCLUSIONS

The space variable Taylor Series method provides a general tool for determining the transfer matrix of a distributed RC network. Although the solution presented here is for an exponentially tapered RC network, the method need not be restricted to either an exponential taper or an RC network. To extend the work to general lines it is only necessary to replace $s_\mathcal{C}$ with $\mathcal{Y}$ and $R$ with $\mathcal{Z}$ in equation (4). The lossless line, $\mathcal{Z} = sL$ and $\mathcal{Y} = s_\mathcal{C}$, might prove to be of particular importance.

The interested reader may check the transfer matrix solution contained here by substituting

\[
\begin{bmatrix}
  \mathcal{V}(x,s) \\
  \mathcal{I}(x,s)
\end{bmatrix} =
\begin{bmatrix}
  \mathcal{A}(x) & \mathcal{B}(x) \\
  \mathcal{C}(x) & \mathcal{D}(x)
\end{bmatrix}
\begin{bmatrix}
  \mathcal{V}(0,s) \\
  \mathcal{I}(0,s)
\end{bmatrix}
\]

in the original differential equation, equation (10).

The matrizant solution, an infinite series of multiple integrals, is shown to yield the network's transfer matrix. This method seems to be devious and is not recommended as a general analytic tool.
VI. REFERENCES


* Portion of thesis written in partial fulfillment of the requirements for the degree of Master of Science in Electrical Engineering at the University of Missouri at Rolla. The work reported in this paper was sponsored jointly by the National Science Foundation, under Grant No. GK-808 and the University of Missouri at Rolla.
Figure 1. 2n Terminal Tapered RC Microcircuit

Figure 2. Equivalent Circuit of Length $\Delta x$ For A 2n Terminal Tapered RC Network
Figure 3. Six-Terminal Exponential RC Microcircuit
APPENDIX A

Check of Solution Given by Equation (25)

The matrix partial differential equation solved was

\[
\frac{\partial}{\partial x} \begin{bmatrix} V(x,s) \\ I(x,s) \end{bmatrix} = \begin{bmatrix} 0 & Re^{2ax} \\ sCe^{-2ax} & 0 \end{bmatrix} \begin{bmatrix} V(x,s) \\ I(x,s) \end{bmatrix}.
\] (A-1)

The solution was of the form

\[
\begin{bmatrix} V(x,s) \\ I(x,s) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V(o,s) \\ I(o,s) \end{bmatrix}
\] (A-2)

where \( A, B, C \) and \( D \) are given by equation (25) as

\[
A = e^{+ax} \left[ \cosh (T x) - a T^{-1} \sinh (T x) \right]
\]
\[
B = e^{+ax} \left[ R (T^T)^{-1} \sinh (T^T x) \right]
\]
\[
C = e^{-ax} \left[ sC T^{-1} \sinh (T x) \right]
\]
\[
D = e^{-ax} \left[ \cosh (T^T x) + a (T^T)^{-1} \sinh (T^T x) \right].
\] (A-3)

Substituting equation (A-1) into equation (A-2) gives

\[
\frac{\partial}{\partial x} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V(o,s) \\ I(o,s) \end{bmatrix} = \begin{bmatrix} 0 & Re^{2ax} \\ sCe^{-2ax} & 0 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V(o,s) \\ I(o,s) \end{bmatrix}
\] (A-4)

or

\[
\begin{bmatrix} \frac{\partial}{\partial x} \quad \frac{\partial}{\partial x} \quad \frac{\partial}{\partial x} \quad \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} V(o,s) \\ I(o,s) \end{bmatrix} = \begin{bmatrix} Re^{2ax} & C & Re^{2ax} & D \\ sCe^{-2ax} & A & sCe^{-2ax} & B \end{bmatrix} \begin{bmatrix} V(o,s) \\ I(o,s) \end{bmatrix}.
\] (A-5)

This equation constitutes the method by which the solution obtained for equation (A-1) was validated.
APPENDIX B

Example Calculation of Network Voltage Gain

The circuit analyzed is shown in Figure 3. The matrix equation relating the terminal variables of this circuit is

\[
\begin{bmatrix}
  V_{11}(s) \\
  V_{21}(s) \\
  I_{11}(s) \\
  I_{21}(s)
\end{bmatrix}
= \begin{bmatrix}
  A & B \\
  C & D
\end{bmatrix}
\begin{bmatrix}
  V_{12}(s) \\
  V_{22}(s) \\
  I_{12}(s) \\
  I_{22}(s)
\end{bmatrix}
\]

(B-1)

where \( A, B, C \) and \( D \) for the exponentially tapered circuit are given by equation (25). The open circuit terminal conditions used in this example solution force \( I_{21}(s), I_{12}(s) \) and \( I_{22}(s) \) to be equal to zero. Therefore equation (B-1) may be written as

\[
\begin{bmatrix}
  V_{11}(s) \\
  V_{21}(s) \\
  I_{11}(s) \\
  0
\end{bmatrix}
= \begin{bmatrix}
  A & B \\
  C & D
\end{bmatrix}
\begin{bmatrix}
  V_{12}(s) \\
  V_{22}(s)
\end{bmatrix} = \begin{bmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22} \\
  c_{11} & c_{12} \\
  c_{21} & c_{22}
\end{bmatrix}
\begin{bmatrix}
  V_{12}(s) \\
  V_{22}(s)
\end{bmatrix}
\]

(B-2)

where the lower case notation indicates scalar elements.

From equation (B-2) the voltage gain in terms of the scalar elements is found to be

\[
\frac{V_{21}(s)}{V_{11}(s)} = \frac{c_{22}}{a_{11}c_{22} - a_{12}c_{21}} = \frac{1}{a_{11} - \frac{a_{12}c_{21}}{c_{22}}}
\]

(B-3)

Thus, to determine the voltage gain the scalar elements of the transfer matrix must be found.
Let
\[ R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \text{ and } a = 1 = d. \tag{B-4} \]

Therefore
\[ \Gamma^2 = (r^T)^2 = s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = (1)^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{B-5} \]
or
\[ \Gamma^2 = (r^T)^2 = \begin{bmatrix} s+1 & -s \\ -s & 2s+1 \end{bmatrix}, \tag{B-6} \]

where \( \Gamma \) was previously defined.

The eigenvalues of \( \Gamma^2 \) are
\[ \begin{align*}
\mu_1 &= 1 + \frac{s}{2} (3+\sqrt{5}) \\
\mu_2 &= 1 + \frac{s}{2} (3-\sqrt{5})
\end{align*} \]

Sylvester's Theorem states that a function of square matrix \( r^2 \) is
\[ F[r^2] = \sum_{r=1}^{r=n} F(\mu_r) [Z_r] \]

where \( Z_r \) is defined by
\[ Z_r = \frac{s^2 r^2 - \mu_r I}{s^2 r^2 (\mu_r - \mu_s)}. \]

Making use of Sylvester's Theorem
\[ Z_1 = \frac{\Gamma^2 - \mu_1 \Gamma}{\mu_1 - \mu_2} = \frac{1}{\sqrt{5}} \begin{bmatrix} \frac{\sqrt{5}-1}{2} & -1 \\
-1 & \frac{\sqrt{5}+1}{2} \end{bmatrix} \]
\[ \mathbf{Z}_2 = \frac{\mu_2 - \mu_1}{\mu_2 - \mu_1} = \frac{1}{\sqrt{5}} \begin{bmatrix} \sqrt{5+1} & 1 \\ +1 & \sqrt{5-1} \end{bmatrix} \]

and thus

\[ \mathbf{C} = e^{-1} \mathbf{sC} \mathbf{\Gamma}^{-1} \mathbf{\sinh} \mathbf{\Gamma} = e^{-1} \mathbf{sC} \left( \mathbf{\Gamma}^2 \right)^{-1/2} \mathbf{\sinh} \sqrt{\mathbf{\Gamma}^2} \]

\[ = e^{-1} \mathbf{sC} \left[ \frac{1}{\sqrt{\mu_1}} \mathbf{\sinh} \sqrt{\mu_1} \mathbf{Z}_1 + \frac{1}{\sqrt{\mu_2}} \mathbf{\sinh} \sqrt{\mu_2} \mathbf{Z}_2 \right] \]

\[ = e^{-1} \mathbf{sC} \left[ \frac{\mathbf{\sinh} \sqrt{\mu_1}}{\sqrt{\mu_1}} \mathbf{C} \mathbf{Z}_1 + \frac{\mathbf{\sinh} \sqrt{\mu_2}}{\sqrt{\mu_2}} \mathbf{C} \mathbf{Z}_2 \right] \]

\[ = e^{-1} \mathbf{sC} \left( \frac{\mathbf{\sinh} \sqrt{\mu_1}}{2 \sqrt{5} \sqrt{\mu_1}} \begin{bmatrix} \sqrt{5+1} & -\sqrt{5-3} \\ -\sqrt{5-3} & 2 \sqrt{5+4} \end{bmatrix} + \frac{\mathbf{\sinh} \sqrt{\mu_2}}{2 \sqrt{5} \sqrt{\mu_2}} \begin{bmatrix} \sqrt{5-1} & -\sqrt{5+3} \\ -\sqrt{5+3} & 2 \sqrt{5-4} \end{bmatrix} \right) \]

\[ \mathbf{A} = e^{+1} \left[ \mathbf{\cosh} \mathbf{\Gamma} - \mathbf{\Gamma}^{-1} \mathbf{\sinh} \mathbf{\Gamma} \right] \]

\[ = e^{+1} \left[ \left( \mathbf{\cosh} \sqrt{\mu_1} - \frac{\mathbf{\sinh} \sqrt{\mu_1}}{\sqrt{\mu_1}} \right) \mathbf{Z}_1 + \left( \mathbf{\cosh} \sqrt{\mu_2} - \frac{\mathbf{\sinh} \sqrt{\mu_2}}{\sqrt{\mu_2}} \right) \mathbf{Z}_2 \right] \]

\[ = e^{+1} \frac{1}{2 \sqrt{5}} \left( \frac{\mathbf{\cosh} \sqrt{\mu_1}}{\sqrt{\mu_1}} \frac{\mathbf{\sinh} \sqrt{\mu_1}}{\sqrt{\mu_1}} \begin{bmatrix} \sqrt{5-1} & -2 \\ -2 & \sqrt{5+1} \end{bmatrix} \right) \]

\[ + \left( \frac{\mathbf{\cosh} \sqrt{\mu_2}}{\sqrt{\mu_2}} - \frac{\mathbf{\sinh} \sqrt{\mu_2}}{\sqrt{\mu_2}} \right) \begin{bmatrix} \sqrt{5+1} & 2 \\ 2 & \sqrt{5-1} \end{bmatrix} \]

From these equations the necessary scalar elements of the transfer matrix are obtained as

\[ c_{22} = e^{-1} \mathbf{sC} \left( \frac{(2 \sqrt{5+4}) \mathbf{\sinh} \sqrt{\mu_1}}{2 \sqrt{5} \sqrt{\mu_1}} + \frac{(2 \sqrt{5-4}) \mathbf{\sinh} \sqrt{\mu_2}}{2 \sqrt{5} \sqrt{\mu_2}} \right) \]

\[ = e^{-1} \mathbf{sC} \left( \frac{(2 \sqrt{5+4}) \mathbf{\sinh} \sqrt{\mu_1}}{2 \sqrt{5} \sqrt{\mu_1}} + (2 \sqrt{5-4}) \frac{\mathbf{\sinh} \sqrt{\mu_2}}{\sqrt{\mu_2}} \right) \]
\[ a_{11} = \frac{e^{1} + 1}{2 \sqrt{5}} \left[ (\cosh \sqrt{\mu_1} - \frac{\sinh \sqrt{\mu_1}}{\sqrt{\mu_1}})(\sqrt{5} - 1) + (\cosh \sqrt{\mu_2} - \frac{\sinh \sqrt{\mu_2}}{\sqrt{\mu_2}})(\sqrt{5} + 1) \right] \]

\[ a_{12} = \frac{e^{-1}}{2 \sqrt{5}} \left[ (\cosh \sqrt{\mu_1} - \frac{\sinh \sqrt{\mu_1}}{\sqrt{\mu_1}})(-2) + (\cosh \sqrt{\mu_2} - \frac{\sinh \sqrt{\mu_2}}{\sqrt{\mu_2}})(2) \right] \]

\[ c_{21} = \frac{e^{-1} \sqrt{5}}{2 \sqrt{5}} \left[ \sinh \sqrt{\mu_1} (-\sqrt{5} - 3) + \sinh \sqrt{\mu_2} (-\sqrt{5} + 3) \right] \]

Previously it was found that

\[ \frac{V_{12}(s)}{V_{11}(s)} = \frac{c_{22}}{a_{11}c_{22} - a_{12}c_{12}} \]  

(B-7)

Using the solutions obtained for \( a_{11}, a_{12}, c_{12} \) and \( c_{22} \) the voltage gain is

\[ \frac{V_{12}(s)}{V_{11}(s)} = \frac{e^{-1}}{2 \sqrt{5}} \left[ \frac{a + b}{20(c + d + e)} \right] \]

where

\[ a = (4+2\sqrt{5}) \frac{\sinh \sqrt{\mu_1}}{\sqrt{\mu_1}} \]

\[ b = (-4+2\sqrt{5}) \frac{\sinh \sqrt{\mu_2}}{\sqrt{\mu_2}} \]

\[ c = (20+8\sqrt{5}) \frac{\sinh \sqrt{\mu_1} \cosh \sqrt{\mu_2}}{\sqrt{\mu_1}} \]

\[ d = (20-8\sqrt{5}) \frac{\sinh \sqrt{\mu_2} \cosh \sqrt{\mu_1}}{\sqrt{\mu_2}} \]

\[ e = -40 \frac{\sinh \sqrt{\mu_1} \sinh \sqrt{\mu_2}}{\sqrt{\mu_1} \sqrt{\mu_2}} \]
APPENDIX C

Derivatives of \( \text{col}[V(x,s), I(x,s)] \)

To develop the Taylor Series expansion of \( \text{col}[V(x,s), I(x,s)] \) it is necessary to obtain the derivatives of \( \text{col}[V(x,s), I(x,s)] \) evaluated at \( x = 0 \).

The first five derivatives used in the expansion given by equation (12) are

\[
\begin{align*}
[V(0,s)]^{(1)} &= K(0) [V(0,s)] \\
[I(0,s)]^{(1)} &= K(0) [I(0,s)] , \\
[V(0,s)]^{(2)} &= [2aM K(0) + K^2(0)] [V(0,s)] \\
[I(0,s)]^{(2)} &= [2aM K(0) + K^2(0)] [I(0,s)] , \\
[V(0,s)]^{(3)} &= [(4a^2 I_{2n} + K^2(0)) K(0) + 2a K^2(0) M] [V(0,s)] \\
[I(0,s)]^{(3)} &= [(4a^2 I_{2n} + K^2(0)) M K(0) + 4a K^2(0) + K^4(0)] [I(0,s)] , \\
[V(0,s)]^{(4)} &= [8a^3 I_{2n} + 4a K^2(0) M K(0) + 4a K^2(0) + K^4(0)] [V(0,s)] \\
[I(0,s)]^{(4)} &= [8a^3 K^2(0) + 4a K^4(0)] M [I(0,s)] , \\
[V(0,s)]^{(5)} &= [16a^4 I_{2n} + 12a^2 K^2(0) M K(0) + (8a^3 K^2(0) + 4a K^4(0))] M [V(0,s)] \\
[I(0,s)]^{(5)} &= [16a^4 K^2(0) + 12a^2 K^4(0)] M [I(0,s)] .
\end{align*}
\]
VITA

The author was born on November 19, 1943, in Cape Girardeau, Missouri. He received his primary and secondary education in Cape Girardeau, Missouri.

He was graduated from the University of Missouri at Rolla with a Bachelor of Science in Electrical Engineering in May of 1965. As an undergraduate, the author was a member of Eta Kappa Nu, honorary Electrical Engineering fraternity; Blue Key, honor fraternity, Institute of Electrical and Electronic Engineers, and Pi Kappa Alpha, social fraternity. He was elected St. Pat of 1965.

Upon completion of summer employment as a project engineer for Central Foundry Division of General Motors Corporation, the author entered graduate school at UMR in September 1965. While attending graduate school, the author served on the faculty of the University of Missouri at Rolla as an Instructor of Electrical Engineering.