An application of the variable gradient method of generating Liapunov functions to discrete systems

Henry N. Peterson

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AN APPLICATION OF THE VARIABLE
GRADIENT METHOD OF GENERATING
LIAPUNOV FUNCTIONS TO DISCRETE
SYSTEMS

BY
HENRY N. PETERSON

A
THESIS
submitted to the faculty of the
UNIVERSITY OF MISSOURI AT ROLLA
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ABSTRACT

The variable gradient method of generating Liapunov functions has been used only for continuous systems. This thesis presents an application of the variable gradient method to non-linear sampled-data control systems.

Certain modifications to the variable gradient method are developed and several examples are presented which demonstrate the application. An attempt was made to synthesize the control system parameters, but due to the complexity of the $\Delta V$ functions, the synthesis in general cannot easily be obtained. To circumvent the problem of determining if the $\Delta V$ functions are negative definite, a set of computer programs were written to determine a finite region of asymptotic stability in the state space. The nonlinear difference equations are solved for several initial conditions to test the region of stability. The region was found to be a valid region, but smaller than the true region.
ACKNOWLEDGMENTS

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CHAPTER I
INTRODUCTION

In the past few years an increasing amount of interest has developed in sampled-data control systems. The increased use of digital computers in control applications has created a demand for useful design techniques. Of central importance in the design of any control system is the determination of system stability. The following is a discussion of a method for determining asymptotic stability by the method of Liapunov for non-linear sampled-data systems.

Only systems of the type shown in Figure 1.1 shall be considered.

\[ r(t) + e(t) \rightarrow e^*(t) \rightarrow h(t) \rightarrow m(t) \rightarrow c(t) \]

Figure 1.1

It should be noted that although the chosen configuration is not completely general, a large number of control systems can be reduced to this configuration. The zero-order hold is used since in most practical systems a sampler is followed by a signal restoration device of some sort.

The system is assumed to be autonomous, i.e. \( r(t) = 0 \) and \( T = \) a constant. This may seem to restrict the practicality of the procedure, as engineers are more concerned with stability when an input is present. Even with this
restriction, the information obtained is useful and will be treated in more detail later in the thesis.

When dealing with the stability problem it is necessary to clearly define the type of stability being sought. In the following only asymptotic stability (in the sense of Liapunov), either global or in a finite region in the state space, is to be determined.

The main difficulty in applying Liapunov's Theorem on stability is in generating a suitable Liapunov Function. The variable gradient method developed for continuous systems by Gibson and Schultz\(^{(2)}\)* has eliminated some of the guesswork in picking a Liapunov Function by providing a systematic procedure for generating the function.

To the best of the author's knowledge, the method has not been applied to sampled-data systems. The following discussion develops a procedure for applying the variable gradient method to the sampled-data control system.

In summary, the problem under consideration is the determination of asymptotic stability of non-linear sampled-data control systems by applying the variable gradient method. Global stability is sought, but if it cannot be readily obtained, a finite region is the state space will be determined where the system is asymptotically stable.

*Numbers in parenthesis designate references in the Bibliography.
CHAPTER II

REVIEW OF LITERATURE

Since 1960 Liapunov's Theorem has received an increasing amount of attention here in the United States; but only in the past few years has there been much work done in applying Liapunov's Theorem to discrete systems.

The paper by Kalman and Bertram\(^{(4)}\) in 1960 was the first paper written from an engineering viewpoint. The paper is written in two parts. The first deals with basic definitions and applications to continuous systems. The second part is devoted entirely to discrete systems. Liapunov's second method is shown to hold for discrete systems and several examples using the theorems are worked. La Salle and Lefschetz\(^{(10)}\) in their book published in 1961 give a very basic treatment of Liapunov's second method. This book considers only continuous systems, but does present some information on the determination of a region of stability in state space. La Salle's research report in 1960\(^{(9)}\) for the Research Institute for Advanced Studies dealt with extensions to Liapunov's second method. In particular, the extent of stability was investigated. Theorems were developed and proved that are useable in determining the region in state space for which a system will be asymptotically stable. More recently a translation of Wolfgang Hahn's\(^{(3)}\) book appeared in 1963. It is written from a mathematical point of view and gives a complete treatment of Liapunov's second method,
including proofs of most of the theorems. Discrete systems and extent of stability are included in his discussions. The extensive bibliography is a useful aid to the researcher.

The variable gradient method was developed in 1961-62 by Gibson and Schultz as part of a research program sponsored by the National Science Foundation. The report of this research\(^{(2)}\) published in 1962, is thorough and readable, and gives the background of other attempts at generating Liapunov functions. In particular, the methods of Ingwerson and Szego are presented, as the variable gradient method was an outgrowth of these two methods. The report gives a complete development of the method for autonomous systems, including numerous examples. Non-autonomous systems are considered, but the results are inconclusive. It should be mentioned that only continuous systems are considered. A condensed version of the method for autonomous systems was published in the AIEE Transactions\(^{(12)}\) in 1962. It contained the essence of the research report. Komo in his Master's Thesis\(^{(7)}\) presented Ingwerson's method as well as the variable gradient approach. Examples are worked and the results of the two methods compared.

Until very recently, the literature on nonlinear sampled-data control systems was very scarce. The reason for this is that the theory of nonlinear sampled-data systems has not been fully developed.

Papers by Shinzo Kodama\(^{(5)(6)}\) in 1962 discuss the problem of finding asymptotic stability for nonlinear sampled-
data control systems. Theorems are presented and examples worked. More recently O'Shea\textsuperscript{(11)} in a paper published in 1964 gave an approximation for the asymptotic stability boundary for discrete-time systems by using an inverse transformation technique on a Liapunov Function. Kuo's book\textsuperscript{(8)}, published in 1963, gives a full treatment of linear sampled-data systems and has several chapters devoted to nonlinear sampled-data systems. A book by DeRusso, Roy and Close\textsuperscript{(1)}, published in 1965, gives a complete treatment of the state variable technique for both continuous and sampled-data systems. A chapter on Liapunov's stability theory and the use of state variables, is included.

The purpose of this thesis is the development of a procedure for determining asymptotic stability which uses most of the above techniques. It is not an entirely new method; rather it is the application of an existing method to a new area where the method has not been previously used.
CHAPTER III

THEORETICAL CONSIDERATIONS

The purpose of this chapter is to present the necessary changes required in the variable gradient method so that it can be applied to discrete systems. Additional theory on the determination of the region of stability in state space is also discussed.

A. Notation

Before beginning a discussion of the theory, certain notations will be adopted. Lower case letters with bars over them will represent row vectors; thus $\vec{x} = (x_1, x_2, x_3, x_4, \ldots, x_n)$. Capital letters with bars over them will represent matrices; thus

$$\bar{A} = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1m} \\
a_{21} & a_{22} & \cdots & a_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nm}
\end{bmatrix}.$$

Vector functions will be represented by capital letters with bars under them; thus

$$\bar{F}(\vec{x}) = \begin{bmatrix}
F_1(\vec{x}) \\
F_2(\vec{x}) \\
F_3(\vec{x}) \\
\vdots \\
F_n(\vec{x})
\end{bmatrix}.$$

Scalar quantities are represented by upper or lower case
letters or symbols with no bars. The scalar function \( V(\bar{x}) \)
means \( V(\bar{x}) = V(x_1, x_2, \ldots, x_n) \). Scalar functions may be
upper or lower case letters or symbols. The only exception
to the vector notation is the gradient of a scalar function;
a vector, which is denoted by \( \text{grad} V \). The transpose of a
vector or matrix is designated by \( \bar{x}' \) or \( \bar{A}' \). The scalar
product of \( \bar{x}, \bar{y} \) is \( \bar{x}'\bar{y} \). The quadratic form associated with
a square matrix \( \bar{A} \) is \( \bar{x}'\bar{A}\bar{x} \). The Euclidean norm of a vector
\( \bar{x} \) is \( ||\bar{x}|| \), where \( ||\bar{x}||^2 = x_1^2 + x_2^2 + \ldots + x_n^2 \).

The small letter \( t \) is reserved for the representation
of continuous time. The capital letter \( T \) is reserved for
the sample period. The time derivative of a function
\( V(t) \) is \( \frac{dV}{dt} = \dot{V} \). The total difference of a function
\[ V[\bar{x}(kT)] = V[x_1(kT), x_2(kT), \ldots, x_n(kT)] \]
is
\[ \Delta V[\bar{x}(kT)] = V \left\{ \bar{x} \left[ (k+1)T \right] \right\} - V[\bar{x}(kT)] \]
\[ = V[x_1(kT) + \Delta x_1(kT), x_2(kT) + \Delta x_2(kT), \ldots, x_n(kT) + \Delta x_n(kT)] \]
\[ - V[x_1(kT), x_2(kT), \ldots, x_n(kT)] \]
where \( k = 0, 1, 2, \ldots, n \)
and \( \Delta x_i(kT) = x_i[(k+1)T] - x_i(kT), i = 1, 2, \ldots, n. \)

The continuous control system can be represented by \( n \)
first order differential equations. Thus the system equa-
tions might be,
\[ x_1 = F_1(x_1, x_2, x_3, \ldots, x_n) \]
\[ x_2 = F_2(x_1, x_2, x_3, \ldots, x_n) \]
\[ \vdots \]
\[ x_n = F_n(x_1, x_2, x_3, \ldots, x_n) \]

or

\[ \dot{x} = F(x) \] (3.1)

where the \( x_i \) are functions of \( t \).

The assumption is made that

\[ \dot{x} = F(\bar{o}) = \bar{o}; \] (3.2)

i.e., the equilibrium point is the origin.

The discrete control system can be represented by \( n \) first order difference equations. Thus the system equations might be

\[ \Delta x_1 = F_1(x_1, x_2, x_3, \ldots, x_n) \]
\[ \Delta x_2 = F_2(x_1, x_2, x_3, \ldots, x_n) \]
\[ \vdots \]
\[ \Delta x_n = F_n(x_1, x_2, x_3, \ldots, x_n) \]

or by

\[ \Delta \bar{x} = F(\bar{x}) \] (3.3)

where \( x_i \) are functions of \( kT \). Again the assumption is made that

\[ \Delta \bar{x} = F(\bar{o}) = \bar{o}. \] (3.4)
B. Liapunov's and Sylvester's Theorems

The following is a listing of the applicable Liapunov's theorems.

**Theorem 1.**

If there exists a real scalar function $V(x)$, continuous with continuous first partial derivatives, such that

1. $V(x) > 0$ for $x \neq 0$
2. $\frac{dV}{dt} \leq 0$ for $x \neq 0$ (at least negative semi-definite)
3. $V(x) \rightarrow \infty$ as $||x|| \rightarrow \infty$
4. $\frac{dV}{dt}$ not identically zero along a solution of the system other than the origin,

then system (3.1) under assumption (3.2), is globally asymptotically stable.

**Theorem 2.**

If there exists a real scalar function $V(x)$ continuous in $x$ with continuous first partial derivatives, such that

1. $V(x) > 0$ for all $x \neq 0$
2. $\Delta V(x) \leq 0$ when $x \neq 0$
3. $V(x) \rightarrow \infty$ when $||x|| \rightarrow \infty$
4. $\Delta V(x)$ not identically zero along a solution of the system other than the origin.

then the system (3.3) under assumption (3.4) is globally asymptotically stable. (This is the discrete equivalent of Theorem 1.)
The following theorem and corollary are helpful in determining if certain functions are positive or negative definite.

**Theorem 3.** (Sylvester's Theorem)

In order that the quadratic form $x'\bar{C}x$ be positive definite it is necessary and sufficient that the principal minors of its determinant, that is the numbers

$$
\begin{vmatrix}
C_{11}
\end{vmatrix} > 0, \quad
\begin{vmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{vmatrix} > 0, \cdots
\begin{vmatrix}
C_{11} & C_{12} & \cdots & C_{1n} \\
C_{21} & C_{22} & \cdots & C_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
C_{1n} & C_{2n} & \cdots & C_{nn}
\end{vmatrix} > 0
$$

be positive.

**Corollary to Theorem 3.**

A necessary and sufficient condition that $x'\bar{C}x$ be negative definite is the conditions of the Theorem 3 on the principal minors with the inequalities of the odd order determinants reversed.

The correctness of the corollary is demonstrated in the following way. Assume that $\bar{C}$ is the matrix of coefficients of a positive definite quadratic form. Then the principal minors must be positive. Now a negative definite form is created by multiplying $\bar{C}$ by -1 i.e., each member of $\bar{C}$ is negative. Now examine the principal minors.
From the rules on determinants we see that an odd number of negative signs can be factored out of an odd ordered determinant and an even number out of an even ordered determinant. Thus the sign of each magnitude is positive or negative as the order of the determinant is even or odd. To determine a condition for a negative definite $\bar{C}$ we force the sign of the left hand side of the magnitudes of Theorem 3 to be positive. Thus the inequalities of the odd order magnitudes are reversed.

C. Review of Variable Gradient Method for Continuous Systems

The variable gradient method is based upon the assumption of a generalized gradient of $V$ (the Liapunov function) in the following form.

$$\nabla V = \begin{vmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{vmatrix} = \begin{vmatrix} \nabla V_1 \\ \nabla V_2 \\ \vdots \\ \nabla V_n \end{vmatrix}$$

The $a$'s are assumed to be made up of a constant and a variable part such that $a_{ij} = a_{ij}(\text{constant}) + a_{ij}(\bar{x})$. In their research report $^{(2)}$ Gibson and Schultz have shown that
certain restrictions may be placed on the a's without impairing the generality of the solution a great deal. The restrictions are:

1. The a_{\alpha\beta}'s are assumed to be functions of x only since the a_{ij}(\vec{x})x_i terms will produce any required cross products.

2. To insure that V is a closed surface one of the a_{\alpha\beta} is forced to be equal to a constant. Usually a_{nn} is set equal to 2.

3. The a_{ij}(\vec{x}) i.e. the variable portion of a_{ij} is assumed not to be a function of x_i.

These restrictions are imposed to insure that V will be quadratic in x_i. If a suitable V function cannot be found under these restrictions they should be removed and an attempt made to find a V using the more general case.

To insure that V may be determined by a line integration of grad V, the following curl equations must be satisfied:

\[ \frac{\partial \text{grad } V_i}{\partial x_j} = \frac{\partial \text{grad } V_j}{\partial x_i} \quad i, j = 1, 2, \ldots, n \quad (3.7) \]

One method of meeting this requirement is to choose a_{ij} = a_{ji} = a constant. This may be too restrictive. If it is, then the curl equations must be satisfied by finding an appropriate function of \( \vec{x} \).

The procedure in using the variable gradient method is set forward by the following theorem:

**Theorem 4** (Gibson and Shultz (2) p. 60)

If for the Equation (3.1) under assumption (3.2) there
exists a real vector function $\nabla V$ with elements $\nabla V_i$, such that

1. \[ \frac{\partial \nabla V_j}{\partial x_j} = \frac{\partial \nabla V_i}{\partial x_i} \]

2. \[(\nabla V)' P(\bar{x}) \leq 0, \text{ but not identically zero on a solution of (3.1) other than the origin and such that the scalar function} \]

\[ V(\bar{x}) = \int_0^{x_1} \nabla V_1 (\sigma_1, 0, 0, \ldots, 0) \, d\sigma_1 \]

\[ + \int_0^{x_2} \nabla V_2 (x_1, \sigma_2, 0, 0, \ldots, 0) \, d\sigma_2 + \ldots \]

\[ + \int_0^{x_n} \nabla V_n (x_1, x_2, \ldots, x_{n-1}, \sigma_n) \, d\sigma_n \]  \hspace{1cm} (3.8)

is continuous with continuous first partials, and

3. $V(\bar{x}) > 0$ for all $\bar{x} \neq 0$

4. $V(\bar{x}) \to \infty$ as $\|\bar{x}\| \to \infty$

then (3.1) is globally asymptotically stable.

Restrictions on the type of equations that may be handled by Theorem 4 are given by the following theorem:

**Theorem 5** (Gibson and Shultz)

If the system described by (3.1) under assumption (3.2) is Lipschitzian (i.e. $F(\bar{x})$ satisfies the Lipschitz condition in a region $R$ if the following condition is satisfied

\[ \|\bar{x}(\bar{\theta}) - \bar{x}(\bar{\delta})\| \leq K \| \bar{\theta} - \bar{\delta} \|, \]

and if the equilibrium state, $\bar{x}_e = 0$, is globally asymptotically stable, then a grad $V$
exists, from which \( V(\vec{x}) \) may be obtained by line integration, and the \( V(\vec{x}) \) so obtained is capable of establishing global asymptotic stability. Notice that Theorem 5 implies continuity of \( F \) in \( \vec{x} \) only not in time. This fact will be used later to establish a corresponding theorem to Theorem 4 for discrete systems.

To summarize the above statements, the following outline for the formal application of the variable gradient method is listed.

1. Assume a gradient in the general form (3.6).
2. From the variable gradient, form \( \frac{dV}{dt} \), as \( \frac{dV}{dt} = (\text{grad } V)' \vec{x} \).
3. Using the Equations (3.7), constrain \( \frac{dV}{dt} \) to be at least negative semi-definite.
4. From the known gradient, determine \( V \) by a line integration (3.8).
5. Invoke the necessary theorem to establish stability.

D. Application of Variable Gradient Method to Discrete Systems

The literature (1), (3), and (4) adequately shows the transfer of the properties of Liapunov's theorems for continuous systems to discrete systems. Thus the purpose here is not to prove this but to state the necessary changes in the variable gradient method so that it may be applied to discrete systems.
Recall that the only continuity condition in the application of the variable gradient to continuous systems was continuity of $F$ in $\bar{x}$. This would seem to indicate that the variable gradient method will apply directly. This is not the case. Consider the result of taking the total difference of a function of several variables.

The total difference of a function of several variables can be written as (see reference (13),

$$
\Delta V(\bar{x}) = \frac{\partial V}{\partial x_1} \Delta x_1 + \frac{\partial V}{\partial x_2} \Delta x_2 + \ldots + \frac{\partial V}{\partial x_n} \Delta x_n
$$

$$
+ \Theta_1(\Delta \bar{x}) \Delta x_1 + \Theta_2(\Delta \bar{x}) \Delta x_2 + \ldots + \Theta_n(\Delta \bar{x}) \Delta x_n
$$

(3.9)

This may be rewritten in a more compact notation as follows:

$$
\Delta V(\bar{x}) = (\text{grad } V)' \Delta \bar{x} + \sum_{i=1}^{n} \Theta_i(\Delta \bar{x}) \Delta x_i
$$

(3.10)

where $\lim_{\Delta \bar{x} \to 0} \Theta_i(\Delta \bar{x}) = 0$

In a discrete system the $\Theta_i(\Delta \bar{x})$ does not go to zero and thus must be taken into account. Notice that $\Delta V$ is analogous to $\frac{dV}{dt}$ for continuous systems. Normally $\frac{\Delta V}{\Delta t}$ would be used, but since we are dealing with autonomous systems, $\Delta t = T$ is a constant, which is removed by multiplication by $T$.

Clearly the variable gradient cannot generate the $\Theta_i(\Delta \bar{x}) \Delta x_i$ terms of $\Delta V$. To use the variable gradient method for discrete systems, it will be convenient to define the following quantity:

$$
\Delta V_1(\bar{x}) = \Delta V - \sum_{i=1}^{n} \Theta_i(\Delta \bar{x}) \Delta x_i = (\text{grad } V)' \Delta \bar{x}
$$

(3.11)
as an intermediate difference in $V$. The $\Delta V_I(\bar{x})$ as indicated by (3.11) can be found by use of the variable gradient. Knowledge of the components of $\text{grad } V$ can be gained by placing constraints of $\Delta V_I(\bar{x})$. However, it is not necessary to force $\Delta V_I(\bar{x})$ to be negative semi-definite. Instead, only those choices of the a's which might simplify $V(\bar{x})$ or $\Delta V(\bar{x})$ are made.

Using the above statements and Theorem 2 we state Theorem 6 which is an analogy of Theorem 4 for discrete systems.

**Theorem 6**

If for the Equation (3.3) under assumption (3.4), there exists a real vector function $\text{grad } V$ with elements $(\text{grad } V)_i$ such that

1. $\frac{\partial \text{grad } V_i(\bar{x})}{\partial x_j} = \frac{\partial \text{grad } V_j(\bar{x})}{\partial x_i}$
2. $V(\bar{x}) > 0$ for all $x(kT) \neq 0$, as determined by a line integration of $\text{grad } V$ (see Theorem 4)
3. $\Delta V(\bar{x}) \leq 0$ for all $x(kT) \neq 0$ as determined by taking the total difference of $V$, and
4. $V(\bar{x}) \to \infty$ as $\|x(kT)\| \to \infty$ then the system is globally asymptotically stable.

Finally, a formal procedure is stated for applying the variable gradient method to discrete systems.

1. Assume a $\text{grad } V(\bar{x})$ in the general form (3.6), as discussed in C above, replacing the continuous variables by the discrete counterparts.
2. Form $\Delta V(\bar{x}) = (\text{grad } V(\bar{x}))' \bar{x}$.

3. Constrain $\Delta V(\bar{x})$ so that the curl equations are met. Also constrain the a's of the gradient so as to eliminate as many indefinite terms as possible. Note that it is not necessary to force $\Delta V(\bar{x})$ to be negative semi-definite.

4. From the known gradient, determine $V(\bar{x})$ by a line integration.

5. Invoke the necessary theorem to establish stability.

E. Extent of Asymptotic Stability in State Space.

In the above discussion, global asymptotic stability is determined if the conditions of the appropriate theorems are met. This insures that the system will be stable under any initial condition. There are times when global stability is not needed or unattainable due to the complexity of the $\Delta V(\bar{x})$ function. When these conditions arise it is desirable to be able to determine a region in state space where the system will be asymptotically stable.

Several authors have investigated this problem. Reference (9) devotes itself entirely to this problem. Reference (2) gives the following theorem.

Theorem 7.

If $V$ (the Liapunov function) is positive definite and

1. One of the surfaces; $V = a$ constant, bounds the region.

2. The gradient of $V$, $(\text{grad } V)$, is not zero anywhere
in the region except at the equilibrium position.

3. \( \frac{dV}{dt} \) is negative or zero inside the region. Then the region is defined by \( V = \text{constant} \).

Hahn, in his book (3), further elaborates on the problem of extent in state space in the following manner. If the system equations are continuous in the state variables and of such a nature that the existence and uniqueness of solutions, as well as their continuous dependence on the initial values is assured and if \( V \) is positive definite, but, \( \frac{dV}{dt} \) is not negative definite throughout the state space, then the region of stability may be found by finding the smallest constant that \( V \) may be set equal to, such that the hypersurface formed in this manner just touches from the inside the surface defined by \( W(x) = \frac{dV}{dt} = 0 \). Conditions that give the point of contact between the surfaces are that the \( \text{grad} \, V \) be parallel to \( \text{grad} \, W \) at the point of contact. One method of determining the point of contact is as follows; let \( \vec{z} \) be the vector that satisfies \( W(x) = 0 \). Then the point of contact is given by

\[
W(\vec{z}) = 0, \quad \left. \frac{\partial V}{\partial x_1} \right|_{x_1=z_1} = \left. \frac{\partial V}{\partial x_2} \right|_{x_2=z_2} = \cdots = \left. \frac{\partial V}{\partial x_n} \right|_{x_n=z_n}
\]

The ratio of partial derivatives is a condition for parallelism of \( \text{grad} \, V \) and \( \text{grad} \, W \).

Again the analogy is made between continuous and discrete systems i.e., if we replace \( \frac{dV}{dt} \) in the above statements...
by $\Delta V(\tilde{x})$ we have the analogous statements for discrete systems.

If $\Delta V(\tilde{x})$ can be expressed as a polynomial in the state variables, the problem of finding the region of stability reduces to that of determining the zeros of a polynomial and then fitting the $V(\tilde{x}) = \text{a constant}$, curve inside the curve defined by $\Delta V(\tilde{x}) = 0$. In the next chapter, example problem (4.3) demonstrates the determination of a region of stability using this method.
CHAPTER IV

EXAMPLES USING THE VARIABLE GRADIENT METHOD

In this chapter several examples are given to demonstrate the application of the variable gradient method to sampled-data control systems.

Example Problem 4.1

Figure 4.1 shows a block diagram of a linear sampled-data control system. The purpose of this example is to demonstrate the correctness of the theorems given in Chapter III on a relatively simple problem.

\[ r(t) = 0 \quad e(t) \quad e^*(t) \quad Z.O.H. \quad h(t) \quad \text{LINEAR ELEMENTS} \]

- \[ G_{ho}(s) = \frac{1-e^{-sT}}{s} \]

Figure 4.1

The system is described by the following state variable difference equations:

\[ x_1[(k+1)T] = x_1(kT) - K[T - (1 - e^{-T}) x_2(kT)] + K(l-e^{-T}) x_2(kT) \]

\[ x_2[(k+1)T] = -(1-e^{-T})x_1(kT) + e^{-T}x_2(kT) \quad (4.1) \]

The above equations will be used to find a stability boundary in the parameter space \( K \) vs. \( T \), using the variable gradient method. The results are verified by applying the
Routh-Hurwitz criterion to obtain a comparison.

To begin the application of the variable gradient method, the first order differences of Equation (4.1) are formed and the substitution $C = (1-e^{-T})$ (4.2) is made.

$$\Delta x_1(kT) = x_1[(k+1)T] - x_1(kT) = -K(T-C)x_1 + KCx_2(kT)$$

$$\Delta x_2(kT) = -Cx_1(kT)Cx_2(kT)$$ (4.3)

To simplify the algebra the following substitutions are made

$$R_1 = K(T-C)$$

$$R_2 = KC$$ (4.4)

It should be noted that $R_1 > 0$ & $R_2 > 0$ for all $K$, $T$, and $C$. Equation (4.3) now becomes

$$\Delta x_1(kT) = -R_1x_1(kT) + R_2x_2(kT)$$

$$\Delta x_2(kT) = -\frac{R_2}{K} [x_1(kT) + x_2(kT)]$$ (4.5)

To simplify the work, the notation $kT$ is not used in the following with the understanding that all variables are discrete functions of time.

The next step is to assume a general gradient of the form (3.5) and apply the assumptions of Chapter III.

$\text{grad } V = \begin{vmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + 2x_2 \end{vmatrix}$ (4.6)

$\Delta V_I$ is now formed.
\[ \Delta V_I = (\text{grad } V)' \delta x = (a_{11}x_1 + a_{12}x_2)(-R_1x_1 + R_2x_2) - \frac{R_2}{K}(x_1 + x_2)(a_{21}x_1 + 2x_2) \]

which simplifies to

\[ \Delta V_I = -(R_1a_{11} + \frac{R_2}{K}a_{21})x_1^2 + (R_2a_{11} - R_1a_{12} - \frac{2R_2 a_{21}}{K})x_1x_2 + (R_2a_{12} - \frac{2R_2}{K})x_2^2 \]  \hspace{1cm} (4.7)

To satisfy the Curl Equations (3.6), let \( a_{12} = a_{21} \). It is convenient at this time to remove the \( x_1x_2 \) term and in so doing knowledge of one of the \( a \)'s is obtained.

Let

\[ a_{11} = \frac{1}{R_2} (R_1 + \frac{R_2}{K})a_{12} + \frac{2R_2}{K} \]

\[ = \frac{1}{K R_2} (KR_1 + R_2) a_{12} + 2R_2 \]  \hspace{1cm} (4.8)

Now substituting (4.8) and \( a_{12} = a_{21} \) into (4.7) and (4.6) gives,

\[ \Delta V_I = -\frac{R_1}{KR_2} (KR_1 + R_2) a_{12} + 2R_2 + \frac{R_2 a_{12}}{K} x_1^2 \]

\[ - (\frac{2R_2}{K} - R_2 a_{12}) x_2^2 \]  \hspace{1cm} (4.9)

and

\[ \text{grad } V = \begin{vmatrix} \frac{1}{KR_2} [(KR_1 + R_2) a_{12} + 2R_2] x_1 + a_{12}x_2 \\ a_{12}x_1 + 2x_2 \end{vmatrix} \]  \hspace{1cm} (4.10)
V is now determined by the following line integration:

\[ V = \int_{0}^{x_1} \frac{1}{KR_2} [(KR_1 + R_2) a_{12} + 2R_2] d\mathcal{Y}_1 + \int_{0}^{x_2} (a_{12} x_1 + 2\mathcal{Y}_2) d\mathcal{Y}_2 \]

\[ = \frac{1}{2KR_2} [(KR_1 + R_2) a_{12} + 2R_2] x_1^2 + a_{12} x_1 x_2 + x_2^2 \quad (4.11) \]

Notice that (4.11) is a quadratic form \( \mathcal{X}' \mathcal{A} \mathcal{X} \) where

\[
\mathcal{A} = \begin{bmatrix}
\frac{1}{2KR_2} [(KR_1 + R_2) a_{12} + 2R_2] & a_{12} \\
\frac{a_{12}}{2} & 1
\end{bmatrix} \quad (4.12)
\]

Sylvester's Theorem (Theorem 3) may be applied to determine the conditions on \( a_{12} \) for \( V \) to be positive definite. Application of this theorem results in the following inequalities:

\[ \frac{1}{2KR_2} [(KR_1 + R_2) a_{12} + 2R_2] > 0 \]

or \( a_{12} > \frac{-2R_2}{KR_1 + R_2} \quad (4.13) \)

and \( \frac{1}{2KR_2} [(KR_1 + R_2) a_{12} + 2R_2] - \frac{a_{12}^2}{4} > 0 \)

or \( -KR_2 a_{12}^2 + 2(KR_1 + R_2) a_{12} + 4R_2 > 0 \)

which results in

\[
\frac{(KR_1 + R_2) - \sqrt{(KR_1 + R_2)^2 + 4KR_2^2}}{KR_2} < a_{12} < \frac{(KR_1 + R_2) + \sqrt{(KR_1 + R_2)^2 + 4KR_2^2}}{KR_2} \quad (4.14)
\]

The final determination of a value for \( a_{12} \) is postponed until \( \Delta V \) has been examined.
To simplify the algebra, the following substitution is made in Equation (4.11).

\[
H = \frac{1}{2KR_2} \left[ (KR_1 + R_2) a_{12} + 2R_2 \right] \tag{4.15}
\]

Clearly \( v \to \infty \) as \( \|x\| \to \infty \) which completes the test on \( v \).

\( \Delta V \) is now formed by incrementing \( v \).

\[
\Delta V = H(x_1 + \Delta x_1)^2 + a_{12}(x_1 + \Delta x_1)(x_2 + \Delta x_2)
+ (x_2 + \Delta x_2)^2 - Hx_1^2 - a_{12}x_1x_2 - x_2^2 \tag{4.16}
\]

or \( \Delta V = 2Hx_1 \Delta x_1 + H \Delta x_1^2 + a_{12}x_1 \Delta x_2 + a_{12}x_2 \Delta x_1
+ a_{12} \Delta x_1 \Delta x_2 + 2x_2 \Delta x_2 + \Delta x_2^2 \tag{4.17} \)

Now substituting Equations (4.5) in (4.17) and collecting like terms gives

\[
\Delta V = (-2HR_1 + HR_1^2 - \frac{a_{12}R_2}{K} + \frac{a_{12}R_1R_2}{K} + \frac{R_2^2}{K^2}) x_1^2
+ [2HR_2 - 2HR_1R_2 - \frac{a_{12}R_2}{K} - a_{12}R_1 + a_{12} \frac{R_1R_2 - R_2^2}{K}
- \frac{2R_2}{K} + \frac{2R_2^2}{K^2} ] x_1x_2 + [HR_2^2 + a_{12}R_2 - \frac{a_{12}R_2^2}{K}
- \frac{2R_2}{K} + \frac{R_2^2}{K^2} ] x_2^2 \tag{4.18} \)

\( \Delta V \) must be at least negative semi-definite. Thus conditions are sought on \( a_{12}, T, \) and \( K \) to insure this. Since \( \Delta V \) is also a quadratic form, the corollary to Theorem 3 may be employed in determining the constraints. An examination of the coefficients of \( \Delta V \) shows that \( a_{12} = 0 \) is the simplest choice for \( a_{12} \). This choice also satisfies the constraints (4.13)
and (4.14) on $a_{12}$.

Substituting $a_{12} = 0$ in Equation (4.18) gives

$$
\Delta V = (-2HR_1 + HR_1^2 + \frac{R_2^2}{K}) x_1^2
+ (2HR_2 - 2HR_1 R_2 - \frac{2R_2^2}{K}) x_1 x_2
+ (HR_2^2 - \frac{2R_2^2}{K} + \frac{R_2^2}{K^2}) x_2^2
$$

(4.19)

The corollary to Theorem 3 is now applied to the coefficients of (4.19). Thus the first principal minor gives

$$
-2HR_1 + HR_1^2 + \frac{R_2^2}{K} < 0
$$

(4.20)

Substituting Equations (4.4) and (4.15) into (4.20) and multiplying by $K^2$ gives

$$
-2K^2(T - C) + K^3(T - C)^2 + K^2C^2 < 0
$$

or $K < \frac{2(T-C) - C^2}{(T-C)^2}$

(4.21)

Substituting (4.2) in (4.21) gives

$$
K < \frac{2(T - 1 + e^{-T}) - (1 - e^{-T})^2}{(T - 1 + e^{-T})^2} = G_1
$$

(4.22)

The second principal minor when Equation (4.15) is substituted into the coefficients of (4.19) gives

$$
\left(\frac{-2R_1 + R_1^2}{K} + \frac{R_2^2}{K^2}\right)\left(\frac{R_2^2}{K} + \frac{R_2^2}{K^2}\right) - \left[\frac{-2R_1R_2}{K} + \frac{2R_2^2}{K^2}\right]^2 > 0
$$

(4.23)
Substituting Equations (4.2) and (4.4) into (4.23) and simplifying gives

\[ K < \frac{2}{T} = G_2 \]  

(4.24)

Clearly if the inequalities (4.22) and (4.24) applied to the parameters \( K \) and \( T \) are met, Theorem 6 is satisfied and the system of Figure 4.1 is globally asymptotically stable.

Now the Routh-Hurwitz stability boundaries are determined. From Figure 4.1 the composite \( G(s) \) is

\[ G_hoG(s) = \left( \frac{1-e^{-sT}}{s} \right) \left( \frac{K}{s(s+1)} \right) \]  

Taking the z-transform gives

\[ G_hoG(z) = K(1 - z^{-1}) \left[ \frac{Tz}{(z-1)^2} - \frac{z}{z-1} + \frac{z}{z-e^{-T}} \right] \]

Simplifying gives

\[ G_hoG(z) = K \left[ \frac{(T-1)e^{-T}z + 1-(T+1)e^{-T}}{(z-e^{-T})(z-1)} \right] \]  

(4.26)

Now the characteristic equation for the system of Figure 4.1 is

\[ K[T(z-e^{-T}) - (z-1)(z-e^{-T}) + (z-1)^2] + (z-1)(z-e^{-T}) = 0 \]

or \( z^2 + [K(T+1) - 2K] z + [e^{-T} + KTe^{-T} - Ke^{-T}] = 0 \)  

(4.27)

To apply the Routh-Hurwitz criterion, the variable \( z \) is mapped into \( r \) by

\[ z = \frac{r + 1}{r - 1} \]  

(4.28)

Substituting (4.28) into (4.27) and letting the coefficients
of $z$ and the constant terms be $A_1$ and $A_2$ respectively gives

\[
\frac{r+1}{r-1} + A_1 \frac{r+1}{r-1} + A_2 = 0
\]

Simplifying gives

\[
(1+A_1 + A_2)r^2 + (2-2A_2)r + (1-A_1+A_2) = 0 \quad (4.29)
\]

from which the following Routh array is formed.

\[
\begin{array}{ccc}
  r^2 & (1+A_1+A_2) & (1-A_1+A_2) \\
  r^1 & 2(1-A_2) & \\
  r^0 & (1-A_1+A_2) \\
\end{array}
\]

The Routh-Hurwitz criterion states that the terms of the first column must be greater than zero. Thus

\[
(1+A_1+A_2) > 0 \quad (4.30)
\]

\[
2(1-A_2) > 0 \quad (4.31)
\]

\[
(1-A_1+A_2) > 0 \quad (4.32)
\]

Inequality (4.30) when $A_1$ and $A_2$ are replaced reduces to

\[
KT(1-e^{-T}) > 0 \quad (4.33)
\]

which is true for all values of $K$ and $T$. Inequality (4.31) when $A_1$ and $A_2$ are replaced reduces to

\[
K < \frac{1-e^{-T}}{1-(T+1)e^{-T}} = G_3 \quad (4.34)
\]

Inequality 4.32 when $A_1$ and $A_2$ are replaced reduces to

\[
K < \frac{2(1+e^{-T})}{T(1+e^{-T}) - 2(1-e^{-T})} = G_4 \quad (4.35)
\]
The boundaries $G_1$, $G_2$, $G_3$ and $G_4$ were programed on the computer. The results are shown in Figure 4.2. The agreement of the boundaries is very good, considering the simple choice made for $a_{12}$. Also shown in Figure 4.2 are points in the parameter space that were tested for stability by solving Equation 4.1 on the computer. In all cases it can be seen that the stable and unstable points agree with the predicted boundaries. The Liapunov boundaries are more restricted, but this is to be expected as Liapunov's Theorem is only a sufficient condition. To obtain the exact boundaries predicted by the Routh-Hurwitz criterion would require that the optimum set of $a$'s in the variable gradient method be generated. At present, a method for doing this is not known.

It should be noted that the variable gradient method generated a quadratic form for a $V$ function in this linear example. This is what should have resulted since it has been shown\(^{(3)}\) that a quadratic Liapunov function is sufficient to prove global asymptotic stability in the linear case.

**Example Problem 4.2**

Figure 4.3 is a block diagram of the nonlinear sampled-data control system to be used in Example Problem 4.2.
FIGURE 4.2
Stability Boundaries in Parameter Space for Example 4.1.

K increments 0.1 units
T increments 0.1 units
• stable point
* unstable point
The state variable equations \* which describe the systems of Figure 4.3 are

\[
x_1[(k+1)T] = x_1(kT) - \frac{KA(T-C)}{B}\tanh[x_1(kT)] + KCx_2(kT)
\]

\[
x_2[(k+1)T] = -AC\tanh[x_1(kT)] + e^{-BT}x_2(kT)
\]

where \[C = \frac{(1-e^{-BT})}{B}\] (4.37)

As in Example Problem 4.1, the first order differences \[\Delta x_1(kT)\] and \[\Delta x_2(kT)\] are formed.

\[
\Delta x_1(kT) = -\frac{KA(T-C)}{B}\tanh[x_1(kT)] + KCx_2(kT)
\]

\[
\Delta x_2(kT) = -AC\tanh[x_1(kT)] - BCx_2(kT)
\]

To simplify the algebra, the following constants are defined:

\[
\begin{align*}
R_1 & = \frac{KA(T-C)}{B} > 0 \\
R_2 & = KC > 0 \\
R_3 & = AC > 0 \\
R_4 & = BC > 0 \\
\end{align*}
\]

(4.39)

\*See Appendix A
\( R_1 > 0 \) implies that \( T - C > 0 \) since \( K, A, T, B \) are all greater than zero by definition.

The condition \( T - C > 0 \) is now proved. Substituting Equation (4.37) in \( T - C > 0 \) gives

\[
T - \frac{(1 - e^{-BT})}{B} > 0 \quad \text{or} \quad TB > 1 - e^{-BT}.
\]

Letting \( z = BT \) gives

\[
z > 1 - e^{-z} \quad (4.40)
\]

This condition is met if the slope of the left side is greater than the right hand side, thus insuring that there is no intersection. Therefore, taking the derivative of (4.40) gives

\[
1 > e^{-z} \quad \text{which is true, since} \quad e^{-z} > 0 \quad \text{for} \quad z > 0. \quad (Q.E.D.)
\]

Equation (4.38) may now be rewritten as

\[
\Delta x_1(kT) = -R_1 \tanh [x_1(kT)] + R_2 x_2(kT)
\]

\[
\Delta x_2(kT) = -R_3 \tanh [x_1(kT)] - R_4 x_2(kT) \quad (4.41)
\]

Again the \( kT \) notation is dropped for convenience and the general grad \( V \) Equation (3.5) is assumed.

\[
\text{grad } V = \begin{vmatrix} a_{11} x_1 + a_{12} x_2 \\ a_{21} x_1 + 2 x_2 \end{vmatrix}
\]

(4.42)

Now \( \Delta V_I = (\text{grad } V)' \cdot \Delta x \)

\[
\Delta V_I = (a_{11} x_1 + a_{12} x_2)(-R_1 \tanh x_1 + R_2 x_2)
\]

\[+ (a_{21} x_1 + 2 x_2)(-R_3 \tanh x_1 - R_4 x_2)\]
Performing indicated operations and simplifying gives
\[ \Delta V_1 = -(R_1 a_{11} + R_3 a_{21}) x_1 \tanh(x_1) \]
\[ - (R_1 a_{12} + 2R_3) x_2 \tanh(x_1) \]
\[ + (R_2 a_{11} - R_4 a_{21}) x_1 x_2 + (R_2 a_{12} - 2R_4) x_2^2 \]
\[ (4.43) \]

To satisfy the curl equation, let
\[ a_{12} = a_{21} = \text{a constant.} \]
\[ (4.44) \]

Also let
\[ a_{11} = \frac{R_4 a_{12}}{R_2} \]
\[ (4.45) \]

to force the \( x_1 x_2 \) term to zero. The determination of \( a_{12} \) is postponed until \( V \) and \( \Delta V \) have been formed. Substituting \( (4.44) \) and \( (4.45) \) into \( (4.42) \) gives

\[ \text{grad } V = \begin{vmatrix} \frac{R_4}{R_2} a_{12} x_1 + a_{12} x_2 \\ a_{12} x_1 + 2x_2 \end{vmatrix} \]
\[ (4.46) \]

\( V \) is now found by the following line integration.
\[ V = \int_0^{x_1} \frac{R_4}{R_2} a_{12} \gamma_1 \, d \gamma_1 + \int_0^{x_2} (a_{12} x_1 + 2 \gamma_2) \, d \gamma_2 \]
\[ V = \frac{R_4}{2R_2} a_{12} x_1^2 + a_{12} x_1 x_2 + x_2^2 \]
\[ (4.47) \]

Since \( V \) is a quadratic form, Sylvester's Theorem applies and gives the following inequality constraints.

1st principal minor
\[ \frac{R_4}{2R_2} a_{12} > 0 \]
\[ (4.48) \]

which is satisfied if \( a_{12} > 0 \)
\[ (4.49) \]
2nd principal minor

\[ \frac{R_4}{2R_2} a_{12} - \frac{a_{12}^2}{4} > 0 \]

or \[ 2R_4 - R_2 a_{12} > 0 \]

or \[ a_{12} < \frac{2R_4}{R_2} \] (4.50)

Substituting for \( R_4 \) and \( R_2 \) from Equation (4.39) gives

\[ a_{12} < \frac{2B}{k} \] (4.51)

Clearly \( V \to \infty \) as \( ||\vec{x}|| \to \infty \) and the conditions on \( V \) are satisfied. Now \( \Delta V \) is formed by incrementing \( V \).

\[ \Delta V = \frac{R_4}{2R_2} a_{12} (x_1 + \Delta x_1)^2 + a_{12} (x_1 + \Delta x_1)(x_2 + \Delta x_2) \]

\[ + (x_2 + \Delta x_2)^2 - \frac{R_4}{2R_2} a_{12} x_1^2 - a_{12} x_1 x_2 - x_2^2 \]

Performing indicated operations and simplifying gives

\[ \Delta V = \frac{R_4}{R_2} a_{12} x_1 \Delta x_1 + \frac{R_4}{2R_2} a_{12} \Delta x_1^2 + a_{12} x_1 \Delta x_2 \]

\[ + a_{12} x_2 \Delta x_1 + a_{12} x_1 \Delta x_2 + 2x_2 \Delta x_2 + \Delta x_2^2 \] (4.52)

Now substituting (4.41) into (4.52) and simplifying gives

\[ \Delta V = - \left( \frac{R_1 R_4}{R_2^2} + R_3 \right) a_{12} x_1 \tanh(x_1) + \left( \frac{R_1}{2R_2} a_{12} + R_1 R_3 a_{12} \right) \]

\[ + R_3^2 \tanh^2(x_1) - (R_1 a_{12} + R_2 R_3 a_{12}) \]

\[ + 2R_3 - 2R_3 R_4 x_2 \tanh(x_1) - (-R_2 a_{12} + \frac{R_2 R_4 a_{12}}{2}) \]

\[ + 2R_4 - R_4^4 x_2^2 \] (4.53)
Let the coefficients of (4.53) be $P_1$, $P_2$, $P_3$ and $P_4$ such that

$$V = -P_1x_1\tanh(x_1) + P_2\tanh^2(x_1) - P_3x_2\tanh(x_1) - P_4x_2^2$$

(4.54)

Substituting $R_1$, $R_2$, $R_3$ and $R_4$ from Equation (4.39) into Equation (4.53) and solving for $P_1$ through $P_4$ gives

$$P_1 = A\alpha_{12}$$

(4.55)

$$P_2 = A^2\left[\frac{K(T^2 - C^2)}{2B}\alpha_{12} + C^2\right]$$

(4.56)

$$P_3 = A[(T - C^2)a_{12} + 2C(1 - BC)]$$

(4.57)

$$P_4 = C(2 - BC)(B - \frac{K\alpha_{12}}{2})$$

(4.58)

Clearly if $\Delta V$ is to be negative semi-definite, the following conditions must be met:

$$P_1 \geq 0, P_2 \leq 0, P_3 = 0, \text{ or the term containing } P_3 \text{ taken care of by some other means, and } P_4 \geq 0.$$  

$P_1 \geq 0$ is true since $a_{12} > 0$ by (4.49). This also makes $P_2 > 0$, thus the $P_2$ term will have to be dealt with by some other means. Now an attempt is made to force $P_3$ to zero.

$$A \left[K(\frac{T - C}{B} + C^2)a_{12} + 2C(1 - BC)\right] = 0$$

or

$$a_{12} = -2C(1 - BC)$$

$$K(\frac{T - C}{B} + C^2)$$

(4.59)

Since $1 - BC = 1 - 1 + e^{-BT} = e^{-BT} > 0$, $a_{12}$ would always be $< 0$, which violates (4.49). Thus $P_3$ is never zero, and some other means must be found to take care of this indefinite term.
Now only $P_4 \geq 0$ remains. From (4.58), if $B - \frac{K a_{12}}{2} \geq 0$, $P_4$ is $\geq 0$. Thus $a_{12} \leq 2\frac{2B}{K}$, which is met by (4.51).

Equation (4.54) is symmetric with respect to the origin. Thus the $-P_3 x_2 \tanh(x_1)$ term will be negative for $x_1$ and $x_2$ of the same sign and positive for $x_1$ and $x_2$ opposite in sign. Also due to symmetry, only the right half plane of the $x_1$-$x_2$ space need be considered in developing constraints. Now consider the case when $x_1$ and $x_2$ are the same sign. Then the $P_3$ and $P_4$ terms are negative for all $x_1$ and $x_2$, and only the $P_1$ and $P_2$ terms must be considered.

If $x_1 \tanh(x_1) \geq \tanh^2(x_1)$ (4.60)

then $\Delta V$ will be negative semi-definite if

$$P_1 \leq P_2$$ (4.61)

Condition (4.60) is now shown to be true for all $x_1$. Consider first the case for $x_1 \geq 0$. Condition (4.60) will be true if the first derivatives meet the same condition, i.e., the curve $x_1 \tanh(x_1)$ never crosses the curve $\tanh^2(x_1)$, except possibly at the origin.

Since $\tanh(x_1) \geq 0$ for $x_1 \geq 0$, a $\tanh(x_1)$ may be cancelled from both sides of (4.60) to give

$$x_1 \geq \tanh(x_1).$$

Taking the first derivative gives

$$1 \geq \text{sech}^2(x_1),$$

which is true for all $x_1 > 0$, since

$$\text{sech}(x_1) \leq 1$$ for all $x_1$. 

Now for the case when $x_1 \leq 0$, condition (4.60) becomes

$$-x_1 \tanh(-x_1) \geq \tanh^2(-x_1).$$

Since $\tanh(-x_1) = -\tanh(x_1)$, division by $\tanh(-x_1)$ gives

$$-x_1 \leq -\tanh(x_1) \quad \text{or} \quad x_1 \geq \tanh(x_1)$$

and obviously the condition is met again, as above. Therefore, (4.60) is true for all $x_1$.

Next consider the case when $x_1$ and $x_2$ are opposite in sign. Then the $P_3$ term is positive, and a new constraint must be found. There are four possibilities for $x_1$ and $x_2$:

1. $x_1 \geq 1$ and $x_2 \geq 1$. Since $\tanh(x_1) \leq 1$, the last two terms of $\Delta V$ will be negative if

$$P_4 \geq P_3.$$  \hspace{1cm} (4.62)

The first two terms are negative by constraint (4.61). Thus $\Delta V$ will be at least negative semi-definite.

2. $x_1 < 1$ and $x_2 \geq 1$. Clearly the constraints (4.61) and (4.62) are sufficient for this case also.

3. $x_1$ is any value and $x_2 < 1$. Clearly the limiting case is when $x_1$ is large enough so that $\tanh(x_1) \approx 1$, since then the $-P_4x_1 \tanh(x_1)$ term obtains a maximum. Any standard table shows that $x_1 = 6.5$ will cause $\tanh(x_1) \approx 1$. Setting $x_1$ to this value in Equation (4.54) gives

$$\Delta V = -6.5P_1 + P_2 + P_3x_2 - P_4x_2^2$$  \hspace{1cm} (4.63)
The sign of the \( P_3 \) term is changed, since \( x_1 \) and \( x_2 \) are opposite in sign. For \( \Delta V \) to be negative semi-definite, Equation (4.63) must meet the following conditions:

\[
-P_4 x_2^2 + P_3 x_2 - 6.5P_1 \leq 0 \tag{4.64}
\]

The left side of inequality (4.64) is a parabola which opens downward due to the negative \( x_2^2 \) term. Thus if it has no real roots, it will always be negative semi-definite. Now the roots are given by

\[
x_2 = \frac{-P_3 \pm \sqrt{P_3^2 + 4P_4(P_2 - 6.5P_1)}}{-2P_4}
\]

To guarantee no real roots, the following must be true:

\[
P_3^2 + 4P_4(P_2 - 6.5P_1) < 0 \tag{4.65}
\]

If inequality (4.65) is met, \( \Delta V \) will be negative semi-definite under possibility 3 above.

The constraints (4.61) and (4.62) have an equality involved in them. To guarantee that \( \Delta V \) is at least negative semi-definite, only one of the equal signs should be used and the remaining constraints should be strict inequalities.

To obtain ranges for the control system parameters, (4.61), (4.62), and (4.65) are examined when (4.55) - (4.58) are substituted for \( P_1 \) through \( P_4 \). A decision is now made to allow the equality to apply only to (4.61), since this yields a simple expression for \( a_{12} \).

Following the above discussion, (4.61) becomes

\[
a_{12} = \frac{AC^2}{T - \frac{KA(T^2 - C^2)}{2B}} \tag{4.66}
\]
Now by (4.49), \( a_{12} \) must be \( > 0 \). Thus in (4.66), the following must be true:

\[
0 < T - \frac{KA(T^2 - C^2)}{2B} \quad \text{or solving for } K \quad K < \frac{2TB}{A(T^2 - C^2)}
\]

(4.67)

To satisfy (4.51), inequality (4.66) must also satisfy the following inequality:

\[
\frac{AC^2}{T - KA(T^2 - C^2)} < \frac{2B}{K} \quad \text{Solving for } K: \quad K < \frac{2B}{TA}
\]

(4.68)

Inequality (4.62) is now examined.

\[
C(2-BC)(B - Ka_{12}) > A[K(T-C + C^2)a_{12} + 2C(1-BC)]
\]

Solving for \( a_{12} \) gives

\[
a_{12} < \frac{BC(2 - BC) - 2AC(1-BC)}{K \left[ C(2 - BC) + A(T-C + C^2) \right]}
\]

For (4.49) to be satisfied \( BC(2 - BC) - 2AC(1 - BC) > 0 \) or solving for \( A \),

\[
A < \frac{B(2 - BC)}{2(1 - BC)}
\]

(4.69)

Also to guarantee that (4.66) can exist, the following must be true:

\[
\frac{AC^2}{T - KA(T^2 - C^2)} < \frac{BC(2 - BC) - 2AC(1 - BC)}{K \left[ C(2 - BC) + A(T-C + C^2) \right]}
\]

Solving this for \( K \) gives

\[
K < \frac{T[BC(2 - BC) - 2AC(1 - BC)]}{A \left\{ C^2 \left[ C(2-BC) + A(T-C + C^2) \right] + \frac{(T^2 - C^2)}{2B} \left[ BC(2-BC) - 2AC(1-BC) \right] \right\}^2}
\]

(4.70)
In general, the determination of ranges for \( T \) and \( B \) are difficult, since both appear as single factors and as the exponent of \( e \). Thus no attempt is made to reduce inequality (4.65), since ranges for \( K \) and \( A \) are provided by (4.61) and (4.62). Instead, \( T \) and \( B \) are determined by trial and error on the computer. In a practical system, usually one or the other or both would be fixed. Thus, the trial and error solution is not as bad as it might seem.

Figure (4.4) is a flow chart of the determination of the system parameters by using (4.65)-(4.70). Table 4.1 lists the results of the computation for the parameters chosen to test this example.

### Table 4.1

**RESULTS OF COMPUTATIONS FOR TEST PARAMETERS**

**OF EXAMPLE PROBLEM 4.2**

<table>
<thead>
<tr>
<th>( T = 0.2 )</th>
<th>( B = 2.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( AB = 2.4918 )</td>
<td>( A ) chosen to be 1.0</td>
</tr>
<tr>
<td>( GB1 = 62.364 )</td>
<td>( GB = 10.963 )</td>
</tr>
<tr>
<td>( GB2 = 20.0 )</td>
<td>( GB = 10.963 )</td>
</tr>
<tr>
<td>( GB3 = 10.963 )</td>
<td>( GB = 10.963 )</td>
</tr>
<tr>
<td>( K = 0.5 )</td>
<td>( K = 10.0 )</td>
</tr>
<tr>
<td>( a_{12} = 0.13695 )</td>
<td>( a_{12} = 0.1618 )</td>
</tr>
<tr>
<td>( z = -0.27596 )</td>
<td>( z = -0.1437 )</td>
</tr>
</tbody>
</table>

It should be noted that the constraints that have been developed are only sufficient; thus they may be more restrictive than necessary to cause stability. Figures 4.5-
4.8 show the phase plane and output vs. time plots for solutions to Equation (4.36), using the parameters of Table 4.1 with $K = 0.5$. Clearly for the initial conditions used the system is stable. It will be stable for all initial conditions since the constraints guarantee global asymptotic stability.

Figures 4.9-4.18 show the solutions for $K = 10.0$ and $K = 50.0$. Notice that for $K = 10.0$, there are increased oscillations in the output. This is expected, since $K = 10.0$ is just inside the $K$ boundary. The solutions for $K = 50.0$ are unstable for most practical systems, but the trajectories will probably eventually go to zero. These solutions demonstrate that the constraints are sufficient only since $K > GB$ and $z > 0$ were true, which should produce an unstable system.

This example has demonstrated the application of the variable gradient method in the determination of global asymptotic stability for a nonlinear sampled-data control system. The constraints on the parameters that were developed are only partial in that a certain amount of trial and error must still be used in determining the system parameters and they are somewhat restrictive as they are only sufficient conditions.

The next example demonstrates the determination of asymptotic stability in a finite region of the state space when global asymptotic stability is unobtainable.
Choose T and B

Compute A boundary from

\[ AB = \frac{AC^2}{T-KA(T^2-C^2)} \frac{2B}{\sqrt{T^2-C^2}} \]

Choose a value for A from \( 0 < A < AB \)

Compute K boundary

GB1 from

\[ GB1 = \frac{2TB}{A(T^2-C^2)} \]

Compute K boundary

GB2 from

\[ GB2 = \frac{2B}{AT} \]

Compute K boundary

GB3 from

\[ GB3 = \frac{TQ}{A \left\{ C^2 \left( \frac{(C^2-BC)+AT-C}{B} \right) + \frac{C^2}{2} + \frac{T^2-C^2}{2B} \right\}} \]

Choose smallest of GB1, GB2, GB3

set = GB

Choose value of K from \( 0 < K < GB \)

Compute \( a_{12} = \frac{AC^2}{T-KA(T^2-C^2)} \frac{2B}{\sqrt{T^2-C^2}} \)

Compute \( P_1 \) through \( P_4 \) from Equations (4.55) - (4.58)

Compute \( z = P_3^2 + 4P_4(P_2 - 6.5P_1) \)

If \( z < 0 \), parameters satisfy constraints

Figure 4.4

FLOW CHART OF CHOICE OF PARAMETERS

EXAMPLE PROBLEM 4.2
FIGURE 4.5.
Phase Plane Plot for Example 4.2.
System parameters are $K=0.5$, $A=1.0$, $B=2.0$, $T=0.2$. Initial conditions are $X_1(kT)=0.5$, $X_2(kT)=0.4$. 
FIGURE 4.6
Phase Plane Plot for Example 4.2.
System parameters are $K=0.5$, $A=1.0$, $B=2.0$, $T=0.2$. Initial conditions are $X_1(kT)=0.5$, $X_2(kT)=-0.4$. 

$X_1(kT)$ increments 0.005 units
$X_2(kT)$ increments 0.005 units
FIGURE 4.7
Phase Plane Plot for Example 4.2.
System parameters are $K=0.5$, $A=1.0$, $B=2.0$, $T=0.2$. Initial conditions are $X_1(kT)=50.0$, $X_2(kT)=50.0$. 

$X_1(kT)$ increments 1.0 units
$X_2(kT)$ increments 1.0 units

$X_2(kT)$

$X_1(kT)$
FIGURE 4.8
Phase Plane Plot for Example 4.2.
System parameters are $K=0.5$, $A=1.0$, $B=2.0$, $T=0.2$. Initial conditions are $X_1(kT)=50.0$, $X_2(kT)=-50.0$. 
FIGURE 4.9
Phase Plane Plot for Example 4.2.
System parameters are $K=10.0$, $A=1.0$, $B=2.0$, $T=0.2$. Initial conditions are $X_1(kT)=0.5$, $X_2(kT)=0.4$. 

$X_1(kT)$ increments 0.05 units
$X_2(kT)$ increments 0.005 units
FIGURE 4.10
Output vs. Time Plot for Example 4.2.
System parameters are $K=10.0$, $A=1.0$,
$B=2.0$, $T=0.2$. Initial conditions are
$X_1(kT)=0.5$, $X_2(kT)=0.4$. 
FIGURE 4.11
Phase Plane Plot for Example 4.2.
System parameters are $K=10.0$, $A=1.0$, $B=2.0$, $T=0.2$. Initial conditions are $X_1(kT)=0.5$, $X_2(kT)=-0.4$. 

$X_1(kT)$ increments 0.5 units
$X_2(kT)$ increments 0.005 units
\[ t = 1.375 \]

\( X_1(kT) \) increments 0.05 units
\( t \) increments 0.2 seconds

**FIGURE 4.12**
Output vs. Time Plot for Example 4.2.
System parameters are \( K=10.0 \), \( A=1.0 \), \( B=2.0 \), \( T=0.2 \). Initial conditions are \( X(kT)=0.5 \), \( X_2(kT)=-0.4 \).
Phase Plane Plot for Example 4.2. System parameters are $K=10.0$, $A=1.0$, $B=2.0$, $T=0.2$. Initial conditions are $x_1(kT)=50.0$, $x_2(kT)=50.0$. 

FIGURE 4.13
Phase Plane Plot for Example 4.2.
System parameters are $K=10.0$, $A=1.0$, $B=2.0$, $T=0.2$. Initial conditions are $x_1(kT)=50.0$, $x_2(kT)=-50.0$. 

$X_1(kT)$ increments 2.0 units
$X_2(kT)$ increments 0.5 units
Figure 4.15
Output vs. Time Plot for Example 4.2.
System parameters are \( K = 50.0 \), \( A = 1.0 \),
\( B = 2.0 \), \( T = 0.2 \). Initial conditions are
\( x_1(kT) = 0.5 \), \( x_2(kT) = 0.4 \).
Output vs. Time Plot for Example 4.2.
System parameters are $K=50.0$, $A=1.0$, $B=2.0$, $T=0.2$. Initial conditions are $X_1(kT)=0.5$, $X_2(kT)=-0.4$. 

FIGURE 4.16
Figure 4.17
Phase Plane Plot for Example 4.2.
System parameters are \( K = 50.0 \), \( A = 1.0 \),
\( B = 2.0 \), \( T = 0.2 \). Initial conditions are
\( X_1(kT) = 50.0 \), \( X_2(kT) = 50.0 \).
FIGURE 4.18
Phase Plane Plot for Example 4.2. System parameters are $K=50.0$, $A=1.0$, $B=2.0$, $T=0.2$. Initial conditions are $X_1(kT)=50.0$, $X_2(kT)=-50.0$. 

$X_1(kT)$ increments 10.0 units 
$X_2(kT)$ increments 0.5 units
Example Problem 4.3

Figure 4.19 is a block diagram of the control system to be used in Example Problem 4.3.

\[ G_{ho} = \frac{1-e^{-sT}}{s}, \quad G_k(kt) = A x_1^2(kt), \quad G(s) = \frac{K}{s(s+B)} \]

Figure 4.19

The state variable equations that describe the system are:

\[
x_1[(k+1)T] = x_1(kt) - \frac{K}{B}(T-C)G_k(kt)x_1(kt) + KCx_2(kt)
\]

\[
x_2[(k+1)T] = -CG_k(kt)x_1(kt) + e^{-BT}x_2(kt)
\]

where \(G_k(kt)\) is a variable gain defined by \(G_k(kt) = A x_1^2(kt)\)

\[(4.72)\]

and \(C\) is defined by \(4.37\).

The first order differences are formed as in the first two examples. Equation (4.72) is substituted in (4.71) and simplifications made giving

\[
\Delta x_1(kt) = -\frac{KA}{B}(T-C)x_1^3(kt) + KCx_2(kt)
\]

\[
\Delta x_2(kt) = -ACx_1^3(kt) - BCx_2(kt)
\]

To simplify the algebra the following definitions are made:

\[ R_1 = \frac{KA(T-C)}{B} \quad (4.74) \]

\[ R_2 = KC \quad (4.75) \]
\( R_3 = AC \) \hspace{1cm} (4.76)
\( R_4 = BC \) \hspace{1cm} (4.77)

Again for convenience the \((kT)\) notation is dropped. The gradient \( V \) is found as before to be

\[
\nabla V = \begin{bmatrix}
a_{11}x_1 + a_{12}x_2 \\
a_{21}x_1 + 2x_2
\end{bmatrix}
\hspace{1cm} (4.78)
\]

Substituting (4.74)-(4.77) into (4.73) \( \Delta V_I \) is found to be

\[
\Delta V_I = (\nabla V)' \Delta \mathbf{x}
\]

\[
= (a_{11}x_1 + a_{12}x_2)(-R_1x_1^3 + R_2x_2) \\
+ (a_{21}x_1 + 2x_2)(-R_3x_1^3 - R_4x_2)
\]

Performing the indicated operations and simplifying gives

\[
\Delta V_I = -(a_{11}R_1 + a_{21}R_3)x_1^4 \\
+ [-(-a_{21}R_1 + 2R_3)x_1^2 + a_{11}R_2 - a_{21}R_4]x_1x_2 \\
- (2R_4 - a_{12}R_2)x_2^2
\hspace{1cm} (4.79)
\]

Some of the \( a_{ij} \)'s will now be determined. To meet the curl equations let

\[
a_{12} = a_{21} = \text{a constant} \hspace{1cm} (4.80)
\]

To eliminate the \( x_1x_2 \) term choose

\[
a_{11} = \frac{1}{R_2} \left[ (a_{21}R_1 + 2R_3)x_1^2 + a_{21}R_4 \right] \text{ or}
\]

when (4.80) is substituted

\[
a_{11} = \frac{1}{R_2} \left[ (a_{12}R_1 + 2R_3)x_1^2 + a_{12}R_4 \right]
\hspace{1cm} (4.81)
Substituting (4.80) and (4.81) into (4.79) and simplifying gives
\[
\Delta V_I = \frac{R_1}{R_2} (a_{12} R_1 + 2R_3) x_1^6 - a_{12} \left( \frac{R_1 R_4}{R_2} + R_3 \right) x_1^4 \\
- (2R_4 - a_{12} R_2) x_2^2
\]
and \(\nabla V\) becomes
\[
\nabla V = \begin{vmatrix}
\frac{1}{R_2} (a_{12} R_1 + 2R_3) & x_1^3 & a_{12} R_4 & x_1 + a_{12} x_2 \\
& 1 & x_1 & a_{12} x_1 + 2x_2 \\
& & & a_{12} x_1 + 2x_2 \\
\end{vmatrix}
\tag{4.82}
\]
The determination of \(a_{12}\) is deferred until \(V\) and \(\Delta V\) are determined.

\(V\) is found from the following line integration:
\[
V = \int_0^{x_1} \frac{1}{R_2} (a_{12} R_1 + 2R_3) \gamma_1^3 + \frac{a_{12} R_4}{R_2} \gamma_1 d \gamma_1 \\
+ \int_0^{x_2} (a_{12} x_1 + 2 \gamma_2) d \gamma_2
\]
Performing the integration gives
\[
V = \frac{1}{4R_2} (a_{12} R_1 + 2R_3) x_1^4 + \frac{a_{12} R_4}{2R_2} x_1^2 + a_{12} x_1 x_2 + x_2^2 \tag{4.83}
\]
\(V\) must be positive definite. Clearly choosing \(a_{12} = 0\) produces this result, but to obtain a more general solution, \(V\) is constrained in the following manner. The last three terms of (4.83) form a quadratic form and Sylvester's Theorem may be applied.

**First Principal Minor**
\[
\frac{a_{12} R_4}{2R_2} > 0 \quad \text{which is true if} \quad a_{12} > 0 \quad \tag{4.84}
\]
since \(R_4\) and \(R_2\) are positive.
Second Principal Minor

\[
\frac{a_{12}R_4}{2R_2} - \frac{a_{12}^2}{4} > 0 \quad \text{or, using (4.84),}
\]

\[2R_4 - R_2a_{12} > 0 \quad \text{and} \quad a_{12} < \frac{2R_4}{R_2} \quad (4.85)
\]

Now if the coefficient of the first term is positive, 

\[V\] is positive definite. Thus

\[
\frac{1}{4R_2} (a_{12}R_1 + 2R_3) > 0 \quad \text{or} \quad a_{12} > -\frac{2R_3}{R_1}
\]

which is met by (4.84).

Notice that (4.84) may now be relaxed to include zero.

Thus:

\[a_{12} \geq 0 \quad (4.86)
\]

Clearly \(V \to \infty\) as \(\|x\| \to \infty\) which completes the tests on \(V\).

To simplify the remaining algebra, define

\[H_1 = \frac{1}{4R_2} (a_{12}R_1 + 2R_3) \quad (4.87)
\]

\[H_2 = \frac{a_{12}R_4}{2R_2} \quad (4.88)
\]

\(\Delta V\) is now found by incrementing (4.83) and substituting (4.87) and (4.88) into the result.

\[
\Delta V = H_1(x_1 + \Delta x_1)^4 + H_2(x_2 + \Delta x_2)^2
\]

\[+ a_{12}(x_1 + \Delta x_1)(x_2 + \Delta x_2) + (x_2 + \Delta x_2)^2
\]

\[- H_1x_1^4 - H_2x_1^2 - a_{12}x_1x_2 - x_2^2 \quad (4.89)
\]

Substituting (4.74)-(4.77) into (4.73), then substituting
(4.73) into (4.89), performing the indicated operations and simplifying gives

$$\Delta V = H_1 R_2^4 x_2^4 + 4H_1 R_2^3 x_1 x_2^3 - 4H_1 R_1 R_2^3 x_1 x_2^3$$

$$+ 6H_1 R_2^2 x_1^2 x_2^2 - 12H_1 R_1 R_2^2 x_1^4 x_2^2 + 6H_1 R_1^2 R_2 x_1^6 x_2^2$$

$$+ (H_2 R_2^2 + a_{12} R_2 - a_{12} R_2 R_4 - 2R_4 + R_4^2) x_2^2$$

$$+ [4H_1 R_2 - 2H_2 R_1 R_2 - a_{12} R_1 + a_{12} (R_1 R_4 - R_2 R_3)$$

$$- 2R_3 + 2R_3 R_4] x_1^3 x_2^2 - 12H_1 R_1 R_2 x_1^5 x_2 + 12H_1 R_1^2 R_2 x_1^7 x_2$$

$$- 4H_1 R_1^3 R_2 x_1^9 x_2 + (-4H_1 R_1 + H_2 R_1^2 + a_{12} R_1 R_3 + R_3^2) x_1^6$$

$$- (2H_2 R_1 + a_{12} R_3) x_1^4 + 6H_1 R_1^2 x_1^8 - 4H_1 R_1^3 x_1^{10}$$

$$+ H_1 R_1^4 x_1^{12}$$

(4.90)

An attempt was made to find constraints on the system parameters, but the complexity of (4.90) thwarted these attempts. Thus if any information is to be obtained about the stability of (4.73), a different approach is needed.

It is clear that for most of the choices that might be made for the system parameters, Equation (4.90) will be indefinite at some point in state space. With this in mind, no attempt is made to find global stability. Instead, the theorems of Section 3.E will be used to obtain expressions which define a region in state space. Since no constraints have been found for the system parameters, the determination of the parameters will be completely trial and error. In addition, since only a range for $a_{12}$ has been found, any
region in the state space will be a function of $a_{12}$ as well as the system parameters.

To apply the ideas of Section 3.E, it is necessary to determine the zeros of $\Delta V$. A computer program* was written to determine the zeros of $\Delta V$.

Once the curve which defines the zeros of $\Delta V$ was found, a program was written to determine the point of contact of $V$ (set equal to a constant which is called PAR) and $\Delta V$ by the method outlined in Section 3.E. To implement this program, a new function called TPAR was derived in the following manner. The ratios of Section 3.E for this example are found to be

$$\begin{align*}
\frac{\partial V}{\partial x_2} &= \frac{\partial \Delta V}{\partial x_2} \\
\frac{\partial V}{\partial x_1} &= \frac{\partial \Delta V}{\partial x_1}
\end{align*}$$

which when rearranged becomes

$$\frac{\partial V}{\partial x_2} \frac{\partial \Delta V}{\partial x_1} - \frac{\partial \Delta V}{\partial x_2} \frac{\partial V}{\partial x_1} = 0$$

(4.92)

The expression on the left is called TPAR. Let the coefficient of $\Delta V$ be $P_1$ through $P_{16}$, the order being the same as (4.90). The indicated operations of (4.92) were performed and the result after simplifying becomes

$$TPAR = (6P_3x_1^3 + 2P_2 - 4a_{12}P_1)x_2^4 + [-2a_{12}P_2x_1 + (3a_{12}P_3 + 8P_5 - 16H_1P_1 - 3a_{12}P_3)x_1^3 + (4P_4 - 8H_2P_1)x_1]x_2^3$$

*See Appendix B for a discussion of the programs.
\[ +[-6H_2P_2x_1^2 + (4a_{12}P_5 + 10P_9 - 12H_1P_2 - 6H_2P_3 \\
- 2a_{12}^2P_5)x_1^4 + (14P_{10} - 12H_1P_3)x_1^6 + 18P_{11}x_1^8]x_2^2 \\
+ [(4a_{12}P_9 + 12P_{13} - 8H_1P_4 - 4H_2P_5)x_1^5 \\
+ (7a_{12}P_{10} + 16P_{14} - 8H_1P_5 - a_{12}P_{10})x_1^7 + (9a_{12}P_{11} \\
+ 20P_{15} - a_{12}P_{11})x_1^9 + (8P_{12} - 4H_2P_4)x_1^3 + 24P_{16}x_1^{11}]x_2^2 \\
+ [(4a_{12}P_{12}x_1^4 + (6a_{12}P_{13} - 2H_2P_9)x_1^6 + (8a_{12}P_{14} \\
- 4H_1P_9 - 2H_2P_{10})x_1^8 + (10a_{12}P_{15} - 4H_1P_{10} - 2H_2P_{11})x_1^{10} \\
+ (12a_{12}P_{16} - 4H_1P_{11})x_1^{12}] = 0 \quad (4.93) \]

Equation (4.93) is written in descending powers of \( x_2 \) for convenience in applying the computer program.

The point of contact between \( V = \text{PAR} = \text{a constant} \) and the curve defining the zeros of \( \Delta V \) are the coordinates \((x_1, x_2)\), which make \( \text{TPAR} = 0 \) and \( \Delta V = 0 \). Thus if the zeros of \( \text{TPAR} \) are found by a computer program and plotted on the same coordinates as the zeros of \( \Delta V \), the intersection of the two curves gives the possible point of contact. Since there will in general be more than one intersection, the intersection in the smallest neighborhood of the origin is taken to be the first point of contact. These coordinates are now substituted into

\[ \text{PAR} = H_1x_1^4 + H_2x_1^2 + a_{12}x_1^2 + x_2^2 \quad (4.94) \]

and a value of \( \text{PAR} \) is determined. To check the results, the roots of

\[ V - \text{PAR} = 0 \quad (4.95) \]

are found and plotted on the same coordinates as \( \Delta V \). The region of asymptotic stability is given by the region defined
by the boundary $V=\text{PAR}$. Since all the above curves are functions of $a_{12}$, each curve (except 4.94) is found for a sequence of $a_{12}$'s, which covers the range for $a_{12}$ defined by (4.85). From these solutions the best point of contact is chosen. This is not the optimum point, just the best one for the sequence of $a_{12}$'s. The choice is made by an inspection of the zeros of $\Delta V$ curves and the intersections of $\text{TPAR}=0$ with it. The decision as to which point of contact to choose depends on the size and shape of the desired region of stability. Usually the largest region in both $x_1$ and $x_2$ directions is desired. Once the point of contact is picked, the region of stability is found as discussed above. The value of PAR is very critical and small variations in the coordinates of the point of contact will cause the $V=\text{PAR}$ curve not to be tangent to the zeros of $\Delta V$ curves. It is felt that this is not a real problem, since the $V$ function which is generated is not in general an optimum. The region found by the methods of Section 3.E is usually well within the true region of stability. Thus the region generated in this manner should be used as an approximation of the true region of stability.

To verify the above statements about the region of stability, a set of system parameters were chosen and the above procedures carried out. The parameters chosen were:

$$K = 0.5, \quad A = 1.0, \quad B = 2.0, \quad T = 0.1$$

(4.96)

Figure 4.20 is a plot of the zeros of $\Delta V$ for $a_{12} = 1.5$, which was the best value of $a_{12}$ for the range $0 < a_{12} < 8.0$ as
determined by (4.85). Figure 4.21 shows the intersection of the zeros of \( \text{TPAR} \) with the zeros of \( \Delta V \). The point of contact was found to be \((4.45, 41.25)\). \( \text{PAR} \) was determined to be 2,436.0. Figure 4.22 is a plot of \( V=\text{PAR} \) and the zeros of \( \Delta V \). Figure 4.23 is an expanded plot of \( V=\text{PAR} \), which is the region of stability. Since \( V=\text{PAR}, \Delta V=0, \) and \( \text{TPAR}=0 \) are all symmetrical with respect to the origin, only the right half plane of the phase space is plotted.

The region defined by Figure 4.23 was tested by solving the Difference Equation (4.71), using the parameters of (4.96) and the initial conditions listed below:

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( 0.5 )</th>
<th>( 4.5 )</th>
<th>( 8.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_2 )</td>
<td>75.0</td>
<td>75.0</td>
<td>15.0</td>
</tr>
<tr>
<td></td>
<td>40.0</td>
<td>35.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>-40.0</td>
<td>-35.0</td>
<td>-15.0</td>
</tr>
<tr>
<td></td>
<td>-75.0</td>
<td>-75.0</td>
<td></td>
</tr>
</tbody>
</table>

Figures 4.24 - 4.34 are phase plane and output versus time plots for the initial conditions of Table 4.2. These results indicate that the region of stability shown in Figure 4.23 is well within the true region of stability. Also notice that the system becomes more unstable as the initial conditions are taken outside the region defined by Figure 4.23.

The region of Figure 4.23 is relatively small. An attempt was made to enlarge the region by another choice of
parameters. The parameters were changed in a logical manner to produce a greater region of stability, i.e., the gain was decreased, the sample period decreased, etc. The parameters chosen were: \( K = 0.125, A = 0.5, B = 4.0, T = 0.05 \) \( (4.97) \)

The procedures used above were again carried out. The results are shown in Figures 4.35 - 4.39. There was some difficulty encountered in finding the appropriate choice for \( a_{12} \), and the method used above was replaced to the extent that each possible point of contact was found and \( V = \text{PAR} \) and the zeros of \( \Delta V \) plotted to determine an appropriate value for \( a_{12} \). The point of contact was found to be \((20.0, 2374.0)\). \( \text{PAR} \) was found to be \( 7.0489 \times 10^6 \). The value of \( a_{12} \) was \( 20.0 \). The range for \( a_{12} \) was \( 0 < a_{12} < 64 \). The determination of the point of contact was made difficult for these parameters by the oscillations in the curve of the zeros of \( T\text{PAR} \). The new region shown in Figure 4.30 is considerably larger than that of Figure 4.23. The system parameters may be constrained so that they cannot be varied, as was done here. Nevertheless, the results indicate that a suitable region could be found with a minimum of variations. The Equations \((4.71)\) were not solved for this set of parameters since it was felt that the results of testing the equations for the parameters \((4.96)\) were sufficient to justify that the procedures employed were valid.

In summary, then, this example has demonstrated the generation of a \( V \) and \( \Delta V \) function, which were used to determine a region of asymptotic stability in state space. While
no constraints were developed for the system parameters as in Example Problem 4.2, it seems that the generation of a suitable region of stability by trial and error choice of the parameters would not be too difficult. This would be particularly easy if some other constraints on the parameters are known, such as the location of the pole being fixed or that only one sample period could be used.

This completes the examples of the application of the variable gradient method to sampled-data control systems. Only second order systems were used, since the examples are to demonstrate the procedure and not to be an exhaustive listing of the possible solutions.

The theorems of Chapter III were developed for $n^{\text{th}}$ order systems. Thus the extension to higher order systems presents no theoretical difficulty. The computer programs written for Example Problem 4.3 can be extended to higher order systems, as it is only a matter of adding DO loops to the programs. The graphical solutions used can be employed up to third order, but for higher order systems they have no meaning. However, if a program could be written to determine the point of contact with a sufficient degree of accuracy, the determination of PAR is insured. Plots could then be made, using the various state variables as parameters and a region in $n$-state space developed. It is conceded that this would probably be costly in terms of computer time, and whether or not the determination of a region of stability was desirable would rest on other considerations, such as usefulness, need to know, etc.
FIGURE 4.20
Zero's of $\Delta V$ Plot for Example 4.3.
System parameters are $K=0.5$, $A=1.0$, $B=2.0$, $T=0.1$. gradV constant $a_{12}=1.5$. 

$X_1(kT)$ increments 0.2 units
$X_2(kT)$ increments 5.0 units
FIGURE 4.21
ΔV and TPAR Plots for Example 4.3.
System parameters are K=0.5, A=1.0, B=2.0, T=0.1. gradV constant $a_{12}=1.5$. 

$X_1(kT)$ increments 0.2 units
$X_2(kT)$ increments 5.0 units
FIGURE 4.22
ΔV and V-PAR=0 Plots for Example 4.3.
System parameters are K=0.5, A=1.0,
B=2.0, T=0.1. gradV constant a_{12}=1.5.
FIGURE 4.23

$V_{\text{PAR}} = 0$ for Example 4.3.

System parameters are $K = 0.5$, $A = 1.0$, $B = 2.0$, $T = 0.1$. $\nabla V$ constant $a_{12} = 1.5$. 

$x_1(kT)$ increments 0.2 units
$x_2(kT)$ increments 1.0 units
FIGURE 4.24
Phase Plane Plot for Example 4.3.
System parameters are $K=0.5$, $A=1.0$, $B=2.0$, $T=0.1$. Initial conditions are $x_1(kT)=0.5$, $x_2(kT)=75.0$. 

$x_1(kT)$ increments 0.1 units
$x_2(kT)$ increments 1.0 units.
FIGURE 4.25
Phase Plane Plot for Example 4.3.
System parameters are \( K = 0.5, \) \( A = 1.0, \)
\( B = 2.0, \) \( T = 0.1. \) Initial conditions are
\( X_1(kT) = 0.5, X_2(kT) = 40.0. \)
FIGURE 4.26
Phase Plane Plot for Example 4.3.
System parameters are $K=0.5$; $\alpha=1.0$, $B=2.0$, $T=0.1$. Initial conditions are $X_1(kT)=0.5$, $X_2(kT)=-40.0$. 

$X_1(kT)$ increments 0.1 units
$X_2(kT)$ increments 1.0 units
Figure 4.27
Phase Plane Plot for Example 4.3.
System parameters are $K=0.5, \ A=1.0, \ B=2.0, \ T=0.1$. Initial conditions are $X_1(kT)=0.5, \ X_2(kT)=-75.0$. 

$X_1(kT)$ increments 0.1 units.
$X_2(kT)$ increments 1.0 units.
Example 4.3. System parameters are $K = 0.5$, $A = 1.0$, $B = 2.0$, $T = 0.1$. Initial conditions are $X_1(kT) = 45$, $X_2(kT) = 75$.0.
FIGURE 4.29
Phase Plane Plot for Example 4.3.
System parameters are $K=0.5$, $A=1.0$, $B=2.0$, $T=0.1$. Initial conditions are $X_1(kT)=4.5$, $X_2(kT)=35.0$. 

$X_1(kT)$ increments 0.5 units
$X_2(kT)$ increments 1.0 units
FIGURE 4.30
Phase Plane Plot for Example 4.3.
System parameters are $K=0.5$, $A=1.0$, $B=2.0$, $T=0.1$. Initial conditions are $X_1(kT)=4.5$, $X_2(kT)=-40.0$. 

$x_1(kT)$ increments 0.5 units
$x_2(kT)$ increments 1.0 units
FIGURE 4.31
Phase Plane Plot for Example 4.3.
System parameters are K=0.5, A=1.0, B=2.0, T=0.1. Initial conditions are $X_1(kT)=0.5$, $X_2(kT)=-75.0$. X_1(kT) increments 0.5 units
X_2(kT) increments 1.0 units
FIGURE 4.32
Phase Plane Plot for Example 4.3.
System parameters are $K=0.5$, $A=1.0$, $B=2.0$, $T=0.1$. Initial conditions are $X_1(kT)=8.0$, $X_2(kT)=15.0$. 

$X_1(kT)$ increments 0.2 units
$X_2(kT)$ increments 1.0 units
Phase Plane Plot for Example 4.3.
System parameters are $K=0.5$, $A=1.0$, $B=2.0$, $T=0.1$. Initial conditions are $X_1(kT)=8.0$, $X_2(kT)=0.0$.
FIGURE 4.34
Phase Plane Plot for Example 4.3.
System parameters are $K=0.5$, $A=1.0$, $B=2.0$, $T=0.1$. Initial conditions are $x_1(kT)=8.0$, $x_2(kT)=-15.0$. 

$x_1(kT)$ increments 0.2 units
$x_2(kT)$ increments 1.0 units
FIGURE 4.35
Zero's of $\Delta V$ for Example 4.3.
System parameters are $K=0.125$, $A=0.5$, $B=4.0$, $T=0.05$. gradV constant $a_{12}=20.0$. 

$X_1(kT)$ increments 1.0 units
$X_2(kT)$ increments 250.0 units
FIGURE 4.36
$\Delta V=0$ and $TPAR=0$ Plots for Example 4.3.
System parameters are $K=0.125$, $A=0.5$, $B=4.0$, $T=0.05$. $\text{grad}V$ constant $a_{12}=20.0$. 

$X_1(kT)$ increments 1.0 units
$X_2(kT)$ increments 250.0 units
FIGURE 4.37
V-PAR=0 and V=0 Plots for Example 4.3. System parameters are K=0.125, A=0.5, B=4.0, T=0.05. gradV constant a_{12} = 20.0.
FIGURE 4.38

V-PAR=0 Plot for Example 4.3.
System parameters are $K=0.125$, $A=0.5$, $B=4.0$, $T=0.05$. $\nabla V$ constant $a_{12}=20.0$. 

$x_1(kT)$ increments 1.0 units
$x_2(kT)$ increments 50.0 units
CHAPTER V

CONCLUSIONS

It has been shown that the variable gradient method of generating Liapunov functions can be extended to the sampled-data system. The modifications necessary to apply the method have been developed and the results of the examples of Chapter IV have verified these modifications.

The desired results of this research were not completely obtained. It has been demonstrated that the variable gradient method can be applied, but it was also desired that the application would be practical. It was also hoped that synthesis of the control system parameters could be done by this method. Example Problem 4.2 shows that it can be done but that even for simple systems the synthesis is only partial and the constraints are complex. This may not be the case with every system, but it is obvious that the $\Delta V$ function will nearly always be difficult to work with. Thus the value of the method seems to be more in analysis rather than in synthesis. Example Problem 4.3 demonstrates a possible alternative to testing $\Delta V$ for negative definiteness. Using the variable gradient method in this manner gives a methodical approach to the determination of asymptotic stability. There also seems to be a possibility of tailoring the region of stability to some desired standard. This at present would be a trial and error procedure, but as was demonstrated by the second set of parameters used in Example
Problem 4.3, it probably would not take many iterations to obtain a desired region.

In summary, then, the variable gradient method has been applied to sampled-data control systems, but the practicality of using it is limited by the complexity of the $\Delta V$ functions. A possible solution to this problem has been demonstrated.

In doing the research for this thesis, several problems were encountered that might provide a basis for further research. The following is a list of these problems:

1. The determination of positive or negative definiteness for complex functions is severely limited, since only Sylvester's Theorem for quadratic forms can be used at present. It may be that this is the only possible theorem, but there may be other possibilities, such as a computer program.

2. The determination of a region of stability as in Example Problem 4.3 is not limited to sampled-data systems. It would be interesting to find out more about how the region of stability varies for variation in the control system parameters, both for continuous as well as discrete systems. The method of Example Problem 4.3 might be the vehicle by which such an investigation could be carried out on a computer.
(3) The constraints of Example Problem 4.2 and the region of Example Problem 4.3 both were depended on $a_{12}$, one of the constants in the variable gradient. It would be very desirable if a method or procedure could be developed which would guarantee an optimum or quasi-optimum set of $a$'s that would in turn generate an optimum or quasi-optimum $V$ function, such that any constraints or regions would be very close to the true constraints or regions. The Liapunov Theorem on instability might be useful in such a procedure.

(4) Some consideration was given to finding a way to modify the $a_{ij}$'s of the variable gradient such that $\Delta V$ was generated by $(\text{grad } V)' \Delta X$. No way was found, but a more thorough examination of the problem might yield a way to modify the $a_{ij}$'s and thus shorten the work involved. The grad $V$ in this case would not be the gradient as normally thought of but would be a matrix, that when multiplied by $\Delta X$ would yield $\Delta V$ and which could be integrated to obtain $V$. 
APPENDIX A

DEVELOPMENT OF DIFFERENCE EQUATIONS FROM BLOCK DIAGRAMS

There are several possible ways in which the difference equations may be formed. The method used in this thesis follows the procedure of Chapter 12 of Kuo's book. The problem of writing the nonlinear difference equations is reduced to linear techniques by linearizing the system between sample points. This is accomplished by treating the nonlinear elements of the block diagram as a variable gain element whose gain changes at the end of each sample period.

Example Problem (4.2) will be used to illustrate the procedure.

The block diagram is repeated here for convenience.

\[ G_{ho} = \frac{1-e^{-sT}}{s}, \quad G_k(t) = \text{Atanh}[x_1(kT)], \quad G(s) = \frac{K}{s(s+B)} \]

Figure A.1

The first step in developing the nonlinear difference equations is to write some of the block diagram equations.

\[ e(t) = r(t) - c(t) = -x_1(t) \text{ since } r(t) = 0, \quad (A.1) \]

\[ e^*(t) = -x_1^*(t) \quad (A.2) \]
The nonlinear elements are described in the following manner. A variable gain element is defined as

\[ G_k(t^+) = \frac{m(t^+)}{h(t^+)} \quad (A.3) \]

but \[ h(t^+) = -x_1(t^+) \quad (A.4) \]

and \[ m(t^+) = A \tanh[h(t^+)] \quad (A.5) \]

Now substituting (A.4) in (A.5) gives

\[ m(t^+) = A \tanh[-x_1(t^+)] = -A \tanh[x_1(t^+)] \quad (A.6) \]

Substituting (A.4) and (A.6) into (A.3) gives

\[ G_k(t^+) = \frac{-A \tanh[x_1(t^+)]}{-x_1(t^+)} = \frac{A \tanh[x_1(t^+)]}{x_1(t^+)} \]

The signal flow graph for the linear elements is now developed. \[ G(S) = \frac{K}{S(S+B)}. \] Dividing the numerator and denominator by the term with the highest degree in \( s \) of the denominator gives

\[ G(s) = \frac{K_s^{-2}}{1+Bs^{-1}} \quad (A.8) \]

but \[ \frac{X_1(s)}{m(s)} = \frac{K_s^{-2}}{1 + Bs^{-1}} \quad (A.9) \]

Now letting \[ U(s) = \frac{m(s)}{1+Bs^{-1}} \quad (A.10) \]

and rearranging terms gives

\[ U(s) = m(s) - U(s)Bs^{-1} \quad (A.11) \]

Also \[ X_1(s) = Ks^{-2}U(s) \quad (A.12) \]
Using (A.11) and (A.12), the signal flow graph is found to be

\[
\begin{align*}
  m(s) & \quad l \quad U(s) \quad s^{-1} \quad X_2(s) \quad s^{-1} \quad K \\
  & \quad -B \quad \frac{X_1(s)}{K} \\
  & \quad X_1(s) \quad X_1(s)
\end{align*}
\]

Figure A.2

The sample and hold elements are treated as a unit and described in the following manner:

\[ H(s) = \frac{1}{s} h(t_0^+) \]  
\[ \text{(A.13)} \]

The \( \frac{1}{s} \) factor is due to \( h(t_0^+) \) being a unit step function for each sample period. Similarly,

\[ x_1(t_0^+) = x_1(t) \]
\[ x_1(s) = \frac{1}{s} x_1(t_0) \]  
\[ \text{(A.14)} \]

and

\[ x_2(t_0^+) = x_2(t) \]
\[ x_2(s) = \frac{1}{s} x_2(t_0) \]  
\[ \text{(A.15)} \]

The nonlinear signal flow graph may now be drawn.

\[
\begin{align*}
  r(t) = 0 \quad h(t_0^+) \quad m(t_0^+) \quad m(s) \quad -B \quad X_2(s) \quad \frac{X_1(s)}{K} \quad X_1(s) \quad C(s)
\end{align*}
\]

Figure A.3
Mason's gain formula is now used to find equations for \( x_1(s) \) and \( x_2(s) \).

To apply Mason's formula, the following is defined:

\[
\Delta = 1 - (-B s^{-1}) = 1 + \frac{B}{s} = \frac{s + B}{s} \tag{A.16}
\]

Now

\[
x_1(s) = \frac{s^{-1}(1 + \frac{B}{s})}{(1 + \frac{B}{s})} - \frac{G_k(to^+)s^{-3}(1)Kx_1(to)}{(s+B)} + \frac{s^{-2}(1)Kx_2(to)}{(s+B)}
\]

which reduces to

\[
x_1(s) = \frac{1}{s}x_1(to) - \frac{G_k(to^+)Kx_1(to)}{s^2(s+B)} + \frac{Kx_2(to)}{s(s+B)} \tag{A.17}
\]

also

\[
x_2(s) = - \frac{G_k(to^+)s^{-1}(1)x_1(to)}{(s+B)} + \frac{s^{-1}(1)x_2(to)}{(s+B)}
\]

which reduces to

\[
x_2(s) = - \frac{G_k(to^+)x_1(to)}{s(s+B)} + \frac{x_2(to)}{(s+B)} \tag{A.18}
\]

Certain terms of (A.17) and (A.18) are expanded by Heaviside's expansion to give

\[
x_1(s) = \frac{1}{s}x_1(to) - G_k(to^+)K \left[ \frac{1}{B^2} + \frac{1}{s} + \frac{1}{(s+B)} \right] x_1(to) + K \left[ \frac{1}{B} - \frac{B}{s+B} \right] x_2(to) \tag{A.19}
\]

and

\[
x_2(s) = -G_k(to^+) \left[ \frac{1}{B} - \frac{B}{s+B} \right] x_1(to) + \frac{x_2(to)}{s+B} \tag{A.20}
\]
The inverse LaPlace transforms are now taken of (A.19) and (A.20) to yield

\[ x_1(t) = \mathcal{L}^{-1}[x_1(s)] = x_1(t_0) - G_k(t_0^+)K[-\frac{1}{B^2} + \frac{1}{B}(t-t_0) + \frac{1}{B^2}e^{-B(t-t_0)}] x_1(t_0) + K[\frac{1}{B} - \frac{1}{B}e^{-B(t-t_0)}] x_2(t_0) \]

(A.21)

and

\[ x_2(t) = \mathcal{L}^{-1}[x_2(s)] = -G_k(t_0^+)\left[\frac{1}{B} - \frac{1}{B}e^{-B(t-t_0)}\right] x_1(t_0) + \left[e^{-B(t-t_0)}\right] x_2(t_0) \]

(A.22)

To make Equations (A.21) and (A.22) difference equations the following substitutions are made:

\[ t_0 = kT \quad \text{and} \quad t = kT + T \]

(A.23)

Also, Equation (A.7) is substituted for \( G(t_0^+) \). The results of these substitutions are

\[ x_1[(k+1)T] = x_1(kT) - \frac{KA}{B}\left[T - \frac{(1-e^{-BT})}{B}\right] \tanh[x_1(kT)] + K\frac{(1-e^{-BT})}{B} x_2(kT) \]

(A.24)

\[ x_2[(k+1)T] = -A\left(\frac{1-e^{-BT}}{B}\right) \tanh[x_1(kT)] + e^{-BT} x_2(kT) \]

(A.25)

Equations (A.24) and (A.25) are the ones used in Example Problem 4.2. Using the same procedure the equations used in Example Problem 4.3 can be developed. The equations of Example Problem 4.1 were found by the same methods, with the exception that the nonlinear elements are not present.
The procedure as outlined above is rather lengthy and cumbersome. Thus a more efficient method would be desirable.
APPENDIX B

COMPUTER PROGRAMS

The following is a discussion of the programs used in this thesis. Since the form a program takes is a function of the programmer, no listing or flow charts are provided. Instead, a discussion of the general procedure and some of the programming difficulties are given for each program.

All programs were written in Fortran II language.

A. The Determination of the Zeros of a Polynomial of Several Variables.

There are many ways in which this problem can be attacked. The procedure used in programming the zeros of a function employs a standard subroutine (ROTPOL) stored on the Mod II 1620 computer. ROTPOL can find all the roots of a single variable polynomial of degree 33. The technique used to implement the program using the ROTPOL subroutine is illustrated by Figure B.1.

Figure B.1
In Figure B.1-A, the curve represents the zeros of a two variable polynomial. To use ROTPOL, the x variable is set equal to a constant b. The program then gives the roots defined by the intersection of Q'Q. In Figure B.1-B, several sections of a surface which represents the zeros of a 3 variable polynomial are shown. The section labeled M represents the zeros for y=b and $0 < x < c$. The line Q'Q has the same function as for A. Thus it can be seen that the determination of the zeros of a polynomial of n variables can be found by using n nested DO loops around the ROTPOL subroutine. Each loop iterates one of the variables, setting it equal to a constant. The sectioning line Q'Q is chosen so the lowest degree variable is used as the variable for ROTPOL. Since ROTPOL gives all the roots, it is necessary to test each root to see if it is complex. Only the real roots are desired and the complex roots are discarded. If the region defined by the roots is to be negative definite, it is also necessary to check the sign on either side of each root along the Q'Q line. If the sign is opposite on opposite sides of the root, it is a valid boundary point. Finally, the valid points are either printed or punched on cards.

The run time for this type of program is dependent on the number of loops, the number of iterations, and the complexity of the subroutine. It is felt that the ROTPOL subroutine could be modified to only find real roots and thus shorten the run time. For the data of Example Problem 4.3, the average run time was 10 minutes.
It should also be noted that this procedure can be used with functions other than polynomials. The subroutine in this case might be a simple iteration, using a variable increment to reduce the run time and to obtain the desired accuracy.

B. Solution of Difference Equations.

The equations used to describe the control systems in Example Problems 4.1-3 are recurrence relationships which lend themselves to easy programing by normal iterative techniques.

To insure that no hidden oscillations occurred, an inter-sampling technique was used. This follows the procedure outlined by Kuo\(^{(8)}\) on page 141. To use this procedure, \(T\) is replaced by \(\Delta T\), where \(T \geq 1\). Thus it is necessary to compute a set of coefficients (the \(P_1\) through \(P_4\) of Example Problems 4.2 and 4.3) for each \(\Delta T\). These are stored in a vector. Now a double loop is written around the recurrence equations. The inner loop computes the value of the equation at each inter-sample point \(T + \Delta T\), using the same \(x(kT)\) as an initial condition. The outer loop iterates the sample period and resets the initial condition to \(x(kT + T)\). The \(x(kT)\) and \(kT\) values are printed or punched on cards. This data is then used to obtain the phase plane plots and output vs. time plots.

C. Program to Determine System Parameters for Example Problem 4.2.

A program was written that followed the flow chart of
Figure 4.4. The only difference was that instead of choosing one value of $A$ or $K$, a set of values were chosen, using a nested DO loop. The values shown in Table 4.1 were computed using this program, but with the iteration loops deleted. This program could be used to test any particular set of parameters or to search for the optimum (largest range for each parameter) set of parameters.
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