Computer-aided design of lossless non-uniform transmission lines

Larry Lee Wesner

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COMPUTER-AIDED DESIGN OF LOSSLESS NON-UNIFORM
TRANSMISSION LINES

by

LARRY LEE WESNER, 1944

A

THESIS

submitted to the faculty of
THE UNIVERSITY OF MISSOURI - ROLLA
in partial fulfillment of the requirements for the
Degree of
MASTER OF SCIENCE IN ELECTRICAL ENGINEERING
Rolla, Missouri
1969

approved by

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ABSTRACT

A numerical iterative design technique is presented for the synthesis of lossless non-uniform transmission lines. A variational approach is used to minimize an error functional to which the system and boundary equality constraints have been adjoined by Lagrangian multipliers. The resultant network optimization conditions lead to an iterative scheme requiring analysis of the non-uniform line, analysis of an adjoint system and calculation of the error gradient in parameter space. A steepest descent procedure is used to optimize parameter values. Finite difference equations are used to provide an approximation model for implementation on the digital computer. A design example and a fortran computer program are provided.
ACKNOWLEDGEMENTS

The author wishes to express gratitude to his advisor Dr. E. G. Bertinolli of the University of Missouri at Rolla for his assistance and guidance throughout the course of this thesis. The author is indebted to the National Science Foundation for a three year Predoctorial Traineeship which made possible the studies concluded by this thesis.

Special thanks is given to my wife Linda, for her help, encouragement and the excellent typing job on the original copy.
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LIST OF SYMBOLS

C - Shunt capacitance of transmission line as a function of distance

d - Length of lossless transmission line

i - Current wave along transmission line as a function of distance and time

I - Propagating electrical current

J - Error functional to be minimized

K - A parameter in reflection coefficient

L - Series inductance of transmission line as a function of distance

m - Space discretization integer

n - Time discretization integer

P - Variable used in definition of Green's theorem

P_l - Variable used in general Ricatti differential equation

Q - Variable used in definition of Green's theorem

Q_l - Variable used in general Ricatti differential equation

r - Reflection coefficient

R_o - Load resistance

R_d - Source resistance

v - Voltage wave along transmission line as a function of distance and time

v_d - Desired output voltage

v_s - Source voltage

V - Propagating electrical voltage

Y - Shunt admittance per unit length

Z - Series impedance per unit length

α - Lagrange multiplier function of distance and time
\( \alpha_c \) - Search parameter scalar for capacitance
\( \alpha_L \) - Search parameter scalar for inductance
\( \beta \) - Lagrange multiplier function of distance and time
\( \gamma \) - A parameter which is a function of \( \beta, \phi \) and \( v \)
\( \epsilon \) - The mean square error of \( v_d \) and \( v_s \)
\( \eta \) - A parameter which is a function of \( \alpha, \gamma \) and \( i \)
\( \xi \) - Problem surface (distance versus time)
\( \phi \) - Lagrange multiplier represents adjoint voltage wave
\( \psi \) - Lagrange multiplier represents adjoint current wave
1.0 INTRODUCTION

1.1 Purpose of this Thesis

The need for lossless non-uniform transmission lines (NUTL) is well established. Their practical application has been predominately in tapered matching sections. In propagation problems one typically encounters waves traveling in cascades of transmission media such as in waveguides and traveling wave tubes. A large class of signal processing systems takes the form of a cascade of elementary 2-port linear transducers; for example, in the classical filter theory cascades of constant-k, m-derived sections. Thus the need for lossless NUTL requires a procedure for the analysis and synthesis of the general line.

The motivation for this work stems from the lack of general synthesis procedures for the lossless case in the literature. Rohrer's [8] work with arbitrary lossy transmission lines can be adapted to the lossless case. In his work a functional is formed which is a measure of the error between the desired and realized output voltage function. This functional is minimized using variational calculus techniques subject to the system and boundary equality constraints and element inequality constraints. The first purpose of this thesis is to develop the mathematics of optimization for the lossless case by Rohrer's method.
Secondly, from a knowledge of the gradient of the error function a steepest-descent procedure is used to increment the parameter values producing an iterative design procedure. A technique for implementing the procedure on a digital computer is presented along with a fortran program.

Finally, the synthesis of a lossless non-uniform delay line is given as an illustrative example.

1.2 Review of Previous Work

There has been a great number of papers on non-uniform transmission line problems. Previous research by many workers described the transformed propagating voltage \( V(x) \) and current \( I(x) \) for a general NUTL by

\[
\frac{dV(x)}{dx} = -Z(x) I(x) \quad (1.1)
\]
\[
\frac{dI(x)}{dx} = -Y(x) V(x) \quad (1.2)
\]

where \( Z(x) \) and \( Y(x) \) are series impedance and shunt admittance per unit and are arbitrary functions of \( x \). Using this with the reflection coefficient

\[
r(x) = \frac{V(x) - I(x) K(x)}{V(x) + I(x) K(x)} \quad (1.3)
\]

where \( K(x) = \sqrt{\frac{Z(x)}{Y(x)}} \), combination of (1.1), (1.2) and (1.3) yielded a generalized Ricatti differential equation,

\[
\frac{dr(x)}{dx} + P_1(x) r(x) + Q_1(x) r^2(x) = Q_1(x) \quad (1.4)
\]

where \( P_1(x) = -2 \frac{\sqrt{Z(x)Y(x)}}{Z(x)Y(x)} \) and \( Q_1(x) = -\frac{dK(x)/dx}{2K(x)} \).
Sugai [14] reduced equation (1.4) to a linear form by two available transforms [15] but the resultant linearized differential equation is of variable coefficient second order hence the problem is unsimplified. In order to make any simplification many workers used approximations of various kinds, such as neglecting the square of the reflection coefficient.

Differentiation of (1.1) yields

\[ \frac{d^2 V(x)}{dx^2} - \frac{dZ(x)/dx}{Z(x)} \frac{dV(x)}{dx} - Y(x)Z(x)V(x) = 0. \]  

Equation (1.5) can be solved by specifying \( V(x) \) and obtaining \( f(Y(x), Z(x)) = 0 \). One shortcoming of this approach is that \( Y(x) \) and \( Z(x) \) are no longer independent. Sugai eliminated the interrelationship among variable coefficients by using a method developed by Hildebrand [4]. Sugai [16] has also applied D'Alembert's technique to convert a specialized Ricatti's non-linear differential equation into a first order linear differential equation.

Schwartz [12] analysed the non-uniform transmission line by transforming their characteristics to corresponding equations for other transmission lines whose solutions are known.

However, the exact solution has never been obtained for a general non-uniform transmission line. Sugai [14] presented exact solutions for special cases, starting with the assumed form of solution of the problem. When all analytical approaches fail to obtain the exact solution,
approximate methods by numerical analysis are available.

Rohrer, Resh, and Hoyt [9] have developed a synthesis
scheme for non-uniform RC transmission lines that can be
implemented on the digital computer by numerical tech-
niques. Through the use of iterative analysis the distrib-
uted parameters of the line are optimized and the output
voltage is realized within some approximation criterion of
the desired output. This approach is ideal for those
problems in which a closed form solution is not expected.
A more general development of this method was reported
by Rohrer [7][8] which will synthesize arbitrary lossy
transmission lines.
2.0 THE MATHEMATICS OF OPTIMIZATION

2.1 Formulation of System and Boundary Equations

A non-uniform lossless transmission line can be represented by a number of series inductance and shunt capacitance elements in a ladder network. The representation is only approximate if the elements are of finite size. The representation can be made exact if each element becomes infinitesimally small, while the number of these infinitesimal elements is allowed to increase without bound.

The differential series elements of such a line are shown in Figure 2.1. In general the voltage and current of the line are functions of distance along the line and of time. The inductance and capacitance may vary with distance down the line, but are time invariant. The change in voltage across the incremental impedance strip, $dx$, can be written

$$\frac{\partial v(x,t)}{\partial x} = L(x) \frac{\partial i(x,t)}{\partial t}$$

or

$$\frac{\partial v(x,t)}{\partial x} - L(x) \frac{\partial i(x,t)}{\partial t} = 0. \tag{2.2}$$

Similarly the differential shunt elements are shown in Figure 2.2. The change in current through this incremental admittance strip, $dx$, is

$$\frac{\partial i(x,t)}{\partial x} = C(x) \frac{\partial v(x,t)}{\partial t}$$

or

$$\frac{\partial i(x,t)}{\partial x} - C(x) \frac{\partial v(x,t)}{\partial t} = 0. \tag{2.4}$$
Figure 2.1. Differential series element of a non-uniform lossless transmission line.
Figure 2.2 Differential shunt element of a non-uniform lossless transmission line.
Of course some capacitance exists along the series arm which would cause a change in current through the differential length, $dx$. However, such capacitance will introduce only second-order infinitesimal terms which may be neglected \([13]\). Likewise, inductance was omitted from Figure 2.2 since it would add only second-order infinitesimal terms to equation (2.3).

The overall non-uniform lossless transmission line with voltage source, source resistance, and load resistance is shown in Figure 2.3 using the conventional symbol for the distributed LC portion of the line.

Writing loop equations around the input and output of the line will give two boundary equations,

\[
v(d,t) + R_d i(d,t) - v_s(t) = 0 \quad (2.5)
\]

and

\[
v(0,t) - R_0 i(0,t) = 0 \quad (2.6)
\]

In addition require that the line be initially relaxed, thus requiring that

\[
v(x,0) = i(x,0) = 0. \quad (2.7)
\]

Denote the desired output voltage as $v_d(t)$, then the error functional is proportional to the error between the output voltage realized and the voltage desired. It is

\[
\int_0^{t_f} \frac{1}{2} [v_d(t) - v(0,t)]^2 \, dt \quad (2.8)
\]

For a given input voltage, input and output resistance, and desired output voltage the NUTL problem has become one of finding $L(x)$ and $C(x)$ such that some error functional is minimized subject to the constraints of time invariance,
Figure 2.3. The non-uniform lossless transmission line.
\[
\frac{\partial L(x)}{\partial t} = 0 \tag{2.9}
\]
\[
\frac{\partial C(x)}{\partial t} = 0 \tag{2.10}
\]

and physical inequality constraints
\[
L_{\text{min}} \leq L(x) \leq L_{\text{MAX}} \tag{2.11}
\]
\[
C_{\text{min}} \leq C(x) \leq C_{\text{MAX}}. \tag{2.12}
\]

It is quite natural to impose bounds on the element values. Size limitations would most probably affect the upper bound and the lower bound might depend upon the frequency at which the line is used.

The error functional is necessary since there is no certainty that the desired output can be realized exactly. This fact suggests that some approximation criterion must be used in a numerical iterative approach.

2.2 Optimization using a Variational Approach

Several methods for solving constrained extrema problems are available for the circuit designer. The use of Lagrange multipliers in a variational approach is suggested by Sage [11] as being direct and amenable to vector formulation. The variational approach is concerned with the minimization or maximization of some index of performance or cost function. If the independent variables of the cost function are actually other functions, then this cost function is called a functional. The mathematics for extremizing functionals is variational calculus and is basic to optimization theory.
Essentially, the procedure is to make the necessary adjustments on the independent variable by using an adjustable multiplying parameter, called a Lagrange multiplier. A new functional is formed by appending the given constraints to the original functional. This new function is then extremized by variational techniques.

In order to append the constraint to the original functional, the constraint must be so arranged so that it is equal to zero. In this manner the value of the functional is unchanged. This is easily accomplished if equality constraints are involved. Inequality constraints, such as encountered in the NUTL problem, are more difficult to apply. One technique for handling inequality constraints is to convert them, through slack variables, into equalities. This, however, usually leads to a system of non-linear equations in the independent variable [17]. Another method (and the one used in this thesis) is to ignore the constraint and find the maximum or minimum. If this turns out to be interior to the boundary of the constraint set, we have the solution. If the parameter violates one of its bounds, it is made equal to the extreme value of that bound (maximum or minimum) and is fixed there.

To minimize a functional subject to constraints, it is necessary that the first variation of the functional, \( J \), be zero. Since the variation plays the same role in variational calculus as the differential in standard
calculus, the first variation of $J$, $\delta J$, can be linearized by using the linear part of $\Delta J$. Thus, if

$$ J = \int_0^t f(x,t) \, dt $$

then

$$ \Delta J = \int_0^t \left[ \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial t} \delta t + \text{higher order terms} \right] \, dt $$

and

$$ \delta J = \int_0^t \left[ \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial t} \delta t \right] \, dt $$

This necessary condition establishes only a relative or local extremum. Sufficient conditions involve the second variation which is avoided in this thesis. As Rohrer [9] has pointed out, sufficient conditions are difficult to work with, requiring that each specific case be dealt with separately. If the error is small, the solution is probably acceptable even if the result is not a global minimum.

A major disadvantage in iterative optimization methods is the inability to distinguish between local and global extrema. Usually, network problems contain a large number of local extrema and the best that can be expected is for convergence to the bottom of whatever valley of the error surface in which the program starts. If local extrema become a problem, the best procedure is to rerun the program using different starting points.

2.3 Formulation of Optimization Conditions

A variational approach to distributed network...
synthesis has been tried by others and has met with initial success [7], [8], [9]. In particular, the iterative network design technique given by Rohrer [7] provides a practical alternative to the impossible analytic solution. Derivation of the network optimization conditions which present a portion of Rohrer's work is given now.

Throughout this paper the existence of the definite integrals defining the functional is assumed, and it is further understood that minimizing functions are to be chosen from the set of all functions having continuous second derivatives on the time interval under consideration. Also, assume that the integral of the functional is at least twice continuously differentiable.

Let the functional to be minimized be defined as

\[ J := \int_0^1 \left[ \frac{1}{2} (v_d(t) - v(0,t))^2 \right] dt. \tag{2.16} \]

The problem is to perform minimization of (2.16) subject to the system equation constraints

\[ \frac{\partial v(x,t)}{\partial x} - L(x) \frac{\partial i(x,t)}{\partial t} = 0 \tag{2.17} \]
\[ \frac{\partial i(x,t)}{\partial x} - C(x) \frac{\partial v(x,t)}{\partial t} = 0 \tag{2.18} \]

and the time invariance constraints

\[ \frac{\partial L(x)}{\partial t} = 0 \tag{2.19} \]
\[ \frac{\partial C(x)}{\partial t} = 0. \tag{2.20} \]

The constraints are adjoined by means of Lagrange
multipliers $\Psi(x,t), \phi(x,t), \alpha(x,t), \beta(x,t)$ and surface integrals:

\[
\delta \left\{ \int_0^t \frac{1}{2} \left[ v_d(t) - \Psi(0,t) \right]^2 \, dt + \int \left\{ \Psi(x,t) \left[ \frac{\partial v(x,t)}{\partial x} \right] \\
- L(x) \frac{\partial i(x,t)}{\partial t} + \phi(x,t) \left[ \frac{\partial i(x,t)}{\partial x} \right] \\
- C(x) \frac{\partial v(x,t)}{\partial t} + \alpha(x,t) \frac{\partial L(x)}{\partial t} \\
+ \beta(x,t) \frac{\partial c(x)}{\partial t} \right\} \, dx \right\} = 0. \tag{2.21}
\]

Note that

\[
\Psi(x,t) \frac{\partial v(x,t)}{\partial x} = \frac{\partial}{\partial x} [\Psi(x,t) v(x,t)]
\]
\[
- v(x,t) \frac{\partial \Psi(x,t)}{\partial x} \tag{2.22}
\]
\[
\Psi(x,t) L(x) \frac{\partial i(x,t)}{\partial t} = \frac{\partial}{\partial t} [\Psi(x,t) L(x) i(x,t)]
\]
\[
- L(x) i(x,t) \frac{\partial \Psi(x,t)}{\partial t} \tag{2.23}
\]
\[
\phi(x,t) \frac{\partial i(x,t)}{\partial x} = \frac{\partial}{\partial x} [\phi(x,t) i(x,t)]
\]
\[
- i(x,t) \frac{\partial \phi(x,t)}{\partial x} \tag{2.24}
\]
\[
\phi(x,t) C(x) \frac{\partial v(x,t)}{\partial t} = \frac{\partial}{\partial t} [\phi(x,t) C(x) v(x,t)]
\]
\[
- C(x) v(x,t) \frac{\partial \phi(x,t)}{\partial t} \tag{2.25}
\]
\[
\alpha(x,t) \frac{\partial L(x)}{\partial t} = \frac{\partial}{\partial t} [\alpha(x,t) L(x)]
\]
\[
- L(x) \frac{\partial}{\partial t} [\alpha(x,t)] \tag{2.26}
\]
\[ \beta(x,t) \frac{\partial C(x)}{\partial t} = \frac{\partial}{\partial t}[\beta(x,t)C(x)] \]

\[-C(x)\frac{\partial}{\partial t} [\beta(x,t)]. \quad (2.27)\]

Substitute the right hand sides of (2.22) through (2.27) into (2.21) and regroup:

\[
\delta J = \delta \left\{ \int_0^T \left[ \frac{1}{2} [v_d(t)-v(0,t)]^2 dt \\
+ \int [\frac{\partial}{\partial x}[\psi(x,t)v(x,t)] \\
+ \frac{\partial}{\partial x}[\phi(x,t)i(x,t)] - \frac{\partial}{\partial t}[\psi(x,t)L(x)i(x,t)] \\
- \frac{\partial}{\partial t}[\phi(x,t)C(x)v(x,t)] + \frac{\partial}{\partial t}[\alpha(x,t)L(x)] \\
+ \frac{\partial}{\partial t}[\beta(x,t)C(x)]} d\sigma + \int \int [\frac{\partial}{\partial t}[\psi(x,t)] \\
- i(x,t) \frac{\partial \phi(x,t)}{\partial x} + L(x)i(x) \frac{\partial \psi(x,t)}{\partial t} \\
+ C(x)v(x,t) \frac{\partial \phi(x,t)}{\partial t} - L(x) \frac{\partial \alpha(x,t)}{\partial t} \\
- C(x) \frac{\partial \beta(x,t)}{\partial t} \} d\sigma \right\} = 0. \quad (2.28) \]

Now consider the first surface integral in equation (2.28). This surface integral can be converted to a line integral if Green's theorem in the plane is applied \[6\]. The requirements of Green's theorem are satisfied since \( \mathcal{L} \) is a finitely decomposable compact region in Euclidean two-space with boundary \( C \) and quantities \( P, Q, \partial P/\partial t \) and \( \partial Q/\partial x \) (as identified below) are defined and continuous. Then

\[
\int_C (P dx + Q dt) = \int \int_{\mathcal{L}} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial t} \right) d\sigma. \quad (2.29) \]
Making the following identifications

\[
Q = \left[ \Psi(x,t) v(x,t) + \phi(x,t) i(x,t) \right]
\]
\[
P = \left[ \Psi(x,t) L(x) i(x,t) + \phi(x,t) C(x) v(x,t) - \alpha(x,t) L(x) - \beta(x,t) C(x) \right]
\]

and applying the theorem in reverse

\[
\int \int \mathcal{E} \left\{ \frac{\partial}{\partial x} [\Psi(x,t) v(x,t)] + \frac{\partial}{\partial x} [\phi(x,t) i(x,t)] \\
- \frac{\partial}{\partial t} [\Psi(x,t) L(x) i(x,t)] \\
- \frac{\partial}{\partial t} [\phi(x,t) C(x) v(x,t)] + \frac{\partial}{\partial t} [\alpha(x,t) L(x)] \\
+ \frac{\partial}{\partial t} [\beta(x,t) C(x)] \right\} \, d\sigma \\
= \int_0^t \left[ \Psi(x,t) L(x) i(x,t) + \phi(x,t) C(x) v(x,t) - \alpha(x,t) L(x) - \beta(x,t) C(x) \right] \bigg|_{t=0} \, dx \\
+ \int_0^t \left[ \Psi(x,t) v(x,t) + \phi(x,t) i(x,t) \right] \bigg|_{x=0} \, dt \\
+ \int_0^t \left[ \Psi(x,t) L(x) i(x,t) + \phi(x,t) C(x) v(x,t) - \alpha(x,t) L(x) - \beta(x,t) C(x) \right] \bigg|_{t=t_f} \, dx \\
+ \int_{t_f}^{t_f} \left[ \Psi(x,t) v(x,t) + \phi(x,t) i(x,t) \right] \bigg|_{x=d} \, dt . \tag{2.32}
\]

Substituting the right hand side of (2.32) into (2.28) yields,
\[ \delta \int_{0}^{t_f} \left\{ \frac{1}{2} [v_d(t) - v(0,t)]^2 \, dt + \Psi(0,t)v(0,t) + \phi(0,t)i(0,t) \right\} \, dt \]
\[ - \delta \int_{0}^{t_f} \left\{ \Psi(\bar{t},t)v(\bar{t},t) + \phi(\bar{t},t)i(\bar{t},t) \right\} \, dt \]
\[ - \delta \int_{0}^{d} \left[ \Psi(x,0)L(x)i(x,0) + \phi(x,0)c(x)v(x,0) - \alpha(x,0)L(x) - \beta(x,0)c(x) \right] \, dx \]
\[ + \delta \int_{0}^{d} \left[ \Psi(x,t_f)L(x)i(x,t_f) + \phi(x,t_f)c(x)v(x,t_f) - \alpha(x,t_f)L(x) - \beta(x,t_f)c(x) \right] \, dx \]
\[ + \int \int_{s} \left[ -v(x,t) \frac{\partial \Psi(x,t)}{\partial x} + c(x)v(x,t) \frac{\partial \phi(x,t)}{\partial t} - i(x,t) \frac{\partial \phi(x,t)}{\partial x} + L(x)i(x,t) \frac{\partial \Psi(x,t)}{\partial t} - L(x) \frac{\partial \alpha(x,t)}{\partial t} - c(x) \frac{\partial \beta(x,t)}{\partial t} \right] \, ds = 0. \quad (2.33) \]

Now the variation of each independent quantity \( i(x,t), v(x,t), L(x) \) and \( C(x) \) over the surface yields four adjoint equations:

\[ \frac{\partial \phi(x,t)}{\partial x} - L(x) \frac{\partial \Psi(x,t)}{\partial t} = 0 \quad (2.34) \]
\[ \frac{\partial \Psi(x,t)}{\partial x} - c(x) \frac{\partial \phi(x,t)}{\partial t} = 0 \quad (2.35) \]
\[ \frac{\partial \alpha(x,t)}{\partial t} - i(x,t) \frac{\partial \Psi(x,t)}{\partial t} = 0 \quad (2.36) \]
\[ \frac{\partial \beta(x,t)}{\partial t} - v(x,t) \frac{\partial \phi(x,t)}{\partial t} = 0, \quad (2.37) \]

respectively.
The variations on the boundaries are constrained by (2.5), (2.6) and (2.7)

\[ \delta v(d,t) + R_d \delta i(d,t) = 0 \]  
\[ \delta v(0,t) - R_o \delta i(0,t) = 0 \]  
\[ \delta v(x,0) - \delta i(x,0) = 0 \]

Thus the voltage and current boundary variations yield the following boundary conditions,

\[ \Psi(d,t) - \frac{1}{R_d} \phi(d,t) = 0 \]
\[ \Psi(0,t) + \frac{1}{R_o} \phi(0,t) = [v_d(t) - v(0,t)] \]

from the pairs \( \delta v(d,t) \) and \( \delta i(d,t) \) and \( \delta v(0,t) \) and \( \delta i(0,t) \) respectively. Also

\[ \phi(x,t_f) = \Psi(x,t_f) = 0 \]

from \( \delta v(x,t_f) \) and \( \delta i(x,t_f) \). The adjoint problem is shown in Figure 2.4.

The remaining boundary variations become

\[ - \int_0^d \left\{ \left[ \alpha(x,t_f) - \Psi(x,t_f)i(x,t_f) \right] \right. \]
\[ + \left[ \Psi(x,0)i(x,0) - \alpha(x,0) \right] \right\} \delta L(x)dx \]
\[ - \int_0^d \left\{ \left[ \beta(x,t_f) - \phi(x,t_f)v(x,t_f) \right] \right. \]
\[ + \left[ \phi(x,0)v(x,0) - \beta(x,0) \right] \right\} \delta C(x)dx = 0 \]

Introduce two new variables \( \eta(x,t) \) and \( \gamma(x,t) \) such that

\[ \frac{\partial}{\partial t}[\eta(x,t)] = \frac{\partial}{\partial t}[\alpha(x,t)] - \frac{\partial}{\partial t}[\Psi(x,t_i) \eta(x,t)] \]

(2.45)
Figure 2.4. Adjoint transmission line problem.
and
\[
\frac{\partial}{\partial t} \psi(x,t) = \frac{\partial}{\partial t} \beta(x,t) - \frac{\partial}{\partial t} \Phi(x,t) \psi(x,t) \tag{2.46}
\]
then (2.44) becomes
\[
\int_0^d \left[ \eta(x,t_f) - \eta(x,0) \right] \delta L(x) dx \\
+ \int_0^d \left[ \gamma(x,t) - \gamma(x,0) \right] \delta C(x) dx = 0. \tag{2.47}
\]
Because of equations (2.45) and (2.46), equations (2.36) and (2.37) can be rewritten as
\[
\frac{\partial \eta(x,t)}{\partial t} + \psi(x,t) \frac{\partial i(x,t)}{\partial t} = 0 \tag{2.48}
\]
\[
\frac{\partial \gamma(x,t)}{\partial t} + \phi(x,t) \frac{\partial v(x,t)}{\partial t} = 0. \tag{2.49}
\]
Integrating equations (2.48) and (2.49)
\[
\eta(x,t_f) - \eta(x,0) = \int_0^{t_f} \psi(x,t) \frac{\partial i(x,t)}{\partial t} dt \tag{2.50}
\]
\[
\gamma(x,t_f) - \gamma(x,0) = \int_0^{t_f} \phi(x,t) \frac{\partial v(x,t)}{\partial t} dt. \tag{2.51}
\]
Equations (2.2), (2.4), (2.5), (2.6), (2.7), (2.34), (2.35), (2.41), (2.42), (2.43), (2.50) and (2.51) are the network optimization conditions and are summarized in Appendix A.
3.0 IMPLEMENTATION ON DIGITAL COMPUTER

3.1 Iterative Procedure

Referring to equations (2.2), (2.4), (2.34) and (2.35) it is clear there are four equations in the six unknowns \( v(x,t) \), \( i(x,t) \), \( \phi(x,t) \), \( \psi(x,t) \), \( L(x) \) and \( C(x) \). While at first the situation may appear unsolvable, this problem can be circumvented if an iterative procedure is used. For instance, choose a nominal \( i \)-th lossless transmission line, \( L_i(x) \) and \( C_i(x) \). Using this \( L_i(x) \) and \( C_i(x) \) solve equations (2.2) and (2.4) subject to boundary conditions (2.5), (2.6), and (2.7) for the unknown \( i \)-th voltage and current waves \( v^i(x,t) \) and \( i^i(x,t) \). With \( v^i(x,t) \) now a known quantity, the error current source function, \( v_d(t) - v^i(0,t) \) is known and the adjoint equations (2.34) and (2.35) can be solved for the unknown \( i \)-th adjoint voltage and current waves, \( \phi^i(x,t) \) and \( \psi^i(x,t) \) subject to boundary conditions (2.41), (2.42) and (2.43).

If the error criterion has not been satisfied, the parameter values can be altered using equations (2.50) and (2.51) in a steepest descent technique, and the entire iteration cycle is repeated for the \( i+1 \)-st transmission line.

3.2 Analysis of the Non-uniform Transmission Line

The first step of the iterative procedure amounts to
the analysis of a lossless NUTL for a given \( L(x) \) and \( C(x) \). For a first guess, \( L(x) \) and \( C(x) \) are normally held constant down the length of the line. Thus, the first analysis problem could be for a uniform lossless line. Regardless of the uniformity or non-uniformity of the line, a general approach must be taken toward the analysis problem.

Inspection of equations (2.2) and (2.4)

\[
\frac{\partial v(x,t)}{\partial x} - L(x) \frac{\partial i(x,t)}{\partial t} = 0 \quad (3.1)
\]

\[
\frac{\partial i(x,t)}{\partial x} - C(x) \frac{\partial v(x,t)}{\partial t} = 0 \quad (3.2)
\]

show them to be hyperbolic simultaneous linear partial differential equations. Computational solution of such a distributed system requires some form of discretization [10]. Space-time discretization is used here because it is particularly suited for computation with digital computers since the discretized system consists of a finite-dimensional system of difference equations.

The size of the space increment is \( \Delta x = d/m \) where \( m \) is an integer and the size of the time increment is \( \Delta t = t_f/n \) where \( n \) is an integer. This covers the strip \( 0 \leq x \leq d, 0 \leq t \leq t_f \) with a net of equal rectangles with sides \( \Delta x \) and \( \Delta t \) as shown in Figure 3.1.

When working on the boundary \( x = d \), it is best to use forward difference approximations in equations (3.1) and (3.2) since only the forward quantities are defined on the net. Thus, the discrete approximations to equations (3.1) and (3.2) are
Figure 3-1. Rectangular net used in time-space discretization model.
\begin{align}
\frac{v_{j+1,k} - v_{j,k}}{\Delta x} - L_j \frac{i_{j,k+1} - i_{j,k}}{\Delta t} &= 0 \quad (3.3) \\
\frac{i_{j+1,k} - i_{j-1,k}}{\Delta x} - C_j \frac{v_{j,k+1} - v_{j,k}}{\Delta t} &= 0 \quad (3.4)
\end{align}

where \( v_{j,k} \equiv v(x,t) \), \( v_{j,k+1} \equiv v(x,t+\Delta t) \) and \( x^e(m-j)\Delta x \) and \( t \equiv k\Delta t \); the other variables are defined similarly.

If this is to represent conditions on the boundary \( x = d \) the appropriate boundary condition must be imposed. The first time derivative of equation (2.5) is

\[ \frac{1}{R_d} \left[ \frac{\partial v_s(t)}{\partial t} - \frac{\partial v(d,t)}{\partial t} \right] - \frac{\partial i(d,t)}{\partial t} = 0 \quad (3.5) \]

Simulate equation (3.5) with finite difference equations,

\[ \frac{1}{R_d} \left[ \frac{v_{k+1}^S - v_k^S}{\Delta t} - \frac{v_{j,k+1} - v_{j,k}}{\Delta t} \right] \]

\[ - \frac{i_{j,k+1} - i_{j,k}}{\Delta t} = 0 \quad (3.6) \]

where "v superscript s" refers to source voltage.

Because this applies to the boundary \( x = d \) set \( j = 0 \) and solve (3.4) and (3.5) for \( v_{0,k+1} \) giving,

\[ v_{0,k+1} = v_{0,k} + \left\{ \frac{1}{R_d} \left[ \frac{v_{k+1}^S - v_k^S}{\Delta t} \right] + i_{0,k} \left[ \frac{1}{\Delta x} + \frac{1}{\Delta t} \right] \right\} \left\{ \left[ \frac{1}{R_d \Delta t} - \frac{C_d}{\Delta t} \right] \right\} (3.7) \]

Solve equation (3.3) and (3.5) for \( i_{0,k+1} \) giving,
Now solve equations (3.7) and (3.8) simultaneously to eliminate $i_{0,k+1}$ from equation (3.7) and $v_{0,k+1}$ from equation (3.8). Finally

$$v_{0,k+1} = \left\{ \left[ \frac{L_d C_d - R_d C_d}{\Delta t} - \frac{L_d}{R_d \Delta t} \right] v_{0,k} - \frac{L_d}{R_d \Delta t} \left[ \frac{v_{k+1} - v_k}{\Delta t} \right] + \frac{v_{l,k}}{\Delta x} \right\} \div \left[ \frac{R_d - L_d}{\Delta t} \right]$$

(3.9)

and

$$i_{0,k+1} = \left\{ \left[ \frac{L_d C_d - R_d C_d}{\Delta t} - \frac{L_d}{R_d \Delta t} \right] \frac{v_{l,k}}{x} + \frac{v_{l,k}}{\Delta x} \right\} \div \left[ \frac{L_d C_d - R_d C_d}{\Delta t} - \frac{L_d}{R_d \Delta t} \right]$$

(3.10)

Referring to Figure 3.2, these last two equations allow calculation of $v(d,t)$ and $i(d,t)$ on the line $k+1$ if the values of $v(d,t)$, $i(d,t)$, $v_{l,k}$ and $i_{l,k}$ on the line $k$ are known. The values of $v(x,t)$ and $i(x,t)$ on the line
\( k = 0 \) are given by equation (2.7), that is \( v(x,0) = i(x,0) = 0 \) since the line is initially relaxed.

Interior to the boundaries it is advantageous to use backward differences, since the number of unknowns in the finite difference equations is reduced. The discrete approximations to (3.1) and (3.2) are

\[
\frac{v_{j,k} - v_{j-1,k}}{\Delta x} - L_j \frac{i_{j,k-1} - i_{j,k-1}}{\Delta t} = 0 \tag{3.11}
\]

\[
\frac{i_{j,k} - i_{j-1,k}}{\Delta x} - C_j \frac{v_{j,k} - v_{j-1,k}}{\Delta t} = 0 \tag{3.12}
\]

Rearranging,

\[
v_{j,k} = v_{j-1,k} + L_j \frac{\Delta x}{\Delta t} \left( i_{j,k} - i_{j-1,k} \right) \tag{3.13}
\]

\[
i_{j,k} = i_{j-1,k} + C_j \frac{\Delta x}{\Delta t} \left( v_{j,k} - v_{j-1,k} \right) \tag{3.14}
\]

Eliminating \( i_{j,k} \) from equation (3.13) yields

\[
v_{j,k} = \left\{ v_{j-1,k} + L_j \frac{\Delta x}{\Delta t} \left[ i_{j-1,k} - i_{j-1,k} \right] - C_j \frac{\Delta x}{\Delta t} v_{j,k-1} \right\} \div \left[ 1 - L_j C_j \frac{\Delta x^2}{\Delta t^2} \right] \tag{3.15}
\]

Eliminating \( v_{j,k} \) from equation (3.14) yields

\[
i_{j,k} = \left\{ i_{j-1,k} + C_j \frac{\Delta x}{\Delta t} \left[ v_{j-1,k} - v_{j-1,k} \right] - L_j \frac{\Delta x}{\Delta t} i_{j,k-1} \right\} \div \left[ 1 - L_j C_j \frac{\Delta x^2}{\Delta t^2} \right] \tag{3.16}
\]

Referring to Figure 3.3, these last two equations allow calculation of \( v(x,t) \) and \( i(x,t) \) on the lines \( j \) and \( k \) if the values of \( v(x,t) \) and \( i(x,t) \) on the lines \( j-1 \) and \( k \) are known and if \( v(x,t) \) and \( i(x,t) \) on the lines \( j \) and \( k-1 \) are known.
indicates \( v(x, t) \) and \( i(x, t) \) are known at node.
- values at node are to be calculated.
- indicates values at white node depend upon quantities known at the black node.

Figure 3.2. Pictorial representation of the calculation of \( v(d,(k+1)\Delta t) \) and \( i(d,(k+1)\Delta t) \).

Figure 3.3. Pictorial representation of the known and unknown quantities in calculating \( v_{j,k} \) and \( i_{j,k} \).
On the boundary \( x = 0 \), boundary condition \((2.6)\) must be satisfied. Once again backward differences are used. The first time derivative of equation \((2.6)\) yields

\[
\frac{\partial v(x,t)}{\partial t} - R_0 \frac{\partial i(x,t)}{\partial t} = 0. \tag{3.17}
\]

The discrete approximation is

\[
\frac{v_{j,k}^{n} - v_{j,k-1}^{n}}{\Delta t} - R_0 \frac{i_{j,k}^{n} - i_{j,k-1}^{n}}{\Delta t} = 0. \tag{3.18}
\]

Solution of equations \((3.18)\) and \((3.11)\) for \(v(d,t)\) yields

\[
v_{m,k}^{n} = \left[ \frac{v_{m-1,k}^{n-1}}{\Delta x} - \left[ \frac{I_m}{R_0 \Delta t} \right] \right] v_{m,k-1}^{n}.
\tag{3.19}
\]

Solution of equations \((3.18)\) and \((3.12)\) for \(i(d,t)\) yields

\[
i_{m,k}^{n} = \left[ \frac{i_{m-1,k}^{n-1}}{\Delta x} - \left[ \frac{C_m R_0}{\Delta t} \right] \right] i_{m,k-1}^{n}.
\tag{3.20}
\]

These last two equations complete the analysis of the line. They allow calculation of \(v(x,t)\) and \(i(x,t)\) on \(x = 0\) if \(v(x,t)\) and \(i(x,t)\) on the lines \(j-1,k\) and \(j,k-1\) are known where \(j = m\). See Figure 3.3.

Now a review of the overall strategy will place the above procedure in perspective. Since the line is initially relaxed, \(v(x,t) = i(x,t) = 0\) along the line \(k = 0\). Sufficient information is available to calculate \(v_{0,k}^{n}\) on line \(k = 1\). Now step-by-step move down line \(k = 1\)
calculating \( v_{1,k'}, v_{2,k} \) and \( i_{1,k'}, i_{2,k} \) and so on until \( j = m \) is reached. Repeat for line \( k = 2 \) and each \( k \) thereafter until \( j = m \) and \( k = n \).

This completes the step-wise analysis of the transmission line. Since \( L(x) \) and \( C(x) \) are represented as functions of \( x \), the procedure is quite general and will work for either uniform or non-uniform lossless lines.

3.3 Analysis of the Adjoint Problem

The analysis of the adjoint problem is approached in a similar manner. One obvious difference does exist. The difference between the desired and realized voltage output is present in boundary condition equation (2.42) as a current source function. This particular boundary condition is applied at \( x = 0 \). Also from equation (2.43) the adjoint line is "initially" relaxed at time \( t_f \).

Referring to Figure 2.4 it should be clear that a solution of the adjoint problem can be effected using the analysis method in section 3.2 by allowing time and distance to run backwards. This is easily accomplished on the digital computer by simply reversing the order of the subscripts on the variables.

Assume the reversal in time and distance has been made. The discrete approximation to equations (2.34) and (2.35) are

\[
\frac{\Phi_{j+1,k} - \Phi_{j,k}}{\Delta x} - L_j \frac{\Psi_{j,k+1} - \Psi_{j,k}}{\Delta t} = 0 \tag{3.21}
\]
The first time derivative of equation (2.42) yields,
\[
\frac{\partial \psi(0,t)}{\partial t} + \frac{1}{R_o} \frac{\partial \phi(0,t)}{\partial t} = \frac{\partial v_a(t)}{\partial t} - \frac{\partial v(0,t)}{\partial t} = \frac{\partial \epsilon(t)}{\partial t}
\]  
(3.23)

where \( \epsilon(t) \) is a new variable defined as \( v_a(t) - v(0,t) \).

Approximating equation (3.23) by finite difference equations yields
\[
\frac{\psi_j, k+1 - \psi_j, k}{\Delta t} + \frac{1}{R_o} \frac{\phi_j, k+1 - \phi_j, k}{\Delta t} = \frac{\epsilon_{k+1} - \epsilon_k}{\Delta t}.
\]  
(3.24)

Equations (3.21) and (3.24) give the adjoint voltage function \( \phi_{0,k+1} \) as
\[
\phi_{0,k+1} = (1 - \frac{\Delta t}{\Delta x}) \phi_{0,k} + \frac{\Delta t}{\Delta x} \phi_{1,k} \\
+ (L_o + R_o) \psi_{0,k} - (L_o + R_o) \psi_{0,k+1} \\
+ R_o (\epsilon_{k+1} - \epsilon_k)
\]  
(3.25)

and equations (3.22) and (3.24) give the adjoint current function \( \psi_{0,k+1} \) as
\[
\psi_{0,k+1} = (1 - \frac{\Delta t}{\Delta x}) \psi_{0,k} + \frac{\Delta t}{\Delta x} \psi_{1,k} + (C_o + \frac{1}{R_o}) \phi_{0,k} \\
- (C_o + \frac{1}{R_o}) \phi_{0,k+1} + \epsilon_{k+1} - \epsilon_k
\]  
(3.26)

Eliminating \( \psi_{0,k+1} \) from equation (3.25) gives
\[ \Phi_{0,k+1} = \left\{ \left( 1 - \frac{\Delta t}{\Delta x} \right) - \left( L_o + R_o \right) \left( C_o + \frac{1}{R_o} \right) \right\} \Phi_{0,k} + \Delta t \frac{\Delta \Phi_{1,k}}{\Delta x} + \left( L_o + R_o \right) \Delta t \Psi_{0,k} - \left( L_o + R_o \right) \Delta t \Psi_{1,k} - L_o \left( \epsilon_{k+1} - \epsilon_k \right) \]

\[ \div \left[ 1 - \left( L_o + R_o \right) \left( C_o + \frac{1}{R_o} \right) \right] \quad (3.27) \]

Equations (3.27) and then equation (3.26) are used to calculate the adjoint current and voltage waves at \( x = 0 \) on the line \( k+1 \).

On the far boundary, \( x = d \), the adjoint boundary condition (2.41) is time differentiated, so as to give

\[ \frac{\partial \Psi(d,t)}{\partial t} - \frac{1}{R_d} \frac{\partial \Phi(d,t)}{\partial t} = 0 \quad (3.28) \]

and then approximated by finite difference equations to yield

\[ \frac{\Psi_{m,k} - \Psi_{m,k-1}}{\Delta t} - \frac{1}{R_d} \frac{\Phi_{m,k} - \Phi_{m,k-1}}{\Delta t} = 0 \quad (3.29) \]

when combined with the adjoint system (3.21) and (3.22), the adjoint voltage and current functions at \( x = d \) are found as

\[ \Phi_{m,k} = \frac{\Phi_{m,k} - \left[ \frac{L_m}{R_d \Delta t} \right] \Phi_{m,k-1}}{\frac{1}{\Delta x} - \frac{L_m}{R_d \Delta t}} \quad (3.30) \]

and similarly

\[ \Psi_{m,k} = \frac{\Psi_{m,k} - \left[ \frac{C_m R_d}{\Delta t} \right] \Psi_{m,k-1}}{\frac{1}{\Delta x} - \frac{C_m R_d}{\Delta t}} \quad (3.31) \]
Analysis interior to the boundaries is accomplished in the same manner as described in section 3.2 except for the reversal of time and distance. With the analysis of the transmission line and the adjoint line completed, \( v(x,t) \), \( i(x,t) \), \( \phi(x,t) \) and \( \Psi(x,t) \) are known quantities at each node of the rectangular net. Hence, sufficient information is available to calculate equations (2.50) and (2.51) which will be used to alter parameter values.

3.4 Convergence and Stability

Whenever finite difference equations are used in a step-by-step manner, consideration should be given to questions of convergence and stability.

The right hand side of the forward difference formulas

\[
\frac{\partial v(x,t)}{\partial x} \approx \frac{v_{j+1,k} - v_{j,k}}{\Delta x} \quad (3.32)
\]

and

\[
\frac{\partial v(x,t)}{\partial t} \approx \frac{v_{j,k+1} - v_{j,k}}{\Delta t} \quad (3.33)
\]

for instance, tend to the respective derivatives as \( \Delta x \) and \( \Delta t \to 0 \) and their convergence is certain if the second derivative of (3.32) with respect to \( x \) and the second derivative of (3.33) with respect to \( t \) exist. In section 2.3 it was assumed that the functions have continuous second derivatives so convergence to the respective derivatives is assured. This also holds true for backward difference formulas.

The satisfaction of these conditions as \( \Delta x \) and
At $\Delta t \rightarrow 0$ does not suffice for the convergence of equations (3.3) and (3.4) or (3.11) and (3.12) to that of the partial differential equations (3.1) and (3.2). Fox [3] and others have shown, however, that if $\left(\frac{\Delta t}{\Delta x}\right)^2 \leq 1$ the method will converge, subject as usual, to the existence of some higher derivatives. Hence, convergence of the analysis method requires $\left(\frac{\Delta t}{\Delta x}\right)^2 \leq 1$.

In addition to convergence, stability, in the sense that inevitable rounding errors must not swamp the true finite difference solution, is required. The Von Neumann's method is a useful and simple technique for finding a stability criterion. Proceed by seeking a special solution of (3.3), (3.4), (3.11) or (3.12) in the form $v_{j,k}$, $i_{j,k}$, $\phi_{j,k}$, or $\psi_{j,k}$ equal to $e^{\alpha k\Delta t} e^{ij\beta \Delta x}$. In order that no solution of this form have real or imaginary parts which grow exponentially in magnitude with increasing $k$, it is necessary that $|e^{\alpha \Delta t}| \leq 1$ for all possible real values of $\beta$.

Substitution into (3.3) gives

$$e^{\alpha \Delta t} = \frac{L_j + \frac{\Delta t}{\Delta x} \left[ (\cos \beta \Delta x - 1) + i \sin \beta \Delta x \right]}{L_j} \quad (3.34)$$

from which $|e^{\alpha \Delta t}| \leq 1$ for all real values of $\beta$ and similarly for (3.4), (3.11) and (3.12). Hence, stepwise stability for the analysis procedure is assured.

3.5 Method of Steepest Descent

After analysis is complete, the iterative procedure outlined earlier calls for altering the parameter values
If the error criterion is not satisfied in such a manner as to minimize the mean square error \( \varepsilon(t) \), perhaps the best known procedure for doing this is the method of steepest descent.

In this method some portion of the error gradient in parameter space is subtracted from the \( i \)-th value of the parameter. Thus, at each iteration the parameter is changed in a direction opposite to the gradient. Calahan [1] and Temes and Calahan [17] have shown that such a change always decreases the error function and results in a first order minimization technique. The advantages of the method are that it allows a rather poor initial guess because convergence is most rapid in regions far removed from the minimum and it does not require calculation of higher order derivatives, which is subject to inaccuracies.

For the lossless non-uniform transmission line

\[
L^{i+1}(x) = L^i(x) - \alpha_L^i \int_0^{t_f} \left[ -\Psi(x,t) \frac{\partial i(x,t)}{\partial t} \right] dt
\]

(3.35)

and

\[
C^{i+1}(x) = C^i(x) - \alpha_C^i \int_0^{t_f} \left[ -\phi(x,t) \frac{\partial v(x,t)}{\partial t} \right] dt
\]

(3.36)

where \( \alpha_L^i \) and \( \alpha_C^i \) are non-negative scalars. The integral factors come from equations (2.50) and (2.51) which determine the error gradients in parameter space.
The partials may be approximated by forward difference equations and the integration may be approximated by the trapezoidal rule. As a word of caution, remember that the adjoint quantities were found by running time and distance backwards and that the adjoint quantities for node \( j, k \) are actually identified as being at node \( m-j, n-k \) in the adjoint analysis. The approximation equations are,

\[
L_j^{i+1} = L_j^i + \alpha_L^i \left[ \psi_{m-j,n}^i \left( i_{j,1}^i - i_{j,0}^i \right) \right. \\
+ \sum_{k=1}^{n-1} \psi_{m-j,n-k}^i \left( i_{j,k+1}^i - i_{j,k}^i \right) \\
+ \left. \psi_{m-j,0}^i \left( i_{j,n}^i - i_{j,n-1}^i \right) \right] 
\tag{3.37}
\]

\[
c_j^{i+1} = c_j^i + \alpha_C^i \left[ \phi_{m-j,n}^i \left( v_{j,1}^i - v_{j,0}^i \right) \right. \\
+ \sum_{k=1}^{n-1} \phi_{m-j,n-k}^i \left( v_{j,k+1}^i - v_{j,k}^i \right) \\
+ \left. \phi_{m-j,0}^i \left( v_{j,n}^i - v_{j,n-1}^i \right) \right] . 
\tag{3.38}
\]
The most important aspect of any optimization procedure is the parameter search. The major portion of the total computational time will be spent in establishing some region in parameter space where a profitable search could be made. Initially the problem is to determine the size of the steps $\alpha_L$ and $\alpha_C$ that will bound some minimum. Then using one of several available procedures, a particular $\alpha_L$ and $\alpha_C$ are sought so that equations (3.35) and (3.36) will give $L^{i+1}(x)$ and $C^{i+1}(x)$ such that $J$ is minimized within the bounded region.

Conceptually, the error surface consists of hills and valleys where the valleys represent various local minima. Once one of the valleys have been bounded, $\alpha_L$ and $\alpha_C$ are refined and the parameters are incremented in such a fashion that the error will descend to the bottom of the bounded valley. However, there is no guarantee that the three-dimensional error surface defined by $J$ (equation 2.16), $L(x)$, and $C(x)$ will be relatively smooth or well-behaved. As will be shown in an example design problem (see section 4.2), many local minima may exist in the non-uniform transmission line problem. There is no assurance that this local minimum is the one desired. If it is not the desired minimum, a large block of computer time has been used and the designer is not much closer to the solution of his problem.
In this thesis a sophisticated search procedure was not used. Instead, large steps in $\alpha_L$ and $\alpha_C$ were made so that a large portion of the error surface could be covered. In this manner, a "feel" for the error contour could be developed. Those minima which were far from the solution could be avoided and the more promising minima could be examined in more detail, saving considerable computer time. Each time a minimum that looked acceptable was found, the program was rerun using starting values within that region of interest.

In calculating the step size of $\alpha_L^k$ and $\alpha_C^k$ for each iteration $k$, within a bounded region, the following scheme proved acceptable. Set $\alpha_L^1$ and $\alpha_C^1$ equal to the arithmetic mean of the bounding values. Then define $\alpha_L^k$ and $\alpha_C^k$ as

$$\alpha_L^k = \alpha_L^1 \left(\frac{1}{2}\right)^{k-1}$$

$$\alpha_C^k = \alpha_C^1 \left(\frac{1}{2}\right)^{k-1}$$
4.0 ILLUSTRATIVE EXAMPLE

So far, the theoretical aspects of this synthesis method have been discussed. In this section an actual design problem will be undertaken in way of illustrating the practical application of the design procedure.

4.1 Statement of Example Problem

Suppose it is desired to synthesize a lossless delay line that is five meters long, excited by a voltage defined by

\[ v_s(t) = 50 \sin (2\pi 5t) \text{, for } 0 \leq t \leq 0.5 \text{ sec (4.1)} \]

and

\[ v_s(t) = 0 \text{, for } 0.5 < t \leq 1.0 \text{ seconds (4.2)} \]

where the time interval of interest is one second.

Specify the source and load resistance, \( R_d \) and \( R_o \) respectively, as 100 ohms each. Let the desired time delay be 0.03 seconds.

First, time and distance must be discretized according to section 3.2. If \( m \) is chosen as 50 and \( n \) as 100, the space and time increments will be of a convenient size and will provide reasonable accuracy. Then

\[ \Delta x = \frac{d}{m} = \frac{5}{50} = 0.1 \text{ meters (4.3)} \]

and

\[ \Delta t = \frac{t_f}{n} = \frac{1}{100} = 0.01 \text{ seconds (4.4)} \]

Note that the ratio

\[ \left( \frac{\Delta t}{\Delta x} \right)^2 = \left( \frac{0.01}{0.1} \right)^2 = 0.01 \text{ (4.5)} \]

is less than one as required for convergence by section 3.4.
Secondly, since the input and output impedances are 100 ohms, a good choice for the initial line is a uniform line with a characteristic impedance of 100 ohms. Let the starting values of \( L(x) \) and \( C(x) \) be 10 millihenries and 1.0 microfarad respectively, for all \( x=j\Delta x \) where \( j=1,2,3,\ldots,m \). Finally, require that \( v_d(t) \) be defined as

\[
v_d(t) = \begin{cases} 
0 & \text{for } t < 0.03 \text{ sec} \\
25 \sin[2\pi5(t-.03)] & \text{for } t > 0.53 \text{ sec}
\end{cases}
\]

(4.6)

and

\[
v_d(t) = 25 \sin[2\pi5(t-.03)] ,
\]

(4.7)

for \( 0.03 \leq t \leq 0.53 \) seconds.

No physical constraints will be imposed on the element values. Physical constraints will limit the size of the error surface, and a large error surface is desired so that various local minima might be located and identified. This will serve to illustrate the nature of the error function.

### 4.2 Results and Comments

This problem was programmed and executed on an IBM Model 360-50 digital computer. A flow-chart showing the major programming steps is given in Appendix B and a detailed fortran program is listed in Appendix C.

It was found that the initial analysis of the transmission line, including the analysis of the adjoint problem, requires about two minutes of CPU time. Subsequent iteration require a disproportionate amount of time. For instance, four iterations, consisting of
analysis, calculation of the error gradient, and parameter adjustment, requires only three minutes of CPU time. Twenty-five such iterations required 9 minutes and 36.62 seconds of CPU time. This was without an extensive search subroutine. If one is used, as much time should be allowed for the search routine as is allotted for analysis and calculation of the error gradient. Clearly, if the initial starting point is far from the desired minimum the computer time required might become excessive.

In attempting to find an acceptable local minimum, a large portion of the error surface was examined. It was found that the error surface contains several small valleys or slight depressions which could conceivably be mistaken for local minimum. However, the least square errors at these locations were sufficiently large as to make the ostensible minima unacceptable. A summary of this investigation is tabulated in Table 4.1. Given in the table are the least square errors, the error gradient with respect to inductance and capacitance, and the corresponding value of \( L(x) \) and \( C(x) \).

As might be expected from the design specifications, the solution to the delay line is a lossless uniform line. It is included herein because it's solution is typical and illustrates the flexibility of the synthesis procedure. Hence in Table 4.1 only one entry for \( L(x) \) and one entry for \( C(x) \) is made since they are constant along the line.

Referring to Table 4.1, an asterisk is used to
<table>
<thead>
<tr>
<th>Near local minimum</th>
<th>Least squares error</th>
<th>Error gradient inductance</th>
<th>Error gradient capacitance</th>
<th>L(x)</th>
<th>C(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>*</td>
<td>237</td>
<td>-13.3</td>
<td>-254,000</td>
<td>0.6 h.</td>
<td>65 μfd</td>
</tr>
<tr>
<td></td>
<td>7887</td>
<td>-13.5</td>
<td>-245,000</td>
<td>0.3 h.</td>
<td>20.4</td>
</tr>
<tr>
<td></td>
<td>8082</td>
<td>-13.2</td>
<td>-250,000</td>
<td>0.3 h.</td>
<td>20.0</td>
</tr>
<tr>
<td></td>
<td>15561</td>
<td>1.5</td>
<td>272,000</td>
<td>22.0 mh.</td>
<td>0.556</td>
</tr>
<tr>
<td>*</td>
<td>6480</td>
<td>12.9</td>
<td>130,000</td>
<td>10.0</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td>5857</td>
<td>15.9</td>
<td>108,000</td>
<td>8.2</td>
<td>0.998</td>
</tr>
<tr>
<td></td>
<td>6258</td>
<td>12.6</td>
<td>129,000</td>
<td>5.0</td>
<td>0.5</td>
</tr>
<tr>
<td></td>
<td>7331</td>
<td>14.3</td>
<td>118,000</td>
<td>4.58</td>
<td>0.5</td>
</tr>
<tr>
<td>*</td>
<td>5534</td>
<td>18.1</td>
<td>94,300</td>
<td>3.59</td>
<td>0.49</td>
</tr>
<tr>
<td></td>
<td>4583</td>
<td>9.2</td>
<td>48,300</td>
<td>3.5</td>
<td>0.5</td>
</tr>
<tr>
<td>*</td>
<td>7137</td>
<td>9.0</td>
<td>46,500</td>
<td>3.49</td>
<td>0.495</td>
</tr>
<tr>
<td></td>
<td>5137</td>
<td>9.1</td>
<td>48,900</td>
<td>3.49</td>
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<tr>
<td>*</td>
<td>7792</td>
<td>9.3</td>
<td>45,600</td>
<td>3.49</td>
<td>0.490</td>
</tr>
<tr>
<td></td>
<td>5540</td>
<td>21.4</td>
<td>77,200</td>
<td>3.0</td>
<td>0.5</td>
</tr>
<tr>
<td>*</td>
<td>5510</td>
<td>21.1</td>
<td>78,400</td>
<td>2.97</td>
<td>0.488</td>
</tr>
<tr>
<td></td>
<td>5703</td>
<td>20.9</td>
<td>78,600</td>
<td>2.97</td>
<td>0.486</td>
</tr>
<tr>
<td></td>
<td>9611</td>
<td>27.4</td>
<td>52,600</td>
<td>2.28</td>
<td>0.498</td>
</tr>
<tr>
<td>*</td>
<td>7760</td>
<td>36.8</td>
<td>27,500</td>
<td>1.38</td>
<td>0.488</td>
</tr>
<tr>
<td></td>
<td>14108</td>
<td>41.2</td>
<td>13,600</td>
<td>0.2</td>
<td>0.497</td>
</tr>
</tbody>
</table>

Table 4.1. Tabulation of error for different values of $L(x)$ and $C(x)$
identify those small depressions or valleys which the program might seek out as local minima. The most obvious minimum occurs for \( L(x)=0.6 \) henries and \( C(x)=65 \) microfarads. The least square error at this location is small enough, compared to the other errors, that it may be regarded as a solution. Figure 4.1 shows the realized output voltage for several values of \( L(x) \) and \( C(x) \). The desired response is most closely approached by the curve for \( L(x)=0.6 \) henries and \( C(x)=65 \) microfarads.

As can be seen from Figure 4.1, the waveform and delay desired have been achieved. The error present in the solution is due to the fact that the maximum desired output was specified to be 25 volts, whereas in the actual realized line, the maximum output was about 28 volts. This produces an error of about 3 volts at each peak. Refinement of the parameter values would tend to minimize this particular error. The error present between the realized and desired response is given in Figure 4.2 for five of the six least square errors identified with an asterisk in Table 4.1.

Classical solution of this problem gives \( L(x)=0.6 \) henries and \( C(x)=60 \) microfarads. Hence the computer solution is very close to the global minimum which is obvious upon inspection of the error curves in Figure 4.2.

In order to investigate a non-uniform line problem, the load resistance, \( R_o \), was increased to 200 ohms. From a knowledge of the previous solution, the desired
Figure 4.1. $v(0,t)$ versus time showing convergence to desired output.
Figure 4.2. Error between desired output and realized output for various local minima.
output was specified to peak at 30 volts. The delay desired remained at 0.03 seconds. The initial line was chosen to be uniform with \( L(x) = 0.4 \) henries and \( C(x) = 40 \) microfarads.

The inductance and capacitance distribution along the realized transmission line is given in Figure 4.3 and 4.4. Some degree of impedance matching was achieved. For instance, \( R_d = 100 \) ohms and

\[
Z_d = \sqrt{\frac{L_d}{C_d}} = \sqrt{\frac{0.49}{0.48(10^{-4})}} = 100 \text{ ohms.} \tag{4.8}
\]

At the output however, \( R_o = 200 \) ohms and

\[
Z_o = \sqrt{\frac{L_o}{C_o}} = \sqrt{\frac{0.65}{0.49(10^{-4})}} = 115 \text{ ohms.} \tag{4.9}
\]

The impedance matching at the output is rather poor. For better matching \( L_o \) must increase and/or \( C_o \) must decrease. Inspection of Figures 4.3 and 4.4 will show that this was indeed, the trend in the parameter values. A more elaborate search procedure for \( \alpha_L \) and \( \alpha_o \) would tend to extrapolate these two curves and better impedance matching would probably occur.

Figure 4.5 illustrates how closely the desired output is achieved. The initial uniform line resulted in about 0.01 seconds of delay, whereas the non-uniform line of the final iteration was practically identical to the desired response. The realized output had a slight undershoot. This undershoot shows up in the error curve as a small peak at about 0.53 seconds. Other
Figure 4.3. Inductance distribution along the lossless non-uniform delay line.

Figure 4.4. Capacitance distribution along the lossless non-uniform delay line.
Initial guess, uniform line with 0.4 henries, 40 μfd

Final iteration, non-uniform line (see Figure 4.3 and 4.4)

--- Desired output
--- Realized output

Figure 4.5. \( v(0,t) \) versus time showing convergence to desired output for the non-uniform lossless delay line.
than this the error curve is quite flat. It is shown in Figure 4.6 along with the error curve of the initial uniform line.

The computer time required for the solution of the non-uniform line was comparable to that of the uniform line. Four iterations required only 3 minutes and 14 seconds of CPU time.
Figure 4.6. Error between desired output and realized output for the initial and final iteration of the lossless non-uniform delay line.
5.0 CONCLUSION

5.1 Concluding Remarks

Using some basic concepts from the calculus of variations has provided the circuit designer with a useful and rather powerful synthesis technique. This variational, time domain method constitutes the basis for a new and different design approach. This method reduces the time domain optimization problem to that of minimizing some error functional given its value and the value of its gradient in the space of its arguments.

Implementation of this method on the digital computer is straightforward and the feasibility of writing a user-oriented design program has been demonstrated. The computational time is small enough to be consistent with the economics of the design, and the cost will be even more attractive as the complexity of the problem increases. In general it can be concluded that the variational method will be most useful in cases where complicated trade-offs are involved and where the complexity of the problem defies the intuition of the designer.

The synthesis scheme presented in this thesis provides the circuit designer with a powerful design tool. In the past, network designers used computer network analysis programs mainly in a cut and try procedure. Obviously, this procedure can be wasteful of
computer time and man-hours. Today, with the presentation of synthesis schemes, such as reported in this thesis, computer-aided network design has evolved from the former cut and try procedures into an orderly approach that directly seeks the optimum network.

5.2 Suggestions for Further Research

The analysis program was approached in a step-wise manner. Other techniques exist for solving the set of simultaneous hyperbolic partial differential equations. If stability is troublesome, there is no better approach than the method of characteristics. By integrating along the characteristic curves, stability is assured since the solution is constantly kept within a region of determinacy. Generally, the characteristics are curved and solution points do not occur at equal intervals in either x or t. If the solution is required on a rectangular grid, the resulting two-dimensional interpolation is formidable. A technique is needed whereby the characteristic method can be used easily on a rectangular net.

In respect to stability, implicit finite difference methods offer an advantage. Usually this method will offer unrestricted stability. However, large systems of simultaneous difference equations must be solved which is a time consuming problem. Investigation is warranted in this area.

It is foreseen that the performance functional and
gradient determination may take large blocks of computer time on large problems. Significant future research will be required on fast integration methods and strategies for quickly descending down the gradient toward an optimum.

The Fletcher-Powell minimization procedure is similar to the method of steepest-descent but its rate of convergence is faster. Its use in this synthesis scheme could be investigated. A parameter search for $\alpha_L$ and $\alpha_0$ would be the most time consuming portion of the synthesis procedure. If they were not accurately determined, it could introduce the danger of divergence in the Fletcher-Powell method. These trade-offs might be interesting to examine.

Since the major portion of total computational time will be the parameter search, highly efficient multi-parameter searches are needed. Investigation into methods to quickly determine the region of parameter search in which a profitable search can be made could be rewarding.
REFERENCES


12. Schwartz, Richard F. "Transformations in the analysis


VITA

Born on December 4, 1944, Larry Lee Wesner received his primary and secondary education in Kansas City, Missouri, graduating valedictorian of his class from Ruskin High School in 1962. He attended the Junior College of Kansas City, Missouri for two years receiving the Associate of Science Degree in Engineering in June, 1964. He then transferred to the University of Missouri at Rolla in September, 1964 and received the Bachelor of Science Degree in Electrical Engineering with first honors in June, 1967. While working on his undergraduate degree he was a cooperative work student with Burns and McDonnell Engineering Consultants in Kansas City, Missouri. He entered the Graduate School of the University of Missouri at Rolla in September, 1967. His graduate studies was conducted under a National Science Foundation Traineeship granted to him. During the summer of 1968 he worked as an electronic engineer at the U.S. Naval Weapons Center, China Lake, California.

The author is a member of Eta Kappa Nu, Tau Beta Pi, Phi Kappa Phi, Phi Theta Kappa, and a student member of IEEE. He is married to the former Linda Sue Thornton.
APPENDIX A. SUMMARY OF OPTIMIZATION CONDITIONS

The network optimization conditions are summarized below. The analysis problem consists of solving equations (A.1) and (A.2) subject to the boundary conditions (A.3) and (A.4) and initial conditions (A.5).

\[
\begin{align*}
\frac{\partial v(x,t)}{\partial x} - L(x) \frac{\partial i(x,t)}{\partial t} &= 0 \\
\frac{\partial i(x,t)}{\partial x} - C(x) \frac{\partial v(x,t)}{\partial t} &= 0
\end{align*}
\] (A.1)

\[
\begin{align*}
v(d,t) + R_d i(d,t) - v_s(t) &= 0 \\
v(0,t) - R_o i(0,t) &= 0 \\
v(x,0) = i(x,0) &= 0
\end{align*}
\] (A.2) (A.3) (A.4) (A.5)

The adjoint analysis problem consists of solving equations (A.6) and (A.7) subject to the boundary conditions (A.8) and (A.9) and "initial" conditions (A.10).

\[
\begin{align*}
\frac{\partial \phi(x,t)}{\partial x} - L(x) \frac{\partial \psi(x,t)}{\partial t} &= 0 \\
\frac{\partial \psi(x,t)}{\partial x} - C(x) \frac{\partial \phi(x,t)}{\partial t} &= 0
\end{align*}
\] (A.6)

\[
\begin{align*}
\psi(d,t) - \frac{1}{R_d} \phi(d,t) &= 0 \\
\psi(0,t) - \frac{1}{R_o} \phi(0,t) &= v_d(t) - v(0,t)
\end{align*}
\] (A.7) (A.8) (A.9)

\[
\begin{align*}
\phi(x,t_f) = \psi(x,t_f) &= 0
\end{align*}
\] (A.10)

The values of the voltage and current waves, \( v(x,t) \) and \( i(x,t) \) and the adjoint voltage and adjoint current waves, \( \phi(x,t) \) and \( \psi(x,t) \) are known from the analysis. The error gradient in parameter space is
calculated from equations (A.11) and (A.12).

\[ \eta(x, t_f) - \eta(x, 0) = \int_{0}^{t_f} \left[ -\Psi(x, t) \frac{\partial i(x, t)}{\partial t} \right] dt \quad (A.11) \]

\[ \gamma'(x, t_f) - \gamma'(x, 0) = \int_{0}^{t_f} \left[ -\phi(x, t) \frac{\partial v(x, t)}{\partial t} \right] dt \quad (A.12) \]

Using equations (A.11) and (A.12) the parameters are updated in an iterative steepest-descent technique.
APPENDIX B. FLOW CHART OF FORTRAN PROGRAM

1. Set initial conditions
2. Read source function
3. Read desired output function
4. Read line information
5. Analysis of transmission line
6. Define current error source function
7. Calculate least square error "FNCT"
8. IF FNCT=ERROR
   a. Output
   b. IF NOT ERROR
      i. Analysis of adjoint line
      ii. Compute $\alpha_L$ and $\alpha_C$
      iii. Calculate error grad.
      iv. Form N+1st parameters
      v. Set limit values
      a. Is $C_{\text{min}} \leq C(x) \leq C_{\text{max}}$?
         i. YES
         ii. Set limit values
         iii. IS $L_{\text{min}} \leq L(x) \leq L_{\text{max}}$?
             i. YES
             ii. Output
             iii. stop
             iv. NO
```
C*****STEEPEST DESCENT ITERATIVE PROCEDURE TO OPTIMIZE
C THE PARAMETERS OF A LOSSLESS NON–UNIFORM TRANS-
C MISSION LINE
C
C SYMBOLS USED IN THIS PROGRAM:
C V : VOLTAGE WAVE
C CUR : CURRENT WAVE
C PHI : ADJOINT VOLTAGE WAVE
C PSI : ADJOINT CURRENT WAVE
C FNCT : LEAST SQUARES ERROR
C EPS : ERROR BETWEEN DESIRED AND REALIZED OUTPUT
C SUM1 : ERROR GRADIENT WITH RESPECT TO INDUCTANCE
C SUM2 : ERROR GRADIENT WITH RESPECT TO CAPACITANCE
C
DIMENSION CUR(51,101), V(51,101), C(51), VS(101),
PHI(51,101), PSI(51,101), ADJL(51), ADJC(51), EPS(101),
VO(101), FNCT(50)
C
C*****SET INITIAL CONDITIONS
DO 100 I=1,51
CUR(I,1)=0.0
V(I,1)=0.0
PHI(I,1)=0.0
100 PSI(I,1)=0.0
C
C*****READ THE SOURCE FUNCTION
READ(1,440) (VS(I), I=1,101)
C
C*****READ THE DESIRED OUTPUT FUNCTION
READ(1,440) (VO(I), I=1,101)
C
C*****READ LINE INFORMATION
READ(1,440) DX, DT, RO, RD
READ(1,440) (C(I), I=1,51)
READ(1,440) (HL(I), I=1,51)
READ(1,440) ERROR
READ(1,440) HLMAX, HLMIN, CMAX, CMIN
ITER=1
1000 CONTINUE
C
C*****ANALYSIS OF THE TRANSMISSION LINE
C*****EVALUATION OF THE COEFFICIENTS FOR THE BOUNDARY
E=(1./DT)*(HL(1)*C(1)-RD*C(1)*HL(1)/RD)
D=(-HL(1)+RD)/DX
H=E-1./DX
F=-C(1)/DX+1./(RD*DX)
P=(1./DX)-(HL(51)/(RO*DT))
Q=(1./DX)-((C(51)*RO)/DT)
C
C*****EVALUATION ON THE BOUNDARY
DO 430 K=2,101
MK=K-1
B=VS(K)-VS(MK)
V(1,K)=(H*V(1,MK)-(HL(1)/(RD*DT))*B+D*CUR(1,MK)
1-D*CUR(2,MK)+V(2,MK)/DX)/E
CUR(1,K)=(H*CUR(1,MK)-C(1)*B/DT+F*V(1,MK)-F*V(2
1,MK)+CUR(2,MK)/DX)/E

C

C*****EVALUATION OF THE INTERIOR VARIABLES
DO 420 J=2,50
MJ=J-1
G=1.-(HL(J)*C(J)*RATIO*RATIO)
V(J,K)=(V(MJ,K)+HL(J)*RATIO*(CUR(MJ,K)-CUR(J,MK
1)-C(J)*RATIO*V(J,MK))/G
420 CUR(J,K)=CUR(MJ,K)+C(J)*RATIO*(V(J,K)-V(J,MK))

C

C*****EVALUATION OF THE BOUNDARY CONDITIONS ON THE
C
FAR END OF THE LINE
V(51,K)=(V(50,K)/DX-(HL(51)/(RO*DT)))*V(51,MK))/P
CUR(51,K)=(CUR(50,K)/DX-(C(51)*RO*CUR(51,MK))/DT
1)/Q
430 CONTINUE
WRITE(3,440)(V(51,K),K=1,101)
WRITE(3,440)(V(1,K),K=1,101)

C

C*****DEFINITION OF THE CURRENT ERROR SOURCE FUNCTION
FNCT(ITER)=0.0
DO 550 I=1,101
J=102-I
EPS(J)=VO(I)-V(51,I)
550 FNCT(ITER)=FNCT(ITER)+EPS(J)*EPS(J)
WRITE(3,555)FNCT(ITER)
555 FORMAT(2X,'LEAST SQUARES ERROR(FNCT)=',El8.8)
WRITE(3,440)(EPS(K),K=1,101)
IF(FNCT(ITER)-ERROR3590,590,570
570 CONTINUE

C

C*****ANALYSIS OF ADJOINT LINE ( TIME AND DISTANCE
C
RUN BACKWARDS )

C

C*****DEFINITION OF L(X) AND C(X) FOR ADJOINT LINE
DO 600 I=1,51
J=52-I
ADJL(I)=HL(J)
600 ADJC(I)=C(J)

C

C*****EVALUATION OF BOUNDARY COEFFICIENTS
ADJL=1.-((DT/DX)
ADJ2=ADJL(1)+RO
ADJ3=ADJC(1)+(1./RO)
ADJ4=ADJ2*ADJ3
ADJ5=1.-ADJ4
ADJP=(1./DX)-(ADJL(51)/(RD*DT))
ADJQ=(1./DX)-(ADJC(51)*RD)/DT
C*****EVALUATION ON THE BOUNDARY
DO 630 K=2,101
   MK=K-1
   PHI(1,K)=((ADJ1-ADJ4)*PHI(1,MK)+(DT/DX)*PHI(2,MK)
   1)+((ADJ2*DT)/DX)*PSI(1,MK)-PSI(2,MK))-ADJ1*(2EPS(K)-EPS(MK))/ADJ5
   PSI(1,K)=ADJ1*PSI(1,MK)+(DT/DX)*PSI(2,MK)+ADJ3*(
   1PHI(1,MK)-PHI(1,K))+EPS(K)-EPS(MK)
C
C*****EVALUATION OF INTERIOR VARIABLES FOR THE ADJOINT
LINE
DO 620 J=2,50
   MJ=J-1
   G=1.-(ADJL(J)*ADJC(J)*RATIO*RATIO)
   PHI(J,K)=(PHI(MJ,K)+ADJL(J)*RATIO*PSI(MJ,K)-PSI
   1(J,MK)-ADJC(J)*RATIO*PHI(J,MK))/G
620 PSI(J,K)=PSI(MJ,K)+ADJC(J)*RATIO*(PHI(J,K)-PHI(J
1,MK))
C
C*****EVALUATION OF ADJOINT PROBLEM ON THE FAR BOUNDARY
PHI(51,K)=(PHI(50,K)/DX-(ADJL(51)/RD*DT))*PHI(5
11,MK))/ADJP
   PSI(51,K)=(PSI(50,K)/DX-(ADJC(51)*RD*PSI(51,MK))
1/DT)/ADJQ
630 CONTINUE
C
C*****INTEGRATION FOR PARAMETER INCREMENT
IF(ITER.NE.1)GO TO 640
   READ(1,802)ALPHAL(1),ALPHAC(1)
802 FORMAT(2E18.8)
640 CONTINUE
C
C*****COMPUTE ALPHAL AND ALPHAC
IF(ITER.EQ.1)GO TO 700
   ALPHAL(ITER)=ALPHAL(1)*(.5**((ITER-1))
   ALPHAC(ITER)=ALPHAC(1)*(.5**((ITER-1)))
700 CONTINUE
   WRITE(3,801)
801 FORMAT(2X,'TIME INTEGRAL(GrADIENT OF ERROR FNCT)
   1')
DO 810 K=2,100
   AP=V(J,K)
   BP=CUR(J,K)
   CP=PHI(52-J,102-K)
   DP=PSI(52-J,102-K)
   EP=V(J,K+1)
   FP=CUR(J,K+1)
   SUM1=SUM1+F1(DP,FP,BP)
810 SUM2=SUM2+F2(CP,EP,AP)
SUM1 = SUM1 + F1F
SUM2 = SUM2 + F2F
WRITE(3,440) SUM1, SUM2

C**** FORMING THE N+1ST LINE PARAMETERS
HL(J) = HL(J) + ALPHA1(ITER) * SUM1
C(J) = C(J) + ALPHA1(ITER) * SUM2
800 CONTINUE

C**** TESTING FOR LIMITING VALUES
DO 900 J = 1, 51
   IF(HL(J) - HLMAX) 902, 901, 901
901 HL(J) = HLMAX
   GO TO 904
902 IF(HL(J) - HLMIN) 903, 903, 904
903 HL(J) = HLMIN
904 IF(C(J) - CMAX) 906, 905, 905
905 C(J) = CMAX
   GO TO 900
906 IF(C(J) - CMIN) 907, 907, 900
907 C(J) = CMIN
900 CONTINUE
WRITE(3,803)
803 FORHAT(2X, 'N+1ST LINE INDUCTANCE PARAMETERS')
WRITE(3,440)(HL(J), J = 1, 51)
WRITE(3,804)
804 FORHAT(2X, 'N+1ST LINE CAPACITANCE PARAMETERS')
WRITE(3,440)(C(J), J = 1, 51)
ITER = ITER + 1
GO TO 1000
590 WRITE(3,591)
591 FORMAT(2X, 'VOLTAGE AT END OF TRANSMISSION LINE 1, V(0,T)')
WRITE(3,440)(V(51,K), K = 1, 101)
WRITE(3,592)
592 FORMAT(2X, 'TRANSMISSION LINE PARAMETERS')
WRITE(3,593)
593 FORMAT(2X, 'INDUCTANCE AT 51 EQUALLY SPACED CO-ORDINATE POINTS')
WRITE(3,440)(HL(J), J = 1, 51)
WRITE(3,594)
594 FORMAT(2X, 'CAPACITANCE AT 51 EQUALLY SPACED CO-ORDINATE POINTS')
WRITE(3,440)(C(J), J = 1, 51)
440 FORMAT(4E18.8)
STOP
END

FUNCTION F1(DP, FP, BP)
F1 = DP * (FP - BP)
RETURN
END
FUNCTION F2(CP, EP, AP)
F2 = CP * (EP - AP)
RETURN
END