1967

Digital filter design and evaluation

David Alvarado Maguin´a

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DIGITAL FILTER DESIGN
AND
EVALUATION
BY
DAVID ALVARADO MAGUIÑA

A
THESIS
submitted to the faculty of the
UNIVERSITY OF MISSOURI AT ROLLA
in partial fulfillment of the requirements for the
Degree of
MASTER OF SCIENCE IN ELECTRICAL ENGINEERING
Rolla, Missouri
1967

Approved by
(Advisor)
This thesis is concerned with the application of two digital filter design techniques\(^1\) to the Pade approximate\(^2\) continuous filters. The \((m, n)\) Pade approximant has a numerator polynomial of degree \(m\) in \(s\) and a denominator of degree \(n\) in \(s\). The \((n-1, n)\) approximates are desirable for linear phase low pass filters, pulse delay circuits without initial condition errors, and pulse forming networks\(^{17}\).

A collection of plots is shown to demonstrate the quality of the approximations in the frequency domain, as far as amplitude and phase are concerned. These low pass approximants are normalized to a cut-off frequency of 1 Hz. With the well-known frequency transformation techniques\(^6\) the high pass frequency approximants can be found in the same way.

Almost all the numerical evaluations were done using the IBM 360/40 digital computer. Four main programs were written: (1) a program to evaluate the roots of polynomials, (2) a generalized program for the evaluation of the terms of a partial fraction expansion for rational fractions of third and fourth order, (3) one program to evaluate and plot standard Z transform approximates, and (4) one program to evaluate and plot bilinear Z transform approximates.
ACKNOWLEDGEMENT

The author wishes to express his gratitude to Dr. Edward C. Bertnolli, Department of Electrical Engineering, for his original idea and guidance in the execution of this work.
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LIST OF SYMBOLS

e = Naperian base, 2.71898
q\pi = 3.14159
j = \sqrt{-1}
T = sampling period, seconds
s = Laplace transformation variable
z = Z transformation variable
W = radian frequency, radians/second
Wc = cut-off frequency, radians/second
\( \otimes \) = multiplier
\(+\) = adder
\( \uparrow \) = unit delay, T seconds
msec = milliseconds
Db = decibel
deg = degree
Hz = Hertz
I. INTRODUCTION

Digital filtering is the process of frequency shaping using digital or sampled data circuits. Depending upon the relation between the filter excitation and response, digital filters are classified as recursive and nonrecursive. In the recursive case, the response depends on the excitation samples and the previous response. In the nonrecursive case the response depends only on the excitation samples. Considering frequency characteristics, digital filters can be classified the same as continuous filters: low pass, high pass, band pass, band stop. This thesis is concerned with the design and evaluation of low pass Pade digital filters. A digital filter is designated by the name of the corresponding continuous filter; Butterworth, Chebyshev, Lerner, Gauss, Pade, etc. The basic mathematical tool for digital filter design is the Z transform theory.

Digital filters are useful for simulation of continuous filters in the low frequency range. When a digital filter is realized with a digital arithmetic unit, additional considerations are necessary to describe the performance of a filter; there are three obvious degradations called quantization noise of digital filters. For digital filters operating in real time the main limitation on frequency is the speed of arithmetic operations. This is being greatly improved by the new high speed digital hardware appearing on the market.
II. REVIEW OF LITERATURE

The most recent and extensive work on digital filter design techniques is that of Gold and Rader\(^1\). References (9, 10, 11, 12, 13, 14, 15) also deal with digital filter design.

From the three digital filter design techniques considered in reference (1), two of them are related to continuous filter design. To design a continuous filter there exists a vast body of literature on network synthesis and on continuous filter design, for instance reference (6). For the present work Storer\(^7\) and Bertnolli\(^2\) provided the necessary background in continuous filtering.

Information on the mathematics of digital filters, the standard Z transform and the bilinear Z transform, is located in textbooks on sampled data control systems\(^3\), \(^8\).

In the area of physical implementation of a digital filter it seems that there is not yet a complete or concise work. The digital components needed for the physical implementation of digital filters are described in the digital computer literature. Because of many arithmetic units needed in a straightforward implementation, it may be advantageous to use a special purpose digital computer\(^4\).
III. THEORY

A. The Pade Approximate Continuous Filters.

The Pade approximates of the unit delay operator, $e^{-s}$, are rational fraction approximations of the Taylor series expansion of $e^{-s}$.

In reference (2) it is shown that the transfer function of P(n-1,n)$e^{-s}$ networks have both amplitude and delay frequency characteristics with $(2n-1)$ order flatness. These $(n-1,n)$ Pade approximates are desirable for linear phase low pass filters, pulse delay circuits without initial condition error, and pulse forming networks\(^{17}\). The transfer function of the P(n-1,n)$e^{-s}$ networks can be written as:\(^{2}\)

$$
\frac{V_2(s)}{V_1(s)} = T(n-1,n) (s) = \frac{A_n(s)}{B_n(s)}
$$

(1)

where

$$
A_n(s) = 2 \frac{n!}{(2n)!} \sum_{1}^{n} \frac{N(2n-N)!}{N! (n-N)!} (-s)^{N-1}
$$

(2)

and

$$
B_n(s) = \frac{n!}{(2n-1)!} \sum_{0}^{n-1} \frac{(2n-2-N)! s^{N+1}}{(N+1)! (n-1-N)!} + 1.
$$

(3)

$V_1(s)$ and $V_2(s)$ are respectively the input and output of a two port network. The P(n-1,n)$e^{-s}$ approximates for $n = 1, 2, 3$ are shown in Table 1. Expressions (1), (2) and (3) may be used to find any $(n-1,n)$ approximate.

The present work considers the Pade approximates up to $n = 3$; but, it is clearly verified that the approximation improves with the increasing values of $n$ as it should.
For purposes of normalization the cut-off frequency of each approximate was found from the relation

$$\left| T(jW_c) \right|^2 = \frac{1}{2}. \hspace{1cm} (4)$$

The value of $W_c$ for each $P(n-1,n)$ approximate ($n = 1, 2, 3$) is shown in Table 1.

B. Linear Digital Filter Theory.

Linear digital filter theory is based on the mathematics of linear difference equations with constant coefficients. An $m$th-order difference equation can be written as:

$$y(nT) = \sum_{i=0}^{r} L_i x(nT - iT) - \sum_{i=1}^{m} K_i y(nT - iT) \hspace{1cm} (5)$$

where

- $y(nT)$ is the sampled response,
- $x(nT)$ is the sampled excitation, $m$ is a number of the previous response samples, $r + 1$ is the number of the previous excitation samples.

Designing a digital filter consists of finding the constants $K_i$ and $L_i$ to fulfill the given requirements.

A useful pictorial representation of (5) consists of time delay units, adders and multipliers (See Fig. 25). The mathematical tool to solve the difference equation (5) is the Z transform calculus.

The Z transform of $x(nT)$ is defined as
\[ X(z) = \sum_{n=0}^{\infty} x(nT) z^{-n} \quad (6) \]

Knowing the Z transform of a particular sequence, the sequence can be found by the inverse Z transform

\[ x(nT) = \frac{1}{2\pi i} \oint X(z) z^{n-1} dz. \quad (7) \]

Modifying the summation index equation (5) can be rearranged as

\[ \sum_{i=0}^{m} K_i y(nT - iT) = \sum_{i=0}^{r} L_i x(nT - iT) \quad (8) \]

where \( K_0 = 1 \).

Applying definition (6) to equation (8) and rearranging, we have

\[ Y(z) = X(z) \frac{\sum_{i=0}^{r} L_i z^{-i}}{\sum_{i=0}^{m} K_i z^{-i}} = X(z)H(z) \quad (9) \]

The inverse Z transform of \( H(z) \) is the impulse response of a digital filter because \( X(z) = 1 \) if \( x(nT) = 1 \) for \( n = 0 \), and \( x(nT) = 0 \) for \( n \neq 0 \).

If \( X(nT) = e^{jnWT} \), then

\[ H(e^{jWT}) = \frac{\sum_{i=0}^{r} L_i e^{-jiWT}}{\sum_{i=0}^{m} K_i e^{-jiWT}} \quad (10) \]

where \( W \) is the radian frequency.

Equation (10) is the response to the sampled sinusoidal input.
Table 1. P(n-1,n)e^{-s} Pade Approximates. Values of $|T(jw)|^2$ for P(n-1,n)e^{-s} Networks. Values of the Cut-Off Frequencies $W_c$.

| n  | $P(n-1,n)e^{-s}$:                                      | $|T(jw)|^2$:                        | $W_c$:  |
|----|-------------------------------------------------------|-----------------------------------|--------|
| 1  | $\frac{1}{1+s}$                                       | $\frac{1}{1+w^2}$                 | 1.00000|
| 2  | $\frac{1 - s/3}{1 + 2s/3 + s^2/6}$                     | $\frac{36 + 4w^2}{36 + 4w^2 + w^4}$| 2.88523|
| 3  | $\frac{1 - 2s/5 + s^2/20}{1 + 3s/5 + 3s^2/20 + s^3/60}$ | $\frac{3,600 + 216w^2 + 9w^4}{3,600 + 216w^2 + 9w^4 + w^6}$ | 4.91246|

* $|T(jw)|$ is the frequency response magnitude of the $P(n-1,n)e^{-s}$ Pade approximate networks.
As \( e^{j\omega T} \) is the locus of the unit circle in the \( z \) plane, the amplitude of the response can be found graphically by the expression

\[
|H(e^{j\omega T})| = \frac{R_1 R_2}{P_1 P_2 P_3}
\]

where, the \( R \)'s are the distances from the zeros of \( H(z) \) to the point of \( z = e^{j\omega T} \) on the unit circle, and the \( P \)'s are the distances from the poles of \( H(z) \) to the same point. From this consideration, \( |H(e^{j\omega T})| \) can be found graphically, but this method has poor accuracy.

A very useful approach to design a digital filter is to take a known analog filter and then using the Z transform calculus find a set of difference equations with an \( H(z) \) resembling the known analog system function. This method is called "the impulse invariance" or standard Z transform technique. Another important approach is the so-called "bilinear Z transform technique".

Another useful approach that makes use of certain known properties of an elemental digital resonator, is known as the "frequency sampling" approach.

The digital filters considered by Rader and Gold include the digital version of the classic Butterworth, Chebyshev, linear phase Lerner, and the comb filters. In the present work the impulse invariance and the bilinear transformation techniques are applied to the Padé \( P(n-1,n) \) approximates of the unit delay \( e^{-s} \).
C. The Standard Z Transform or Impulse Invariance Technique Applied to Pade Approximant Filters.

This technique is based on the impulse invariance concept. This concept implies that the impulse response of a digital filter $H(z)$ is equal to the sampled impulse response of a given continuous filter. The $Z$ transforms corresponding to certain continuous system functions are tabulated in textbooks of sampled data control systems such as references 3 and 8. To find the $Z$ transform, $H(z)$, the continuous filter function is generally expanded in partial fractions; each term in the expansion is readily transformed from tables.

The steps considered to design and evaluate the low pass Pade digital filters using the standard $Z$ transform techniques are described below as steps 1, 2 and 3.

1. Normalization

For purpose of better presentation and comparison, the $P(n-1,n)$ Pade approximates (Table 1) were normalized. To get the normalization, the following variable substitution was made

$$s \rightarrow \frac{s_{W_n}}{W_n}$$

where, $W_c$ is the radian cut-off frequency of the respective $P(n-1,n)$ Pade approximant. These values of $W_c$ are in Table 1. $W_n$, the radian normalizing cut-off frequency, in this case was equal to $2\pi$ because the digital filters were normalized to $f = 1$Hz.
2. Z Transform for Impulse Invariance

To do this, the normalized $P(n-1,n)e^{-s}$ were expanded in partial fractions, transformed and then collected.

(a) The partial expansion was done with the aid of the digital computer program appearing in Appendix A.

(b) To find the Z transforms the following relations were used:

\[
\frac{A_i}{s + s_i} \Rightarrow \frac{A_i}{1 - e^{-s_i T} z^{-1}} \quad (13)
\]

\[
\frac{s + a}{(s+a)^2 + b^2} \Rightarrow \frac{1 - e^{-aT \cos(bT)} z^{-1}}{1 - 2e^{-aT \cos(bT)} z^{-1} + e^{-2aT} z^{-2}} \quad (14)
\]

\[
\frac{b}{(s+a)^2 + b^2} \Rightarrow \frac{e^{-aT \sin(bT)} z^{-1}}{1 - 2e^{-aT \cos(bT)} z^{-1} + e^{-2aT} z^{-2}} \quad (15)
\]

(c) The respective standard Z transforms ($H(z)$) are collected in Table 2, and the values of their coefficients are in Table 3.

3. Evaluation

The evaluation consisted of finding the frequency response of the $P(n-1,n)$ Pade digital filters. To find the frequency response, $z$ in $H(z)$ was replaced by $e^{j\omega T}$.

The sampling period $T$ was chosen less than the maximum established by the Shannon sampling theorem $^{18}$. This theorem states that "if a signal contains no frequency higher than $W_c$, it is completely characterized by values of the signal measured
at intervals of time \( T = \frac{1}{2}(2\pi / W_c)^3 \). As \( W_c \) was normalized to 1 Hz, then \( T = \frac{1}{2}(2\pi / 2\pi) = 500 \text{ msec} \).

The evaluation was made with \( T = 100 \text{ msec} \) and \( T = 10 \text{ msec} \) for the \( P(n-1,n) \) Pade digital filter with \( n = 1, 2 \) and 3.

The digital computer program used is in Appendix B. The results of this evaluation are shown in Figs. (1-12).

**Example 1.**

Design of the \( P(1,2) \) Pade digital low-pass filter with cut-off frequency 1 Hz.

1. Normalization - \( W_c \) will be equal to \( 2\pi \). From Equation (12) we have \( s > s \frac{W_c}{W_n} \). From Table 1, for \( P(1,2) \), \( W_c = 2.88523 \), which in this case is equal to \( W_n \). Then \( s > 2.275 \text{ s} \).

2. From Tables 2 and 3 we have (reducing the numerical values to the precision of a slide rule)

\[
H(z) \text{ of } P(1,2) = -0.692 e^{-0.692T} \frac{\cos(0.169T)z^{-1}}{1-2e^{-0.692T} \cos(0.169T)z^{-1} + e^{-2x0.692T}z^{-2}} + 2.4 e^{-0.692T} \frac{\sin(0.169T)z^{-1}}{1-2e^{-0.692T} \cos(0.169T)z^{-1} + e^{-2x0.692T}z^{-2}}
\]

(16)

Again, a more accurate expression of \( H(z) \) is in Tables 2 and 3.

3. The frequency response for \( T = 10 \text{ msec} \) is obtained by making \( z = e^{jWT} \) in Equation (16), and evaluating \( H(z) \) at frequencies ranging from 0 Hz up to 2 Hz with a 0.01 Hz step. The result is shown in Figs. (7,8).

The conclusions drawn from the observation of the plots appear in sections III-E and IV.
Table 2. Z Transformation for the $P(n-1,n)e^{-s}$ Pade Approximates*.

<table>
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<tr>
<th>$P(n-1,n)$:</th>
<th>$Z(P(n-1,n))$:</th>
</tr>
</thead>
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<tr>
<td>$P(0,1)$</td>
<td>$\frac{A_0}{1 - z^{-1}e^{-a_0T}}$</td>
</tr>
<tr>
<td>$P(1,2)$</td>
<td>$\frac{A_0}{1 - z^{-1}e^{-a_0T}} + \frac{A_1}{1 - 2z^{-1}e^{-a_1T}\cos(b_1T) + z^{-2}e^{-2a_1T}}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{1 - e^{-a_1T}\cos(b_1T)z^{-1}}{1 - 2z^{-1}e^{-a_1T}\cos(b_1T) + z^{-2}e^{-2a_1T}}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{1 - 2z^{-1}e^{-a_1T}\cos(b_1T) + z^{-2}e^{-2a_1T}}{1 - 2z^{-1}e^{-a_1T}\cos(b_1T) + z^{-2}e^{-2a_1T}}$</td>
</tr>
<tr>
<td>$P(2,3)$</td>
<td>$\frac{A_0}{1 - z^{-1}e^{-a_0T}} + \frac{A_1}{1 - 2z^{-1}e^{-a_1T}\cos(b_1T) + z^{-2}e^{-2a_1T}}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{1 - e^{-a_1T}\cos(b_1T)z^{-1}}{1 - 2z^{-1}e^{-a_1T}\cos(b_1T) + z^{-2}e^{-2a_1T}}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{1 - 2z^{-1}e^{-a_1T}\cos(b_1T) + z^{-2}e^{-2a_1T}}{1 - 2z^{-1}e^{-a_1T}\cos(b_1T) + z^{-2}e^{-2a_1T}}$</td>
</tr>
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*The values of the parameters appearing in this table are in Table 3.*
Table 3. Parameters of the Standard Z Transformation of the 
P(n-1,n)e⁻ˢ Pade Approximates.

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<tr>
<th>P(n-1,n)</th>
<th>A₀</th>
<th>a₀</th>
<th>A₁</th>
<th>a₁</th>
<th>B₁</th>
<th>b₁</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(0,1)</td>
<td>1.0</td>
<td>1.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>P(1,2)</td>
<td>0.0</td>
<td>0.0</td>
<td>-0.69319</td>
<td>0.69319</td>
<td>2.45079</td>
<td>0.16988</td>
</tr>
<tr>
<td>P(2,3)</td>
<td>3.72481</td>
<td>0.74053</td>
<td>-3.11412</td>
<td>0.54578</td>
<td>1.69040</td>
<td>0.12641</td>
</tr>
</tbody>
</table>
FIG. 1.  P(0.1) FREQUENCY RESPONSE OF THE PADE LOW-PASS FILTERS WITH T=300 MSEC., AND Fc = 1 HZ:

++ CONTINUOUS VERSION, XX DIGITAL VERSION OBTAINED WITH THE IMPULSE INVARIANCE TECHNIQUE.
FIG. (2) P(0.1) PHASE RESPONSE OF THE PADE LOW-PASS FILTERS WITH T=100 MSEC., AND FC=1.0 Hz. 
++ CONTINUOUS VERSION; XX DIGITAL VERSION OBTAINED WITH THE IMPULSE INVARIANCE TECHNIQUE
FIG. (3) $P(0.4)$ FREQUENCY RESPONSE OF THE PODE LOW-PASS FILTERS WITH $T=40$ MSEC., AND $F_C=1$ KHZ

++ CONTINUOUS VERSION; XX DIGITAL VERSION OBTAINED WITH THE IMPULSE INVARIANCE TECHNIQUE.
FIG. (4) P (o,j) PHASE RESPONSE OF THE PARD LOW-PASS FILTERS WITH $T = 10$ MSEC., AND $FC = 1$ HZ.

+ CONTINUOUS VERSION; X DIGITAL VERSION OBTAINED WITH THE IMPULSE INVARIANCE TECHNIQUE
FIG. (5) P(1,2) FREQUENCY RESPONSE OF THE PADE LOW-PASS FILTERS WITH T=100 MSEC., AND FC.=1 HZ

**CONTINUOUS VERSION; XX DIGITAL VERSION OBTAINED WITH THE IMPULSE INVARINANCE TECHNIQUE**
FIG. (6) P(1,2) PHASE RESPONSE OF THE PAZE LOW-PASS FILTERS WITH T=100SEC, AND FC=1 HZ.

++ CONTINUOUS VERSION; XX DIGITAL VERSION OBTAINED WITH THE IMPULSE INVARIANCE TECHNIQUE
FIG. (7) P(1,2) FREQUENCY RESPONSE OF THE PAIR LOW-PASS FILTERS WITH T=10 MSEC., AND FC=1.0 HZ

++ CONTINUOUS VERSION; XX DIGITAL VERSION OBTAINED WITH THE IMPULSE INVARINACE TECHNIQUE
FIG. (8) P(1,2) PHASE RESPONSE OF THE PADE LOW-PASS FILTERS WITH $T_0 = 10$ SEC., AND $f_0 = 1$ Hz.

++ CONTINUOUS VERSION; XX DIGITAL VERSION OBTAINED WITH THE IMPULSE INVARIANCE TECHNIQUE
FIG. (9) P(2,3) FREQUENCY RESPONSE OF THE PADE LOW-PASS FILTERS WITH T=100 MSEC., AND FC.=1.0 HZ.

++ CONTINUOUS VERSION; XX DIGITAL VERSION OBTAINED WITH THE IMPULSE INVARIANCE TECHNIQUE.
FIG. (10) P (2.3) PHASE RESPONSE OF THE PADE LOW-PASS FILTERS WITH T=100 MSEC., AND FC=1.0 Hz.

+++ CONTINUOUS VERSION; XX DIGITAL VERSION OBTAINED WITH THE IMPULSE INVARIANCE TECHNIQUE
FIG. (11) P (2,3) FREQUENCY RESPONSE OF THE PADE LOW-PASS FILTERS WITH T = 10MSEC., AND FC = 1.0 Hz

+ CONTINUOUS VERSION; XX DIGITAL VERSION OBTAINED WITH THE IMPULSE INVARINACE TECHNIQUE
Fig. 12: P(2,3) phase response of the Prate low-pass filters with T = 10msec., and FC = 1 Hz. *

* Continuous version; xx digital version obtained with the impulse invariance technique.
D. The Bilinear Z Transform Design Technique applied to Pade Digital Filters

This technique has the advantage that the bilinear transformation maps the imaginary axis of the s-plane into the unit circle in the plane \( p = \frac{z^{-1}}{z+1} \) (reference 1). This transformation is obtained making the substitution

\[
s = \frac{z^{-1}}{z+1}
\]

in \( H(s) \), the \( P(n-1,n) \) Pade approximate.

If \( W_{DT} \) is the digital frequency variable, and \( W_A \) the analog frequency variable, then the functions \( H(W_A) \) and \( H'(W_{DT}) \) will have the same values for

\[
\frac{jW_A}{e^{jW_{DT}} - 1} = \frac{\frac{jW_{DT}}{2}}{e^{jW_{DT}} + 1} \left( \frac{\frac{jW_{DT}}{2}}{-e^{jW_{DT}}} \right) \]

or \( W_A = \tan \frac{W_{DT}}{2} \)

Equations (17) and (18) lead to a technique for designing a digital filter from a continuous filter. The procedure is as follows:

1. Call \( W_{Di,T} \) critical frequencies and ranges (passband or stopband, maximum attenuation point, etc.), compute the new sets of frequencies \( W_{Ai} \) by the relation

\[
W_{Ai} = \tan \frac{W_{Di,T}}{2}
\]

2. Design a transfer function \( H(s) \) with the properties of the digital filter at the new frequencies and ranges.

3. Make the substitution (17) in \( H(s) \) and arrange \( H(z) \) as a ratio of polynomials.
The application of these steps to the design of low-pass Pade digital filters, using the bilinear transformation technique, consisted of the following steps:

(1). For purposes of comparison with the standard $Z$ transform technique applied to the Pade filters, the cut-off frequency ($W_D$) was chosen equal to $2\pi$, and the sampling period ($T$) equal to 100 msec and 10 msec. The values of the analog cut-off frequency are

(a) for $T = 100$ msec.

$$W_A = \tan \frac{2\pi \times 0.1}{2} = 0.32492$$

(b) for $T = 10$ msec.

$$W_A = \tan \frac{2\pi \times 0.01}{2} = 0.03141$$

(2) Using the expression (12) with $W_n = W_A$, new $P(n-j,n)e^{-s}$ Pade approximates $H(s)$ were obtained.

(3) $H'(z)$ was found using equation (17).

As in the impulse invariance technique the evaluation of these filters consisted in finding their frequency response; i.e., making $z = e^{j\omega T}$ in $H'(z)$. The results of this evaluation are plotted in Figs. (13 - 24).
Example 2

Design a \( P(1,2) \) Pade low-pass filter with a cut-off frequency of 1 Hz.

1. Determination of \( W_A \). Equation (18) establishes that 
   \[
   W_A = \tan \frac{W_D T}{2}.
   \]
   Since \( W_D = 2\pi \times 1 \), and \( T = 100 \) msec.,
   \[
   W_A = \tan \frac{2\pi \times 10^{-1}}{2} = 0.325.
   \]

2. Change of variable. If in equation (15) \( W_A = W_n \), and from Table 1, \( W_c = 2.88533 \), then
   \[
   s = \frac{W_c}{W_A} = 8.90 \text{ s}
   \]

From Table 1, the new Pade \( P(1,2) \) will be
\[
P(1,2)e^{-s} = \frac{1-2.97s}{1+5.92s+2.96s^2}
\]

(19)

3. Finding \( H(z) \). Equation (17) establishes that
   \[
   s = \frac{z-1}{z+1}.
   \]

Substituting in equation (20) and rearranging,
\[
H(z) = \frac{3.97 - 2.97z^{-1}}{9.88 - 3.92z^{-1} - 1.96z^{-2}(1 + z^{-1})}
\]

The frequency response is obtained making \( z^{-1} = e^{-j \omega T} \) in equation (20). Figs. (17, 18) are the plots of this result.
FIG. (13) $P(0,1)$ FREQUENCY RESPONSE OF THE RAOE LOW-PASS FILTERS WITH $T=100$ MSEC., AND FC=1 Hz

++ CONTINUOUS VERSION; XX DIGITAL VERSION OBTAINED WITH THE BILINEAR TRANSFORMATION TECHNIQUE
FIG. (4) P(0.1) PHASE RESPONSE OF THE PADE LOW-PASS FILTERS WITH T=10ΜSEC., AND FC=1.0 Hz. 

+++ CONTINUOUS VERSION; XX DIGITAL VERSION OBTAINED WITH THE BILINEAR TRANSFORMATION TECHNIQUE
FIG. (16) $P(0,1)$ FREQUENCY RESPONSE OF THE PADE LOW-PASS FILTERS WITH $T=10$ MSEC., AND $F_C=1$ Hz

- Continuous version; $xx$ digital version obtained with the Bilinear Transformation Technique.
FIG. (16) $\phi(\omega)$ PHASE RESPONSE OF THE PAOE LOW-PASS FILTERS WITH $\tau = 10$ SEC., AND $f_0 = 1$ Hz.

++ CONTINUOUS VERSION; XX DIGITAL VERSION OBTAINED WITH THE BILINEAR TRANSFORMATION TECHNIQUE.
Fig. (17) P(1,2) frequency response of the PADE low-pass filters with T=100msec., and fc=1.0Hz

++ Continuous version; xx digital version obtained with the bilinear transformation technique
FIG. 4.8  P(1,2) PHASE RESPONSE OF THE PAIR, LOW-PASS FILTERS WITH T=100mSEC., AND FC=1.0Hz.:

++ CONTINUOUS VERSION; XX DIGITAL VERSION OBTAINED WITH THE BILINEAR TRANSFORMATION TECHNIQUE
Fig. (19) P(1,2) frequency response of the Pade low-pass filters with T=10 msec., and fc=1 Hz
++ Continuous version; xx digital version obtained with the bilinear transformation technique.
FIG. (2b) P (4,2) PHASE RESPONSE OF THE PHASE LOW-PASS FILTERS WITH τ = 10 MS, AND FC = 1 Hz.

++ CONTINUOUS VERSION; XX DIGITAL VERSION OBTAINED WITH THE BILINEAR TRANSFORMATION TECHNIQUE
Fig. (21) P(2,3) Frequency response of the Padé low-pass filters with T=100msec, and fc=1 Hz

+ Continuous version; xx digital version obtained with the bilinear transformation technique.
FIG. (22)  P(2,3) PHASE RESPONSE OF THE PADE LOW-PASS FILTERS WITH T=100MSEC., AND FC=1.0Hz.

++ CONTINUOUS VERSION; X DIGITAL VERSION OBTAINED WITH THE BILINEAR TRANSFORMATION TECHNIQUE
FIG. (28) FREQUENCY RESPONSE OF THE PADE LOW-PASS FILTERS WITH \( T = 10 \) MSEC., AND \( FC = 1 \) HZ

**+** CONTINUOUS VERSION; **XX** DIGITAL VERSION OBTAINED WITH THE BILATERAL TRANSFORMATION TECHNIQUE.
FIG. (24) $P(2,3)$ PHASE RESPONSE OF THE PADE LOW-PASS FILTERS WITH $T=10$ Microsec., AND $F_C=1$ Hz.

++ CONTINUOUS VERSION; XX DIGITAL VERSION OBTAINED WITH THE BILINEAR TRANSFORMATION TECHNIQUE

The evaluation consisted of finding the frequency and phase responses of the Pade digital filters, all normalized to the cut-off frequency of 1 Hz. Each one of these results was compared with the frequency and phase responses of the corresponding P(n-1,n) Pade continuous filter. The results of this comparison are shown in Figs. (1-24).

It is verified with Figs. 1, 5 and 9 that the Pade continuous filters frequency response magnitude approach better to the ideal low-pass filter with the increase in values of n. Considering arbitrarily the frequency response magnitude at -0.1 db, the corresponding frequencies of each of the Pade continuous filters are, 0.14 Hz for P(0,1), 0.34 Hz for P(1,2), and 0.44 Hz for P(2,3). The linear portion of the phase response does not change appreciably with the increase in values of n, for all the three continuous Pade filters considered in this thesis it lies approximately in the frequency range of 0 Hz to 0.62 Hz. The phase reaches -180 degrees at f > 2 Hz for P(0,1), at 1.46 Hz for P(1,2), and at 0.64 Hz for P(2,3). All these results are shown in Table 4.

The comparison between the Pade continuous filters and the Pade digital filters obtained by the impulse invariance technique in the frequency range of 0 Hz to 1 Hz, using the sampling period of 100 msec, is shown in Table 4b. This sampling period for practical considerations was chosen 1/5th the maximum required by the sampling theorem. For P(0,1) the maximum error is 0.14 db and occurs at 1 Hz, and its shape approximates very closely
the continuous one. For \( P(1,2) \) the maximum error is 5.37 db and occurs at 1 Hz, and its shape does not approximate the continuous one. For \( P(2,3) \) the maximum error is -4.79 db, and occurs at 0.7 Hz, an error of 2 db occurs at 1 Hz, and its shape does not approximate the continuous one.

The cause of the high discrepancies mentioned above is the so-called folding problem. To solve this difficulty the sampling period was decreased arbitrarily to 1/10th its original value, i.e. \( T = 10 \) msec. Again, the comparison of the continuous and digital Pade filters in the frequency range from 0 Hz to 1 Hz is shown in Table 4b. For \( P(0,1) \) the maximum error is 0.01 db and occurs at 1 Hz. For \( P(1,2) \) the maximum error is 0.43 db and occurs at 1 Hz. For \( P(2,3) \) the maximum error is -0.36 db, an error of -0.20 db occurs at 1 Hz. The shape of all these three approximations are very close to the shape of the continuous ones. The sampling period to obtain a 0 db error at the frequency of 1 Hz can be found by solving the frequency response magnitude function.

The phase response compares quite favorably for both values of the sampling period. However, for \( T = 10 \) msec all of the phase responses are identical.

The comparison of the continuous Pade filter with the Pade digital filter obtained by the bilinear \( Z \) transformation technique shows that for \( T = 100 \) msec all the \( P(n-1,n) \) approximates have an error of 0.06 db at 0.7 Hz, and no error at 1 Hz. All the digital approximate shapes are very close to those of the corresponding continuous case. These properties can be observed
in Fig. (13, 17, 21). For the sampling period of 10 msec all the approximates coincide. See Figs. (15, 19, 23). As far as phase response is concerned, for \( T = 100 \) msec the approximates are very close, and for \( T = 10 \) msec they coincide. See Figs. (14, 15, 18, 20, 22, 24).

For the bilinear \( Z \) transformation technique the cause of the discrepancy with the continuous version is the so-called frequency warping.\(^5\) From the evaluation made in this thesis it can be concluded that this discrepancy is almost inappreciable even for the sampling period of 100 msec.

The \( \text{Pade} \) digital filters obtained by the two methods considered in this thesis improve their approximation to the continuous case with the increasing values in the sampling period. However, it converges more rapidly for the bilinear \( Z \) transformation technique than for the impulse invariance technique.

In general the \( \text{Pade} \) digital filter that best approximates the continuous \( \text{Pade} \) filter is obtained by the bilinear \( Z \) transformation technique.

The digital computer programs used for the evaluation, and for the plotting of the results that appear in Figs. (1-24) are in Appendices B and C.
Table 4a. Conclusion Drawn from the Evaluation of the Pade Continuous Filters.

<table>
<thead>
<tr>
<th>P(n-1,n)</th>
<th>Frequency response mag.</th>
<th>Phase Response</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>f for 0.10 db</td>
<td>l*</td>
</tr>
<tr>
<td>P(0,1)</td>
<td>0.14 Hz</td>
<td>1,3,13,15</td>
</tr>
<tr>
<td>P(1,2)</td>
<td>0.34 Hz</td>
<td>5,7,17,19</td>
</tr>
<tr>
<td>P(2,3)</td>
<td>0.44 Hz</td>
<td>9,11,21,23</td>
</tr>
</tbody>
</table>

Table 4b. Conclusions Drawn from the Comparison of the Continuous and Digital Pade Filters Obtained by the Impulse Invariance Technique.

<table>
<thead>
<tr>
<th>P(n-1,n)</th>
<th>Samp. period in msec</th>
<th>Max. error from 0-1 Hz</th>
<th>Error at 1 Hz in db</th>
<th>Fig.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mag. in db</td>
<td>f in Hz</td>
<td></td>
<td></td>
</tr>
<tr>
<td>P(0,1)</td>
<td>100</td>
<td>0.14</td>
<td>0.14</td>
<td>1</td>
</tr>
<tr>
<td>P(1,2)</td>
<td>100</td>
<td>5.37</td>
<td>5.37</td>
<td>5</td>
</tr>
<tr>
<td>P(2,3)</td>
<td>100</td>
<td>-4.79</td>
<td>2</td>
<td>9</td>
</tr>
<tr>
<td>P(0,1)</td>
<td>10</td>
<td>0.01</td>
<td>0.01</td>
<td>3</td>
</tr>
<tr>
<td>P(1,2)</td>
<td>10</td>
<td>0.43</td>
<td>0.43</td>
<td>7</td>
</tr>
<tr>
<td>P(2,3)</td>
<td>10</td>
<td>-0.36</td>
<td>-0.20</td>
<td>11</td>
</tr>
</tbody>
</table>

* 1 Frequency interval, in Hz, of the linear portion of the phase response.

* 2 Frequency in Hz at which the phase response is -180 degrees.
F. Simulation

The simulation of a digital filter is the computation process or algorithm by which a sequence of numbers (acting as the input sampled signal) is transformed into a second sequence of numbers designated the output samples\(^5\). The computer program widely used in the simulation of sampled data systems is the so-called BLODI\(^5,10\) (block diagram compiler). This compiler accepts as input a description of the sampled-data system block diagram written in the BLODI language\(^5\). The block diagram needed to implement a digital filter consists of digital units (adders, multipliers, and time delays) arranged according to the realization scheme chosen.

The canonical realization schemes for digital filters are direct, parallel and cascade. The advantage and limitations of any one of them depends on the type of digital filter to be implemented\(^5\).

Figs. (25) and (26) are respectively the parallel realization scheme for the low-pass Pade digital filters obtained with the impulse invariance technique, and the cascade realization scheme for the same filters but obtained with the bilinear Z transform technique. A comparison between them shows that for the first technique the number of multipliers required is less than the number of multipliers required for the second technique.
Fig. (25) Realization Schemes for the Pade Digital Filters Obtained with the Impulse Invariance Technique; (a) for \( n = 1 \), (b) for \( n = 2 \), (c) for \( n = 3 \).

The schemes for \( n \) greater than three are combinations of blocks alike to (a) and (b).
\[ H(z) = A'(1 + z^{-1}) \]

(a)

\[ H(z) = \frac{A'(1 + A_0 z^{-1})(1 + A_1 z^{-1})}{(1 + B_0 z^{-1})(1 + B_1 z^{-1})} \]

(b)

\[ H(z) = A' \frac{(1 + A_0 z^{-1})}{(1 + B_0 z^{-1})} \times \frac{(1 + A_1 z^{-1})}{(1 + B_1 z^{-1})} \times \frac{(1 + A_2 z^{-1})}{(1 + B_2 z^{-1})} \]

(c)

Fig. (26) Cascade Realization Schemes of the Pade Digital Filters Obtained with the Bilinear Z Transformation Technique. (a) for \( n=1 \), (b) for \( n=2 \), (c) for \( n=3 \).
IV. CONCLUSION

A. Results

From the two digital filter design techniques used in this work, the bilinear transformation technique gives better results than the impulse invariance technique applied to the same \( P(n-1,n) \) Pade filter, and with the same sampling period. For 10 msec. sampling period, the impulse invariance technique gives a competitive result with the bilinear transformation technique for a 100 msec sampling period, both techniques applied to the same \( P(n-1,n) \) Pade filter.

The great departure of the digital filters obtained by the impulse invariance technique from the continuous version for \( n = 2 \) and 3, and \( T = 100 \) msec is due to the folding problems. This abnormality disappears for \( T = 10 \) msec. For the digital filters obtained with the bilinear transformation technique, the inherent frequency warping is not apparent.

The execution time for the computer simulation of the digital filter obtained by each design technique considered in the present work, depends on the realization scheme chosen. For instance, the parallel scheme of the digital filter obtained with the impulse invariance technique has fewer multipliers than the cascade scheme for the filter obtained with the bilinear \( Z \) transformation technique. This means that the execution time for the first approximation is less than the execution time for the second one.
B. Suggestions for Further Development

As the main use of digital filters is in the area of analog filter simulation, it would be convenient to have an assembler language like BLODI\textsuperscript{5} that could be used in the same fashion as the ECAP (electric circuit analysis program).

The great advantages of a digital filter with respect to its continuous version is a greater degree of accuracy and relatively small size, especially in the low frequency ranges. Its major actual shortage is its speed, but increasing speeds and decreasing costs of microelectronic digital circuitry make possible the conception of real-time digitized systems which perform filtering operations usually performed by analog hardware\textsuperscript{1}. For these reasons it would be interesting and important to implement an algorithm to design an optimum real-time digitized system to implement any digital filter.


APPENDIX A

The computation of the roots of a polynomial was necessary to find the poles of the $P(n-1,n)e^{-s}$ Pade approximates for their partial fraction expansion required to find the Z transformation. This was done using the scientific subroutine package of the Computer Science Center;

SUBROUTINE POLRT (Section II-1).

This subroutine uses the Newton-Raphson method iterative technique. It finds the real and complex roots of polynomials up to the 36th degree.

A partial fraction program was written to compute the partial fraction expansion for rational fractions of 3rd and 4th degree of known poles. Besides the partial fraction expansion, this program computes the coefficients of the Z transformation corresponding to the $P(n-1,n)e^{-s}$ Pade approximates.

A 3rd degree rational fraction can have only three different combinations of roots, but a 4th degree rational fraction can have around 13 possible combinations; for this reason it is cumbersome to write a partial fraction expansion generalized program. The present program was written with the aid of a flow chart which is not presented here.
THE PROGRAM 0. ALVARADO

PARTIAL FRACTION EXPANSION

N1 IS THE DEGREE OF THE NUMERATOR POLYNOMIAL, A(I, J) COEFFICIENT

N0 IS THE DEGREE OF THE DENOMINATOR POLYNOMIAL, L(I, J) COEFFICIENT

N3 IS THE REAL AND IMAGINARY PARTS OF A COMPLEX ROOT.

THE VALUES OF N1 ARE 4, 4, 3, 3, AND THE VALUES OF N0 AND N3 FOR

M=2 ARE THUS OBTAINED WITH THE SUBROUTINE PRINT

DIMENSION A(4, 4), X(I, 4), PR(4, 4), PI(4, 4), N1(4), XN(4), PRN(4),

PIN(4), AK(4, 4), AI(4)

SMP(XI, XZ, X3, X4) = XI + XZ + X3 + X4
SMI(XI, XZ) = XI + XZ
GC(X, F1, F2, F3) = (X - F1) * (X - F2) * (X - F3)
Q4(X, F1, F2, F3, F4) = H*F1*X**3+F2*X**2+F3*X+F4
R4(X, F1, F2, F3, H) = (X - F1) * (X - F2) * (H * X - F3)
READ(1, 200) (N1(I), I=1, 4)

200 FORMAT (4I3)
READ(1, 100) (A(I, J), X(I, J), PR(I, J), PI(I, J), J=1,4), I=1,4)

100 FORMAT (4F7.4)
WRITE(3, 300)
300 FORMAT (' THE L ARE POLYNOMIALS, A1, A2, ... ARE FOR SIMPLE ROOTS

A11, A22, ... ARE FOR EQUAL ROOTS, F11, F12, F12 FOR IMAGINARIES')
WRITE(3, 301)
301 FORMAT (' THE VALUES OF FS ARE AS FOLLOWS

F1, F2, F3, F4 FOR COMPLEX ROOTS, G1, G2, G3, G4, COEFF. OF A QUAD.')
WRITE(3, 303)
98 FORMAT (' AK(K) REPRESENT 2 REAL IF CODE=1 AND 3 EQUALS IF CODE=

1E=21)
WRITE(3, 375)
375 FORMAT (' RA1, RB1, ... ARE FOR CONVENIENCE')
DO 900 J=1,4
N=N1(J)
L=J
SR=SMPI(X1(1, J), X2(1, J), X3(1, J), X4(1, J))
ST=SMPI(P1(1, J), P2(3, J))
DO 910 I=1, N
IF(X(I, J)) 102, 101, 102
102 XN(I)=1,
GO TO 103
101 XN(I)=0.
103 IF(PR(I, J)) 105, 104, 105
105 PRN(I)=1.
GO TO 906
104 PRN(I)=0.
906 IF(P1(I, J)) 908, 907, 908
908 PIN(I)=1.
GO TO 910
907 P1N(I)=0.
910 CONTINUE
IF(N-4)190,911,900
911 IF(SR)<13,912,913
912 IF(PR(1,J)+PR(3,J))915,914,915
915 IF(APS(P(3,J))900,916,917
917 G1=PR(1,J)**2+PS(1,J)**2
920 G2=-2.*PR(1,J)
G3=-2.*PP(3,J)
G4=-2.*PP(1,J)**2
D11=A(2,J)-A(3,J)-A(4,J)*(G3-G4**2)
D12=G1-G2**2
D21=A(2,J)-A(3,J)*G3+A(4,J)*G4*G3
D22=-G2*G4*G6*G3
F3=(D11*D22-D12*D21)/(D22*(G2-G4)-D12*(G1-G3))
F4=(D11-D22*G2-G4)/(D12*(G2-G4))
F5=F3+F4
F6=F2+F4
F7=F3+F4
F8=F2+F4
400 FORMAT(2X,'(I2,2X,'RA1=;'F15.5,2X,'RB1=;'F15.5,2X,'RA2=;'F15.5,'RB2=;'F15.5)
401 CONTINUE
GO TO 900
916 M1=1
707 E1=A(1,J)
E2=A(2,J)
E3=A(3,J)
E4=A(4,J)
IF(M1<2)721,706,708
708 Q=04(X1(1,J),E4,E3,E2,E1,1.)
R=G(C(X1(1,J)+X1(2,J),-2*PR(1,J),PR(1,J)**2+PI(1,J)**2)
A1=Q1/P
Q=04(X1(2,J),E4,E3,E2,E1,1.)
R=G(C(X1(2,J)+X1(1,J),-2*PR(1,J),PR(1,J)**2+PI(1,J)**2)
A2=Q2/P
G1=PR(1,J)**2+PR(2,J)**2
G2=-2.*PR(1,J)
```
F1 = (E1 + A1 * X1(2, J) + G1) + A2 * X1(1, J) + G1) / (X1(1, J) * X1(2, J))
F2 = F4 - A1 - A2
WRITE (3, 500) L, A1, A2, F1, F2, G1, G2
500 FORMAT(2X, 'L', 17X, A1, A2, 2X, 'X', F1, 11X, G1, 11X, G2) / (I2, E1, 5, 5, J)
G0 TN 900
721 C = P1(3, J)
G1 = PR(1, J)**2 + PI(1, J)**2
G2 = 2. * P2(1, J)
F1 = ((C - G1) * (G1 * E3 - E1) + G2 * (G1 * (E2 - E4 - C) - G2 * F1)) / ((G1 - C)**2 -
2C**2)
F12 = (F2 - F11 - F1) / G2
F2 = F4 - F12
WRITE (3, 600) L, F11, F12, F1, F2, G1, G2
(F1, F13, 5, 6F15, 5, J)
G0 TN 900
914 IF (PI(1, J) - PI(3, J))$23, 900, 923
923 F11 = (A(3, J) * D(1, J) - A(1, J)) / (PI(1, J) - PI(3, J))
F12 = A(3, J) - F11
F21 = A(4, J) * PI(3, J) - A(2, J) / (PI(1, J) - PI(3, J))
F22 = A(4, J) - F21
WRITE (3, 650) L, F11, F12, F21, F22
650 FORMAT(2X, 'L', 17X, F1, 11X, F2, 11X, F21, 11X, F22) / (I2, F13, 6, 3F15, 5, J)
G0 TN 900
913 IF (SI) 931, 930, 931
931 M = 3
932 IF (X(1, J) - X(2, J)) 934, 933, 934
934 K = 1
G0 TN 707
933 M = 2
711 F1 = A(1, J) + A(2, J) * X1(1, J) + A(3, J) * X1(1, J)**2 + A(4, J) * X1(1, J)**3
F2 = A(2, J) + 2. * X1(1, J) * A(3, J) + 3. * X1(1, J)**2 * A(4, J)
F3 = A(3, J) + 3. * X1(1, J) * A(4, J)
F4 = A(4, J)
IF (M = 3) 900, 300, 800
801 G1 = -2. * X1(1, J) * PR(1, J) + PR(1, J)**2 + PI(1, J)**2 + X1(1, J)**2
A11 = F1 / G1
G2 = 2. * (X1(1, J) - PR(1, J))
A12 = F2 - A11 * G2 / G1
F1 = F3 - A11 - A12 * G2
F2 = F4 - A12
WRITE (3, 700) L, A11, A12, F1, F2, G1, G2
G0 TN 900
800 IF (M = 3) 741, 740, 742
742 A11 = F1
```
A22=F2
A33=F3
A44=F4
WRITE(3,750)L,A11,A22,A33,A44
750 FORMAT(2X'IL'7X'A11''11X'A22'11X'A33'11X'A44'/(12,F13.5,2F15.5))
GO TO 900
741 G1=(X1(1,J)-X1(3,J))*(X1(1,J)-X1(4,J))
K=3
A11=(F2-A11)*(-X1(4,J))
A22=(F2-A11)/(-X1(4,J))
A33=(F2-A22)/(-X1(4,J))
WRITE(3,850)L,A11,A22,A33
850 FORMAT(2X'IL'7X'A11''11X'A22'11X'A33'/(12,F13.5,2F15.5))
GO TO 707
740 A11=F1/(-X1(4,J))
A22=(F2-A11)/(-X1(4,J))
A33=(F3-A22)/(-X1(4,J))
WRITE(3,851)L,A11,A22,A33
851 FORMAT(2X'IL'7X'A11''11X'A12'11X'A13'/(12,F13.5,2F15.5))
GO TO 707
930 SYN=SMP(XN(1),XN(2),XN(3),XN(4))
IF(SYN-1.)900,911,772
772 M=2
IF(SYN-2.*XN(1))900,932,774
774 IF(SYN-3.)900,775,776
776 IF(X1(1,J)-X1(4,J))775,777,776
776 M=4
GO TO 711
781 M=3
GO TO 711
706 CONTINUE
DO 15 I=K,4
X=PR(I,J)
 1 DO =PR(I,J)
IF(1-1)1,1,7
 1 D1=PR(I,J)
GO TO 11
11 IF(1-3)4,3,3
 3 D2=PR(2,J)
GO TO 12
 4 D2=PR(3,J)
GO TO 12
12 IF(1-4)6,5,6
 5 D3=PR(3,J)
GO TO 13
 6 D3=PR(4,J)
13 G=4(X,E4,E3,E2,E1).L.
 4 R=4(X,D1,D2,D3,L.)
A1=0/R
99 FORMAT(5X,'L='L2,5X,'I='L2,5X,'A1='F12.5)
15  WRITE(3,99)L, I, A
   GO TO 900
190  SX0=SMR(XN(1),XN(2),XN(3),0.)
   TF(SXN=1.) 900, 870, R71
871  IF(SXN=2.) 000, 872, 873
872  K1=1
   KODE=1
   GO TO R75
873  AK((1,J))=(A(1,J)+A(2,J)*XI(1,J)+A(3,J)*XI(1,J)**2)/(-XI(3,J))
   AK((2,J))=(2.*X1(1,J)*A(3,J)+A(2,J)-AK(1,J))/(-XI(3,J))
   K1=3
   KODE=2
875  DO 26 K=K1, 3
   IF(K=2) 23, 24, 25
23  R=R4(X1(1,J), XI(2,J), XI(3,J), -1., 0.)
   GO TO 26
24  R=R4(X1(2,J), XI(1,J), XI(3,J), -1., 0.)
   GO TO 26
25  R=R4(X1(3,J), XI(1,J), XI(2,J), -1., 0.)
   GO TO 26
26  A1(K)=0./R
   WRITE(3,111) I, KODE, (A1(K), K=K1, N)
111  FORMAT(2X'H', 2X'H', KODE', 5X'A1(K), K=K1, N)
870  IF(PR((1,J))=1) 113, 112, 113
113  G1=PR((1,J))**2+PT((1,J))**2
   G2=-2.*PR((1,J))
   A1=(A(1,J)+A(2,J)*XI(3,J)+A(3,J)*XI(3,J)**2)/(G1+G2*X1(3,J)+X1(3,J)**2)
   F1=(A1+G1-A(1,J))/XI(3,J)
   F2=A(3,J)-A1
   WRITE(3,115)L, A1, F1, F2, G1, G2
115  FORMAT(2X'H', 7X'A1', 1X'F1', 1X'F2', 11X'G1', 11X'G2', /I2, F13.5, 4F15.5)
   IF(L=3) 900, 351, 352
351  H1=3.0
   GO TO 361
352  H1=24.
361  RA1=PR((1,J))
   RR1=RA1*(F1+F2+PR((1,J))
   WRITE(3,373)L, RA1, RR1
373  FORMAT(2X'H', 7X'L'=I2, 2X, 2F20.5)
   GO TO 900
180  IF(PR((1,J))=000, 112, 113
112  A1=A((1,J)/(PR((1,J))**2+PT((1,J))**2)
   F2=A(3,J)-A1
   F1=A(2,J)+A1)**2.*PR((1,J))
   WRITE(3,125)L, A1, F1, F2, G1, G2
125  FORMAT(2X'H', 7X'A1', 1X'F1', 11X'F2', 11X'G1', 11X'G2', /I2, F13.5, 4F15.5))
   900  CONTINUE
STOP
END
APPENDIX B

THESIS PROGRAM O. ALVARADO

PADE DIGITAL FILTER EVALUATION PROGRAM FOR THE IMPULSE INVARIANCE

THFJ S PROGRAM

BO, R1, A1, R1, A, B ARE THE UNNORMALIZED COEFFICIENTS VALUES OF

TABLE 3, AND WC(J) THE VALUES OF WC FROM TABLE 1.

DIMENSION WC(3), YS(101), PS(101), Y(101), P(101), F(101)

C COMPLEX CONJG, CMPLX, S, FA, FB, Z
READ(1, 101) (WC(I), I = 1, 3)

101 FORMAT(3F12.5)

CALL PENDOS('ALVARADO', O', 10, 1)

DO 50 J = 1, 3
W1 = WC(J)
DO 45 K = 1, 2
T = 10. ** (-K)
WA = TAN(3.14159 * T)
W00 = W1 / WA
DO 55 N = 1, 101
F(N) = 0.02 * (N - 1)

50 CONTINUE

DO 30 M = 2, 3
IF(M - 2) 21, 4, 3

4 WS = W1 * F(N)
S = CMPLX(0., WS)
GO TO 20

3 W = 2. ** 3.14159 * F(N)
S = W00 ** (2. - 1.)/(2. + 1.)
GO TO 20

20 IF(J - 2) 25, 26, 27

25 FA = 1.
FB = 1. * S
GO TO 33

26 FA = 2. * (3. - S)
GO TO 33

FB = 60. * (36. + (9. + S) * S) * S

33 FAM = SORT((REAL(FA)) ** 2 + (AIMAG(FA)) ** 2)
FRM = SORT((REAL(FB)) ** 2 + (AIMAG(FB)) ** 2)
IF(N - 1) 21, 10, 11

10 SL = 20. * ALOG10(FAM / FRM)
11 GAIN = 20. * ALOG10(FAM / FRM) - S1
ANG1 = ATAN2(AIMAG(FA), REAL(FA))
ANG2 = ATAN2(AIMAG(FB), REAL(FB))
ANGL = ANGL - ANG2
PHASE = ANGL * 180. / 3.14159
IF(180. - 90 * (PHASE)) 77, 78, 78

77 IF(PHASE) 87, 21, 88
87 PHASE=360.+PHASE
  GO TO 78
88 PHASE=PHASE-360.
  IF(M-2)<21,6,7
  6 YS(N)=GAIN
      PS(N)=PHASE
      GO TO 30
  7 Y(N)=GAIN
      P(N)=PHASE
  30 CONTINUE
55 CONTINUE
  CALL NEWPLT(1.0,6.5,11.0)
  CALL ORIGIN(0.,0.)
  CALL XSCALE(0.,2.,8.5)
  CALL TSCALE(0.,2.,8.5)
  CALL YSCALE(-10.,5.,5.)
  CALL TPLT(Y,101,1,4)
  CALL TAXIS(0.:1)
  CALL YAXIS(1.)
  CALL SYM(0.,-3.5,0,07,'FIG.(  ) D( , ) FREQUENCY RESPONSE OF THE P
1ADE LOW-PASS FILTERS WITH T= MSEC., AND FC=1.07*0.94)
  CALL SYM(0.,-3.8,0,07,'++ CONTINUOUS VERSION: XX DIGITAL VERSION 0
1BTAINED WITH THE BILINEAR TRANSFORMATION TECHNIQUE',0.,94)
  CALL TPLT(YS,101,1,3)
  DO 11 I=1,21
      X=(I-1)*0.1
      XIN=XSTOIN(X)
1 CALL NUM(XIN,0.125,0.07,X,0,0.1)
  1 CALL NUM(XIN,0.125,0.07,X,0,0.1)
  DO 11 I=1,16
      Y1=(6-I)*1.
      YIN=YSTOIN(Y1)
  2 CALL NUM(-0.75,YIN,0.07,Y1,0,0,0.,0.
  CALL ENDPLT
  CALL NEWPLT(1.0,5.5,11.0)
  CALL ORIGIN(0.,0.)
  CALL TSCALE(0.,2.,8.5)
  CALL XSCALE(0.,2.,8.5)
  CALL YSCALE(-10.,5.,5.)
  CALL TPLT(P,10,1,4)
  CALL TAXIS(0.1)
  CALL YAXIS(10.)
  CALL SYM(0.,-2.7,0,07,'FIG.(  ) D( , ) PHASE RESPONSE OF THE PADE
1LOW-PASS FILTERS WITH T= MSEC., AND FC=1.07*0.91)
  CALL SYM(0.,-3.0,0,07,'++ CONTINUOUS VERSION: XX DIGITAL VERSION 0B
1TAINED WITH THE BILINEAR TRANSFORMATION TECHNIQUE',0.,94)
  CALL TPLT(PS,101,1,3)
  DO 18 I=1,21
      X=(I-1)*0.1
XIN = XSTO(IN(X))
8 CALL NUM(XIN, 0.125, 0.07, X, 0.0, 1)
   DO 9 I = 1, 19, 2
   YI = (10 - I) * 20.
9 CALL NUM(-0.40, YIN, 0.07, Y1, 0.0, 0.)
   CALL ENDPLT
45 CONTINUE
50 CONTINUE
21 CALL LSTPLT
   CALL EXIT
   END
APPENDIX C

THESIS PROGRAM D. ALVARADO

PAGE DIGITAL FILTER EVALUATION PROGRAM FOR THE BILINEAR TRANSFORMATION

C

DIMENSION WC(3), P(101), Y(101), PS(101), YS(101)
COMPLEX X, FH, FH1, FH2, F1(20), FH21, Z, CONJG, CMPLX, C, FA, FB, F1, S
E1(BO, R, I, J, Z) = 30/11. - EXP(-R1*Z) * Z
READ(1, 250) (WC(J), J = 1, 3)
CALL PENPOSI(ALVARADO, D', 10, 1)
P12 = 3. 14159265

DO 50 J = 1, 3
READ(1, 250) BO, R1, A1, B1, A, R
W1 = WC(J)
RO = P12/W1
BO = BO*RO
P1 = R1*RO
A1 = A1*RO
IF (2 - J) 71, 71, 70

71 B = B*RO
A = A*RO
R1 = R1*RO
B1 = B1*RO/B

DO 70 K = 1, 2
T = 10. * ** (1 - K)
WRITE(3, 149) J
FORMAT(3X, 'J=', I2)
WRITE(3, 152) T
FORMAT('J=', F15.5)

DO 55 N = 1, 101
F = 0.02*(N - 1)
W = F*W1
U = P12*F
Z = CMPLX(COS(U*T), -SIN(U*T))
S = CMPLX(0., W)
TF = Z1.33, 34, 35

33 FA = 1.
FR = 1. + S
GO TO 25

34 FA = 2. * (3. - S)
FR = 6. + (4. + S)*S
GO TO 25

35 FA = 3. *(20. + (-8. + S)*S)
FR = 60. + (36. + (0. + S)*S)*S
25 FAM = SQRT(REAL(FA)**2 + AIMAG(FA)**2)
FRM = SQRT(REAL(FB)**2 + AIMAG(FB)**2)
IF(2-J) 5, 4, 5
5 FH1=F1(B0,R1,T,Z)
   GO TO 7
4 FH1=0.
7 IE(2-.1)11,11,10
10 FH2=0.
   GO TO 15
11 C=7*EXP(-A*T)
   D=A1*COS(B*T)+B1*SIN(B*T)
   FH20=A1+C*D
   FH21=1.-2.*C*COS(B*T)+C**2
   FH2=FH20/FH21
15 FH=FH)+FH2
   X=CONJG(FH)
   RMAG=REAL(X*FH)
   IF(N-1)21,20,22
20 Z1=10.*ALOG10(RMAG)
   S1=20.*ALOG10(FAM/FRM)
22 GAIN=10.*ALOG10(FAM/FRM)-Z1
   GAINS=20.*ALOG10(FAM/FRM)-S1
   RIM=AIMAG(FH)
   RE=REAL(FH)
   ANGL=ATAN2(RIM,RE)
   PHASE=ANGL*180./3.14159
   ANG1=ATAN2(AIMAG(FA),REAL(FA))
   ANG2=ATAN2(AIMAG(FB),REAL(FB))
   PHASS=(ANG1-ANG2)*180./3.14159
   IF(180.-ABS(PHASS))44,45,47
44 IF(PHASS)46,21,47
46 PHASS=360.*PHASS
   GO TO 45
47 PHASS=-360.+PHASS
45 Y(N)=GAIN
   P(N)=PHASE
   YS(N)=GAINS
   PS(N)=PHASS
55 CONTINUE
   CALL NEWPLOT(1.0,6.5,11.0)
   CALL ORIGIN(0.,0.)
   CALL XSCLF(0.,2.,8.5)
   CALL TSCLF(0.,2.,8.5)
   CALL YSCLF(-10.,5.,5.)
   CALL TPLT(Y,101,1.4)
   CALL TAXIS(0.1)
   CALL YAXIS(1.)
   CALL SYM(0.,-3,5.,.07);"FIG. ( ) FLY'( . ) FREQUENCY RESPONSE OF THE P
18E LOW-PASS FILTERS WITH T= MSEC., AND FC.=1.HZ.,0.,94)
   CALL SYM(0.,-3,9.,.07);"CONTINUOUS VERSION: XX DIGITAL VERSION 0
18TAIRED WITH THE IMPULSF INVARIANCE TECHNIQUE",0.,89)
CALL TPLT(YS, 101, 1, 3)
DO 1 I = 1, 21
   X = (I - 1) * 0.1
   XIN = XSTOIN(X)
1 CALL NUM(XIN, 0.125, 0.07, X, 0, 0, 1)
DO 2 I = 1, 16
   Y1 = (6 - I) * 1.
   YIN = YSTOIN(Y1)
2 CALL NUM(-0.25, YIN, 0.07, Y1, 0, 0, 0.7)
CALL ENDPLT
CALL NEWPLT(1.0, 0.5, 5, 11.)
CALL ORIGIN(0., 0.)
CALL YSCALE(0., 7., 9.5)
CALL XSCALE(0., 2., 8.5)
CALL YSCALE(-180., 180., 5.)
CALL TPLT(P, 101, 1, 4)
CALL VAXIS(0.1)
CALL VAXIS(10.)
CALL SYM(-2.7, 0.7, 'FIG. (1) P(, ) PHASE RESPONSE OF THE PADE
LOW-PASS FILTERS WITH T = MSEC... AND FC=1.HZ. '0., 91.)
CALL SYM(-3.0, 0.7, '++ CONTINUOUS VERSION: XX DIGITAL VERSION DB
1TAINED WITH THE IMPULSE INVARIANCE TECHNIQUE', 0., 89)
CALL TPLT(PS, 101, 1, 3)
DO 8 I = 1, 21
   X = (I - 1) * 0.1
   XIN = XSTOIN(X)
8 CALL NUM(XIN, 0.125, 0.07, X, 0, 0, 1)
DO 9 I = 1, 19, 2
   Y1 = (10 - I) * 0.
   YIN = YSTOIN(Y1)
9 CALL NUM(-0.4, YIN, 0.07, Y1, 0, 0, 0.7)
CALL ENDPLT
80 CONTINUE
90 CONTINUE
21 CALL LSPLT
CALL EXIT
END
VITA

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