A comparison of proportional and nonproportional damping in multistory shear buildings

Edwin C. Bailey

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A COMPARISON OF PROPORTIONAL AND NONPROPORTIONAL DAMPING IN MULTISTORY SHEAR BUILDINGS

BY

EDWIN BAILEY

A THESIS submitted to the faculty of THE UNIVERSITY OF MISSOURI AT ROLLA in partial fulfillment of the requirements for the Degree of MASTER OF SCIENCE IN CIVIL ENGINEERING Rolla, Missouri 1968

Approved by

John L. Best (advisor) Frankl y. Cheng

Billy E. Gillett
ABSTRACT

The purpose of this study was to determine whether proportional damping could be used to satisfactorily simulate the more practical nonproportional damping condition in multistory shear buildings. To accomplish this one computer program was written to analyze proportionally damped systems and another to analyze nonproportionally damped systems; each giving as output the frequencies, mode shapes, displacements, shears and moments and the maximum shears and moments.

Shear buildings of two, three and four stories were studied in which the damping coefficients were varied from one to ten percent of critical damping. Although the programs could handle various other types, only linearly decreasing forcing functions were used. For the proportional system, the damping matrix was taken proportional to the stiffness matrix such that a good approximation to the nonproportional damping matrix was obtained. The resulting shears at one particular time and the maximum shears were then compared. For the structures investigated it was found that the proportionally damped systems, in comparison to the nonproportionally damped systems, yielded conservative values for the shears examined whereas the maximum shears were almost identical to those in the nonproportional system.
ACKNOWLEDGEMENT

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LIST OF SYMBOLS

A list of symbols is written here for convenience. In all cases a capital letter in brackets, [ ], represents a matrix and its lower case subscripted equivalent represents an element of that matrix. Also, the symbol [ ] represents a diagonal matrix.

\[ [A] = \text{coefficient matrix in the equations of motion for a nonproportionally damped system.} \]

\[ [B] = \text{coefficient matrix in the equations of motion for a nonproportionally damped system.} \]

\[ [C] = \text{damping matrix.} \]

\[ c_i = \text{damping coefficient of i-th story.} \]

\[ c_r = \text{critical damping coefficient.} \]

\[ [D] = -[B]^{-1}[A]. \]

\[ E = \text{modulus of elasticity.} \]

\[ [F] = \text{forcing function matrix.} \]

\[ f_i = \text{forcing function of i-th story.} \]

\[ f_{1i} = \text{constant forcing function.} \]

\[ f_{2i} = \text{time dependent forcing function.} \]

\[ f_{3i} = \text{amplitude of sinusoidal forcing function.} \]

\[ f_{5i} = \text{amplitude of cosinusoidal forcing function.} \]

\[ [2H] = \text{uncoupled damping matrix for a proportionally damped system.} \]

\[ [I] = \text{identity matrix.} \]

\[ I_i = \text{moment of inertia of the columns in the i-th story.} \]

\[ [K] = \text{stiffness matrix.} \]

\[ k_i = \text{stiffness of i-th story.} \]

\[ L_i = \text{length of i-th column.} \]
$L'_i$ = length vector.

$M_i$ = moment at point $i$.

$[M]$ = mass matrix.

$m_i$ = mass of $i$-th story.

$n$ = number of stories or degrees of freedom in a building.

$[P]$ = circular frequency matrix for a nonproportionally damped system.

$[P^2]$ = circular frequency squared matrix for a proportionally damped system.

$P_i$ = circular frequency of $i$-th mode.

$[Q]$ = transformation matrix.

$[q_i]$ = $i$-th normalized mode shape.

$R$ = "real part of."

$t$ = time.

$V_i$ = shear in $i$-th story.

$w_i$ = frequency of sinusoidal and cosinusoidal forcing function.

$[X]$ = displacement matrix.

$x_i$ = displacement of $i$-th story.

$[X_i]$ = $i$-th mode shape for a proportionally damped system.

$[X_{ji}]$ = matrix whose columns are the mode shapes of a proportionally damped system.

$[X']$ = uncoupled displacement matrix for a proportionally damped system.

$[\dot{X}]$ = velocity matrix.

$\dot{x}_i$ = velocity of $i$-th story.

$\ddot{x}_i$ = acceleration of $i$-th story.

$[\ddot{X}']$ = uncoupled velocity matrix for a proportionally damped system.
\[ [X_0] \] = initial displacement matrix.
\[ x_{0i} \] = initial displacement of \( i \)-th story.
\[ [X'0] \] = uncoupled initial displacement matrix for a proportionally damped system.
\[ \dot{X}_0 \] = initial velocity matrix.
\[ \dot{x}_{0i} \] = initial velocity of \( i \)-th story.
\[ [X'0] \] = uncoupled initial velocity matrix for a proportionally damped system.
\[ [Y] \] = a matrix containing the coupled velocities and displacements for a nonproportionally damped system.
\[ [Y0] \] = a matrix containing the initial coupled velocities and displacements for a nonproportionally damped system.
\[ [Y_i] \] = \( i \)-th mode shape for a nonproportionally damped system.
\[ [Z] \] = a matrix containing the uncoupled velocities and displacements for a nonproportionally damped system.
\[ [Z0] \] = a matrix containing the initial uncoupled velocities and displacements for a nonproportionally damped system.
\[ \lambda_i \] = \( 1/p_i \)
\[ \mu_i \] = real part of \( \lambda_i \).
\[ \nu_i \] = imaginary part of \( \lambda_i \).
\[ \alpha_i \] = real part of \( p_i \).
\[ \beta_i \] = imaginary part of \( p_i \).
\[ [\xi_i] \] = matrix containing real part of \( [Y_i] \).
\[ [\eta_i] \] = matrix containing imaginary part of \( [Y_i] \).
I. INTRODUCTION

Dynamic consideration in the analysis and design of structures has become increasingly important through the years. A significant factor to be taken into account when considering structural vibrations is that of damping or the reduction in amplitude of a body caused by a gradual conversion of the vibration energy into heat. Of the various types of damping, the most commonly used for engineering purposes is viscous damping, wherein the damping force is proportional to the velocity of the vibratory movement. One problem then is to determine this constant of proportionality. The standard practice is to use the critical damping condition as a measure of the relative amount of damping in a particular system. Thus, the damping coefficient in each story is taken as some percentage of critical damping, $c_r$, where

$$c_r = 2\sqrt{\frac{k}{m}}$$

If the damping matrix is taken proportional to the mass, stiffness, or a linear combination of the two, a transformation matrix can be constructed which will uncouple the equations of motion. On the other hand, if the damping coefficients are taken as some percentage of critical damping, or i.e., if they are nonproportional, then this transformation matrix will not uncouple the damping matrix. For nonproportional damping a procedure somewhat similar to that followed for proportional damping is used to construct the transformation which uncouples the damping matrix. However, in this case the problem is complicated
by the fact that the circular frequencies would be complex variables and thus the transformation matrix would contain complex elements. Therefore, a particular problem would be greatly simplified if it could be reduced to one with proportional damping.

With this in mind, it can be seen that the stiffness matrix, if divided by the proper constant, results in a reasonable approximation of the nonproportional damping matrix. To find this constant each nonzero element of the tridiagonal stiffness matrix is divided by its corresponding element in the nonproportional damping matrix. The constant is then determined by simply taking the average of those results, from which the proportional damping matrix is obtained. Thus, the purpose of this paper is to determine whether it is feasible to make the damping matrix proportional to the stiffness matrix.

To accomplish this two computer programs were written, one to analyze proportionally damped systems and the other to analyze nonproportionally damped systems. Shear buildings of two, three, and four stories were investigated in which damping ranged from one to ten percent of critical. The maximum shear and shear at one particular time in each story were then compared for both cases. For this study no consideration was given to joint rotations.
II. REVIEW OF LITERATURE

Using a reduced form of the differential equations of motion introduced by Frazer, Duncan, and Collar (5), Foss (4) presented a method which uncoupled the equations of motion for nonproportionally damped systems. However, the application of this method resulted in complex frequencies and mode shapes which complicated the problem. Because of this added difficulty, further work was carried out to develop various restrictions on the form of the damping matrix so that the undamped and damped systems could still be uncoupled by the same transformation.

Caughey (1) stated in one of his earlier papers on classical normal modes that, "A necessary and sufficient condition that a damped dynamic system possess classical normal modes is that the damping matrix be diagonalized by the same transformation which uncouples the undamped system." He then went on to derive a relationship between transformed forms of the stiffness and damping matrices which established a criteria for uncoupling the damping matrix. Later, DaDeppo (3) described a form of the damping matrix more general than the proportional case, that could be resolved into normal modes of vibration for which the equations of motion could be uncoupled. In a following paper, Caughey (2) presented additional conditions that must be adhered to by the stiffness and damping matrices if a system is to possess classical normal modes.

The above articles contained additional references on this subject but because of their limited distribution the author was not able to obtain them.
III. GENERAL THEORY

The analysis of a vibrating multi-story building is an extremely complicated problem. To precisely compute the motion of such a system except by actual measurement is beyond reason. Therefore, it is desireable to introduce assumptions that will reduce the amount of computations, even though there is a small sacrifice in accuracy. With this in mind the following "shear building" assumptions are made:

1. The total mass of the structure and the external forces are concentrated at the floor levels;
2. The girders of the structure are infinitely rigid as compared to the columns; and
3. All columns are inextensible.

These assumptions lead to the following simplifications:

1. From assumption 1 the problem is reduced from one of a structure with an infinite number of degrees of freedom to a structure which has as many degrees of freedom as it has lumped masses at the story levels.
2. From assumption 2 it follows that the joints are fixed against rotation.
3. From assumption 3 the deformations are fixed such that the rigid girders will remain horizontal.
4. From assumptions 2 and 3 it follows that the columns retain vertical slopes at their ends and will thus act as fixed-end beams which are subject to settlement of the supports in a direction perpendicular to the beam axis, that is, the
Fig. 1a
n-STORY SHEAR BUILDING

Fig. 1b
SPRING-MASS SYSTEM
floors are assumed to move parallel to each other with the relative deflection in any story being dependent only on the shear in that story. In the analysis of an n-story tall structure, as in Fig. la, it is convenient to assume that the motion of any one story is resisted only by the spring forces brought into play by the relative displacements from the two neighboring stories. This implies that the sets of columns between floors act like simple springs connecting the masses of the floors, as in Fig. lb.

The spring constant, $k_i$, is the shear force necessary to produce a unit relative horizontal displacement between the ends of all the columns of the i-th story. To obtain the expression for $k_i$, the slope-deflection equation is written between points A and B for the column in Fig. 2.

Since rotation of the ends of the column is not permitted, as explained previously, we have:
Due to symmetry

\[ M_A = M_B = M \]

and

\[ V_A = V_B = V \]

The summation of moments at point A yields

\[ V = \frac{2M}{L} = \frac{12EI\Delta}{L^3} \]

For a unit displacement or \( \Delta = 1 \), we have

\[ V_i = k_i = \frac{12EI_i}{L_i^3} \]

in which \( E \) is the modulus of elasticity, in pounds per square inch; \( I_i \) denotes the sum of the moments of inertia of all columns of the \( i \)-th story, in inches to the fourth; and \( L_i \) refers to the length of the columns of the \( i \)-th story, in inches.

For the special case where the first story columns are pinned at the foundations we use a modified slope-deflection equation and obtain from Fig. 3:

\[ M_A = \frac{3EI\Delta}{L^2} \]
Again

\[ M_A = M \]

and

\[ V_A = V_B = V \]

The summation of moments at A now yields,

\[ V = \frac{M}{L} = \frac{3EI\Delta}{L^3} \]

For \( \Delta = 1 \), we have

\[ V_i = k_i = \frac{3EI_i}{L_i^3} \tag{2} \]

If some of the columns in the first story are fixed and some pinned, \( k_i \) is obtained by multiplying \( E/L_i^3 \) by the sum of the products of the moment of inertia of each column and the constant 12 or 3 depending on whether the column is fixed or pinned.

To develop the equations of motion for the spring-mass system in Fig. 1b, we refer to the free body diagram of the \( i \)-th mass in Fig. 4.
Using D'Alembert's principle, which can be written as:

\[ \Sigma (\text{forces in } x\text{-direction}) - m \dddot{x} = 0 \]

in which the quantity \(-m \dddot{x}\) is the product of the mass of the body, in pound seconds per inch, and negative acceleration, we obtain

\[ f_i + k_{i+1}(x_{i+1} - x_i) - k_i(x_i - x_{i-1}) - m_i \dddot{x}_i = 0 \]  \hspace{1cm} 3a)

or

\[ m_i \dddot{x}_i - k_i x_{i-1} + (k_{i} + k_{i+1})x_i - k_{i+1}x_{i+1} = f_i \]  \hspace{1cm} 3b)

For free vibration, the forcing function \(f_i\), in pounds, equals zero and Eq. 3b takes the form

\[ m_i \dddot{x}_i - k_i x_{i-1} + (k_{i} + k_{i+1})x_i - k_{i+1}x_{i+1} = 0 \]  \hspace{1cm} 4)

Assuming a solution of the form

\[ x_i = X_i \cos (pt - \rho) \]  \hspace{1cm} 5)

where \(X_i\) is the displacement amplitude of the i-th mass in inches, \(p\) is the circular frequency in radians per second and \(\rho\) is the phase angle in radians, and substituting into Eq. 4...
we obtain

\[-m_i p^2 X_i \cos(pt - \rho) - k_i X_{i-1} \cos(pt - \rho)\]

\[+ (k_i + k_{i+1}) X_i \cos(pt - \rho) - k_{i+1} X_{i+1} \cos(pt - \rho) = 0\]

Since \(\cos(pt - \rho)\) cannot equal zero or the solution would be trivial, we have

\[-m_i p^2 X_i - k_i X_{i-1} + (k_i + k_{i+1}) X_i - k_{i+1} X_{i+1} = 0\]  \hspace{1cm} 7a)

or

\[-k_i X_{i-1} + (k_i + k_{i+1}) X_i - k_{i+1} X_{i+1} = m_i p^2 X_i\]  \hspace{1cm} 7b)

Thus, for the n-story building in Fig. 1a we obtain the following set of n simultaneous homogenous equations

\[(k_1 + k_2) X_1 - k_2 X_2 = p^2 m_1 X_1 \]

\[-k_2 X_1 + (k_2 + k_3) X_2 - k_3 X_3 = p^2 m_2 X_2 \]

\[-k_n X_{n-1} + k_n X_n = p^2 m_n X_n \]

\[8)\]

The set of Eqs. 8 may be expressed in the matrix form

\[[K][X_i] = p_i^2 [M][X_i]\]  \hspace{1cm} 9a)

If we premultiply each side of Eq. 9a by \([M]^{-1}\) and simplify, we obtain

\[[M^{-1}K - p_i^2][X_i] = 0\]  \hspace{1cm} 9b)

in which \([K], [M]\) and \([X_i]\) are matrices defined as follows
Stiffness matrix $[K] = \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & (k_2 + k_3) & -k_3 \\ -k_n & k_n \end{bmatrix}$

Mass matrix $[M] = \begin{bmatrix} m_1 \\ m_2 \\ m_n \end{bmatrix}$

and

Displacement amplitude matrix $[X_i] = \begin{bmatrix} x_{1i} \\ x_{2i} \\ x_{ni} \end{bmatrix}$

In Eq. 9b, the quantity $p$ may take on as many values as there are equations in the system and for every value of $p_i$ there is a corresponding vector $[X_i]$. This problem is called an eigenvalue problem in which the quantities $p_i^2$ are called eigenvalues and the relationship among the amplitudes, or elements of $[X_i]$, are called eigenfunctions. The vector $[X_i]$ corresponding to a particular natural mode of vibration is called an eigenvector or mode shape. The significance of a natural mode is explained in the following: "If the initial conditions are properly imposed it is possible to cause vibration in any one of several natural modes which are characteristic
of the structure. In a natural mode each point in the structure executes harmonic motion about a position of static equilibrium, every point passing through its equilibrium position at the same instant and reaching its extremum at the same instant. Obviously, then, the frequency of the oscillation is the same at every point and this is the natural frequency of the structure in the particular mode involved. If we consider the structure at an instant when all the points in it reach an extremum and, hence, are momentarily stationary, we visualize the structure in a particular deformation configuration that is a peculiar property of a natural mode. It is convenient, then, to think of a natural mode as a deflected configuration in which the motion of each point is harmonic and in which the vibration has a specific natural frequency associated with that mode."(7) It can be shown that for our ideal structure, the number of natural modes is equal to its number of degrees of freedom or to the number of stories.

To solve the set of Eqs. 9b we note from Cramer's rule that there can only be nontrivial values for \([X_i]\) if the determinate of the coefficients of \([X_i]\) vanish or i.e.,

\[
|M^{-1}K - p_i^2| = 0
\]

Upon expanding the determinate and solving the resulting \(n\)-th order equation in \(p_i^2\), we obtain the \(n\) values of \(p_i\). When the \(n\) \(p_i\)'s are inserted into Eq. 9b, there results \(n\) sets of \([X_i]\), each containing \(n\) values. Because the determinate of the coefficient matrix is equal to zero, Eq. 11, the rank of the coefficient matrix is less than \(n\). Therefore, there are an
infinite number of solutions to \([X_1]\) and we obtain only ratios among the elements of \([X_1]\) by assigning an arbitrary value to one of them. "The physical interpretation of this mathematical result is that the deflected configuration of a structure which describes a natural mode is defined by known ratios among the amplitudes of motion at the various points. The actual amplitude of motion of the system as a whole is arbitrary and not a property of a natural mode."(7)

A. Derivation of the Equations of Motion with Damping Considered

Eq. 9a was derived by considering the vibrations to persist once being initiated without the application of external forces. In reality such a vibration without decrease in amplitude is never realized. The presence of damping forces causes the dissipation of energy that progressively reduces the amplitude of vibration and ultimately stops the motion when all energy initially stored in the system has been dissipated.

To facilitate mathematical manipulations it is customary to assume that damping is viscous, i.e., that the damping forces are proportional to the rate of change of displacements or

\[ D(t) = c\dot{x} \]

Where \(D(t)\) is some function of time, \(c\) is a damping constant, in pound seconds per inch, and \(\dot{x}\) is the velocity of the moving mass, in inches per second. This expression is considered to be very good for representing the resistance to motion of air surrounding a body which moves at a low speed, or the internal frictional resistance of most solid materials.(7)
One question which may arise is whether damping depends on absolute or relative velocities. In this paper, damping forces depend on the relative velocities between masses except in the case when there is no ground motion. Then the damping coefficient in the first story is a function of the absolute velocity of the first mass.

\[ c_i \]

Fig. 5

i-TH DASHPOT

It is customary to represent viscous damping by a dashpot as shown in Fig. 5. In much the same way as the spring constant \( k_i \) characterizes the elastic spring, the damping coefficient \( c_i \) characterizes the dashpot. Thus, the spring-mass-dashpot system for the multi-story building in Fig. 1a is shown in Fig. 6.

\[ k_i \begin{pmatrix} x_i \\ \dot{x}_i \end{pmatrix} = \begin{pmatrix} f_i \\ f_i' \end{pmatrix} + \begin{pmatrix} m_i \ddot{x}_i \\ m_i \ddot{x}_i \end{pmatrix} \]

Fig. 6

SPRING-MASS-DASHPOT SYSTEM

\[ \begin{align*}
k_i(x_i - x_{i-1}) & \quad - \\
\dot{c}_i(\dot{x}_i - \dot{x}_{i-1}) & \quad - \\
m_i \ddot{x}_i & \quad - \\
\end{align*} \]

Fig. 7

FREE BODY OF i-TH MASS IN FIG. 6
Again we draw, in Fig. 7, the free body diagram for the i-th mass in Fig. 6. Using D'Alembert's principle we obtain the following damped equation of motion

\[ f_i + k_{i+1}(x_{i+1} - x_i) + c_{i+1}(\dot{x}_{i+1} - \dot{x}_i) - k_i(x_i - x_{i-1}) \]

\[ - c_i(\dot{x}_i - \dot{x}_{i-1}) - m_i \ddot{x}_i = 0 \]  \hspace{1cm} 12a)

or

\[ m_i \ddot{x}_i - c_i \dot{x}_{i-1} + (c_i + c_{i+1})\dot{x}_i - c_{i+1}\dot{x}_{i+1} - k_i x_{i-1} \]

\[ + (k_i + k_{i+1})x_i - k_{i+1}x_{i+1} = f_i \]  \hspace{1cm} 12b)

Written in matrix notation, Eq. 12b takes the form

\[ [M][\ddot{X}] + [C][\dot{X}] + [K][X] = [F] \]  \hspace{1cm} 13)

in which \([K]\) and \([M]\) are defined in Eq. 10 and the matrices \([X]\), \([C]\) and \([F]\) are defined as follows

\[ [X] = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \]  \hspace{1cm} 14)

Displacement matrix \([X]\) =

\[ [C] = \begin{bmatrix} c_1 + c_2 & -c_2 & \cdots & -c_n \\ -c_2 & c_2 + c_3 & \cdots & -c_n \\ \vdots & \vdots & \ddots & \vdots \\ -c_n & -c_n & \cdots & c_n \end{bmatrix} \]  \hspace{1cm} 15)

Damping matrix \([C]\) =
Forcing matrix \([F]\) =
\[
\begin{bmatrix}
  f_1 \\
  f_2 \\
  \vdots \\
  f_n
\end{bmatrix}
\]

It is customary to define the damping of a system as a percentage of the coefficient of critical damping \((\text{cr})\) in which
\[
\text{cr}_i = \frac{2}{\sqrt{k_i m_i}}
\]

A great amount of work has been devoted to the determination of the percentages for a given material and a given type of structure. Numerous sources list numerical values of the percent of critical damping for these cases. \((8,9)\)

B. Solution of the Differential Equations of Motion for a Proportionally Damped System

"In formulating equations of motion, it is natural to use coordinates in which the motion of the structure can be interpreted most readily. However, the most logical choice of coordinates for physical interpretation is not always the best choice for solution of the equations. Therefore, it is useful to consider coordinate transformation in dynamic systems." \((7)\)

In our case we shall investigate the possibility of transforming the matrix \([X]\) into a matrix \([X']\), such that the coefficient matrices, \([M]\), \([K]\), and \([C]\), of Eq. 13, will become a diagonal matrix. To do this we must consider Eq. 9a.

First, let us consider any two mode shapes for the problem, denoted by \([X_i]\) and \([X_j]\) and their corresponding circular frequencies as \(p_i\) and \(p_j\). Using Eq. 9a we may write
\[ [K][X_i] = p_i^2[M][X_i] \quad 17 \]
\[ [K][X_j] = p_j^2[M][X_j] \quad 18 \]

Now postmultiply the transpose of Eq. 17 by \([X_j]\) and premultiply Eq. 18 by \([X_i]^T\) to give

\[ [KX_i]^T[X_j] = p_i^2[MX_i]^T[X_j] \quad 19 \]
\[ [X_i]^T[KX_j] = p_j^2[X_i]^T[MX_j] \quad 20 \]

Since the matrix \([M]\) is diagonal, \([M] = [M]^T\). Also, from the reversal law for transposed matrix products, \([KX_i]^T = [X_i]^T[K]^T\), and from the reciprocity theorem of Maxwell, \([K] = [K]^T\). In this case, the left-hand sides of Eqs. 19 and 20 are equal and thus the right-hand sides must be equal. From this

\[ (p_j^2 - p_i^2)[X_i]^T[M][X_j] = 0 \quad 21 \]

If \(p_i\) and \(p_j\) are not equal, it must follow that

\[ [X_i]^T[M][X_j] = 0 \quad 22 \]

By definition two matrices are said to be orthogonal if the product of one and the transpose of the other is equal to zero. Therefore, we say that the mode shapes \([X_i]\) and \([X_j]\) are orthogonal with respect to the weighing matrix \([M]\).

If \([X_i] = [X_j]\) or if the mode shapes are the same, then \(p_i = p_j\). Thus, Eq. 21 is satisfied regardless of the value of \(p\). This being the case, it must follow that \([X_i]^T[M][X_i] \neq 0\) necessarily. Defining the length vector, \(L'\), as \(\sqrt{[X]^T[X]}\), we have in our case the length in terms of the weighting matrix
\[ M \] as

\[ L_i' = [X_i]^T [M] [X_i] \] \hspace{1cm} (23)

Dividing the left and right sides by \( L_i' \), we have

\[ 1 = \frac{[X_i]^T [M] [X_i]}{L_i} \] \hspace{1cm} (24)

or

\[ 1 = [q_i]^T [M] [q_i] \] \hspace{1cm} (25)

in which the quantity \( [q_i] = \frac{[X_i]}{L_i} \) is called a normalized or unit vector.

Now we define a new matrix \([Q]\) as

\[
[Q] = \begin{bmatrix}
q_{11} & q_{12} & \cdots & q_{1n} \\
q_{21} & q_{22} & \cdots & q_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
q_{n1} & q_{n2} & \cdots & q_{nn}
\end{bmatrix}
\] \hspace{1cm} (26)

noting that the elements of any column are the components of one of the unit mode shapes.

Similarly we define a matrix \([X_{ji}]\) such that

\[
[X_{ji}] = \begin{bmatrix}
x_{11} & x_{12} & \cdots & x_{1n} \\
x_{21} & x_{22} & \cdots & x_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
x_{n1} & x_{n2} & \cdots & x_{nn}
\end{bmatrix}
\] \hspace{1cm} (27)

where each column contains the components of a mode shape.

Using Eq. 27, we may rewrite Eq. 9a in the all inclusive form
where

\[
[K][X_{ji}] = [M][X_{ji}][P^2]
\]  

Dividing Eq. 28 by \(L_i\), we have

\[
\frac{[K][X_{ji}]}{L_i} = \frac{[M][X_{ji}][P^2]}{L_i}
\]

But

\[
\frac{[X_{i}]}{L_i} = [q_i]
\]

therefore

\[
[K][Q] = [M][Q][P^2]
\]

Now if we premultiply through by \([Q]^T\) and recognize that

\[
[Q]^T[M][Q] = [I]
\]

since from Eqs. 22 and 25

\[
[q_i]^T[M][q_j] = \begin{cases} 
0, & i \neq j \\
1, & i = j
\end{cases}
\]

we find that

\[
[Q]^T[K][Q] = [P^2]
\]

Next we consider the damping matrix, \([C]\), in Eq. 15. If it is premultiplied by \([Q]^T\) and postmultiplied by \([Q]\), the resulting matrix is
Comparing this triple product with Eqs. 33 and 35 it becomes apparent that the matrix \([2H]\) will be diagonal only when the damping matrix \([C]\) is proportional to the mass matrix \([M]\), or to the stiffness matrix \([K]\), or to a linear combination of the two.

Referring again to the basic equations of motion

\[
[M][\ddot{X}] + [C][\dot{X}] + [K][X] = [F]
\]

and applying the coordinate transformation

\[
[X] = [Q][X']
\]

from which

\[
[\dot{X}] = [Q][\dot{X}']
\]

and

\[
[\ddot{X}] = [Q][\ddot{X}']
\]

we have

\[
[M][Q][\ddot{X}'] + [C][Q][\dot{X}'] + [K][Q][X'] = [F]
\]

If Eq. 40 is premultiplied by \([Q]^T\) we obtain

\[
\]

Now using Eqs. 33, 35 and 36 we find

\[
[\ddot{X}'] + [2H][\dot{X}'] + [P^2][X'] = [Q]^T[F]
\]

Hence, transformation (37) caused the equations of motion to become uncoupled; namely only one unknown, \(\dot{X}'_i\), \(\ddot{X}'_i\) and \(X'_i\), appears in any one of Eqs. 42.
For a general forcing function, \( f_i \), of the form

\[
f_i = f_{1i} + f_{2i}(t) + f_{3i}\sin w_it + f_{5i}\cos w_it \quad 43)
\]

with \( i = 1, 2, \ldots, n \), the \( i \)-th uncoupled equation is

\[
\ddot{x}_i' + 2h_i\dot{x}_i' + p_i^2x_i' = CSTF_i + TDF_i(t) + \sin_i + \cos_i \quad 44)
\]

in which

\[
h_i = h_{ii}
\]

\[
CSTF_i = q_{1i}f_{1i} + q_{2i}f_{2i} + \ldots + q_{ni}f_{ni}
\]

\[
TDF_i = q_{1i}f_{2i} + q_{2i}f_{2i} + \ldots + q_{ni}f_{2i}
\]

\[
\sin_i = q_{1i}f_{3i}\sin w_it + q_{2i}f_{3i}\sin w_{2i} + \ldots + q_{ni}f_{3i}\sin w_{ni}
\]

\[
\cos_i = q_{1i}f_{5i}\cos w_it + q_{2i}f_{5i}\cos w_{2i} + \ldots + q_{ni}f_{5i}\cos w_{ni}
\]

Eq. 44 is a nonhomogeneous equation and as such has a general solution consisting of the homogeneous solution (right-hand side of the differential equation equal to zero) plus the particular solution. Thus

\[
x_i' = x_{hi}' + x_{pi}'
\]

The particular solution to Eq. 44 is

\[
x_{pi}' = A_i' + B_i'(t) + C_{li}\sin w_{1i}t + C_{2i}\sin w_{2i}t + \ldots + C_{ni}\sin w_{ni}t + D_{li}\cos w_{1i}t + D_{2i}\cos w_{2i}t + \ldots 45)
\]

+ D_{ni}\cos w_{ni}t
in which

\[
A'_i = \frac{(CSTF_i - \frac{2h_i \text{TDF}_i}{p_i^2})}{p_i^2}
\]

\[
B'_i = \frac{\text{TDF}_i}{p_i^2}
\]

\[
C'_{ji} = q_{ji} \frac{(f_{5j} w_j^2 + f_{3j} (p_j^2 - w_j^2))}{((p_j^2 - w_j^2)^2 + (w_j 2h_i)^2)}
\]

and

\[
D'_{ji} = q_{ji} f_{5j} - C'_{ji} w_j 2h_i
\]

\[
(p_j^2 - w_j^2)
\]

with \( j = 1, 2, \ldots, n \)

For the homogeneous equation

\[
\ddot{x}'_i + 2h_i \dot{x}'_i + p_i^2 x'_i = 0
\]

we investigate solutions of the form

\[
x'_i = Ae^{Nt}
\]

Substituting Eq. 47 into Eq. 46 and simplifying, we obtain

\[
N^2 + 2h_i N + p_i^2 = 0
\]

The two solutions of this auxiliary equation may be found, through the use of the quadratic formula, to be

\[
N = -h_i \pm \sqrt{h_i^2 - p_i^2}
\]
The final solution, \( x'_i \), will depend on whether \( h_i^2 \) is greater than, equal to, or less than \( p_i^2 \).

To determine the constants of integration the uncoupled initial displacements, \( x'_0_i \), and initial velocities, \( \dot{x}'_0_i \), will be used. These initial conditions can be found by multiplying Eq. 37 and Eq. 38 by \([Q]^{-1}\) which results in

\[
[X'_0] = [Q]^{-1}[X_0]
\]

and

\[
[\dot{X}'_0] = [Q]^{-1}[\dot{X}_0]
\]

If Eq. 33 is postmultiplied by \([Q]^{-1}\), a simple expression for \([Q]^{-1}\) is found to be

\[
[Q]^{-1} = [Q]^T[M]
\]

Thus, for \( h_i^2 > p_i^2 \) the solution is

\[
x'_i = e^{\frac{-h_i t}{2}} \left( A_i e^{\frac{\sqrt{h_i^2 - p_i^2}}{2} t} + B_i e^{\frac{-\sqrt{h_i^2 - p_i^2}}{2} t} \right) + x'_{pi}
\]

in which

\[
B_i = ((x'_0_i - A'_i - D'_li - D'_2i - \ldots - D'_ni)(\sqrt{h_i^2 - p_i^2} - h_i) + B'_i + C'_liW_l + C'_2iW_2 + \ldots + C'_niW_n - x'_0_i)/2\sqrt{h_i^2 - p_i^2}
\]

and

\[
A_i = x'_0_i - A'_i - D'_li - D'_2i - \ldots - D'_ni - B_i
\]

For \( h_i^2 = p_i^2 \) the solution is

\[
x'_i = e^{\frac{-h_i t}{2}} (A_i + B_i t) + x'_{pi}
\]
in which

\[ A_i = x'_{o_i} - A'_i - D'_1i - D'_2i - \ldots - D'_ni \]

and

\[ B_i = x'_{o_i} + A_i h_i - B'_i - C'_1i w_1 - C'_2i w_2 - \ldots - C'_ni w_n \]

For \( h_i^2 < p_i^2 \) the solution is

\[ x'_i = e^{-h_it} \left( A_i \sin \sqrt{p_i^2 - h_i^2} t + B_i \cos \sqrt{p_i^2 - h_i^2} t \right) \]

\[ + x'_{pi} \]

in which

\[ B_i = x'_{o_i} - A'_i - D'_1i - D'_2i - \ldots - D'_ni \]

and

\[ A_i = \frac{\left( x'_{o_i} + B_i h_i - B'_i - C'_1i w_1 - C'_2i w_2 - \ldots - C'_ni w_n \right)}{\sqrt{p_i^2 - h_i^2}} \]

From the above equations the \( n \) uncoupled displacements can be calculated at any time, \( t \). Then using Eq. 37 the actual displacements can be determined.

C. Equations of Motion for Nonproportional Damping

On page 20 it was shown that the transformation matrix \([Q]\) would uncouple the equations of motion only if the damping matrix \([C]\) is proportional either to the mass matrix \([M]\), to the stiffness matrix \([K]\) or to a linear combination of the two. If, on the other hand, the more realistic case is used where the damping coefficients are taken as some percentage of
critical damping, as in Eq. 16, the transformation matrix \([Q]\) will no longer uncouple the equations of motion as the damping matrix would now be nonproportional. It is possible, however, to extend the methods discussed previously and to construct a transformation which will uncouple the equations of motion even though the damping is nonproportional. The method requires at the outset the solution of the homogeneous equations to give the free-vibration response of the system. From this solution one may construct the uncoupling transformation and proceed to the solution of the nonhomogeneous equations for the forced response. However, it can be shown that the unit mode shapes, which will again make up this transformation matrix, will be complex, with its components differing in phase as well as in amplitude. It follows that two bits of information are required to determine each component or \(2n\) equations are required to determine all components of an \(n\)-degree-of-freedom system in each mode. Therefore, to the \(n\) equations of motion (13), must be added another \(n\) equations giving a system of \(2n\) equations to be solved in the case of nonproportional damping. It has been found (4) that the additional \(n\) equations can be supplied by making the substitution

\[
[M][\ddot{X}] - [M][\dot{X}] = 0
\]

Eqs. 13 and 51 are combined to give the following matrix equation of order \(2n\)

\[
\begin{bmatrix}
[0] & [M] & [\ddot{X}]
\end{bmatrix}
+ \begin{bmatrix}
-[M] & [0] & [\dot{X}]
\end{bmatrix}
= \begin{bmatrix}
0
\end{bmatrix}
\]

\[
[M][C][\ddot{X}] + [0][K][\dot{X}] = [F]
\]
or

\[ [A][\dot{Y}] + [B][Y] = [Y] \]  

in which

\[
[A] = \begin{bmatrix} [0] & [M] \\ [M] & [C] \end{bmatrix}
\]

\[
[B] = \begin{bmatrix} -[M] & [0] \\ [0] & [K] \end{bmatrix}
\]

\[
[Y] = \begin{bmatrix} [\dot{X}] \\ [X] \end{bmatrix}
\]

\[
[Y] = \begin{bmatrix} [0] \\ [P] \end{bmatrix}
\]

The great advantage of this formulation lies in the fact that the matrices \([A]\) and \([B]\), both of order \(2n\), are real and symmetric. Therefore, to solve Eq. 53 techniques very similar to those used previously may be employed.

We will now consider the homogeneous equation obtained by setting the right side of Eq. 53 equal to zero.

\[ [A][\dot{Y}] + [B][Y] = 0 \]

Assuming solutions of the form

\[ y = C'e^{pt} \]

we have

\[ \dot{[Y]} = p[Y] \]  

or
\[
\ddot{X} = p\dot{X} \\
\dot{X} = p[X]
\]

Eq. 55 is then written in terms of the unknown number \( p \) and unknown vector \( [Y] \) as follows

\[
p[A][Y] = -[B][Y] \tag{58}
\]

If Eq. 58 is premultiplied by \([B]^{-1}\) and simplified, it may be written in the form

\[
[D - \lambda][Y] = 0 \tag{59}
\]

where

\[
[D] = -[B]^{-1}[A] = -\begin{bmatrix}
-[M]^{-1} & [0] \\
[0] & [-1]
\end{bmatrix}
\begin{bmatrix}
[M] & [C] \\
[K]
\end{bmatrix}
\]

\[
= \begin{bmatrix}
[0] & [IJ] \\
-[K]^{-1} & [M] - [K]^{-1} [C]
\end{bmatrix}
\]

and

\[
\lambda = 1/p
\]

Again, as in Eq. 9b, we have an eigenvalue problem. Thus, for a nontrivial solution, the determinant of the coefficient matrix must vanish or

\[
|D - \lambda| = 0 \tag{60}
\]

This leads to a set of \( 2n \) roots or eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_{2n} \). For a critically damped or overdamped mode these roots will be real and negative and for an underdamped mode they will be complex with a negative real part. If there are complex eigenvalues
they will occur in conjugate pairs. Thus, if the r-th and s-th eigenvalues are complex conjugates, they may be written as

\[ \lambda_r = \mu_r + iv_r \]
\[ \lambda_s = \bar{\lambda}_r = \mu_r - iv_r \]

where \( \mu_r \) and \( \nu_r \) are the real and imaginary parts, respectively, and \( i \) is the unit imaginary number. Using the identity

\[ \mu_r^2 + \nu_r^2 = (\mu_r - iv_r)(\mu_r + iv_r) \]

the complex value \( p_r \), corresponding to the complex eigenvalue \( \lambda_r \), can be found from the relation

\[ p_r = 1/\lambda_r = \alpha_r + i\beta_r \]

in which

\[ \alpha_r = \frac{\mu_r}{\mu_r^2 + \nu_r^2} \]

and

\[ \beta_r = \frac{-\nu_r}{\mu_r^2 + \nu_r^2} \]

Corresponding to each eigenvalue \( \lambda_r \) there exists an eigenvector \([Y_r]\) having 2n components. There are 2n of these eigenvectors, \( r = 1, 2, \ldots, 2n \). For a pair of complex conjugate eigenvalues as given by Eq. 61, the corresponding eigenvectors are complex conjugates, thus

\[ [Y_s] = [\bar{Y}_r] \]
Therefore, if all eigenvalues of a system are complex, in which case they occur in conjugate pairs, all eigenvectors will be complex and will also occur in conjugate pairs.

D. Solution of the Differential Equation of Motion for a System with Nonproportional Damping

"The eigenvectors of a nonproportionally damped system are orthogonal just as are those for an undamped or proportionally damped system, the proof of which proceeds in the same manner. Consider the r-th and s-th eigenvectors $[Y_r]$ and $[Y_s]$, both of which satisfy Eq. 58. First we write that equation for the r-th mode and premultiply both sides by the transposed vector $[Y_s]^T$, thus

$$p_r[Y_s]^T[A][Y_r] = -[Y_s]^T[B][Y_r]$$

Again, using the reversal law for transposed matrix products and recalling that $[A]$ and $[B]$ are symmetrical matrices, we transpose both sides as

$$p_r[Y_r]^T[A][Y_s] = -[Y_r]^T[B][Y_s]$$

If Eq. 64 is subtracted from 63 the result is

$$(p_r - p_s)[Y_r]^T[A][Y_s] = 0$$

If the eigenvalues $p_r$ and $p_s$ are different, the following
orthogonality property relates the two eigenvectors

\[ [Y_r]^T[A][Y_s] = 0 \]  \hspace{1cm} (66)

From either Eq. 63 or 64 it follows that these vectors are also orthogonal with respect to the matrix \([B]\).

\[ [Y_r]^T[B][Y_s] = 0 \]  \hspace{1cm} (67)

For underdamped modes in which the eigenvalues are complex, the above orthogonality criteria relate two modes having different frequencies of vibration. Furthermore, it is emphasized that the relationship applies also to two conjugate complex eigenvectors which are associated with a single mode, for the only requirement that Eqs. 66 and 67 hold is that the two complex eigenvalues associated with that mode be different. The fact that they are different is seen as follows. If

\[ p_r = \alpha_r + i\beta_r \]

and

\[ p_s = \alpha_r - i\beta_r \]

then

\[ p_r - p_s = (\alpha_r + i\beta_r) - (\alpha_r - i\beta_r) = 2i\beta_r \neq 0 \]

For underdamped systems it is useful to derive the orthogonality relationships in terms of the real and imaginary parts of the eigenvectors. We let

\[ [Y_r] = [\xi_r] + i[\eta_r] \]

\[ [Y_s] = [\xi_r] + i[\eta_s] \]  \hspace{1cm} (68)
Eq. 66 may be written as

\[(\xi_r^T + i\eta_r^T)[A](\xi_s + i\eta_s) = 0\]  \hspace{1cm} (69)

By expanding the products and noting that the real and imaginary parts must vanish separately, the two following equations are obtained.

\[\xi_r^T[A][\xi_s] - \eta_r^T[A][\eta_s] = 0\]  \hspace{1cm} (70)

\[\xi_r^T[A][\eta_s] + \eta_r^T[A][\xi_s] = 0\]

Next we consider an eigenvector \([Y_s]\) that is orthogonal not only to \([Y_r]\) which results in Eq. 70, but also to the conjugate eigenvector \([\overline{Y}_r]\). Thus,

\[(\xi_r^T - i\eta_r^T)[A](\xi_s + i\eta_s) = 0\]

This separates into the two equations

\[\xi_r^T[A][\xi_s] + \eta_r^T[A][\eta_s] = 0\]  \hspace{1cm} (71)

\[\xi_r^T[A][\eta_s] - \eta_r^T[A][\xi_s] = 0\]

Considering the first equation of (70) together with the first one of (71) leads us to the following results.

\[\xi_r^T[A][\xi_s] = 0\]  \hspace{1cm} (72)

\[\eta_r^T[A][\eta_s] = 0\]

The second equations of (70) and (71) yield

\[\xi_r^T[A][\eta_s] = 0\]  \hspace{1cm} (73)

\[\eta_r^T[A][\xi_s] = 0\]
One additional relationship is useful. It is derived from Eq. 70 considering \([Y_s]\) as the conjugate of \([Y_r]\), i.e.,

\[ [\xi_s] = [\xi_r] \]

and

\[ [\eta_s] = -[\eta_r] \]

In this case the first equation of (70) yields the following result.

\[ [\xi_r]^T[A][\xi_r] = -[\eta_r][A][\eta_r] \quad (74) \]

Eqs. 72, 73, and 74 express orthogonality of the eigenvectors in terms of their real and imaginary parts. These relationships are developed with matrix \([A]\) as the weighting matrix starting with Eq. 66. It is easy to see that if we start with Eq. 67 we can show that the same relationships hold with matrix \([B]\) as the weighting matrix.\(^{(7)}\)

Referring again to Eq. 65, it is evident that this equation is satisfied whenever \(p_r = p_s\). Thus, \([Y_r]^T[A][Y_r] \neq 0\) necessarily. (This is similar to the procedure followed on page 17.) We can now define the length vector in terms of the weighting matrix \([A]\) as

\[ L_r' = [Y_r]^T[A][Y_r] \quad (75) \]

Dividing both sides by \(L_r'^2\), we have

\[ 1 = \frac{[Y_r]^T[A][Y_r]}{L_r'^2} \]
or

\[ l = [q_r]^T[A][q_r] \]

in which \([q_r]\) is the \(r\)-th normalized eigenvector. Therefore, using the above relation and Eq. 66 it follows that

\[ [q_r]^T[A][q_s] = \begin{cases} 0, & r \neq s \\ 1, & r = s \end{cases} \quad 76) \]

and from this relation and Eq. 63 or 64

\[ [q_r]^T[B][q_s] = \begin{cases} 0, & r \neq s \\ -p_r, & r = s \end{cases} \quad 77) \]

In a manner similar to that described previously for an undamped or proportionally damped system, the eigenvectors which define the modes of free vibration of the system may be used to construct a transformation of coordinates in which the equations of motion are uncoupled. (4) For a nonproportionally damped system that transformation is given by

\[ [Y] = [Q][Z] \quad 78) \]

The transformation matrix \([Q]\) is constructed column-by-column using the 2n normalized eigenvectors; \([q_1], [\overline{q}_1], \ldots, [q_r], [\overline{q}_r], \ldots, [q_n], [\overline{q}_n]\). Thus, the matrix is of order 2n. Where complex eigenvectors exist, both the eigenvector \([q]\) and its conjugate \([\overline{q}]\) are used.

\[ [Q] = \begin{bmatrix} [q_1][\overline{q}_1] & \ldots & [q_r][\overline{q}_r] & \ldots & [q_n][\overline{q}_n] \end{bmatrix} \]
\[
\begin{bmatrix}
q_{1,1} & q_{1,2} & q_{1,r} & q_{1,2n} \\
q_{2,1} & q_{2,2} & q_{2,r} & q_{2,2n} \\
q_{j,1} & q_{j,2} & q_{j,r} & q_{j,2n} \\
q_{2n,1} & q_{2n,2} & q_{2n,r} & q_{2n,2n}
\end{bmatrix}
\]

When this transformation is applied to the nonhomogeneous equation 53, the following result is obtained:

\[
[Q]^T [A][Q][\dot{Z}] + [Q]^T [B][Q][Z] = [Q]^T [Y]
\]

From Eqs. 76 and 77 it follows that

\[
[Q]^T [A][Q] = [I]
\]

and

\[
[Q]^T [B][Q] = [-P]
\]

Therefore, Eq. 80 can be written as

\[
\dot{Z} - [P][Z] = [Q]^T [Y]
\]

When expanded Eq. 83 results in 2n uncoupled equations of the form

\[
\dot{z}_r - p_r \bar{z}_r = [q_r]^T [Y]
\]

and

\[
\dot{\bar{z}}_r - \bar{p}_r \bar{z}_r = [\bar{q}_r]^T [Y]
\]

in which \(\bar{z}_r\) represents the complex conjugate of \(z_r\). It can be verified (10) that the solutions of Eqs. 84 and 85 will also be complex conjugates or
\[ z_r = a + ib \]
\[ \overline{z_r} = a - ib \]

Assuming we have solved for the uncoupled responses, we may proceed through the coordinate transformation to find the response to the generalized time-dependent forces \([F(t)]\) in terms of the generalized displacements, \(\mathbf{x}_r\), and velocities, \(\dot{\mathbf{x}}_r\). The transformation Eq. 78 may, for this purpose, be written in the convenient form

\[ y_j(t) = q_{j,1}z_1 + q_{j,2}\overline{z}_1 + q_{j,3}z_2 + q_{j,4}\overline{z}_2 + \ldots + q_{j,2n-1}z_{2n-1} + q_{j,2n}\overline{z}_{2n} \]

where \(j=1, 2, \ldots, 2n\). Focusing our attention on the first two products in Eq. 86 and recalling that \(q_{j,1}\) and \(q_{j,2}\) are complex conjugates as are \(z_1\) and \(\overline{z}_1\), it is evident that the real parts of the two products will be identical and the imaginary parts will cancel when the two terms are summed. A similar statement can be made for the remaining terms. From this it can be seen that \(y_j(t)\) is real and may be written as twice the real part of the complex number on the right side of Eq. 86. Thus, we may write

\[ y_j(t) = 2R(q_{j,1}z_1 + q_{j,3}z_2 + q_{j,5}z_3 + \ldots + q_{j,2n-1}z_{2n}) \]

where \(R\) stands for "the real part of." Next, using the transformation connected with Eq. 53 relating \([Y]\) and \([X]\), which is rewritten for convenience

\[ [Y] = \begin{bmatrix} [\dot{X}] \\ [X] \end{bmatrix} \]
we see the first $n$ values of $y_j(t)$ are the generalized velocities and the second $n$ values are the generalized displacements.

We now turn our attention to the solution of the $r$-th uncoupled equation. For a general forcing function, $f_s$, of the form

$$f_s = f_{1s} + f_{2s}(t) + f_{3s} \sin w_st + f_{5s} \cos w_st$$

we get

$$z_r - p_r z_r = q_{n+1,r}(f_{1r} + f_{2r}t + f_{3r} \sin w_1 t + f_{5r} \cos w_1 t)$$

$$+ q_{n+2,r}(f_{1r} + f_{2r}t + f_{3r} \sin w_2 t + f_{5r} \cos w_2 t) + ...$$

The solution to this equation is

$$z_r = \frac{(e^{p_r t} - 1)}{p_r}(q_{n+1,r}f_{1r} + q_{n+2,r}f_{2r} + ... + q_{2n,r}f_{1r})$$

$$+ \frac{(e^{p_r t} - p_r t - 1)}{p_r^2}(q_{n+1,r}f_{2r} + q_{n+2,r}f_{2r} + ... + q_{2n,r}f_{2r})$$

$$+ \frac{q_{n+1,r}f_{3r}}{(p_r^2 + w_1^2)}(e^{p_r w_1 t} - p_r \sin w_1 t - w_1 \cos w_1 t)$$

$$+ \frac{q_{n+2,r}f_{3r}}{(p_r^2 + w_2^2)}(e^{p_r w_2 t} - p_r \sin w_2 t - w_2 \cos w_2 t) + ...$$

$$+ \frac{q_{2n,r}f_{3r}}{(p_r^2 + w_n^2)}(e^{p_r w_n t} - p_r \sin w_n t - w_n \cos w_n t)$$
\[ \begin{align*}
&+ \frac{q_{n+1}r_f s_{l+1}}{\left( \frac{p_r}{2} + w_1^2 \right)} \left( e^{prt} p_r - p_r \cos w_1 t + w_1 \sin w_1 t \right) \\
&+ \frac{q_{n+2}r_f s_{l+2}}{\left( \frac{p_r}{2} + w_2^2 \right)} \left( e^{prt} p_r - p_r \cos w_2 t + w_2 \sin w_2 t \right) + \ldots \\
&+ \frac{q_{2n}r_f s_{l+2n}}{\left( \frac{p_r}{2} + w_n^2 \right)} \left( e^{prt} p_r - p_r \cos w_n t + w_n \sin w_n t \right) + z_r(0)e^{prt}
\end{align*} \]

The value \( z_r(0) \) can be found by using the initial displacements, \( x_0 \), and initial velocities, \( \dot{x}_0 \). If Eq. 78 is multiplied by \( [Q]^{-1} \) we have

\[ [Z] = [Q]^{-1}[Y] \]

or

\[ [Z_0] = [Q]^{-1}[Y_0] = [Q]^{-1} \begin{bmatrix} \dot{x}_0 \\ x_0 \end{bmatrix} \]

A simple method for finding \( [Q]^{-1} \) can be derived by postmultiplying Eq. 81 by \( [Q]^{-1} \), which results in

\[ [Q]^{-1} = [Q]^T[A] \]

Using the initial conditions and the desired forcing functions, Eq. 91 can be solved and the uncoupled displacement can then be determined.
IV. DISCUSSION

For this investigation, shear buildings of two, three and four stories (illustrated in Fig. 8 with the assumed masses, stiffnesses and forcing functions) were studied in which proportional and nonproportional damping were compared. In the nonproportional case the damping coefficients were taken as various percentages of critical damping. For proportional damping, the damping coefficients were assumed to be proportional to the stiffnesses such that they reasonably approximated the nonproportional coefficients. To determine whether nonproportionally damped systems could be reasonably approximated by proportionally damped systems, the damping coefficients were varied from one to ten percent of critical and the shears at one particular time and maximum shears were then compared. To assure that the time increments were small enough to obtain the maximum shear, increments of 0.1, 0.05 and 0.01 seconds were initially tried. The resulting shears were then plotted and it was found that the 0.05 second increment was sufficient to yield the maximum shears, for all practical purposes, for the particular forcing functions used in this study.

Figs. 9, 10 and 11 are graphs of the shears in the various stories of the structures investigated. These shears were calculated 1.25 seconds after the application of the forcing functions, which, in this case, happens to be the time at which the forcing functions ceased. The circled points on these graphs represent shears for the nonproportionally damped system, the
points enclosed by triangles represent the proportionally damped system and the number near each set of lines denotes the story.

The maximum shears in each story of the structures in Fig. 8 are shown in Figs. 12, 13 and 14. The dashed lines in these figures designate the shears for the undamped system. The two left most lines represent the maximum shears in nonproportional and proportional systems, respectively, with ten percent critical damping. The next line corresponds to one percent critical damping in both proportionally and nonproportionally damped systems since the shears in these systems are almost identical for such small degrees of damping. Although they are not shown, the maximum shears for the remaining percentages of critical damping are almost equally distributed between these lines. Thus, these three lines represent the ranges over which the maximum shear varies for damping from one to ten percent of critical in proportional and nonproportional systems. In all instances the proportional values were slightly greater than the nonproportional values.

The data presented in these graphs was found by using two computer programs. These programs, one of which was written for proportionally damped systems and the other for nonproportionally damped systems, can be found in Appendixes A and B, respectively. Also included in each of these Appendixes are an explanation of the program and two example problems. These problems involve the two story shear building shown in Fig. 8. In the first, the structure is assumed to be undamped and in the second one percent critical damping is assumed. In these
examples the calculations of the shears were carried out for 2.5 seconds. It was discovered that for this study this was a sufficient length of time to assure that the maximum shear would be obtained.
Fig. 8

STRUCTURES INVESTIGATED

\[ f = 52000 - 41600t \text{ lb.} \]
\[ m = 832 \text{ lb. sec.}^2/\text{in.} \]
\[ k_1 = 166000 \text{ lb./in.} \]
\[ k_2 = 234000 \text{ lb./in.} \]
Fig. 9
SHEAR IN TWO-STORY SHEAR BUILDING AT T = 1.25 SECS.
VS.
PERCENT DAMPING
Fig. 10

SHEAR IN THREE-STORY SHEAR BUILDING AT T = 1.25 SECS.

VS.

PERCENT DAMPING
Fig. 11
SHEAR IN FOUR- STORY SHEAR BUILDING AT T = 1.25 SECS.
VS.
PERCENT DAMPING
MAXIMUM SHEAR IN TWO-STORY SHEAR BUILDING
Fig. 13

MAXIMUM SHEAR IN THREE- STORY SHEAR BUILDING
V. CONCLUSION

From the graphs of the shears at 1.25 seconds, a few observations can be made. In general, proportional damping yields conservative results compared to nonproportional damping, with the shears being more nearly equal for lower percentages of critical damping. For the most part, the shears determined from proportional damping become less conservative as the building height is increased. In fact the only case in which proportional damping is unconservative, about 0.5 percent, occurs in the first story of the four story building. This may be an indication that proportionally damped systems may not be very good approximations to the shears in certain stories of a building whose height exceeds, for example, four stories. However, this does not necessarily mean that the maximum shears in that story cannot be reasonably approximated by the proportional condition.

From the graphs of the maximum shears, it is evident that the ranges of these shears diminish in the upper stories and in the taller buildings for both proportional and nonproportional damping. This shows that damping in general has a lesser effect on the maximum shear in these cases. In the discussion a statement is made concerning the proportional values of maximum shear being slightly greater than the nonproportional values. From the standpoint of using the maximum shear for design it would seem to follow that proportional damping is a good approximation of nonproportional damping, especially in the upper stories and in taller structures. Another interesting point is that the
maximum shear in the first stories increases as building height increases but the maximum shear in the top stories is approximately 45 kips in all three buildings. No reason can be given to explain this except that it is just a peculiarity due to the particular forcing function used.

Based on the structures considered it can be said that proportional damping results in conservative approximations to the shear at 1.25 seconds but gives good approximations to the maximum shears. However, there is no reason to believe that this would be true in general. Before this could be said, further investigations would have to be made in which more extensive considerations are given to the various structural properties and forcing functions.
BIBLIOGRAPHY


VITA

Edwin C. Bailey, Jr., was born at St. Louis, Missouri on October 15, 1942 to Mr. and Mrs. Edwin C. Bailey, Sr. He received his primary education at schools in St. Louis and graduated from Normandy High School in May, 1961. He received his college education from the Normandy Residence Center in St. Louis (now the University of Missouri at St. Louis) and the University of Missouri at Rolla. He received a Bachelor of Science Degree in Civil Engineering from the University of Missouri at Rolla in May, 1966.

He is a member of the American Society of Civil Engineers, Missouri Society of Professional Engineers, National Society of Professional Engineers and the Engineers Club of St. Louis.

On June 27, 1964, Mr. Bailey married Miss Mary Jo Powers.
APPENDIX A

DESCRIPTION OF PROGRAM FOR PROPORTIONALLY DAMPED SYSTEM

The following is a discussion of the coding of data for and limitations of the computer program.

The first card is a title card which will be printed with the output for the convenience of the user. On the second card, in the first field of five right justified, is specified the number of degrees of freedom in the problem, \( n \). The next fields of ten and five, respectively, contain the length and increments of time over which the displacements are to be computed. The following seven fields of five contain control values, \( Z_1 \) through \( Z_7 \), for the matrices \( [X_0] \), \( [\dot{X}_0] \), \( [C] \), \( [F_1] \), \( [F_2] \), \( [F_3] \) and \( [F_5] \), respectively. If all the elements in one of these matrices are zero, then some number other than zero is placed in its respective field and the elements of that matrix need not be read as they will be set equal to zero. If the elements are not zero the field should be left blank and the values will be read from the appropriate cards. The next field of ten contains the value \( \tau \) which will be discussed later. The next four sets of cards contain the diagonal elements of the mass matrix, the elements of the stiffness matrix and, finally, the stiffness and lengths of each story. The remaining sets of cards define \( [X_0] \), \( [\dot{X}_0] \), \( [C] \), \( [F_1] \), \( [F_2] \), \( [F_3] \), \( [W] \) and \( [F_5] \) if the control values have stipulated that they be read.

We now refer to the value \( \tau \). This is the duration of the forcing function of the form

\[
f = f_1 - f_2(t)
\]
which is shown in Fig. 15

![Fig. 15](image)

**Fig. 15**

**TIME DEPENDENT FORCING FUNCTION**

Thus,

\[ \tau = \frac{f_1}{f_2} \]

When the time \( t \) is exactly equal to \( \tau \), the forcing functions \( f_1 \) and \( f_2 \) are set equal to zero and the uncoupled velocities, \( \dot{x}'o \), are computed. The values \( x'o(\tau) \) and \( \dot{x}'o(\tau) \) are then used as initial conditions with the new forcing functions and what is essentially a new problem is continued.

A few points should be noted in connection with this type of forcing function. Since the time \( t \) must be exactly equal to \( \tau \), the time increment \( dt \) should be such that some integer multiple of \( dt \) is equal to \( \tau \). Second, the program was written on the assumption that, although the forcing functions may not be equal, they have the same duration \( \tau \). Finally, if the forcing function is not of this form the field reserved for \( \tau \) should be left blank.

One other limitation in the program concerns the cosinusoidal forcing function. Unlike the sinusoidal forcing function, the cosinusoidal forcing function must be used in combination
with the sinusoidal forcing function and must have the same
frequency.

Line by line explanation of computer program:

1 to 2: DIMENSION and REAL statements.

3: READ information card and transfer control to stop
statement if there is no more data.

4: READ n, time span (TIME), time increment (DT), con­
trol values (Z1 through Z7) and τ (TAU).

5 to 8: Initialize [M] and [M1] to zero.

9: READ diagonal elements of [M].

10 to 12: READ [K], stiffness, [K1], and length, [L], of each
story.

13 to 50: READ [X0], [X0], [C], [F1], [F2], [F3], [F4] and
[F5] if the control values have designated that
they be read and if not set them equal to zero.

51 to 70: WRITE important input information. Note, in line
55 the diagonal elements of [M1] are set equal to
the diagonal elements of [M]. Thus, these two
matrices are identical. This is done because [M]
is destroyed in one of the subroutines used, but
is needed later in the program.

71 to 74: Subroutine ARRAY changes the indicated matrices
from double subscript storage to single subscript
or vector storage, or vice versa, depending on
whether the first value in the argument list is 2
or 1, respectively. This is necessary as all the
subroutines in the Scientific Subroutine Package,
supplied by IBM, use this type of storage. Note
in line 73 that \([K]\) is changed and the result is called \([Q]\).

75: This subroutine calculates the eigenvalues, \([PSQ]\), or i.e., the diagonal elements of \([P^2]\), and eigenvectors or mode shapes, \([EIVECT]\), of the matrix product \([M]^{-1}[K]\) with \([Q] = [K]\). This is the subroutine in which \([MlJ]\) is destroyed. It is important to note that \([Q]\), or \([K]\), is destroyed and is replaced by the eigenvectors, normalized with respect to the weighting matrix \([MlJ]\), which is very convenient.

76 to 77: \([EIVECT]\) and \([Q]\) are converted to double subscript storage to simplify their output.

78 to 79: Circular frequencies, \([P]\), are found from \([PSQ]\).

80 to 87: WRITE \([P]\), \([EIVECT]\) and \([Q]\).

88 to 92: Check for resonance.

93: WRITE literal.

94: \([Q]\) is changed back to single subscript storage.

95: \([Q]^T\) is formed.

96 to 97: \([2H]\) is formed and is temporarily called \([C]\).

98: The storage mode of \([C]\) is changed and the resulting vector, \([HX2]\), contains its diagonal elements.

99: \([Q]^{-1}\) is formed.

100 to 101: \([X'0]\) and \([\dot{X}'0]\) are formed.

102 to 103: \([CSTF]\) and \([TDF]\) are formed.

104: \([Q]^T\) is changed to double subscript storage.

105 to 114: The maximum shears, \([VMAX]\), and moments, \([MMAX]\), are initialized to zero and \(\sqrt{h_i^2 - p_i^2}\) is calculated.
115 to 117: The relative time (T), absolute time (T1), and control value (CONTRL) are initialized to zero.

118 to 137: The coefficients of the uncoupled equations are calculated.

138 to 150: [X'] is calculated.

151: [X] is calculated.

152 to 156: The shears, [V], and moments, [MOM], are calculated.

157 to 162: [VMAX] and [MMAX] are found.

163 to 164: WRITE T1, [X], [V] and [MOM].

165: If the problem is over write [VMAX] and [MMAX] and if not proceed.

166: If the forcing function is of the form fl - f2t,
then at T = τ, fl and f2 must be set equal to zero.
Thus, this statement checks for a forcing function
of this form or for a value for τ.

167: Check for T = τ.

168: Set CONTRL to cause transfer of control when line
189 is reached so new coefficients of the uncoupled
equations will be calculated.

169 to 186: Initialize conditions for the new problem.

187 to 188: Increment the time.

189: Transfer control to calculate either the new coeffi-
cients of the uncoupled equations or the uncoupled
displacements.

190: WRITE [VMAX] and [MMAX].

191: Start new problem.
DIMENSION C(20,20), F1(20), F2(20), F3(20), F4(20), F5(20), X(20), 
IY(20), YI(20), DX(30), X1(20), Y1(20), X2(20), Y2(20), 
F1(20), F2(20), F3(20), F4(20), F5(20), X1(20), X2(20), 
Y1(20), Y2(20), A(20), AR(20), CE(20), CF(20), TDF(20), DUM(20), 
400T(20), SI(20), CO(20), V(20), VMAX(20)

REAL M(20,20), M1(20,20), K1(20,20), K2(20,20), L(20), NNUM, MGM(20), 
WMAX(20)

C 71 THROUGH 77 ARE CONTROL VALUES
C 71 FOR XO, INITIAL DISPLACEMENTS
C 72 FOR C, DAMPING COEFFICIENTS
C 73 FOR E1, CONSTANT FORCING FUNCTION
C 74 FOR E2, TIME DEPENDENT FORCING FUNCTION
C 75 FOR E3 AND E4, AMPLITUDE AND FREQUENCY OF SINUSOIDAL FORCING FUNCTION
C 76 FOR F3 AND F4, AMPLITUDE AND FREQUENCY OF COSINUSOIDAL FORCING FUNCTION

1 READ(1,300,UNIT=74)
DO 3 J=1,N
DO 4 I=1,N
M(I,J)=0.0
M(I,I)=1.0
C READ DIAGNOSTIC ELEMENTS OF MASS MATRIX, M.
READ(1,300) (M(I,I),I=1,N)
C READ STIFFNESS MATRIX, K, BY ROWS OR COLUMNS SINCE SYMMETRIC
READ(1,300) (K(I,J),I=1,N,J=1,N)
READ(1,300) (K(I,I),I=1,N)
READ(1,300) (L(I,I),I=1,N)
IF(71) 4,5,4
READ(1,400) (X(I),I=1,N)
GO TO 7
DO 6 I=1,N
X(I)=0.0
6 IF(72) 8,9,8
READ(1,500) (DX(I),I=1,N)
GO TO 11
DO 10 I=1,N
DX(I)=0.0
10 IF(74) 12,13,12
READ(1,600) ((C(I,J),I=1,N,J=1,N)
GO TO 15
DO 14 I=1,N
C(I,J)=0.0
14 IF(76) 16,17,16
READ(1,700) (F(I),I=1,N)
GO TO 10
DO 18 I=1,N
F(I)=0.0
18 F(I)=0.0
10 IF(75) 20,21,20
20 READ(1,700) (F2(I),I=1,N)
      GO TO 23
21 DO 22 J=1,N
22 F2(I)=0.0
      DO 23 J=1,N
23 IF(70) 24,25,24
24 READ(1,700) (F3(I),I=1,N)
      READ(1,700) (F4(I),I=1,N)
      GO TO 27
25 DO 26 J=1,N
26 F4(I)=0.0
      DO 27 J=1,N
27 IF(77) 28,29,28
28 READ(1,700) (F5(I),I=1,N)
      GO TO 31
29 DO 30 J=1,N
30 F5(I)=0.0
31 WRITE(3,1000)
   WRITE(3,1000) N,TIME,DT,TAU
   WRITE(3,1000) M(I,J),J=1,N
32 WRITE(3,1000) (M(I,J),J=1,N)
   WRITE(3,1200)
   DO 33 J=1,N
33 WRITE(3,1200) (K(I,J),J=1,N)
   WRITE(3,1900)
   DO 34 J=1,N
34 WRITE(3,1900) (C(I,J),J=1,N)
   WRITE(3,5000)
   WRITE(3,5000) (I,L(I),I=1,N)
35 WRITE(3,5000) (I,XO(I),I=1,N)
   WRITE(3,1600)
   WRITE(3,1600) (I,DXO(I),I=1,N)
   WRITE(3,2000)
   WRITE(3,2000) (I,EF(I),E3(I),E2(I),E1(I),I=1,N)
36 CALL AARRAY(2,N,N,20,20,1,M1)
37 CALL AARRAY(2,N,N,20,20,41)
38 CALL AARRAY(2,N,N,20,20,44)
39 CALL AARRAY(2,N,N,20,20,1,P1)
40 CALL AARRAY(2,N,N,20,20,1,PQ)
41 CALL AARRAY(2,N,N,20,20,1,PO)
   WRITE(3,1700)
   WRITE(3,1700) (I,PO(I),I=1,N)
42 CALL AARRAY(2,N,N,20,20,1,PO)
43 CALL AARRAY(2,N,N,20,20,1,PO)
44 CALL AARRAY(2,N,N,20,20,1,PO)
   WRITE(3,2000)
   WRITE(3,2000) (I,PO(I),I=1,N)
37 WRITE (2, 2500) I, (EIVECT(J, I), J=1, N)
38 WRITE (3, 2600)
39 DO 38 I = 1, N
40 WRITE (3, 2700) (O(I, J), J=1, N)

C CHECK FOR REASONANCE
41 DO 40 I = 1, N
42 IF (O(I) = F4(I)) 40, 30, 40
43 WRITE (3, 3100) I
44 GO TO 1

45 CONTINUE
46 WRITE (2, 2800)
47 CALL ARAY(2, N, 1, 20, 20, 0, 0)
48 CALL CNTRA(0, OT, N, N)
49 CALL CSORD(O, C, K, N, N, N)
50 CALL SMTR(E, HY??, N, N)
51 CALL CSORD(O, INV, YO, PSP, N, N, N)
52 CALL CMUD(O, INV, DXO, DSP, N, N, 1)
53 CALL CSROD(O, E, F1, CSTF, N, N)
54 CALL CMUD(O, F2, TDF, N, N, 1)
55 CALL ARAY(1, N, N, 20, 20, OT, OT)
56 DO 43 I = 1, N
57 WMX(I) = 0.0
58 WMX(I) = 0.0
59 UNM(I) = UNN(I) * UNR(I) / 4. - PSQ(I)
60 IF (UNM(I) <= 41, 42, 42
61 NNM = NNM - UNM(I)
62 SOT(I) = SORT(NNM)
63 GO TO 43
64 CONTINUE
65 T = 0.0
66 T1 = 0.0
67 CONT = 0.0

C CALCULATE COEFFICIENTS OF UNCOPLED EQUATIONS
68 DO 42 I = 1, N
69 AP(I) = CSTF(I)-UNN(I) * TDF(I) / PSQ(I)
70 B(I) = TDF(I) / PSQ(I)
71 SUMS1 = 0.0
72 SUMS1 = 0.0
73 SUMC = 0.0
74 DO 45 J = 1, N
75 SUMS1 = SUMS1 + S(I, J) * UNN(I) * TDF(I)
76 CONT = CONT + S(I, J) * UNN(I) * TDF(I)
77 S(I, J) = S(I, J) * UNN(I) * TDF(I) / (PSQ(I) - F4(J) * F4(J))
45 SUMCN = SUMCN + CN(I,J)
46 IF (SUM(I)) 45,47,48
47 P(I) = XPO(I) - AP(I) - SUMCN
48 A(I) = (XPO(I) + HX2(I)*R(I)/2. - BP(I) - SUMSI)/SQT(I)
49 GO TO 40
40CONTINUE
C CALCULATE UNCOUPLED DISPLACEMENTS
50 DO 55 I=1,N
51 SUMSI = 0.0
52 SUMCN = 0.0
53 DO 54 J=1,N
54 SUMSI = SUMSI + S(I,J)*SIN(F4(I,J)*T)
55 SUMCN = SUMCN + COS(F4(I,J)*T)
56 IF (SUM(I)) 52,53,54
57 XP(I) = EXP(-HX2(I)*T/2.)*A(I)*SIN(SQT(I)*T) + P(I)*COS(SQT(I)*T)
58 AP(I) = EXP(-HX2(I)*T/2.)*A(I)*SIN(SQT(I)*T) + P(I)*COS(SQT(I)*T)
59 GO TO 55
60 CONTINUE
C M = CUSD0(I,XP,X,N,N,1)
61 V(I) = K(I)*X(I)
62 DO 64 J=1,N
63 V(I) = K(I)*(X(I) - X(I-1))
64 DO 68 T=1,N
65 VH(I) = V(I)*V(I)/2.
66 DO 60 I=1,N
67 IF (ABS(VH(I)) = ABS(V(I))) 91,92,93
68 VH(I) = V(I)
69 IF (ABS(VH(I)) = ABS(MOM(I))) 93,94,95
70 VH(I) = MOM(I)
71 CONTINUE
WRITE(3,2000) T1
WRITE(3,2000) (T, X(I), I, V(I), I, MOM(I), I=1,N)
73 T(I)=T(I)-.0005 73,73,59
74 T(I)=T(I)-.0005 60,72,60
75 T(I)=T(I)-.0005 60,72,60
76 T(I)=T(I)-.0005 63,63,72
77 CONTINUE
78 TAU = 0.0
DO 70 I = 1, N
XPO(I) = XP(I)
CSTR(I) = 0.0
TDF(I) = 0.0
SUMC(I) = 0.0
SIMST = 0.0
DO A(I) = 1, N
SUMST = SUMST + S(I, J) * F4(J) * COS(F4(J) * T)
CSTR = CSTR + C04(I, J) * F4(J) * SIN(F4(J) * T)
DO 67 I = 1, N
67 XPO(I) = EXP(-H*X2(I)*T/2.)*C04(I) * COS(S04(I)*T) - A(I) * SIN(S04(I)*T) + B(I) * COS(S04(I)*T)) + BP(I)
2+SUMST-SUMC
CD TO 70
68 XPO(I) = EXP(-H*X2(I)*T/2.)*(-A(I) * H*X2(I)/2. + B(I) * (-1. - H*X2(I)*T/2.))
1+BP(I) + SUMST-SUMC
CD TO 70
69 XPO(I) = EXP(-H*X2(I)*T/2.)*(-A(I) * H*X2(I)/2. + B(I) * EXP(-1-S04(I)*T)) - (H*X2(I)/2.)*A(I) * EXP(S04(I)*T) + B(I) * EXP(-S04(I)*T))
2+BP(I) + SUMST-SUMC
70 CONTINUE
71 T = T + DT
72 T1 = T + DT
73 WRITE(3, 3200) (I, VMAX(I), I, MMAX(I), I = 1, N)
74 STOP
100 FORMAT(15, F10.3, F5.3, F5.0, F10.3)
200 FORMAT(PE10.4)
300 FORMAT(PE10.4)
400 FORMAT(PE10.4)
500 FORMAT(PE10.4)
600 FORMAT(PE10.4)
700 FORMAT(PE10.4)
800 FORMAT(PE10.4)
900 FORMAT(PE10.4)
1000 FORMAT(T1, 10X, 'ORDER OF MATRIX = ', Z2, 10X, 'TIME SPAN = ', F8.3, 10X, 'SECS')
1100 FORMAT(T1, 10X, 'TIME INCREMENT = ', F6.3, 10X, 'SECS')
1200 FORMAT(T1, 10X, 'INITIAL DISPLACEMENTS (IN.)')
1300 FORMAT(T1, 10X, 'INITIAL VELOCITIES (IN. /SEC.)')
1400 FORMAT(T1, 10X, 'DAMPING MATRIX (LB. SEC. /IN.)')
1500 FORMAT(T1, 10X, 'STIFFNESS MATRIX (LB. /IN.)')
1600 FORMAT(T1, 10X, 'MASS MATRIX (LB. SEC. /IN.)')
0212 2000 FORMAT('I0//I0X:FORCING FUNCTIONS (LR,))
0213 2100 FORMAT('I0//I0X:(',I2,'')='F10.2, ',F10.2,'+',',F10.2,'+')
0214 2200 FORMAT('I0//I0X:CIRCULAR FREQUENCIES (RAD./SEC.))
0215 2300 FORMAT('I0//I0X:(',I2,'')='F8.4')
0216 2400 FORMAT('I0//I0X:MODE SHAPES')
0217 2500 FORMAT('I0//I0X:MODE SHAPE',',',I2,'='(',I0X,'F8.4))
0218 2600 FORMAT('I0//I0X:MATRICES')
0219 2700 FORMAT('I0//I0X,F8.4')
0220 2800 FORMAT('I0//I0X:DISPLACEMENTS (IN.)', '6X:SHAPES (LR,)', '15X'
0221 2900 FORMAT('I0//I0X:TIME= ',F6.3,' SECS')
0222 3000 FORMAT('I0//I0X:(',I2,'')='F12.6, PX, V(',I2,')='F16.6, 7X, M(',I2,')='F18.6')
0223 3100 FORMAT('I0//I0X:FORCED: P(',I2,') EQUALS FORCING FREQUENCY')
0224 3200 FORMAT('I0//I0X:MAX(',I2,')='F16.6, 15X, MMAX(',I2,')='F16.6')
0225 3300 FORMAT('I0//I0X:STORY HEIGHT (IN.)')
0226 3400 FORMAT('I0//I0X:(',I2,')='F7.2)
0227 3500 FORMAT("")
UNDAMPED

ORDER OF MATRIX = 2

TIME SPAN = 2.500 SECS

TIME INCREMENT = 0.050 SECS

TAU = 1.250 SECS

MASS MATRIX (LB. SEC^2./IN.)

832.0000 0.0
0.0 416.0000

STIFFNESS MATRIX (LB./IN.)

400000.0000 -234000.0000
-234000.0000 234000.0000

DAMPING MATRIX (LB. SEC./IN.)

0.0 0.0
0.0 0.0
STORY HEIGHT (IN.)
L(1) = 144.00
L(2) = 144.00

INITIAL DISPLACEMENTS (IN.)
XO(1) = 0.0
XO(2) = 0.0

INITIAL VELOCITIES (IN./SEC.)
DXO(1) = 0.0
DXO(2) = 0.0

FORCING FUNCTIONS (LB.)
F(1) = 52000.00 + (-41600.00)T + 0.0 SIN(0.0 T) + 0.0 COS(0.0 T)
F(2) = 24000.00 + (-20800.00)T + 0.0 SIN(0.0 T) + 0.0 COS(0.0 T)

CIRCULAR FREQUENCIES (PAD./SEC.)
P(1) = 30.3558
P(2) = 11.0360
### Mode Shapes

**Mode Shape 1**
- $0.5386$
- $0.8480$

**Mode Shape 2**
- $0.6167$
- $0.7872$

### Q Matrix
- $-0.0232$  $0.0257$
- $0.0364$  $0.0328$

### Displacements (in.)

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1% CRITICAL DAMPING

ORDER OF MATRIX = 2

TIME SPAN = 2.500 SECS

TIME INCREMENT = 0.050 SECS

TAU = 1.250 SECS

MASS MATRIX (LB. SEC²/IN.)

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STIFFNESS MATRIX (LB./IN.)

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DAMPING MATRIX (LB. SEC./IN.)

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STORY HEIGHT (IN.)

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L(2) = 144.00
INITIAL DISPLACEMENTS (IN.)
\[ x_0(1) = 0.0 \]
\[ x_0(2) = 0.0 \]

INITIAL VELOCITIES (IN./SEC.)
\[ \dot{x}_0(1) = 0.0 \]
\[ \dot{x}_0(2) = 0.0 \]

FORCING FUNCTIONS (LP.)
\[ F(1) = 52000.00 + (-41600.00)T + 0.0 \sin(0.0 T) + 0.0 \cos(0.0 T) \]
\[ F(2) = 26000.00 + (-20800.00)T + 0.0 \sin(0.0 T) + 0.0 \cos(0.0 T) \]

CIRCULAR FREQUENCIES (RAD./SEC.)
\[ p(1) = 30.3550 \]
\[ p(2) = 11.0360 \]

MODE SHAPES
MODE SHAPE 1
\[-0.5336 \]
\[0.8430 \]
MODE SHAPE 2
\[0.6167 \]
\[0.7872 \]
O MATRIX

\[-0.0232 0.0257 \]
\[0.0364 0.0320 \]

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<tr>
<td>0.800 S</td>
<td>0.550494</td>
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<tr>
<td>0.850 S</td>
<td>0.590679</td>
<td>98052.562500</td>
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<tr>
<td>0.900 S</td>
<td>0.635960</td>
<td>83989.312500</td>
</tr>
<tr>
<td>TIME</td>
<td>X(1)</td>
<td>V(1)</td>
</tr>
<tr>
<td>----------</td>
<td>--------------</td>
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</tr>
<tr>
<td>0.950 SECS</td>
<td>0.306471</td>
<td>50874.113281</td>
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<tr>
<td>1.000 SECS</td>
<td>0.044824</td>
<td>7440.804687</td>
</tr>
<tr>
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<td>-0.197250</td>
<td>-32743.507812</td>
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<td>-58322.250000</td>
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<tr>
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<td>-50200.464844</td>
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<tr>
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<tr>
<td>TIME</td>
<td>X(1)</td>
<td>V(1)</td>
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<tr>
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<td>------</td>
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<tr>
<td>1.300 SECS</td>
<td>0.191673</td>
<td>16877.777344</td>
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<td>0.292323</td>
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<tr>
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<td>0.371245</td>
<td>61626.710937</td>
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<td>0.055704</td>
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<td>X(1)</td>
<td>V(1)</td>
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<tr>
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<td>X(2)</td>
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<tr>
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<td>0.297472</td>
<td>0.396114</td>
</tr>
<tr>
<td>2.100 SFCs</td>
<td>0.126645</td>
<td>0.166645</td>
</tr>
<tr>
<td>2.150 SFCs</td>
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<td>-0.105240</td>
</tr>
<tr>
<td>2.200 SFCs</td>
<td>-0.261718</td>
<td>-0.339436</td>
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<tr>
<td>2.250 SFCs</td>
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<td>2.300 SFCs</td>
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<td>----------</td>
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</tr>
<tr>
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<td>SFC</td>
<td>0.239260</td>
</tr>
<tr>
<td></td>
<td></td>
<td>VMAX(1)= 134555.062500</td>
</tr>
</tbody>
</table>
APPENDIX B

DESCRIPTION OF PROGRAM FOR NONPROPORTIONALLY DAMPED SYSTEMS

The coding of data for this program is identical to that of the previous program except for one point. Here the damping matrix must be read after the stiffness matrix whereas in the first program the inclusion of the damping matrix in the data set was optional. Thus, the control values Z3 through Z6 now correspond to the matrices \([F_1], [F_2], [F_3] \text{ and } [F_5]\), respectively.

Also, the limitations of the first program apply here with one addition. Because the method used to calculate the eigenvalues cannot solve a matrix of order two or less, one degree of freedom problems cannot be solved with this program. (6)

Before the logic of the program can be fully understood, one point regarding the complex arithmetic should be discussed. This concerns the complex conjugate, \(\bar{z}_i\). From Eq. 87 it is obvious that the conjugates are not needed in the calculations of the displacements or velocities. From this it follows that it is not necessary to find any of the conjugates values. Thus, the conjugate terms in \([P], [Q], [Q]^{-1}, [Z0], [Y]\), etc., are not calculated in the program.

Line by line description of program:

1 to 3: DIMENSION, REAL and COMPLEX statements. Note that \([SC3]\) is in the REAL statement so that it can be initialized to zero.

4 to 64: These statements are almost identical to lines 3 to 70 in the first program except that \([M1]\) is not needed here.
The required constants, $2n$, $n^2$ and $n+1$, are formed.

[CMJ, [K] and [C] are changed to single subscript storage.

[A] is formed.

[Y0] is formed.

[K] is destroyed and is replaced by $[K]^{-1}$ if the determinate of $[K]$ is nonsingular.

[D] is formed. Note that [SC3] is set equal to zero for the next problem.

[A] and [D] are changed to double subscript storage.

The eigenvalues, [ALAMDA], of [D] are calculated. The subroutine used can find real or complex eigenvalues of a nonsymmetric matrix that is destroyed in the process. Therefore, [D] is saved in lines 5 through 7 of the subroutine as it is needed to calculate the eigenvectors. Note the order of [D] is destroyed in line 44.

Order of [D] is restored.

The eigenvectors, [Q], of [D] are calculated using a modified form of Gaussian elimination with pivotal condensation. Referring to the definition of [D] it is seen that the second $n$ eigenfunctions can be determined directly from the first $n$ eigenfunctions. Therefore, since [D] is of order $2n$, only $n-1$ eigenfunctions need to be computed because one of them is assumed. Here both the real and imaginary parts of eigenfunction $n-1$ are arbitrarily assumed to be one. In line 15 the order of [D] is again destroyed.
96: The order of [D] is restored.
97 to 121: The diagonal elements of [P], called [P] here, are formed from [ALAMDA]. [Q]\(^{-1}\) is formed with ALSQ equal to \(L'_i\). [Q] is normalized. [ZO] is formed. The constant terms of the uncoupled equations are formed.
122 to 127: WRITE [P], [Q] and literal. Note that the conjugate values are not written although they exist.
128 to 129: The relative time (T) and absolute time (T1) are initialized to zero.
130 to 137: [Z] is calculated.
138 to 142: [Y] and then [X] are calculated.
143 to 158: These statements are identical to lines 152 to 167 in the first program.
159 to 164: Initialize conditions for the new problem.
165 to 166: Increment time.
167: Transfer control to calculate uncoupled displacements.
168: WRITE [VMAX] and [MMAX].
169: Start new problem.
DIMENSION C(20,20), F1(20), F2(20), F3(20), F4(20), F5(20), Y0(40),
  SC1(20), SC2(20), SC4(20,20), SC5(30,40), SC6(20,40), A1(40,40),
  X(40,40), X1(20), X2(20), X0(20), Y(20), V(20), VM(20), VMX(20)
REAL M(20,20), K(20,20), K1(20), L(20), MOM(20), SC3(20,20)

1/40*0.07, VMX(20)

COMPLEX P(20), ALAMDA*16(20), O*16(40,20), QINV(20,40), CSTF(20),
  TSF(20), T(20), Y(20), Y0(20), SUM3, SUM5, QPW(20,20), EPT, PSQ,
  2ALCO

C 71 THROUGH 76 ARE CONTROL VALUES
C 71 FOR Y0, INITIAL DISPLACEMENTS
C 72 FOR DX0, INITIAL VELOCITIES
C 73 FOR F1, CONSTANT FORCING FUNCTION
C 74 FOR F2, TIME DEPENDENT FORCING FUNCTION
C 75 FOR F3 AND F4, AMPLITUDE AND FREQUENCY OF SINUSOIDAL FORCING FUNCTION
C 76 FOR F5 AND F6, AMPLITUDE AND FREQUENCY OF COSINUSOIDAL FORCING FUNCTION

1 READ(1,000,END=58)
2 READ(1,100) N, TIME, DT, M1, M2, M3, M4, M5, M6, TAU
3 DO 2 I=1,N
4 DO 2 J=1,N
2 M(I,J)=0.0

C READ DIAGONAL ELEMENTS OF MASS MATRIX, M.
READ(1,200) (M(I,I),I=1,N)

C READ STIFFNESS MATRIX, K, BY ROWS OR COLUMNS SINCE SYMMETRIC
READ(1,300) (K(I,J),I=1,N,J=1,N)
READ(1,300) (K(I,J),I=1,N,J=1,N)
READ(1,300) (L(I,I),I=1,N)

1 IF(71) 4,5,4
4 READ(1,400) (XD0(I),I=1,N)
GO TO 7
5 DO 6 I=1,N
6 XD0(I)=0.0
7 IF(72) 8,9,8
8 READ(1,500) (DX0(I),I=1,N)
GO TO 11
9 DO 10 J=1,N
10 DX0(I)=0.0
11 IF(73) 16,17,16
16 READ(1,700) (F1(I),I=1,N)
GO TO 19
17 DO 18 J=1,N
18 F1(I)=0.0
19 IF(74) 20,21,20
20 READ(1,700) (F2(I),I=1,N)
GO TO 23
21 DO 22 J=1,N
22 F2(I)=0.0
23 IF(75) 24,25,24
24 READ(1,700) (F3(I),I=1,N)
25 READ(1,700) (F4(I),I=1,N)
<table>
<thead>
<tr>
<th>Line</th>
<th>Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>0037</td>
<td>Go to 27</td>
</tr>
<tr>
<td>0038</td>
<td>Go to 28 I=1,N</td>
</tr>
<tr>
<td>0039</td>
<td>C3(I)=0.0</td>
</tr>
<tr>
<td>0040</td>
<td>F4(I)=0.0</td>
</tr>
<tr>
<td>0041</td>
<td>IF (I6 26,20,28</td>
</tr>
<tr>
<td>0042</td>
<td>READ(1,700) (F5(I),I=1,N)</td>
</tr>
<tr>
<td>0043</td>
<td>Go to 31</td>
</tr>
<tr>
<td>0044</td>
<td>Go to 30 I=1,N</td>
</tr>
<tr>
<td>0045</td>
<td>F5(I)=0.0</td>
</tr>
<tr>
<td>0046</td>
<td>WRITE(2,800)</td>
</tr>
<tr>
<td>0047</td>
<td>WRITE(3,800) N,TIME,DT,TAU</td>
</tr>
<tr>
<td>0048</td>
<td>WRITE(3,1000)</td>
</tr>
<tr>
<td>0049</td>
<td>Go to 22 I=1,N</td>
</tr>
<tr>
<td>0050</td>
<td>WRITE(2,1100) (M(I,J),J=1,N)</td>
</tr>
<tr>
<td>0051</td>
<td>WRITE(3,1200)</td>
</tr>
<tr>
<td>0052</td>
<td>Go to 33 I=1,N</td>
</tr>
<tr>
<td>0053</td>
<td>WRITE(3,1300) (K(I,J),J=1,N)</td>
</tr>
<tr>
<td>0054</td>
<td>WRITE(3,1400)</td>
</tr>
<tr>
<td>0055</td>
<td>Go to 34 I=1,N</td>
</tr>
<tr>
<td>0056</td>
<td>WRITE(3,1500) (C(I,J),J=1,N)</td>
</tr>
<tr>
<td>0057</td>
<td>WRITE(2,5000)</td>
</tr>
<tr>
<td>0058</td>
<td>WRITE(3,5100) (1,L(I),I=1,N)</td>
</tr>
<tr>
<td>0059</td>
<td>WRITE(3,1400)</td>
</tr>
<tr>
<td>0060</td>
<td>WRITE(3,1500) (1,X(I),I=1,N)</td>
</tr>
<tr>
<td>0061</td>
<td>WRITE(3,1600)</td>
</tr>
<tr>
<td>0062</td>
<td>WRITE(3,1700) (1,DX(I),I=1,N)</td>
</tr>
<tr>
<td>0063</td>
<td>WRITE(3,2000)</td>
</tr>
<tr>
<td>0064</td>
<td>WRITE(3,2100) (I,F1(I),F2(I),F3(2),F4(I),F5(I),F4(I),I=1,N)</td>
</tr>
<tr>
<td>0065</td>
<td>IF 2=2*NSO</td>
</tr>
<tr>
<td>0066</td>
<td>NSO=NSO+1</td>
</tr>
<tr>
<td>0067</td>
<td>MP=MP+1</td>
</tr>
<tr>
<td>0068</td>
<td>CALL ARRAY(2,N,N,20,20,M,N)</td>
</tr>
<tr>
<td>0069</td>
<td>CALL ARRAY(2,N,N,20,20,K,K)</td>
</tr>
<tr>
<td>0070</td>
<td>CALL CTF(SC3,M,SC5,N,N,0,0,N)</td>
</tr>
<tr>
<td>0071</td>
<td>CALL CTF(SC5,M,SC6,N,N,0,0,N)</td>
</tr>
<tr>
<td>0072</td>
<td>CALL CTF(SC6,M,SC7,N,N,0,0,N)</td>
</tr>
<tr>
<td>0073</td>
<td>CALL CTF(SC7,M,SC8,N,N,0,0,N)</td>
</tr>
<tr>
<td>0074</td>
<td>CALL CTF(SC8,M,SC9,N,N,0,0,N)</td>
</tr>
<tr>
<td>0075</td>
<td>CALL MRK(K,N,0,0,0,0,0,0)</td>
</tr>
<tr>
<td>0076</td>
<td>IF (DET) 25,35,36</td>
</tr>
<tr>
<td>0077</td>
<td>WRITE(3,6000)</td>
</tr>
<tr>
<td>0078</td>
<td>Go to 35</td>
</tr>
<tr>
<td>0079</td>
<td>CALL GMPND(K,M,SC3,N,N,N)</td>
</tr>
<tr>
<td>0080</td>
<td>CALL GMPND(K,C,SC4,N,N,N)</td>
</tr>
<tr>
<td>0081</td>
<td>Go to 37 I=1,NSO</td>
</tr>
<tr>
<td>0082</td>
<td>M(I,J)=SC3(I,J)</td>
</tr>
<tr>
<td>0083</td>
<td>C(I,J)=SC4(J,I)</td>
</tr>
<tr>
<td>0084</td>
<td>SC2(I,J)=0.0</td>
</tr>
</tbody>
</table>
37 SC4(I,1)=0,0
38 DO 39 I=1,NS0,N1
39 SC4(I,1)=1,
CALL CSTF(SC3,SC4,SC5,N,N,0,0,N)
CALL CSTF(SC5,C,SC6,SC7,0,0,0,N)
CALL CSTF(SC6,SC7,0,0,0,0,N)
CALL ARRAY(1,N2,N2,0,40,N,N)
CALL ARRAY(1,N2,N2,0,40,N,1)
CALL CFIVTS(N2,0,ALAMDA,0)
m2=m2
CALL CFIVTS(N2,0,ALAMDA,0)
DO 43 I=1,N
ALSO=(0,0,0,0)
CSTF(I)=(0,0,0,0)
TOP(I)=(0,0,0,0)
70(I)=(0,0,0,0)
VMAX(I)=0,0
VMAX(I)=0,0
P(I)=1./ALAMDA(I)
DO 40 J=1,N2
INV(I,J)=(0,0,0,0)
DO 30 I=1,N2
QINV(I,J)=Q(I,J)*A(I,J)
ALSO=ALSO+QINV(I,J)*Q(J,I)
ALSO=CALSO(ALSO)
DO 41 J=1,N2
GINV(I,J)=QINV(I,J)/ALSO
G(J,I)=G(J,I)/ALSO
QINV(I,J)=Z0(I,J)*QINV(I,J)*G(J)
PSO=P(I)*P(I)
DO 42 J=1,N
CSTF(I)=CSTF(I)+G(N+J,I)*FO(I)
TOP(I)=TOP(I)+G(N+J,I)*F2(I)
QINV(I,J)=Q(N+J,I)/PSO+F4(J)*F4(J)
CSTF(I)=CSTF(I)/P(I)
TOP(I)=TOP(I)/PSO
WRITE(3,2200)
WRITE(3,2200) (I,P(I),I=1,N)
WRITE(3,2200)
DO 44 I=1,N2
WRITE(3,2200) (Q(I,J),J=1,N)
WRITE(3,2200) Y=C
T=C
DO 45 J=1,N
FPT=CFEXP(P(I)*T)
SUM3+(0,0,0,0)
45 GO TO 47
0131 \( \text{SUM}^*\text{F}3 = (0,0,0,0) \)
0134 DO 46 J = 1, N
0135 \( \text{SUM}^*\text{F}4 = \text{SUM}^*\text{F}3 + \text{CM}H(I,J) * \text{F}3(J) \times (\text{EPT}^*\text{F}4(J) - P(I) * \text{SIN}(\text{F}4(J) * T) - \text{F}4(J) * \text{COS}(\text{F}4(J) * T)) \)
0136 46 \( \text{SUM}^*\text{F}4 = \text{SUM}^*\text{F}4 + \text{CM}H(I,J) * \text{F}5(J) * (\text{EPT}^*P(I) - P(I) * \text{COS}(\text{F}4(J) * T) + \text{F}4(J) * \text{SIN}(\text{F}4(J) * T)) \)
0137 47 \( T(I) = (\text{STF}(I) * (\text{EPT} - 1.) + \text{TDF}(I) * (\text{EPT} - P(I) * T - 1.) + \text{SUM}^*\text{F}3 + \text{SUM}^*\text{F}5 + ZO(I) * \text{EPT} \)
0138 DO 40 I = 1, N
0139 \( \text{V}(I) = (0,0,0,0) \)
0140 DO 40 J = 1, N
0141 40 \( \text{V}(I) = \text{V}(I) + \text{D}(I+N, J) * \text{Z} \)
0142 DO 40 I = 2, N
0143 \( \text{V}(I) = \text{K} * \text{V}(I) * \text{X}(I) \)
0144 DO 40 I = 2, N
0145 \( \text{V}(I) = \text{K} * \text{V}(I) * (\text{X}(I) - X(I-1)) \)
0146 DO 40 I = 1, N
0147 \( \text{M}^*\text{M}(I) = \text{V}(I) * L(I) / 2. \)
0148 DO 94 I = 1, N
0149 \( \text{IF}(\text{ABS}(\text{M}^*\text{M}(I)) - \text{ABS}(\text{V}(I))) \) 01, 02, 03
0150 \( \text{V}^*\text{MAX}(I) = \text{V}(I) \)
0151 \( \text{IF}(\text{ABS}(\text{M}^*\text{M}(I)) - \text{ABS}(\text{V}^*\text{MAX}(I))) \) 03, 04, 05
0152 \( \text{V}^*\text{MAX}(I) = \text{V}^*\text{MAX}(I) \)
0153 \( \text{WRITE}(3,2300) T1 \)
0154 \( \text{WRITE}(3,3300) (I, X(I), I, V(I), I, \text{M}^*\text{M}(I), I = 1, N) \)
0155 \( \text{IF}(\text{T1} \neq T1 - .0005) \) 57, 57, 57
0156 52 \( \text{IF}(\text{TAU}) \) 53, 54, 55
0157 53 \( \text{IF}(\text{ABS}(\text{TAU} - T1) - .0005) \) 54, 55, 56
0158 54 \( \text{T} = \text{T} \)
0159 DO 56 I = 1, N
0160 \( \text{CM}H(I) = 7(I) \)
0161 \( \text{STF}(I) = 7(I) \)
0162 \( \text{TDF}(I) = (0,0,0,0) \)
0163 \( \text{T} = 0,0 \)
0164 \( \text{T} = \text{T} + \text{DT} \)
0165 \( \text{GO TO} 45 \)
0166 \( \text{GO TO} 46 \)
0167 \( \text{GO TO} 45 \)
0168 \( \text{IF}(\text{T} \leq T1) \) 58
0169 \( \text{STOP} \)
0170 \( \text{STOP} \)
0171 \( \text{STOP} \)
0172 100 \( \text{FORMAT}(5,F10.3,F5.3,6F5.0,F10.3) \)
0173 200 \( \text{FORMAT}(R10.4) \)
0174 300 \( \text{FORMAT}(F10.4) \)
0175 400 \( \text{FORMAT}(R10.4) \)
0176 500 \( \text{FORMAT}(F10.4) \)
0177 600 \( \text{FORMAT}(R10.4) \)
0178 700 \( \text{FORMAT}(R10.4) \)
0179 800 \( \text{FORMAT}(R10.4) \)
0170      000 FORMAT(10,'10X;ORDER OF MATRIX='I2//11X;TIME SPAN='1,FR.3,' SECS'
1//11X;TIME INCREMENT='1,FR.4,' SECS'/'11X;TAU='1,FR.4,' SECS')
0180      1000 FORMAT(0//10X;MASS MATRIX (LB. SEC.2//IN.11))
0190      1100 FORMAT(0//10X;STIFFNESS MATRIX (LB./IN.11))
0200      1200 FORMAT(0//10X;DAMPING MATRIX (LB. SEC./IN.11))
0210      1300 FORMAT(0//10X;INITIAL DISPLACEMENTS (IN.11))
0220      1400 FORMAT(0//10X;INITIAL VELOCITIES (IN./SEC.11))
0230      1500 FORMAT(0//10X;FORCING FUNCTIONS (LB.,11))
0240      1600 FORMAT(0//10X;FORCED VELOCITIES (FR.,11))
0250      1700 FORMAT(0//10X;FORCED DISPLACEMENTS (FR.11))
0260      1800 FORMAT(0//10X;FORCED STIFFNESS (FR./IN.11))
0270      1900 FORMAT(0//10X;CONVERT TO SINGULAR MATRICES)
0280      2000 FORMAT(0//10X;SINGULAR MATRICES)

END

SUBROUTINE CFTWLS(N2,N0,ALAMDA)
DIMENSION A(40,40), GAMMA(3), RHO(3), FN(40), D(40,40)
REAL G,E

C DOUBLE OR
do 5 i=1,N2
do 6 j=1,N2
g(i,j)=d(i,j)
5   eps(1)=10,E-6
6   t=0

c GAUSSIAN ELIMINATION - UPPER HESENBERG
m=2-2
do 345 kp=1,ml
cw=abs(a(kp+1,kp))
kop=kp+1
hp=kp+2
do 315 i=m0,n2
t=(cw(a(i,kp))-ck) 315,310,310
0017 310  CK=ARS(A(I,KR))
0018  KOW=1
0019  315  CONTINUE
0020  DO 320  J=KP,N2
0021  SV=A(KP+1,J)
0022  A(KP+1,J)=A(KOW,J)
0023  320  A(KOW,J)=SV
0024  DO 325  I=MP,N2
0025  325  FN(I)=A(I,KP)/A(KP+1,KR)
0026  DO 330  J=KP,N2
0027  US=A(KP+1,J)
0028  DO 335  I=MP,N2
0029  330  A(I,J)=A(I,J)-FN(I)*US
0030  DO 335  I=1,N2
0031  SV=A(I,KP+1)
0032  A(I,KP+1)=A(I,KOW)
0033  335  A(I,KOW)=SV
0034  DO 340  I=MP,N2
0035  DO 340  I=1,N2
0037  345  CONTINUE
C  CHOOSE ORIGIN SHIFTS
0038  SW1=0.0
0039  SW2=0.0
0040  SWF=0.0
0041  IF (A(N2-1,N2-1)-EPSLN) 25,15,15
0042  IF (A(N2,N2-1)-EPSLN) 20,75,75
0043  20  WRITE(3,400) A(N2,N2)
0044  NO=NO-1
0045  GO TO 40
0046  25  IF (A(N2,N2-1)-EPSLN) 35,30,30
0047  30  SW2=1.0
0048  GO TO 75
0049  35  WRITE(3,400) A(N2,N2)
0050  WRITE(3,400) A(N2-1,N2-1)
0051  NO=N2-2
0052  40  IF (NO-2) 50,50,45
0053  45  GO TO 10
0054  50  IF (NO-1) 55,55,60
0055  55  WRITE(3,400) A(N2,N2)
0056  GO TO 275
0057  60  SW=1.0
0058  IF (A(N2,N2-1)-EPSLN) 70,75,75
0059  70  WRITE(3,400) A(N2,N2)
0060  WRITE(3,400) A(N2-1,N2-1)
0061  GO TO 275
C  FIND EIGENVALUES OF LOWER SUBMATRIX
0062  75  R=-A(N2,N2)-A(N2-1,N2-1)
C = A(N2-1, N2-1) * A(N2, N2) - A(N2, N2-1) * A(N2-1, N2)

0 = C * 9 - 1 * C

G = SQRT(A(A(N1, I1) - 10, E-9) / 2.

IF (G) 90, 36, 35

P0 = PTPN(TCP2 + E**2)

SW = 1.0

GO TO 90

95 PTPN = C**2 - E**2

00 PTPN = 2.0

IF (SW) 100, 06, 105

05 IF (SW2) 125, 128, 100

100 SW = 0.0

N1 = N2 - 2

105 IF (SW) 110, 110, 120

110 RT = C + E

WRITE(3, 400) RT

WRITE(3, 400) PT2

IF (SW) 140, 40, 275

120 SW = 0.0

170 IF = I + 1

175 AMPA(I) = DCMPLX(G, E)

180 IF (SW) 40, 40, 275

128 N = N2 - 3

126 N = N2 - 2

K0 = N

IF (N2 - 3) 127, 127, 129

C CHECK SUBDIAGONAL ELEMENTS FOR DECOMPOSITION

127 K0 = 1

128 GO TO 155

129 DO 140 MP = 1, NP

140 MP = N

130 IF (ABS(A(I+1, I)) - 10, E-9) 145, 145, 130

140 IF (ABS(A(I+1, I)) * A(I+2, I+1) * (ABS(A(I+1, I+1)) + A(I+2, I+2) - PTSM) + ABS

2/(A(I+3, I+2)) / (A(I+1, I+1) * (A(I+1, I+1) - PTSM) + A(I+1, I+2) * A(I+2, I+1) +

3 PTSM) - 10, E-8) 145, 145, 140

140 KP = I

145 CONTINUE

140 MP = 1, KP

140 MP = 1, KP

134 IF (ABS(A(I+1, I)) - 10, E-9) 155, 155, 150

150 KP = I
0101 CONTINUE
0102    N1 = N2 - 1
0103    C  I = F + 1, J = F, N2
0105    IF(1-I-KP) 160, 165, 160
0111    160  GAMMA(1) = A(KP, KP) * (A(KP, KP) - RTSM) + A(KP+1, KP) * A(KP+1, KP) + RTPN
0110    GAMMA(2) = A(KP+1, KP) * (A(KP+1, KP) - RTSM)
0110    GAMMA(3) = A(KP+1, KP) * A(KP+1, KP) + A(KP+2, KP+1)
0110    A(KP2+2, KP) = 0.0
0116    GO TO 180
0117    165  GAMMA(1) = A(I, I-1)
0118    GAMMA(2) = A(I+1, I-1)
0119    IF(I-(N2-2)) 170, 170, 175
0110    170  GAMMA(3) = A(I+2, I-1)
0110    GO TO 180
0110    175  GAMMA(3) = 0.0
0111    180  F = SORT(GAMMA(1)**2 + GAMMA(2)**2 + GAMMA(3)**2)
0112    IF(GAMMA(1)) 105, 105, 190
0113    190  F = -F
0114    105  IF(F) 200, 105, 200
0115    106  PHO(1) = 0.0
0116    PHO(2) = 0.0
0117    ALPHA = 2.
0118    GO TO 205
0119    200  PHO(1) = GAMMA(2) / (GAMMA(1) + F)
0121    PHO(2) = GAMMA(3) / (GAMMA(1) + F)
0110    ALPHA = 2. / (A. + PHO(1)**2 + PHO(2)**2)
0110    205  IF(I-KO) 210, 225, 210
0110    210  IF(I-KO) 215, 220, 215
0110    215  A(I, I-1) = -F
0116    GO TO 225
0111    220  A(I, I-1) = A(I, I-1)
0112    225  DO 240 J = I, N2
0113    226    ZM = ALPHA * (A(J, J) + PHO(1) * A(J+1, J))
0114    227       IF(I-(N2-2)) 230, 230, 235
0115    230    ZM = ZM + A(J) * (PHO(2) * (J+2, J))
0116    231    A(I, J) = A(I+2, J)-PHO(2) * ZM
0117    232    LO = I+2
0118    235    A(I, J) = A(I, J) - ZM
0119    240    A(I, J) = A(I, J) - PHO(1) * ZM
0112    243  DO 255 J = KO, LO
0113    244        ZM = ALPHA * (A(J, J) + PHO(1) * A(J, J+1))
0114    245        IF(I-(N2-2)) 245, 245, 250
0115    245    ZM = ZM + A(J) * (PHO(2) * A(J+1, I+2))
0116    246    A(J, I) = A(J, I) - PHO(2) * ZN
0117    250    A(J, I) = A(J, I) - ZN
0118    255    A(J, I+1) = A(J, I+1) - PHO(1) * ZN
0119    255    IF(I-(N2-3)) 260, 260, 265
0110    260    ZM = ALPHA * PHO(2) * A(I+3, I+2)
A(I+3,I) = -7N
A(I+3,I+1) = -RHO(1) * 7N
A(I+3,I+2) = A(I+3,I+2) - RHO(2) * 7N

26 CONTINUE
GO TO 10

275 RETURN

400 FORMAT (5X,F18.3)
END

SUBROUTINE CEIVTS(N,A1,AL,X)
DIMENSION A1(40,40)
COMPLEX A1(160), X(40,20), R(40), AL(20), ATMP, TMP, SUM, C, D
N2 = N
N = N2 - 1
DO 72 I = 1, N2
X(N2+I) = (1.D0,1.D0)
DO 72 I = 1, N
R(I) = (-1.D0,1.D0) * A1(N2+I,N2) + AL(N1) * A1(N2+I,1)
DO 72 I = 1, N
1 A(I,J) = A1(N2+I,J) + AL(N1) * A1(N2+I,N2+J)
2 A(1,1) = A(1,1) - AL(N1) * AL(N1)
M = 0
M = N1
IF(N1) 3,3,6
3 X(1,11) = 6(1)/A(1,1)
GO TO 77

6 DO 50 J = 1, N1
1MAY = J
C = A(J,J)
M = M + 1
M = M1
DO 5 J = M1, N
IF(CARDS(C) - CARDS(A(J,M))) 4,5,5
4 C = A(J,M)
IMAX = J
CONTINUE
IF(CARDS(A(IMAX,M)) - 5.5E-4) 30,10,10
10 IF(1MAY - J) 20,40,20
20 DO 7C I = M, N
ATMP = A(M,I)
A(M,I) = A(IMAX,I)
70 A(IMAX,I) = ATMP
TMP = R(M)
20 DO 7C I = M, N
R(I) = R(I) - A(I,M) * R(M)/A(M,M)
70 DO 50 I = M, N
50 D = D + 1
50 CONTINUE
50 A(I,L)=A(I,L)-A(I,I)*A(M,L)/A(M,M)
1F(CAPS(A(N,N))=5.E-4) 30,45,45
30 WRITE(3,400)
GO TO 73
45 X(M,1)=A(1,1)/A(M,N)
| =N+1
DO 71 J=1,N
SUM=(0.000,0.000)
DO 75 K=1,J
71 SUM=SUM+A(N-I,J)*X(J,11)
75 SUM=SUM+A(N-I,1)*D
71 X(N-I,11)=(A(N-I)-SUM)/A(N-I,N-J)
77 DD INDEX I=1,ND2
70 X(ND2+J,11)=A(L,11)*X(I,11)
72 CONTINUE
73 RETURN
460 FORMAT(10' ', NO SOLUTION)
END
UNDAAMPED
ORDER OF MATRIX = 2
TIME SPAN = 2.500 SECS
TIME INCREMENT = 0.050 SECS
TAU = 1.250 SECS

MASS MATRIX (LB. SEC^2./IN.)
\[
\begin{pmatrix}
932.000000 & 0.0 \\
0.0 & 416.000000
\end{pmatrix}
\]

STIFFNESS MATRIX (LB./IN.)
\[
\begin{pmatrix}
400000.000000 & -234000.000000 \\
-234000.000000 & 234000.000000
\end{pmatrix}
\]

DAMPING MATRIX (LB. SEC./IN.)
\[
\begin{pmatrix}
0.0 & 0.0 \\
0.0 & 0.0
\end{pmatrix}
\]
STORY HEIGHT (IN.)
L(1) = 144.00
L(2) = 144.00

INITIAL DISPLACEMENTS (IN.)
X0(1) = 0.0
X0(2) = 0.0

INITIAL VELOCITIES (IN./SEC.)
DX0(1) = 0.0
DX0(2) = 0.0

FORCING FUNCTIONS (LB.)
F(1) = 52000.00 + (-41600.00)T + 0.0 \sin(0.0 T) + 0.0 \cos(0.0 T)
F(2) = 26000.00 + (-20800.00)T + 0.0 \sin(0.0 T) + 0.0 \cos(0.0 T)

CIRCULAR FREQUENCIES
P(1) = 0.0 + i -0.3035911E-02
P(2) = 0.0 + i -0.11036008E-02
### O Matrix

\[
\begin{bmatrix}
-0.0639399 + i 0.0639399 & 0.0427495 + i -0.0427495 \\
0.1002678 + i -0.1002678 & 0.0545636 + i -0.0545636 \\
-0.002070 + i -0.002070 & 0.0038736 + i 0.0038736 \\
0.0033031 + i 0.0033031 & 0.0049441 + i 0.0049441
\end{bmatrix}
\]

### Displacements (IN.)

<table>
<thead>
<tr>
<th>TIME = 0.0 SECS</th>
<th>X(1)</th>
<th>V(1)</th>
<th>M(1)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
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<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
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</tbody>
</table>

### Shears (LR.)

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<th>TIME = 0.050 SECS</th>
<th>X(1)</th>
<th>V(1)</th>
<th>M(1)</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>0.073987</td>
<td>12281.792969</td>
<td>884289.000000</td>
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<tr>
<td></td>
<td>0.076939</td>
<td>600.791016</td>
<td>49736.937500</td>
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### Moments (IN.-LR.)

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<tr>
<th>TIME = 0.100 SECS</th>
<th>X(1)</th>
<th>V(1)</th>
<th>M(1)</th>
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</thead>
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<tr>
<td></td>
<td>0.260190</td>
<td>43191.539062</td>
<td>3109790.000000</td>
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<tr>
<td></td>
<td>0.295160</td>
<td>8416.859375</td>
<td>606013.500000</td>
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</table>

### TIME = 0.150 SECS

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<th>X(1)</th>
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<th>M(1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.484979</td>
<td>80838.375000</td>
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<tr>
<td>0.602583</td>
<td>27051.511710</td>
<td>1947708.000000</td>
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### TIME = 0.200 SECS

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<th>X(1)</th>
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<th>M(1)</th>
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<td>0.600562</td>
<td>114633.312500</td>
<td>9253598.000000</td>
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<tr>
<td>0.884087</td>
<td>45784.753996</td>
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</tr>
<tr>
<td>TIME</td>
<td>X(1)</td>
<td>X(2)</td>
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<tr>
<td>---------</td>
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<tr>
<td>0.250SEC</td>
<td>0.816674</td>
<td>1.031804</td>
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<tr>
<td>0.300SEC</td>
<td>0.814440</td>
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<tr>
<td>0.350SEC</td>
<td>0.646364</td>
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<tr>
<td>0.400SEC</td>
<td>0.417356</td>
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<tr>
<td>0.450SEC</td>
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<td>0.186367</td>
</tr>
<tr>
<td>0.500SEC</td>
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<td>-0.116638</td>
</tr>
<tr>
<td>0.550SEC</td>
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<td>-0.267091</td>
</tr>
<tr>
<td>TIME</td>
<td>X(1)</td>
<td>V(1)</td>
</tr>
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<tr>
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<tr>
<td>0.750</td>
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<td>0.800</td>
<td>0.666924</td>
<td>94111.000000</td>
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<tr>
<td>0.850</td>
<td>0.609620</td>
<td>101396.937500</td>
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<tr>
<td>0.900</td>
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<tr>
<td>TIME = 0.050 SECS</td>
<td>X(1) = 0.215687</td>
<td>V(1) = 52403.972656</td>
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<tr>
<td>X(2) = 0.382850</td>
<td>V(2) = 15716.332031</td>
<td>M(2) = 1131575.000000</td>
</tr>
</tbody>
</table>

| TIME = 1.000 SECS | X(1) = 0.038324 | V(1) = 6361.792969 | M(1) = 458049.062500 |
| X(2) = 0.054760 | V(2) = 3845.003652 | M(2) = 276912.750000 |

| TIME = 1.050 SECS | X(1) = -0.217127 | V(1) = -36043.035156 | M(1) = -2505098.000000 |
| X(2) = -0.267534 | V(2) = -10859.741719 | M(2) = -781866.500000 |

| TIME = 1.100 SECS | X(1) = -0.376490 | V(1) = -62497.335037 | M(1) = -4499808.000000 |
| X(2) = -0.400619 | V(2) = -26706.226562 | M(2) = -1922848.000000 |

| TIME = 1.150 SECS | X(1) = -0.411765 | V(1) = -68352.875000 | M(1) = -4921407.000000 |
| X(2) = -0.544484 | V(2) = -31055.476562 | M(2) = -2236056.000000 |

| TIME = 1.200 SECS | X(1) = -0.323447 | V(1) = -59692.219750 | M(1) = -3865839.000000 |
| X(2) = -0.407506 | V(2) = -19600.894531 | M(2) = -1417744.000000 |

<p>| TIME = 1.250 SECS | X(1) = -0.130179 | V(1) = -21609.660156 | M(1) = -1555895.000000 |
| X(2) = -0.149093 | V(2) = -4191.972556 | M(2) = -301872.000000 |</p>
<table>
<thead>
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<th>TIME</th>
<th>X(1)</th>
<th>V(1)</th>
<th>M(1)</th>
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<tr>
<td>TIME</td>
<td>X(1)</td>
<td>V(1)</td>
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**VMAX** (2) = 50340.382812
17 CRITICAL DAMPING
ORDER OF MATRIX= 2

TIME SPAN= 2.600 SECS

TIME INCREMENT= 0.050 SECS

TAU= 1.250 SECS

MASS MATRIX (LB. SEC^2./IN.)

\[
\begin{bmatrix}
322.000000 & 0.0 \\
0.0 & 416.000000
\end{bmatrix}
\]

STIFFNESS MATRIX (LB./IN.)

\[
\begin{bmatrix}
400000.000000 & -234000.000000 \\
-234000.000000 & 234000.000000
\end{bmatrix}
\]

DAMPING MATRIX (LB. SEC./IN.)

\[
\begin{bmatrix}
432.367920 & -197.325099 \\
-197.325099 & 197.325099
\end{bmatrix}
\]

STORY HEIGHT (IN.)

L(1) = 144.00
L(2) = 144.00
INITIAL DISPLACEMENTS (IN.)
\[ \begin{align*}
 y_0(1) &= 0.0 \\
 x_0(2) &= 0.0 
\end{align*} \]

INITIAL VELOCITIES (IN./SEC.)
\[ \begin{align*}
 \dot{y}_0(1) &= 0.0 \\
 \dot{x}_0(2) &= 0.0 
\end{align*} \]

FORCING FUNCTIONS (LR.)
\[ \begin{align*}
 f(1) &= 52000.00 + (-41600.00)T + 0.0 \sin(0.0 T) + 0.0 \cos(0.0 T) \\
 f(2) &= 26000.00 + (-20800.00)T + 0.0 \sin(0.0 T) + 0.0 \cos(0.0 T) 
\end{align*} \]

CIRCULAR FREQUENCIES
\[ \begin{align*}
 p(1) &= -0.41415751E00 + 1.0 -0.30353043E02 \\
 p(2) &= -0.2837592E01 + 1.0 -0.11035764E02 
\end{align*} \]

Q MATRIX
\[ \begin{align*}
 0.0629594 + 1 -0.0650115 & -0.0424581 + 1 0.0430404 \\
 -0.0629594 + 1 0.1014364 & -0.0541053 + 1 0.0550192 \\
 0.0021131 + 1 0.0021031 & -0.0038710 + 1 -0.0038765 \\
 -0.0021257 + 1 -0.0033006 & -0.0049485 + 1 -0.0049399 
\end{align*} \]
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