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Infinite beams on an elastic foundation

Shi-peing Chang

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INFINITE BEAMS ON AN ELASTIC FOUNDATION

BY

SHI-PEING CHANG, 1937

A

THESIS

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ABSTRACT

In this paper a partial differential equation for the analysis of beams resting on an elastic foundation under dynamic load is presented. A solution for this equation is derived, and solutions for various boundary conditions of infinite beams resting on an elastic foundation are discussed. The application of the equation is demonstrated by an example of a moving load traveling along an infinite beam.
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NOTATIONS

\( \alpha \)
- parameter of integration

\( \Delta \)
- deformation per unit length

\( \rho \)
- radius of curvature

\( \xi \)
- reduced coordinate in x-direction of beam defined as \( \xi = \frac{x - \nu t}{l} \)

\( \eta \)
- transformation of t defined as \( \eta = t \)

\( \Delta \)
- stress

\( \varepsilon \)
- strain

\( b \)
- width of beam

\( D \)
- denominator of the solution defined as \( D = \alpha^2 - 2n^2\alpha^2 + 1 \)

\( E \)
- modulus of elasticity

\( g \)
- gravitational force acceleration

\( I \)
- moment of inertia

\( (I) \)
- integral

\( i \)
- imaginary unit

\( k \)
- constant of foundation

\( \lambda \)
- characteristic length defined as \( \lambda = \frac{EI}{k} \)

\( M \)
- bending moment per unit length of beam

\( n \)
- defined by \( n^2 = \frac{\omega^2}{g} \frac{\nu^2}{2k\lambda^2} < 1 \)

\( p, q \)
- applied load

\( P, Q \)
- Fourier coefficients

\( t \)
- time

\( v \)
- velocity

\( w \)
- unit weight of beam

\( x \)
- horizontal axis of beam
vertical axis of beam or deflection
I. INTRODUCTION

The problem of beams resting on an elastic foundation is often found in the design of structural members of buildings, railroads, airports, highways, and other structures.

Due to the advancement of engineering, many think today that the solution of this problem by the classic equation, which was derived nearly a century ago (18), (11) and is still being used, is not satisfactory because it does not consider the existing dynamic action.

In recent decades the development of mathematics and the theory of elasticity have made unprecedented strides that make possible the solution of some difficult dynamic problems.

In this paper an equation of motion for beams resting on elastic foundations and influenced by dynamic loads is obtained. The derivation of this equation is modeled after the more exact plate theory of Reissner (11). It is assumed throughout this paper that the depth of the beam may be neglected, the deflection is very small compared to the dimensions of the beam, and the beam is loaded within the elastic limit. Other assumptions are that the foundation has no mass and the properties of the foundation obey Winkler's assumption (14), that at every point the reaction of the foundation is proportional to the deflection at the corresponding point. These assumptions have been found
(6) to give usable results. With these assumptions, a general partial differential equation of motion will be derived.
II. REVIEW OF LITERATURE

The problem of the elastic beam on an elastic foundation has been popular in scientific and engineering literature ever since Zimmerman (18) presented his solution for the analysis of the railroad track in 1888. This solution was based on Winkler's (14) assumption that the deflection at any point is proportional to the foundation pressure at that point, and does not depend on the pressure at any other point of the foundation.

S. P. Timoshenko (12) was the first to use the solution in this country when he found the strength of rails. Westergaard (16) used the solution to explain cracking in concrete pavements.

Hetenyi (6) has done a great job in his famous book, Beams on Elastic Foundation, on the theoretical development of equations for various boundary conditions.

To explain the fact that the rate of change of the deflection is a function of time for a beam resting on a viscoelastic medium, which is defined as a material whose force-deflection relations are functions of time, Freudenthal and Lorsch (5) put a velocity term \( \frac{\partial y}{\partial t} \) into the classical equation

\[
EI \frac{\partial^4 y}{\partial x^4} + ky = p
\]

so that the equation becomes

\[
EI \frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial x^2} + ky = p.
\]

To deal with the particular case of transverse vi-
Vibration of a beam which lies on an elastic foundation and is subjected to a periodic longitudinal load, \( p_0 + p_t \cos \varphi \), Bolotin (1) has derived a partial differential equation

\[
EI \frac{\partial^4 y}{\partial x^4} + (p_0 + p_t \cos \varphi t) \frac{\partial^2 y}{\partial x^2} + ky + m \frac{\partial^2 y}{\partial t^2} = 0.
\]

where \( y(x, t) = f_k(t) \sin \frac{k\pi x}{L} \), \( k = 1, 2, 3, \ldots \)

and \( f_k(t) \) are as yet undetermined functions of time.

The equations presented by either Freudenthal and Lorsch or Bolotin are satisfactory for individual boundary conditions, but are not satisfactory to solve the general problem of a moving load traveling along a beam which is resting on an elastic foundation. The reason is because a moving load applied to a beam will subject the beam to a vibration force and a static force.

In order to solve this problem, this paper presents a general partial differential equation and its solution, which are found by means of a Fourier integral.
In deriving the equations based on the dynamics of beams, the assumptions of ordinary beam theory are used:

1. The material is isotropic, that is, any part of the material that is large enough to contain a considerable number of grains will display the same properties of overall stress and strain regardless of the directions in which the part has been cut or is loaded (9).

2. Stresses are below the proportional limit which means that the ratio of stress to strain is a constant.

3. Deflections are very small compared to the dimensions of the beam.

4. A plane cross-section before bending remains plane after bending.

5. The beam is initially straight.

6. Shearing deflections are neglected.

7. The neutral axis of a beam in bending is the locus of the centroids of the cross-sectional area.

Accepting the above-mentioned assumptions (17), the deformation of the neutral axis, \( y \), can be represented by Figure 1.

\[
\frac{Cd\theta}{2} = \Delta
\]

\[
E = \frac{\varepsilon}{\varepsilon}
\]

\[
\varepsilon = \frac{2\Delta}{dx}
\]
Fig. 1 Deflection of Beam on Rigid Supports.
\[ \Delta = \frac{\Delta d\alpha}{2E} \]

\[ \frac{Cd\theta}{2} = \frac{\Delta dx}{2E} \]

\[ \Delta = \frac{MC}{I} \]

\[ \frac{Cd\theta}{2} = \frac{MCdx}{2EI} \]

then

\[ d\theta = \frac{Mdx}{EI} \]

Furthermore, since

\[ \rho \cdot d\theta = dx \]

or

\[ d\theta = \frac{dx}{\rho} \]

\[ \frac{1}{\rho} = \frac{M}{EI} \] (1)

The coordinate geometry of the neutral axis of this beam is represented by Figure 2, from which

\[ \rho \cdot d\theta = ds \]

For small slopes \( ds \) is approximately equal to \( dx \), hence

\[ \rho \cdot d = dx \] (2)

From Equation 1 substituted into Equation 2

\[ d\theta = \frac{M}{EI} \cdot dx \] (3)

As \( dx \) approaches zero

\[ \tan \theta = \frac{dy}{dx} \]

or in radians,

\[ \theta = \frac{dy}{dx} \] (4)

Differentiating both sides with respect to \( x \), the equation will be
Fig. 2 The Coordinate Geometry of the Elastic Curve
\[
\frac{d^2 y}{dx^2} = \frac{M}{EI} \quad (5)
\]

In order to discuss the behavior of dynamic forces applied to the infinite beam resting on an elastic foundation, one must assume that \( k \), the modulus of subgrade reaction is a constant, which may be estimated from Table 1, the foundation has no mass, and the infinite beam is supported along its entire length by the foundation and subjected to a vertical force, \( p \), acting in the principal plane of the symmetrical cross section (Figure 3).

If \( y \) is the deflection at a point, by Winkler's assumption, it can be found that

\[
p = -bk
\]

where \( b \) is width of the beam. Assuming a unit width beam, i.e. that \( b \) equals one, the equation becomes,

\[
k = -\frac{p}{y}
\]

An infinitely small element enclosed between two vertical cross-sections a distance \( dx \) apart in the beam is under consideration and a load, \( p \) lbs/in., is applied as in Figure 4. The sign convention is that the upward acting forces are positive. Therefore,

\[
\Sigma F = 0
\]

\[
V - (V + dV) + kydx - pdx = 0
\]

Whence

\[
-dV + kydx - pdx = 0
\]

that is

\[
\frac{dV}{dx} = ky - p
\]
Fig. 3 Deflection of a Beam Resting on an Elastic Foundation.
By the relation
\[ v = \frac{dM}{dx} \]
then
\[ \frac{d^2M}{dx^2} = ky - p. \] (6)

Substituting Equation 5 into Equation 6,
\[ EI \frac{d^2}{dx^2} \frac{d^2y}{dx^2} = - \frac{d^3M}{dx^3} \]
or
\[ EI \frac{d^4y}{dx^4} = - \frac{d^3M}{dx^3} \]
that is
\[ EI \frac{d^4y}{dx^4} = - ky + p. \] (7)

This is the equation for the deflection curve of a beam supported on an elastic foundation and subjected to static loads. If a beam is loaded by vibration only, then
\[ p = - \frac{w}{g} \frac{\partial^2 y}{\partial t^2} \]
\[ \frac{w}{g} \]

is the mass of a unit length of the beam, so that Eq. 7 becomes
\[ EI \frac{\partial^4y}{\partial x^4} + \frac{w}{g} \frac{\partial^2 y}{\partial t^2} + ky = 0. \] (8)

Now, a moving load, \( p \), traveling along a beam is to be considered. The design of a beam which is resting on an elastic foundation and which is subjected to a moving load must be based on the consideration of the action of both vibration and static forces (8) so that the load term must be added into Equation 8, which becomes
\[ EI \frac{\partial^4y}{\partial x^4} + \frac{w}{g} \frac{\partial^2 y}{\partial t^2} + ky = p. \] (9)

This is the equation for beams on elastic foundation considering dynamic loads. This equation satisfies the boundary conditions of not only infinite beams resting on an
Figure 4  Behavior of Forces Applied to Beam Resting on an Elastic Foundation
elastic foundation but also finite beams resting on an elastic foundation.
IV. DERIVATION OF SOLUTION

In order to solve the general equation for beams on an elastic foundation, a load, \( p \), moving along an infinite beam resting on an elastic foundation at a velocity, \( v \), is considered. At time \( t \), the distance along the beam to the original axis is (Figure 5)

\[
\xi = \frac{x - vt}{l}, \quad \eta = t
\]

where \( l \) is characteristic length defined by \( l = \sqrt{\frac{EI}{k}} \).

By changing coordinates from \( x \) to \( \xi \) of the equation,

\[
EI \frac{\partial^4 y}{\partial \xi^4} + \frac{W}{8} \frac{\partial^2 y}{\partial \xi^2} + ky = p
\]

it may be observed:

\[
\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial \xi} \left( \frac{\partial y}{\partial x} \right) + \frac{\partial^2 x}{\partial \eta^2} \frac{\partial y}{\partial x} = \frac{1}{l^2} \frac{\partial^2 y}{\partial \xi^2} + o
\]

\[
\frac{\partial^2 y}{\partial x^4} = \frac{\partial}{\partial \xi} \left( \frac{\partial^2 y}{\partial x^2} \right) = \frac{1}{l^4} \frac{\partial^4 y}{\partial \xi^4}
\]

\[
\frac{\partial^4 y}{\partial x^4} = \frac{\partial}{\partial \xi} \left( \frac{\partial^3 y}{\partial x^3} \right) = \frac{1}{l^4} \frac{\partial^4 y}{\partial \xi^4}
\]

Assuming the stress and deflection patterns set up move with the load at a constant velocity, \( y \) is a function of \( \xi \) only. Consequently, \( \frac{\partial y}{\partial \eta} \) is zero and

\[
\frac{\partial^2 y}{\partial \xi^2} = \frac{\partial}{\partial \xi} \left( -\frac{v \partial y}{\partial \xi} \right) \frac{\partial y}{\partial \xi} = \frac{v^2 \partial^2 y}{l^2 \frac{\partial \xi^2}{\partial \xi^2}}
\]

and by substituting Equation b and Equation d into Equation 9, the differential equation will be
\[
\frac{EI}{k^4} \frac{\partial^4 y}{\partial \xi^4} + \frac{W}{8} \frac{v^2}{k^2} \frac{\partial^2 y}{\partial \xi^2} + ky = p.
\]

(10)

Dividing Equation 10 by \( k \) the equation becomes

\[
\frac{\partial^4 y}{\partial \xi^4} + \frac{W}{8} \frac{v^2}{k^2} \frac{\partial^2 y}{\partial \xi^2} + y = \frac{p}{k}.
\]

(11)

Let

\[
n^2 = \frac{W}{2k^2} \frac{v^2}{k^2}
\]

and Equation 11 becomes

\[
\frac{\partial^4 y}{\partial \xi^4} + 2n^2 \frac{\partial^2 y}{\partial \xi^2} + y = \frac{p}{k}.
\]

(13)

Both \( p \) and \( y \) are functions of \( \xi \).

If, then, \( p \) and \( y \) are absolutely integrable, \( y(\xi) \)
and \( p(\xi) \) may be represented by a Fourier transform (13)
consisting of integration with respect to a parameter \( \alpha \) so
that

\[
y(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\alpha) e^{i\alpha \xi} d\alpha
\]

(14)

and

\[
p(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} P(\alpha) e^{i\alpha \xi} d\alpha
\]

(15)

Where \( \alpha \) is independent of \( \xi \) and

\[
P(\alpha) = \int_{-\infty}^{\infty} p(\xi) e^{-i\alpha \xi} d\alpha
\]

(16)

Substituting Equation 15 and the second and fourth deri-
vatives of Equation 14 into Equation 13, then the equation
becomes

\[
(\alpha^4 - 2n^2 \alpha^2 + 1)Y(\alpha) = \frac{1}{k} p(\alpha)
\]

so that

\[
Y(\alpha) = \frac{1}{k} \frac{P(\alpha)}{\alpha^4 - 2n^2 \alpha^2 + 1}
\]

(17)

Now a load of magnitude \( q(x, t) \) uniformly di-
over a length of 2d moving with a constant velocity, \( v \),
traveling on the beam is considered (Figure 5). The applied load is
\[
p(x, t) = \frac{q}{2d}
\]
where \(-d < x - vt < d\).

Transforming the coordinate this load becomes
\[
p(\xi) = \frac{q}{2d}
\]
where \(-\frac{d}{l} < \xi < \frac{d}{l}\).

From Equation 16
\[
P(\alpha) = \int_{-\frac{d}{l}}^{\frac{d}{l}} 0d\xi + \int_{-\frac{d}{l}}^{\frac{d}{l}} \frac{q}{2d} e^{-i\alpha \xi} d\xi + \int_{-\frac{d}{l}}^{\frac{d}{l}} 0d\xi
\]
\[
= \frac{q}{2\alpha id} (e^{i\alpha d/l} - e^{-i\alpha d/l})
\]
\[
= \frac{q}{d2i\alpha} (e^{i\alpha d/l} - e^{-i\alpha d/l}).
\]

Since \(\sin \alpha d/l = \frac{e^{i\alpha d/l} - e^{-i\alpha d/l}}{2i}\)
\[
P(\alpha) = \frac{q}{d} \frac{\sin \alpha d/l}{\alpha}
\]
(18)

Substituting Equation 18 into Equation 17, one finds
\[
Y(\alpha) = \frac{q \sin \alpha d/l}{k \alpha d (\alpha^2 - 2n^2 d^2 + 1)}
\]
(19)

From Equation 14
\[
y(\xi) = \frac{q}{2\pi ka} \int_{-\infty}^{\infty} \frac{\sin \alpha d/l}{\alpha (\alpha^2 - 2n^2 d^2 + 1)} e^{i\alpha \xi} d\alpha
\]
or the real form may be used
\[
y(\xi) = \frac{q}{\pi ka} \int_{0}^{\infty} \frac{\sin \alpha d/l \cos \alpha \xi}{\alpha (\alpha^2 - 2n^2 d^2 + 1)} d\alpha
\]
(20)

Let
\[
\frac{dy}{dx} = \frac{q}{\pi ka} \int_{0}^{\infty} \frac{-\alpha \sin \alpha d/l \sin \alpha \xi}{\alpha (\alpha^2 - 2n^2 d^2 + 1)} d\alpha
\]
and
\[
\begin{align*}
\frac{\partial^2 y}{\partial x^2} &= -\frac{q}{\pi kd} \int_0^\infty \frac{\sin \alpha \frac{d\theta}{d\alpha} \sin \frac{\theta}{2}}{\alpha^4 - 2n^2\alpha^2 + 1} \, d\alpha \\
\frac{\partial^2 z}{\partial x^2} &= -\frac{q}{\pi kd} \int_0^\infty \frac{\alpha \sin \alpha \frac{d\theta}{d\alpha} \cos \frac{\theta}{2}}{\alpha^4 - 2n^2\alpha^2 + 1} \, d\alpha \\
\frac{M}{EI} &= \frac{\partial^2 y}{\partial x^2} = \frac{1}{\lambda^2} \frac{\partial^2 y}{\partial \lambda^2}
\end{align*}
\]

then
\[
\begin{align*}
M &= \frac{qEI}{\pi kd^2} \int_0^\infty \frac{\alpha \sin \alpha \frac{d\theta}{d\alpha} \cos \frac{\theta}{2}}{\alpha^4 - 2n^2\alpha^2 + 1} \, d\alpha \\
E^2 &= \frac{EI}{k}
\end{align*}
\]

so that
\[
M = \frac{qL^2}{\pi d} \int_0^\infty \frac{\alpha \sin \alpha \frac{d\theta}{d\alpha} \cos \frac{\theta}{2}}{\alpha^4 - 2n^2\alpha^2 + 1} \, d\alpha \tag{21}
\]

Let
\[
\phi(\alpha) = \frac{\sin \alpha \frac{d\theta}{d\alpha}}{\alpha}
\tag{22}
\]

then Equation 20 and Equation 21 become
\[
\begin{align*}
y(\xi) &= \frac{q}{\pi k} \int_0^\infty \frac{\phi(\alpha) \cos \frac{\theta}{2}}{\alpha^4 - 2n^2\alpha^2 + 1} \, d\alpha \\
M(\xi) &= \frac{qL^2}{\pi} \int_0^\infty \frac{\phi(\alpha) \alpha \cos \frac{\theta}{2}}{\alpha^4 - 2n^2\alpha^2 + 1} \, d\alpha \
\end{align*}
\tag{23}
\tag{24}
\]

These equations are the particular solutions of Eq. 9, and are the general solutions of Equation 10.
V. DISCUSSION

The two important elements to be discussed in this section are deflection and moment.

A. Deflection:

When the case of a concentrated load applied to the beam is considered, d in Equation 23 approaches zero. Thus in Equation 22, \( \phi(x) \) approaches \( \frac{1}{l} \) and

\[
y(x) = \frac{q}{\pi k l} \int_{0}^{l} \frac{\cos \alpha \xi}{\alpha^4 - 2n^2 \alpha^2 + 1} d\alpha
\]

is evolved.

It may be desirable to investigate

\[ D = \alpha^4 - 2n^2 \alpha^2 + 1; \]

the denominator, for it is necessary to understand every symbol thoroughly.

(1) When \( n = 1 \) then

\[ D = \alpha^4 - 2n^2 \alpha^2 + 1 = (\alpha^2 - 1)^2 \]

If \( \alpha = \pm 1 \) in this equation \( y \) will become infinite, so that it does not exist.

(2) When \( n > 1 \) then \( D \) has real roots therefore the deflection cannot be defined.

(3) The only case for which the deflection is defined everywhere occurs when the denominator does not have a real root, that is when \( n < 1 \). This means that

\[ y < \frac{2k g l^2}{w} \]

In the above discussion it can be seen that

\[ y_c = \frac{2k g l^2}{w} \]

will be the critical condition for the solution. Whenever
the velocity is larger than or equals to $v_c$ the deflection of the beam becomes infinitely large.

For an example, a rail with a cross section as shown in Figure 9, rests on ballast having a modulus $k = 230 \times 6.5$ lb/in.. The modulus of elasticity of the rail is $30 \times 10^6$ psi, $I_{N.A} = 112$ in$^4$, and weight of the rail is $3.78$ lb/in.. Find the maximum velocity of the train that the rail can endure.

For

$$\frac{w}{g} = \frac{3.78}{12 \times 32.4} = 0.00973 \text{ lb-sec}^2/\text{in.}$$

$$k = \sqrt{\frac{EI}{k}} = \sqrt{\frac{30 \times 10^6 \times 112}{230 \times 6.5}}$$

$$= 1,497 \text{ in}^2$$

$$v_c = \sqrt{\frac{2 \times 1,500 \times 1,497}{0.00973}}$$

$$= \sqrt{461,000,000}$$

$$= 21,500 \text{ in/sec.}$$

$$= 1,220 \text{ mph.}$$

It is obvious that no train will, as yet, move at this speed. But if the combined action of the velocity and vertical loading are taken into consideration the influence of velocity on the deflection and moment of beams may well be large.

The wave produced by the traveling load may be represented by the equation

$$y = y_0 \sin \frac{x}{a}$$

where $y_0$ is the deflection of the beam subjected to a static
load.

If the beam is subjected to a uniform load, and the load is stopped somewhere on the beam, for which \( \xi = 0 \), Equation 23 becomes

\[
y = \frac{q}{\pi k} \int_{\alpha}^{\infty} \frac{\phi(\alpha)}{\alpha^4 - 2n^4 \alpha^2 + 1} \, d\alpha \quad 0 < n < 1
\]  

(27)

The argument might be extended to the maximum deflection under a concentrated load. It enters the situation when both \( d \) and \( \xi \) approach zero. From equation 23 then

\[
y = \frac{q}{\pi kl} \int_{\alpha}^{\infty} \frac{d\alpha}{\alpha^4 - 2n^4 \alpha^2 + 1} \quad 0 < n < 1
\]  

(28)

This integration involves a known Tshebyscheff polynomial (7) that may be evaluated by the residue calculus which is based on the theory of analytic function, in particular, the residue theorem.

The complex function

\[
\frac{1}{z^4 - 2n^2 z^2 + 1}
\]

is integrated over the contour Figure 6 and \( R \) approaches \( \infty \). It can be shown that the integral over \( C_R \) approaches 0. By the residue theorem,

\[
\oint \frac{dz}{z^4 - 2n^2 z^2 + 1} = 2\pi i \Sigma \text{Residues in } \Delta
\]

(29)

and

\[
\left( \int_{-R} + \int_{C_R} \right) = 2\pi i \Sigma \text{Res.}
\]

Letting \( R \) approach infinity, then

\[
\int_{-\infty}^{\infty} \frac{d\alpha}{\alpha^4 - 2n^4 \alpha^2 + 1} + 0 = 2\pi i \Sigma \text{Res.}
\]

The solution of the integral is
Fig. 6 Residue Curve
and the deflection equation for concentrated load is

\[ y = \frac{\sqrt{2} \rho}{\pi k \sqrt{4l - \eta ^2}} = \frac{\sqrt{2} \rho}{4 k \sqrt{4l - \eta ^2}} \quad 0 < n < 1 \]  

(30)

When the velocity is zero,

\[ y_o = \frac{\sqrt{2} \rho}{4 k \lambda} \]  

(31)

A graph of the variation of \( y/y_o \) with \( n \) is given in Figure 7. From this figure it can be seen that the ratio \( y/y_o \) approaches infinity as \( n \) reaches a value very close to unity. This is a main point which should be discussed here. The actual deflection will never become infinite in any case, but it must be borne in mind, however, that the beam is assumed to be an undamped elastic system which neglects the resistance to the velocity. In most physical cases the effect of damping would certainly tend to reduce the deflection. Also it can be seen in the numerical example stated before that the critical velocity will never be reached. In our calculation \( v_c = 1220 \) mph. This is greater than ten times the maximum speed of a normal locomotive, consequently, in the problem where \( n \) equals 0.1, the influence on the ratio of deflection is 0.01. This is so small that it may be neglected here.

B. Bending moment:

When a uniformly distributed moving load is applied
Fig. 7 Diagram for the Variation of $\frac{y}{y_0}$ with $n$. 
to beams the bending moment under such load can be found from Equation 24. It is

$$ M_0(0) = \frac{qL^2}{\pi} \int_0^{\infty} \frac{\phi(\omega) \omega^2}{\omega^4 - 2n^2 \omega^2 + 1} \, d\omega \quad 0 < n < 1 \quad (32) $$

This integral converges to a finite value.

As a concentrated load is to be considered, then $d$ approaches zero, and Equation 32 becomes

$$ M(0) = \frac{qL}{\pi} \int_0^{\infty} \frac{\omega^2}{\omega^4 - 2n^2 \omega^2 + 1} \, d\omega \quad 0 < n < 1 \quad (33) $$

or

$$ M(0) = \frac{qL}{\pi} \int_0^{\infty} \frac{1/\omega^2}{1 - 2n^2 \omega^2 + 1/\omega^2} \, d\omega $$

From this equation it can be seen that the integral will not remain defined, so that a solution does not exist. It will be shown later that by changing the integral into another form a divergent solution will be obtained.

The static condition of Equation 32 for a concentrated load and $n = 0$ is considered and

$$ M_0 = \frac{qL}{\pi} \int_0^{\infty} \frac{\omega^2}{\omega^4 + 1} \, d\omega \quad (34) $$

To find the bending moment under the concentrated moving load, one may go back to Equation 32, reduced by Equation 34, and

$$ M_0(0) - M_0 = \frac{qL^2}{\pi} \int_0^{\infty} \frac{\phi(\omega) \omega^2}{\omega^4 - 2n^2 \omega^2 + 1} \, d\omega - \frac{qL}{\pi} \int_0^{\infty} \frac{\omega^2}{\omega^4 + 1} \, d\omega $$

$$ = \frac{qL^2}{\pi} \int_0^{\infty} \left[ \frac{\phi(\omega) \omega^2}{\omega^4 - 2n^2 \omega^2 + 1} - \frac{\omega^2}{\omega^4 + 1} \right] \, d\omega $$
\[
\frac{q l^2}{\pi} \int_0^\infty \frac{\alpha^2 (d(\alpha) (\alpha^4 + 1) - (\alpha^2 n^2 + 1))}{(\alpha^4 + 1) (\alpha^4 - 2n^2 \alpha^2 + 1)} \, d\alpha
\]

\[
= \frac{q l^2}{\pi} \int_0^\infty \frac{\alpha^2 (\sin(\alpha) \cos(\alpha) - (\alpha^2 n^2 + 1))}{(\alpha^4 + 1) (\alpha^4 - 2n^2 \alpha^2 + 1)} \, d\alpha
\]

provided \(0 < n < 1\)

Equation 35 remains convergent.

Now, let \(d\) approach 0; that is, \(\sin(\alpha)\) approaches \(\alpha\).

Since \(\varphi(\alpha) < 1/\alpha\) when the loaded area is finite, Equation 35 can be simplified to give an upper bound for the increase in \(M(0)\), this bound being approached as the loaded area decreases. Thus

\[
M(0) = M_0 < \frac{q l^2}{\pi} \int_0^\infty \frac{\alpha^2 (\alpha^4 + 1) - \alpha^2 (\alpha^2 n^2 + 1)}{(\alpha^4 + 1) (\alpha^4 - 2n^2 \alpha^2 + 1)} \, d\alpha
\]

so that

\[
M(0) - M_0 < \frac{2n^2 q l^2}{\pi} \int_0^\infty \frac{\alpha^4}{(\alpha^4 + 1) (\alpha^4 - 2n^2 \alpha^2 + 1)} \, d\alpha
\]

Dividing both sides of Equation 36 by \(M_0\) it becomes

\[
\frac{M(0)}{M_0} < 1 + \frac{281n^2}{\pi M_0} \int_0^\infty \frac{\alpha^4}{(\alpha^4 + 1) (\alpha^4 - 2n^2 \alpha^2 + 1)} \, d\alpha
\]

\[
0 < n < 1
\]

From (4)

\[
M_0 = \frac{f_2}{4} \frac{l}{q}
\]

hence Equation 37 may then be written

\[
\frac{M(0)}{M_0} < 1 + \frac{41n^2}{\pi} \int_0^\infty \frac{\alpha^4}{(\alpha^4 + 1) (\alpha^4 - 2n^2 \alpha^2 + 1)} \, d\alpha
\]

\[
0 < n < 1
\]
FIGURE 8
Diagram for the Variation of (I) with n.
Let \( (I) = \int_0^\infty f(\alpha) d\alpha = \int_0^\infty \frac{\alpha^4}{(\alpha^4 + 1)(\alpha^4 - 2n^2\alpha^2 + 1)} \, d\alpha \) (40)

then

\[
\frac{M(0)}{M} < 1 + \frac{4n^2}{\pi^2} (I)
\] (41)

(1) can be found from Figure 8 which is plotted by the autoplotter from the output data of approximate solutions of Equation 40 (Table 3). The solutions are found by Simpson's law with different \( n \) values and six various upper bound values which are from zero to ten, twenty, forty, eighty, one hundred sixty, and three hundred twenty. It is to be noted from Table 3 that the increase in (I) is very small as the value of the upper bound of the integral increases from forty to three hundred twenty. It shows that the integral may be written

\[
(I) = \int_0^\infty f(\alpha) d\alpha = \int_0^N f(\alpha) d\alpha + \int_N^\infty f(\alpha) d\alpha
\]

and \( \int_N^\infty f(\alpha) d\alpha \) in the integral is very small when \( N \) is larger than forty and it may be neglected.

To explain the use of the graphs and equations of the solution, a problem utilizing the rail previously treated is considered (Figure 9). It is assumed:

(1) A concentrated load of 40,000 lbs. traveling with a velocity of 122 mph is applied.

(2) \( l^2 = 1,497 \) in²

(3) \( k = k'xb = 230 \times 6.5 = 1,500 \) lb/in²

\( (k' \) is evaluated from Table 1.)

(4) \( v_c = 1,220 \) mph
Figure 9  A Rail Resting on Ballast
Find

a. The maximum rail deflection.

b. The maximum bending moment.

From the given data, it is calculated that

\[ n = \frac{v}{v_c} = \frac{122}{1220} = 0.1 \]

\[ n^2 = 0.01 \]

Substituting the value of \( n \) into Equation 30, the deflection under the load is

\[ y = \frac{1.414 \times 40,000}{4 \times 1.500 \times 497 \times 0.995} \]

\[ = 0.252 \text{ in.} \]

From Equation 38,

\[ M_c = \frac{\sqrt{2}}{4} \times 40,000 \times \sqrt{1497} \]

\[ = 546,000 \text{ in}-\text{lbs.} \]

By means of Figure 8 and Equation 41,

\[ M_5 = 546,000 \times (1 + 0.0018 \times 0.49) \]

so that

\[ M = 551,000 \text{ in}-\text{lbs.} \]

When the velocity is disregarded, \( n = 0 \). Using \( n = 0 \) in Equation 30 and 41, the deflection is 0.250 in. and the moment is 546,000 in-lbs. Comparison of these values to the values for \( v = 122 \text{ mph.} \) of \( y = 0.252 \text{ in.} \) and \( M = 551,000 \text{ in}-\text{lbs.} \) shows that the influence of velocity is not large and may be neglected in this case.

From the above calculations, it may be seen that obtaining the solutions by use of the graphs is simple.
VI. CONCLUSION

From the theoretical point of view Equation 9 shows that it may be possible to obtain solutions for various boundary conditions of not only infinite beams resting on an elastic foundation subjected to dynamic load but also finite beams. So far as infinite beams resting on an elastic foundation subjected to an uniformly distributed moving load and concentrated moving load are concerned, the particular solutions of Equation 9 presented in this study show that in ordinary cases the influence of the velocity of the loads on the deflection and moment of the beam is small and may be neglected.

It should be pointed out, however, that the mass of foundation will reduce the critical velocity, if it is considered in this study, and due to the uncertainty of foundation properties it is necessary to verify the solutions in this paper by experiment.
Table 1. Modulus of Subgrade Reaction

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<th>500</th>
<th>800</th>
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<td>Poor Subgrade</td>
<td>Fair to good subgrade</td>
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<td>Good subbase</td>
<td>Good base</td>
<td>Best base</td>
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<td>GW</td>
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Table 2. Program for Solution of Equation (40)

*LIST PRINTER

*ALL STATEMENT MAP
C C***62975CSX021 S P CHANG 07/08/65 FORTRAN II
C APPROXIMATE SOLUTION OF THE MOMENT INTEGRATION
S=0.0
PRINT 101
DO 5 K=1,10
   B=10.
   S2=S**2
   DO 4 J=1,6
      B4=B**4
      A=B4/((B4+1.)*(B4-2.)*(S2)*(B**2)+1.)
      N=B
      Y=0
      X=1.
      DO 1 I=2,N,2
         X4=X**4
         Y=Y+X4/((X4+1.)*(X4-2.*S2)*(X**2)+1.0))
      1 X=X+1.
      Z=0
      X=2.
      DO 2 I=3,N,2
         X4=X**4
         Z=Z+X4/((X4+1.0)*(X4-2.*S2)*(X**2)+1.0))
      2 X=X+1.
      SUM=(4.*Y+2.*Z+A)/3.
   PRINT 100,S,B,SUM
   PUNCH 100,S,B,SUM
   4 B=B*2.
   5 S=S+0.1
      101 FORMAT(3X,1HM,5X,1HB,10X,3HSUM)
      100 FORMAT(3X,F3.1,3X,F5.0,3X,E14.7)
   CALL EXIT
END
### Table 3. Output Data of Equation (40)

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BIBLIOGRAPHY


VITA

Shi-peing Chang was born on June 26, 1937, in Fukien, China, the son of Mr. and Mrs. Shih-juh Chang. He received his primary education in FooChow, Fukien, China, secondary education in Taiwan, China, and was granted a B. S. degree in Civil Engineering from Taiwan Christian College in 1960. After graduation, he served one and one-half years as a second lieutenant with the Chinese Air Force Engineering Corps in Taiwan, China.

In October 1961 he worked as a junior engineer in the Taiwan Provincial Water Conservancy Bureau until he came to the United States.

In September 1964 he was enrolled as a graduate student at the University of Missouri at Rolla, Rolla, Missouri, majoring in Civil Engineering.