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A study of a model-referenced adaptive control system

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A STUDY OF A MODEL-REFERENCED
ADAPTIVE CONTROL SYSTEM

BY
JAMES WILLIAM GRAHAM III, 1945

A

THESIS

submitted to the faculty of
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A model-referenced adaptive controller for a first-order plant with time-varying parameters is developed using Lyapunov's second method. The system is studied with the aid of analogue computer simulation and an oscillatory mode is encountered. A method of reducing the oscillations by addition of a compensating network is proposed. The compensation is obtained through the use of linearization and root locus techniques. The compensated system shows a much improved error response over the original system.
ACKNOWLEDGEMENTS

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LIST OF SYMBOLS

a  model dynamic parameter
K_m  model d.c. gain parameter
b  plant dynamic parameter
K_s  plant d.c. gain parameter
K  controller feedback parameter for adjustment of plant dynamics
K_v  controller feedforward parameter for adjustment of plant d.c. gain
\mu_1  gain of the feedforward parameter adjustment loop
\mu_2  gain of the feedback parameter adjustment loop
\theta_m  model output
\theta_s  plant output
r  command input
e  system error, \theta_m - \theta_s
x_1  misalignment of the model and controlled plant
d.c. gain parameters, \begin{align*} K_m - K_v K_s \end{align*}
\begin{align*} x_2 \end{align*}  misalignment of the model and controlled plant
dynamic parameters, a - (b + K_s K)
q  output of the lead compensator
c, d  parameters of the lead compensator
I. INTRODUCTION

A. The Adaptive Problem

The middle 1950s represented a remarkable upswing in the growth of automatic control and particularly in the development of adaptive systems. In 1958 Aseltine, Mancini, and Sarture\(^1\) presented a survey of the adaptive control work dating back to 1951. The number and variety of publications in this period alone enabled them to organize adaptive systems into five general classifications according to the manner in which adaptation was achieved.

The general adaptive problem is to maintain a desired response by developing within a system the ability for self-adjustment under the influence of changing conditions. The problems of meeting specified performance criteria in flight control systems is a prime example. Of the papers cited by Aseltine\(^1\), et al, the majority of applications dealt with flight systems, and this is similarly true today. One of the problems facing those who design flight systems is dynamic parameter variation. The relatively new model-referenced adaptive control system seems ideally suited to this problem.
The general model-referenced system consists of a model, plant, and controller as shown in Figure 1. The plant represents a system with varying parameters and the model represents the ideal plant from the designer's viewpoint. The controller is the mechanism which adapts the plant to match the model, resulting in zero error between plant and model outputs.

B. History of the Design of Model-Referenced Adaptive Control Systems

The first model-referenced system and design criteria were proposed by Whitaker, Yamron, and Kezer\(^2\) in 1958. Osburn, Whitaker, and Kezer\(^3\) subsequently published a paper in which simplification of the control mechanisms and a reduction in convergence time of adaptation were improvements over the original proposal. The system was designed to minimize the integral of the error squared. Control signals for parameter adjustment were based on functions of the error and time rates of change of the plant parameters. This method of design is referred to in later literature as the "M. I. T. Rule."

Just prior to Osburn's paper, Kalman and Bertram\(^4\) presented their paper, "Control System Analysis and Design Via the Second Method of Lyapunov," which marked the introduction of Lyapunov's theories into Western literature. Lyapunov's second method can be used for determination of stability or asymptotic stability in the large, and is well
Figure 1. General Model-Referenced System
known to be an important addition to control theory. Parks used Lyapunov's second method to show that systems developed by the "M. I. T. Rule" are unstable for certain inputs.

Grayson, in 1961, presented a thesis in which Lyapunov's second method was used to design a model-referenced control system. Many papers quickly followed. Shackcloth and Butchart presented a synthesis technique using Lyapunov's second method which was redesigned by Parks. The culmination of these and others came in 1967 when Shackcloth presented a general design criteria for model-referenced adaptive systems using Lyapunov's second method. He has shown that stability in the large is guaranteed and asymptotic stability is guaranteed under certain conditions. The adaptive control loops are defined based on the error, the input, and the plant output and its derivatives. Within a period of months a similar design was offered by Zemlyakov and Rutkovskii. The main difference in the papers was the manner in which a Lyapunov function was chosen, the latter method being the same as that chosen by Monopoli and used later in this paper.

C. Choice of a Problem and Summary of Approach

The first-order model-referenced adaptive system of Figure 2 was chosen for study since nowhere in the available literature was there evidence of a simulation study in which both plant parameters, $K_s$ and $b$, were assumed
Figure 2. First-Order Model-Referenced System - Adaptive Control Unspecified
different from the model. Of the simulations\textsuperscript{5,9} known to the author, only one plant parameter was assumed variable, or fixed and different from the model, while the other parameter was fixed and identically matched to the corresponding model parameter. Since it was desired to make all plant parameters variable, or at least fixed and different from the model, the available analogue computer limited simulation studies to systems not greater than first order.

Once the first-order system had been chosen, an analogue simulation was undertaken as a check against the general theory presented by Shackcloth\textsuperscript{9}. Two observations from the simulation are noted here. First, the system contained oscillatory modes; and second, after the system reached the steady-state for step inputs, the adaptive control gains, $K$ and $K_v$ of Figure 2, tended to drift away from their ideal values.

The use of adaptive control in flight systems and the possibility of effectively using model-referenced adaptive control in this area has been mentioned previously. In this light, it was decided that the oscillations previously observed could have damaging effects of a magnitude that would render this approach useless. However, if the oscillations could be removed or at least highly damped, then a significant improvement will have been made particularly from the standpoint of maintaining stability in an aircraft.
The basic organization for the remainder of the paper will take the following form. The first-order model-referenced adaptive system will be synthesized according to present theory. Following this is a section devoted to linearizing the system, followed by a section in which compensation techniques are applied to the linearized system. A study of several first-order systems showing the desired improvement completes the main body of the paper.
II. A FIRST-ORDER MODEL-REFERENCED
ADAPTIVE CONTROL SYSTEM

A. Synthesis by Lyapunov's Second Method

In model-referenced systems, adaptive control loops must be determined such that the plant and model outputs become the same. Shackcloth has developed a synthesis technique for finding the control equations of a linear, nth-order model-referenced system in which the plant gain and all dynamic parameters are assumed constant or slowly time-varying. His method is used here to determine the control of the first-order system of Figure 2 in which $K_s$ and $b$ are assumed slowly time-varying.

The necessary control for a first-order linear system is obtained from the variable gains $K$ and $K_v$ of Figure 2. This is seen by comparing the transfer functions, $H_m$ and $H_p$, of the model and plant. For the model

$$H_m(s) = \frac{K_m}{s+a} \quad (1)$$

and for the plant

$$H_p(s) = \frac{K_vK_s}{s+b+K_sK} \quad (2)$$
If the plant and model are to be identical, or matched, it is required that their transfer functions be equal or from (1) and (2) that

\[ K_m = K_v K_s \]

and

\[ a = b + K_s K \]

The control equations for determining \( K \) and \( K_v \) are now developed using Lyapunov’s second method.

1. Mathematical Development

Equations (1) and (2) may be rewritten as

\[ (s+a)\theta_m = K_m r \] (3)

and

\[ (s+b+K_s K)\theta_s = K_v K_s r \] (4)

The error between the model and plant is defined as

\[ e = \theta_m - \theta_s \] (5)

although an equally valid choice for error is \( \theta_s - \theta_m \).

Now

\[ (s+a)e = (s+a)(\theta_m - \theta_s) \]

Appropriate substitutions from equations (3) and (4) yield

\[ (s+a)e = (K_m - K_v K_s)r - \left[ a - (b + K_s K) \right] \theta_s \] (6)

Let \( x_1 = K_m - K_v K_s \) and \( x_2 = a - (b + K_s K) \). Then

\[ \dot{e} = -ae + x_1 r - x_2 \theta_s \]

which becomes in matrix notation

\[ \dot{e} = Ae + Bx \]

where

\[ A = [-a], \ B = \begin{bmatrix} r & -\theta_s \end{bmatrix}, \ e = [e], \ \text{and} \ x^r = [x_1 \ x_2]. \]
The following choice of a Lyapunov function, $V$, is taken from Monopoli:

$$V = e^T Pe + x^T N x$$  \hspace{1cm} (7)

where $P$ and $N$ are positive definite matrices and in the first-order case

$$P = \begin{bmatrix} p_{11} \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} n_1 & 0 \\ 0 & n_2 \end{bmatrix}.$$  

Equation (7) is a logical choice for a Lyapunov function, in that both terms are quadratic forms which go to zero when the plant is matched to the model. Differentiation of (7) with respect to time yields

$$\dot{V} = \dot{e}^T Pe + \dot{e}^T Pe + \dot{x}^T N x + \dot{x}^T N \dot{x}$$

$$= e^T A^T Pe + x^T B^T Pe + e^T P A \dot{e} + e^T P B \dot{x} + \dot{x}^T N x + \dot{x}^T N \dot{x}$$

$$= e^T (A^T P + P A) \dot{e} + x^T (B^T Pe + N \dot{x}) + (e^T P B + \dot{x}^T N) \dot{x}$$  \hspace{1cm} (8)

If

$$\dot{x} = -N^T B^T Pe$$  \hspace{1cm} (9)

then (8) reduces to

$$\dot{V} = e^T (A^T P + P A) \dot{e} = -e^T Q e$$

$\dot{V}$ is negative definite if $Q$ is chosen as a positive definite matrix. The term $e^T Pe$ is positive definite, since $P$ is positive definite, and $x^T N x$ is a quadratic form which is positive definite; therefore, $V$ is positive definite, and asymptotic stability is assured. The choice of $V$ and $\dot{x}$ result in a $\dot{V}$ which ensures that $e$ will go to zero but does not guarantee that $x$ will go to zero.

A solution for $P$ can be obtained from $A^T P + P A = -Q$. This solution is unique, and $P$ will be positive definite.
if $A$ is a stable matrix. The model in Figure 2 is stable provided $a > 0$; hence $A$ is stable for a stable model. Substituting for $P$ and $A$

$$-Q = A^T P + PA = -2ap_{11}$$

Choosing $Q = [2a]$, equation (10) yields $p_{11} = 1$. The adaptive control can now be formed from equation (9).

$$\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
-K_s k_v \\
-K_s k
\end{bmatrix} = -\frac{1}{n_1n_2} \begin{bmatrix}
n_2 & 0 \\
0 & n_1
\end{bmatrix} \begin{bmatrix}
r \\
-p_{11}
\end{bmatrix}$$

(11)

In the differentiation of $x$ it has been assumed that $K_s$ and $b$ are constant or may be approximated as constant.

Simplifying (11)

$$\begin{bmatrix}
\dot{k}_v \\
\dot{k}
\end{bmatrix} = \frac{1}{n_1n_2K_s} \begin{bmatrix}
n_2r \\
n_1\theta_s
\end{bmatrix}$$

The adaptive control equations for the system of Figure 2 are

$$\dot{k}_v = \mu_1 e$$

(12a)

and

$$\dot{k} = -\mu_2 \theta_s e$$

(12b)

where

$$\mu_1 = \frac{1}{n_1K_s} \quad \text{and} \quad \mu_2 = \frac{1}{n_2K_s}$$

Although $K_s$ may be slowly time-varying, the adaptive loop gains $\mu_1$ and $\mu_2$ are constant if $n_1$ and $n_2$ are chosen inversely proportional to $K_s$. This choice of $n_1$ and $n_2$ appears to add terms involving $\dot{N}$ to the differentiated Lyapunov function, $\dot{V}$, of equation (8). Since $K_s$ has been
approximated as constant, so that \( \dot{K}_s \) equals zero, \( \dot{N} \) may also be approximated as zero and equation (8) is unchanged. The first-order system with the adaptive control specified in equation (12) is shown in Figure 3.

A summary of assumptions made in this development are:

(a) the model is stable in order to ensure that \( P \) is positive definite,
(b) \( K_s \) and \( b \), the plant parameters, are approximately constant in order that \( x \) may be differentiated to the form of equation (11), and
(c) since \( N \) must be positive definite and \( \mu_1 \) and \( \mu_2 \) are constant, then \( K_s \) cannot change sign.

2. Analogous Simulation

An analogous simulation study, in which both \( K_s \) and \( b \) are not necessarily matched to the model, was undertaken to gain a better understanding of the system. Due to the limitations of the available analog computer only one of the plant parameters could be varied with time. The model parameters were arbitrarily selected as \( K_s = 2 \) and \( a = 1 \) for use in the simulation. In Appendix A is the analog simulation drawing for use on an Electronic Associates, Inc. TR-48 computer. The following cases are typical of many that were simulated.

Case 1: \( r = 1 \quad b = 0 \quad K_v(0) = 0 \)

\[ \mu_1 = 1 \quad K_s = 1 \quad \theta_m(0) = 0 \]

\[ \mu_2 = 1 \quad K(0) = 0 \quad \theta_s(0) = 0 \]

The model-plant error and \( K \) are shown in Figure 4.
Figure 3. First-Order Model-Referenced System - Adaptive Control Specified
Case 2: Case 1 was repeated except $\mu_1 = \mu_2 = 20$. The error and K are shown in Figure 5.

Case 3: $r = 1 \quad b = .5 \quad K_v(0) = 2$

$\mu_1 = 1 \quad K_s = 1 \quad \theta_m(0) = 2$

$\mu_2 = 1 \quad K(0) = 1 \quad \theta_s(0) = 2$

The error and K are shown in Figure 6.

Case 4: Case 3 was repeated except $\mu_1 = \mu_2 = 20$. The error and K are shown in Figure 7. Cases 3 and 4 are equivalent to a step change in b after the plant has been adapted to the model.

Case 5: $r = 1 \quad b = 0 \quad K_v(0) = 2$

$\mu_1 = 1 \quad K_s = 1 \quad \theta_m(0) = 2$

$\mu_2 = 1 \quad K(0) = 1 \quad \theta_s(0) = 2$

In this case the plant and model are identical. A plot of K versus $K_v$ is shown in Figure 8.

It is noted that only fixed plant parameters were used and that simulations involving a time-varying plant are presented in a later section.

3. Results

The results presented in this section are primarily general observations since at this time only a general knowledge of the adaptive process and any associated problems is being sought.

In cases one through four, there are three very notable features which may be observed from Figures 4 through 7. First, there are oscillatory modes in the system as noted
Figure 4. System Error, $e(t)$, and $K(t)$ for Case 1
Figure 5. System Error, $e(t)$, and $K(t)$ for Case 2
Figure 6. System Error, $e(t)$, and $K(t)$ for Case 3
Figure 7. System Error, $e(t)$, and $K(t)$ for Case 4
Figure 8. Drift of Adaptive Parameters for Case 5
by the damped sinusoidal error response. Second, an increase in the adaptive loop gains, $\mu_1$ and $\mu_2$, caused an increase in the frequency of oscillation and a decrease in the peak value of the error. Third, upon reaching zero error the adaptive gains, $K$ and $K_v$, were not at their ideal values, e.g. in case one, ideal values would be $K = 1$ and $K_v = 2$, whereas Figure 5 shows $K = 0.73$ and $K_v$ was found to be 1.46. This apparent conflict with the theory is not mentioned by Shackcloth\textsuperscript{9}. The reasons for it are related to the d.c. gains of the model and plant and the use of a constant input; this will be discussed further in part B. Case 5 was included to show that a drift problem exists. Figure 8 shows that beginning with a perfectly adapted system, the values of $K$ and $K_v$ drift in a manner that keeps the d.c. gain of the plant equal to that of the model. Once again the constant input was found to be a contributing factor. The drift problem will not be discussed further until the final section.

B. Linearization of the System

The preceding investigation has shown that oscillatory modes do exist and that increasing the adaptive loop gains increases the frequency of oscillation. The purpose here is to linearize the system in an attempt to isolate the oscillatory modes for further study. The model-referenced system of Figure 3 is nonlinear due to multipliers employed
in the adaptive control; therefore, a linear analysis of the system in its present form cannot be used. A nonlinear system can, however, be linearized and subsequently analyzed about a known equilibrium point. Such a point for the chosen system occurs when the plant and model are matched or when \( x_1 = x_2 = e = 0 \). The system will be linearized about the known equilibrium point and then verified by analogue simulation.

1. Mathematical Development

The state equations for the system of Figure 3 may be written directly as

\[
\begin{align*}
\dot{\theta}_m &= -a\theta_m + K_r r \\
\dot{\theta}_s &= -b\theta_s + K_s (K_r r - K\theta_s) \\
\dot{e} &= -\mu_2 \theta_s e - \mu_2 \theta_s (\theta_m - \theta_s) \\
\dot{K}_v &= \mu_1 e r = \mu_1 r (\theta_m - \theta_s)
\end{align*}
\]

The method for linearizing the system may be found in DeRusso, Roy, and Close\textsuperscript{13}. The technique used is basically a Taylor series expansion about the equilibrium point in which only the linear terms are retained. The system is then studied in terms of deviation from the equilibrium point.

Let \( z \) represent the deviation of the state variables from the equilibrium point. The known equilibrium, \( x_1 = x_2 = e = 0 \), implies for \( x_1 = x_2 = 0 \) that \( a = K_s K + b \) and \( K_m = K_v K_s \) and for \( e = 0 \) that \( \theta_m = \theta_s \). The model output at equilibrium is \( K_m r / a \).
Then

\[
\mathbf{z} = \begin{bmatrix}
\Theta_m - \frac{K_mr}{a} \\
\Theta_s - \frac{K_mr}{a} \\
a - b \\
K - \frac{K_m}{K_s} \\
K_v - \frac{K_m}{K_s}
\end{bmatrix}
\]  \hspace{1cm} (13)

The linearized system is described by

\[
\dot{\mathbf{z}} = \mathbf{J}_{eq} \mathbf{z}
\]

called the "variational equations" of the linearized system. \( \mathbf{J}_{eq} \) is the Jacobian matrix of the state equations evaluated at the equilibrium point. The Jacobian is determined\(^1\) as

\[
\mathbf{J} = \begin{bmatrix}
-a & 0 & 0 & 0 \\
0 & -(b + K_sK) & -K_s\Theta_s & K_sr \\
-\mu_2\Theta_s & \mu_2\Theta_s & 0 & 0 \\
\mu_1 r & -\mu_1 r & 0 & 0
\end{bmatrix}
\]  \hspace{1cm} (15)

At equilibrium, equation (15) becomes

\[
\mathbf{J}_{eq} = \begin{bmatrix}
-a & 0 & 0 & 0 \\
0 & -a & -K_s\Theta_{s_{eq}} & K_sr \\
-\mu_2\Theta_{s_{eq}} & \mu_2\Theta_{s_{eq}} & 0 & 0 \\
\mu_1 r & -\mu_1 r & 0 & 0
\end{bmatrix}
\]

where \( \Theta_{s_{eq}} \) is the plant output at equilibrium. \( \mathbf{J}_{eq} \) is used to determine the eigenvalues and corresponding eigenvectors for the linearized system. The eigenvalues
are found from \( \lambda I - J_{eq} = 0 \), where

\[
\begin{bmatrix}
\lambda + a & 0 & 0 & 0 \\
0 & \lambda + a & S\theta_{eq} & -K_s r \\
\mu_2 \theta_{eq} & -\mu_2 \theta_{eq} & \lambda & 0 \\
-\mu_1 r & \mu_1 r & 0 & \lambda \\
\end{bmatrix}
\]

\[
= \lambda(\lambda + a)\left[\lambda^2 + a\lambda + S\left(\mu_1 r^2 + \mu_2 \theta_{eq}^2\right)\right] \quad (16)
\]

In symbolic form the equations are very unwieldy, therefore a particular example is studied in order to show the usefulness of the linearized system. Let the model parameters be \( a = 1 \) and \( K_m = 2 \), the plant parameters be \( b = 0 \) and \( K_s = 1 \), and the adaptive loop gains be \( \mu_1 = \mu_2 = 20 \).

The input will be \( r = 1 \), and equilibrium values are \( \theta_{eq} = 2 \), \( S = 1 \), and \( K_v = 2 \). Then

\[
\left| \lambda I - J_{eq} \right| = \lambda(\lambda + 1)(\lambda^2 + \lambda + 100) = 0 \quad (17)
\]

Solving (17), the eigenvalues are

\[
\lambda_1 = 0 \\
\lambda_2 = -1 \\
\lambda_3 = -0.5 - j10 \\
\lambda_4 = -0.5 + j10
\]

The eigenvectors are found from \( \text{Adj}[\lambda I - J_{eq}] \). The column vectors of \( \text{Adj}[\lambda I - J_{eq}] \) are proportional and evaluation of any column of \( \text{Adj}[\lambda I - J_{eq}] \) at \( \lambda = \lambda_i \) gives the corresponding eigenvector. The matrix \( \text{Adj}[\lambda I - J_{eq}] \) is given in equation (18) on the following page.

The modes of dynamic behavior in a linear system are expressed in terms of motion along the eigenvectors, hence
\[\text{Adj}[\lambda I - J_{eq}] = \begin{bmatrix}
\lambda^3 + \lambda^2 + 100\lambda & 0 & 0 \\
100\lambda & \lambda^3 + \lambda^2 & -2(\lambda^2 + \lambda) \\
-40(\lambda^2 + \lambda) & 40(\lambda^2 + \lambda) & \lambda^3 + 2\lambda^2 + 21\lambda + 20 \\
20(\lambda^2 + \lambda) & -20(\lambda^2 + \lambda) & 40(\lambda + 1) \\
& & \lambda^3 + 2\lambda^2 + 81\lambda + 80
\end{bmatrix} \quad (18)\]
the matrix whose columns are the eigenvectors is termed the modal matrix. Once the modal matrix is known, time-solutions of the state variables may be found for any set of initial conditions chosen. Substitution of the various eigenvalues into (18) yields

\[
M = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 1 & -0.5 - j10 & -0.5 + j10 \\
1 & 0 & 40 & 40 \\
2 & 0 & -20 & -20 \\
\end{bmatrix}
\]

(19)

Solutions of \(z(t)\) may now be calculated.

If \(z(t)\) is known, \(\theta_m(t)\) and \(\theta_s(t)\) are known and hence \(c(t) = \theta_m(t) - \theta_s(t)\) may be found. The following transformation will simplify calculations. Let

\[
z = My
\]

(20a)

Then

\[
\dot{z} = \dot{My}
\]

(20b)

Substitution of (20) into (14) yields

\[
\dot{My} = J_{eq}My
\]

or

\[
\dot{y} = M^{-1}J_{eq}My
\]

(21)

The form

\[
\Lambda = M^{-1}J_{eq}M
\]

where \(\Lambda\) is a diagonal matrix composed of the eigenvalues of \(J_{eq}\) is well known\(^{13}\). Equation (21) then becomes

\[
\dot{y} = \Lambda y
\]

(22)

where
The solution of equation (22) is

\[ \Lambda = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -0.5 - j10 & 0 \\ 0 & 0 & 0 & -0.5 + j10 \end{bmatrix} \]

The solution of equation (22) is

\[ \mathbf{y}(t) = e^{\Lambda t} \mathbf{y}(0) = \begin{bmatrix} y_1(0)e^{\lambda_1 t} \\ y_2(0)e^{\lambda_2 t} \\ y_3(0)e^{\lambda_3 t} \\ y_4(0)e^{\lambda_4 t} \end{bmatrix} \]

The system response is dependent on how strongly each mode is excited which is dependent on \( \mathbf{y}(0) \). Since it is desired to study the oscillatory mode, which is due to the complex pair of eigenvalues, \( \mathbf{y}(0) \) is chosen as

\[ \mathbf{y}^T(0) = 0.01 \begin{bmatrix} 0 & 0 & -0.625 - j1.03 & -0.625 + j1.03 \end{bmatrix} \] (23)

The reasons for this selection will be discussed in the next section on analogue verification. Now using equation (20a), \( \mathbf{z}(t) \) is as shown on the following page in equation (24). Using equation (13) a solution for \( e(t) \) is

\[ e(t) = \theta_m(t) - \theta_s(t) = z_1(t) - z_2(t) \] (25)

And

\[ e(t) = 0.242 e^{-0.5 t} \cos(10t + 33.8^\circ) \] (26)

This solution of \( e(t) \) is the result of exciting only the last two modes of the modal matrix. The steps intermediate to (25) and (25) may be found in Appendix B.
\[ z(t) = My(t) = 0.01 \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 1 & -0.5 - j10 & -0.5 + j10 \\
1 & 0 & 40 & 40 \\
2 & 0 & -20 & -20 \\
\end{bmatrix} \begin{bmatrix}
0 \\
0 \\
-0.625 + j1.03 e^{-0.5 + j10 t} \\
-0.625 - j1.03 e^{-0.5 - j10 t} \\
\end{bmatrix} \]
2. **Analogue Verification**

The analytic solution obtained for \( e(t) \) will be verified by an analogue simulation of the system of Figure 3 and the values for the example being studied. The simulation drawing used in conjunction with an EAI TR-48 analogue computer is in Appendix A. First, however, the particular choice of \( y(0) \) in equation (23) will be explained.

If the oscillatory mode is the only one to be excited, a possible form of \( y(0) \) is

\[
y^T(0) = \begin{bmatrix}
0 & 0 & a + j\beta & a - j\beta
\end{bmatrix}
\]

Thus there are infinitely many choices of \( a \) and \( \beta \) which should yield solutions containing only the oscillatory mode. However, it was found that while the error went to zero for initial conditions corresponding to (27) the model and plant were not always perfectly matched, that is \( x_1 \) and \( x_2 \) were not zero. From (19), (20a), (27), and (13)

\[
z(0) = \begin{bmatrix}
0 \\
-a + 20\beta \\
80a \\
-40a
\end{bmatrix} = \begin{bmatrix}
\theta_m(0) - 2 \\
\theta_s(0) - 2 \\
K(0) - 1 \\
K_v(0) - 2
\end{bmatrix}
\]

This shows \( \theta_m(0) = 2 \) for any \( a \) and \( \beta \). For the constant input assumed, the model output equals two for all time which means the dynamic model parameter, \( a \), is not a determining factor in the adaptation process. The only requirement for adaption is that the controller match the plant and model d.c. gains. From equations (1) and (2) the d.c. gains
are \[ H_{m}(0) = \frac{K_{m}}{a} \quad \text{and} \quad H_{p}(0) = \frac{K_{v}K_{s}}{b + K_{s}K} \] (29)

For the example chosen, (29) becomes

\[ H_{m}(0) = 2 \quad \text{and} \quad H_{p}(0) = K_{v}/K \]

The error will be zero whenever the ratio of \( K_{v} \) to \( K \) is equal to two. Then with a constant input and initial conditions corresponding to (27) the equilibrium is a set of points represented by the line \( K_{v} = 2K \). This is the same line along which \( K \) and \( K_{v} \) drifted as shown earlier in Figure 8. For each pair of values of \( K(0) \) and \( K_{v}(0) \) there is an initial error, \( e(0) = e_{m}(0) - e_{s}(0) \), that causes the system to end at a particular point on the equilibrium line once the error becomes zero. There are then a set of points in the state-space of \( K, K_{v}, \) and \( e \) that represent initial conditions which will give a perfect match of the model and plant. Using the analogue simulation, one such point was found by trial and error to be \( e(0) = 0.2, K(0) = 0.5, \) and \( K_{v}(0) = 2.25 \). Since \( e_{m}(0) = 2 \) the initial conditions on the state variables in the simulation are

\[
\begin{bmatrix}
\theta_{m}(0) \\
\theta_{s}(0) \\
K(0) \\
K_{v}(0)
\end{bmatrix} =
\begin{bmatrix}
2 \\
1.8 \\
.5 \\
2.25
\end{bmatrix}
\]

(30)

From equations (13) and (30), \( z(0) \) becomes

\[ z^{T}(0) = \begin{bmatrix} 0 & -.2 & -.5 & .25 \end{bmatrix} \]

and from equation (28) \( a = -.00625 \) and \( \beta = -.0103 \). Substitution of \( a \) and \( \beta \) into equation (27) then gives \( y(0) \) of
equation (23). As a check, \( y(0) \) may be found from equation (20a) as

\[
y(0) = M^{-1}z(0)
\]

(31)
The solution of \( y(0) \) by this method is shown in Appendix B.

The analogue simulation of the chosen example with initial conditions of (30) was performed and a record of \( e(t) \) and \( K(t) \) is shown in Figure 9. Equation (26), the analytical solution is shown in Figure 10. The results of the two error solutions of Figures 9 and 10 show a very good correlation of peak values, and while the periods of oscillation are not identical, they are very close.

In summary, the nonlinear adaptive system was linearized about a known equilibrium point. A particular example was chosen for study and both analytical and analogue simulation results were obtained for eigenvector excitation of the oscillatory mode. It was necessary, however, to find appropriate initial conditions by trial and error. The calculated system error and the simulated system error were in very close agreement. It is concluded then that the linear model is a good representation of the system in the neighborhood of the equilibrium point. Further, since the desired improvement in the system is to remove or highly damp any oscillations, the verification of the linear model suggests the use of linear compensation techniques for this purpose.
Figure 9. System Error and K(t) for Eigenvector

Excitation of a Model-Referenced System
Figure 10. Calculated System Error for Eigenvector Excitation of the Linearized System
C. Compensation of the Linear System

Since the linearized model of the system has been verified, a logical solution to the oscillatory problem is the use of linear compensation techniques. A root locus of the linear system suggests a specific type of compensation. The compensation is added to the system and the system error is calculated and verified for the particular example studied previously.

1. Mathematical Development

Referring to equation (16), the only design parameters available for which the choice is arbitrary once the model is selected are $\mu_1$ and $\mu_2$. A root locus may be plotted for
\[
\lambda^2 + a\lambda + K_S(\mu_1 r^2 + \mu_2 \theta_S^2) = 0
\]
which is obtained from equation (16). It should be noted that only the complex pair of roots, which represent the oscillatory mode, are affected by $\mu_1$ and $\mu_2$. Let $\mu_1 = \mu_2 = \mu$, then (32) becomes
\[
\lambda^2 + a\lambda + \mu K_S(r^2 + \theta_S^2) = 0
\]
Equation (33) can be written as
\[
\frac{K_S(r^2 + \theta_S^2)}{1 + \mu \frac{1}{\lambda(\lambda + a)}} = 0
\]
This is the proper form for plotting a root locus\(^4\) where $\mu$ is the gain to be selected for operation at any point on the locus. Let $K_S = r = a = 1$ and $\theta_S = 2$. These are the values of the example studied in the last section. Then
equation (34) becomes

\[ 1 + \mu \frac{5}{\lambda(\lambda + 1)} = 0 \quad (35) \]

The root locus of (35) is shown in Figure 11. For increasing values of \( \mu \), it is seen that frequency increases and damping decreases. It might be suggested by the root locus that for extremely small values of \( \mu \), there would be no oscillation; however, as noted earlier low gains in the adaptive loops cause an increase in peak error of the system. Low values of \( \mu \) also decrease the signal from which \( K \) and \( K_v \) are derived. Correspondingly, there is an increase in the time required for adaptation of plant to model. The most important observation is that large values of \( \mu \) are necessary for adequate tracking of the model by the plant but the result is high frequency oscillations in the system.

Under an assumption of large values of \( \mu \), the root locus in Figure 11 shows that improved damping can only be obtained by drawing the complex roots further into the left half plane. This can be done by addition of appropriate poles and zeros.

A passive lead compensator adds one pole and zero, the zero being closer to the origin. The transfer function of the lead network is

\[ G(s) = \frac{ds + c}{s + c} \quad \text{where} \quad d > 1. \]

Because oscillations occur in both adaptive loops the input to the lead compensator was chosen as the system error and
Figure 11. Root Locus of the Linearized Model-Referenced System
the output was then used as the error signal for deriving \( K \) and \( K_v \). The output of the lead is denoted by "q."

Figure 12 shows the original system of Figure 3 with the lead compensation added. The procedure from this point is very similar to the preceding section.

The state equations of the new system are

\[
\begin{align*}
\dot{\theta}_m &= -a\theta_m + K_m r \\
\dot{\theta}_s &= -b\theta_s + K_s (K_v r - K\theta) \\
\dot{K} &= -\mu_2 \theta_s q \\
\dot{K}_v &= \mu_1 r q \\
q &= -cq + ce + de \\
\end{align*}
\]

\( (36a) \quad (36b) \quad (36c) \quad (36d) \quad (36e) \)

The compensated system is now linearized about the equilibrium \( x_1 = x_2 = e = q = 0 \). The variational equations are

\[
\dot{w} = J_{eq} w \quad (37)
\]

where

\[
\begin{bmatrix}
\theta_m - \frac{K_m r}{a} \\
\theta_s - \frac{K_m r}{a} \\
a - b \\
\frac{K}{K_s} \\
K_v - \frac{K_m}{K_s} \\
q
\end{bmatrix}
\]

\( (38) \)

and the Jacobian evaluated at equilibrium for the
Figure 12. First-Order Model-Referenced System with Lead Compensation
compensated system is

\[ J_{eq} = \begin{bmatrix} -a & 0 & 0 & 0 & 0 \\ 0 & -a & -K_S \theta_{seq} & K_S r & 0 \\ 0 & 0 & 0 & 0 & -\mu_2 \theta_{seq} \\ 0 & 0 & 0 & 0 & \mu_1 r \\ c - da & da - c & dK_S \theta_{seq} & -dK_S r & -c \end{bmatrix} \]

Eigenvalues are then determined from

\[
\left| \lambda I - J_{eq} \right| = \lambda(\lambda + a)\{ (\lambda + c)[\lambda^2 + a\lambda + K_S(\mu_1 r^2 + \mu_2 \theta_{seq}^2)] + (d - 1)\lambda K_S(\mu_1 r^2 + \mu_2 \theta_{seq}^2) \} = 0 \quad (39)
\]

The example studied for the original adaptive system is used in studying the compensated system in order to facilitate a comparison of the two systems. The values to be used are

\[
K_m = 2 \quad a = 1 \quad \theta_{seq} = 2 \quad \mu_1 = 20 \\
K_S = 1 \quad b = 0 \quad r = 1 \quad \mu_2 = 20
\]

The choice of \( c \) and \( d \) can be seen from a root locus of the linearized compensated system. Let \( \mu_1 = \mu_2 = \mu \). The substitution of values, except \( \mu \), into equation (39) yields

\[
\lambda(\lambda + 1)\{ (\lambda + c)[\lambda^2 + \lambda + 5\mu] + 5\mu(d - 1)\lambda \} = 0 \quad (40)
\]

Note that a designer now has three parameters he may select, these being \( \mu \), \( c \), and \( d \). Now let \( \mu = 20 \) as in the original system. A root locus may be obtained from

\[
(\lambda + c)[\lambda^2 + \lambda + 100] + 100(d - 1)\lambda = 0 \quad (41)
\]

and will be plotted assuming \( d \) as the variable. The proper root locus equation is

\[
1 + (d - 1)\frac{100\lambda}{(\lambda + c)(\lambda^2 + \lambda + 100)} = 0 \quad (42)
\]
For values of \( c = 10, 15, \) and \( 20 \) the loci are shown in Figure 13. A choice of \( c \leq 20 \) was made to eliminate scaling problems in the analogue simulation. From Figure 13 it is seen that for \( c = 20 \) the best damping can be achieved. The root locus shows that increasing \( d \) causes the pole at \( c \) to move toward the origin which increases the time constant of the mode due to \( c \); therefore, the response time of this mode increases. The selection of \( d \) was based on obtaining a good damping ratio while maintaining a response time for the mode due to \( c \) which is lower than that for the mode due to the complex roots. For the example under consideration, \( d = 2.9 \) appears to satisfy both of these requirements, and the root locus predicts the oscillatory mode will have a time constant of \( 1/5.5 \) and a frequency of \( 13.1 \) radians/second. As seen from the root locus the time constant of the oscillatory mode is a function of \( \mu, a, c, \) and \( d \) whereas the time constant of the oscillatory mode in the original system is found as \( 2/a \).

A compensator has been specified and a solution of the system error will be pursued as a check against the root locus predictions of frequency and time constant. Substitution of values into (39) yields

\[
\begin{align*}
\lambda (\lambda + 1) \left( (\lambda + 20) \left( \lambda^2 + \lambda + 100 \right) + 190\lambda \right) &= 0 \\
\lambda (\lambda + 1) \left( \lambda^3 + 21\lambda^2 + 310\lambda + 2000 \right) &= 0 \\
\lambda (\lambda + 1)(\lambda + 10)(\lambda^2 + 11\lambda + 200) &= 0
\end{align*}
\]

(43)

The eigenvalues from equation (43) are
Figure 13. Root Locus of the Linearized Compensated Model-Referenced System
\[ \lambda_1 = 0 \quad \lambda_4 = -5.5 - j13.03 \]
\[ \lambda_2 = -1 \quad \lambda_5 = -5.5 + j13.03 \]
\[ \lambda_3 = -10 \]

The modal matrix is found from \( \text{Adj} [\lambda I - J_{eq}] \) which is shown in Appendix B. The \( i \)th column of \( M \) is the eigenvector corresponding to \( \lambda_i \).

\[
M = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 1 & 190 & 9(4.5 - j13.03) & 9(4.5 + j13.03) \\
1 & 0 & 684 & 684 & 684 \\
2 & 0 & -342 & -342 & -342 \\
0 & 0 & 171 & 17.1(5.5 + j13.03) & 17.1(5.5 - j13.03)
\end{bmatrix}
\]  

(44)

To simplify calculations the following transformation is made

\[ w = Mv \]  

(45)

Then, as shown earlier

\[ \dot{v} = \Lambda v \]  

(46)

where

\[
\Lambda = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & -10 & 0 & 0 \\
0 & 0 & 0 & -(5.5 + j13.03) & 0 \\
0 & 0 & 0 & 0 & -(5.5 - j13.03)
\end{bmatrix}
\]  

(47)

The solution of equation (46) is

\[ v(t) = e^{\Lambda t} v(0) \]  

(48)

As before, trial and error was used to find the following initial conditions for eigenvector excitation of the oscillatory mode in which the system reached the assumed
equilibrium point.

\[
\begin{bmatrix}
\theta_m(0) \\
\theta_s(0) \\
K(0) \\
K_v(0) \\
q(0)
\end{bmatrix}
= \begin{bmatrix}
2 \\
1.8 \\
.5 \\
2.25 \\
.26
\end{bmatrix}
\] (49)

Using these values equation (38) becomes

\[
\begin{bmatrix}
0 \\
-.2 \\
-.5 \\
.25 \\
.26
\end{bmatrix}
\] (50)

Equation (45) may be rewritten and solved as

\[
v(0) = M^{-1}w(0) = \begin{bmatrix}
0 \\
0 \\
.000018 \\
-.00037 - j.000736 \\
-.00037 + j.000736
\end{bmatrix}
\]

\(M^{-1}\) may be found in Appendix B. The solution for \(v(t)\) is found from equation (48) as

\[
v(t) = \begin{bmatrix}
0 \\
0 \\
.000018e^{-10t} \\
(-.00037 - j.000736)e^{-(5.5 + j13.03)t} \\
(-.00037 + j.000736)e^{-(5.5 - j13.03)t}
\end{bmatrix}
\] (51)

The system error is the difference of model and plant
outputs or from equation (38)
\[ e(t) = w_1(t) - w_2(t) \]  
(52)

Appendix B contains the solution of \( e(t) \) from equations (44), (45), and (51). The error due to eigenvector excitation of the oscillatory mode only is
\[ e(t) = 0.206 e^{-5.5t} \cos(13.03t + 7.7^\circ) \]  
(53)

It is noted that the time constant and frequency of oscillation for this solution were correctly predicted by the root locus of Figure 13.

2. Analogue Verification

As a check for accuracy of calculations based on the linearized model of the compensated system, a simulation of the system of Figure 12 was made using the values of the example being studied and the initial conditions of equation (49). The simulation drawing used in conjunction with an EAI TR-48 analogue computer may be found in Appendix A. The system error, \( e(t) \), and \( K(t) \) were recorded as shown in Figure 14. To facilitate comparisons the calculated system error, equation (53), was simulated and its time solution is shown in Figure 15.

In comparing Figures 14 and 15 it is seen that once again there is an excellent correlation between calculations based on a linear representation of the system and results obtained from an analogue simulation of the actual system. The most important observations come from comparing the system error of the original and compensated systems which
Figure 14. System Error and $K(t)$ for Eigenvector Excitation of the Compensated System
Figure 15. Calculated System Error for Eigenvector Excitation of the Linearized Compensated System
Figure 16. System Error of the Original and Compensated Systems for Eigenvector Excitation
are shown in Figure 16. The systems were identical, including initial conditions, except for the addition of compensation. The compensated system has a heavily damped error response. The peak overshoot has been significantly reduced and the response time has been decreased by a factor of approximately ten as compared to the original system. The same conclusions are found by comparing the system error of equations (26) and (53).

In summary, it has been found that large adaptive loop gains, \( \mu_1 \) and \( \mu_2 \), are necessary for good parameter tracking but this creates higher frequencies of the oscillatory mode. A root locus of the original linearized system has shown that the system may be improved by drawing the complex roots further into the left half plane. Compensation was added in the form of a lead network. The study of an example showed that the compensation significantly improved the oscillation problem. An increase in damping was predicted by the root locus of the compensated system and verified both by an analytic solution and an analogue simulation.

D. Study of System Performance

The example studied for the linearized system and the linearized compensated system was selected to simplify the mathematics involved in calculating system error. This section provides two systems in which parameter variation was more general. Due to the number of multipliers on the
available analogue computer only one of the plant parameters, $K_s$ and $b$, could be continuously time-varying. The plant dynamic parameter, $b$, was arbitrarily chosen as the parameter to be varied continuously. In the first system the plant has a sinusoidal variation of $b$ and of the input $r$. The second system has pseudo-random variation of $b$ and $r$ and step changes in $K_s$.

Although no restrictions are placed on the input one of the assumptions in the development of the model-referenced adaptive control was that plant variations would be "slowly" time-varying. It was noted that for higher adaptive gains, $\mu_1$ and $\mu_2$, the system adapts more rapidly. In view of these two points the value of $\mu_1$ and $\mu_2$ and the rate at which $b$ varies were chosen to demonstrate the improvement in system error response with the addition of compensation. For the two types of variation selected both the original and the compensated systems were simulated on an analogue computer.

1. First-Order System with Sinusoidal Variation of $r$ and $b$
   a. Original System

   The values used in the system of Figure 3 were:

   $K_m = 2 \quad \mu_1 = 20 \quad b(t) = .5 + \sin 1t \quad a = 1$
   $K_s = 1 \quad \mu_2 = 20 \quad r(t) = \sin .45t$

   The initial conditions were:

   $K_v(0) = 2 \quad K(0) = .5 \quad \theta_m(0) = 0 \quad \theta_s(0) = 0$
The time plots of $b$ and $r$ are shown in Figure 17. Plots of $x_1$, $x_2$, and $e$ are shown in Figure 18.

b. **Compensated System**

The values and initial conditions used in the system of Figure 12 were the same as the original system. The compensator had parameter values of $c = 20$ and $d = 2.9$; and $q(0)$ was zero. The time plots of $x_1$, $x_2$, and $e$ are shown in Figure 19. An expanded portion of the error is shown in Figure 20 for both the original and compensated systems.

2. **First-Order System with Pseudo-Random Variation of $r$ and $b$ and Step Changes in $K_s$**

a. **Original System**

The values for the model and adaptive loop gains used in the system of Figure 3 were:

\[ a = 1, \quad K_m = 2, \quad \mu_1 = \mu_2 = 20 \]

The variation of $r$, $b$, and $K_s$ are shown in Figure 21. The pseudo-random signals, $r$ and $b$, were generated using a digital computer. The method used is presented in Appendix C. The initial conditions were:

\[ K_v(0) = 1, \quad K(0) = .5, \quad \theta_m(0) = 0, \quad \theta_s(0) = 0 \]

and the resulting system error is shown in Figure 22.

b. **Compensated System**

The values and initial conditions for the system of Figure 12 were the same as the original system. The compensator had parameter values of $c = 20$ and $d = 2.9$; and
Figure 17. Input, r, and Dynamic Plant Parameter, b, for Sinusoidal Variation
Figure 13. $x_1(t)$, $x_2(t)$, and $e(t)$ for the Original System

with Sinusoidal Variation of $r$ and $b$
Figure 19. $x_1(t)$, $x_2(t)$, and $e(t)$ for the Compensated System with Sinusoidal Variation of $r$ and $b$
Figure 20. System Error for the Original and Compensated Systems with Sinusoidal Variation of \( r \) and \( b \)
Figure 21. Pseudo-Random Variation of \( r \) and \( b \) and Step Changes in \( K_s \)
Figure 22. System Error for the Original System with Pseudo-Random Variation of $r$ and $b$ and Step Changes in $K_s$
Figure 23. System Error for the Compensated System with Pseudo-Random Variation of $r$ and $b$ and Step Changes in $K_s$
Figure 24. System Error for both Original and Compensated Systems with Pseudo-Random Variation of \( r \) and \( \theta \) and Step Changes in \( K_3 \)
q(0) was zero. The system error is shown in Figure 23 and
an expanded portion of system error for both the original
and compensated systems is shown in Figure 24.

3. Comments

For the original system with sinusoidal variations the
error response is very oscillatory as noted from Figure 18.
The tracking of $K_m$ and $a$, denoted by $x_1$ and $x_2$ in Figure 18,
are also oscillatory. In Figure 19 the corresponding $e$, $x_1$
and $x_2$ for the compensated system show a definite improve­
ment in that oscillations are heavily damped without an
increase in peak values. The error for both systems may be
compared directly from Figure 20 which again points out the
improvement obtained from compensation.

Although the tendency to oscillate is not as prevalent
in the system with pseudo-random variations, it can be seen
from Figure 22 that high frequency oscillations do occur in
the original system. These oscillations have been heavily
damped in the compensated system as noted from Figure 23.
The error peaks for both systems are very large. In the
original system the maximum error is approximately \(12\frac{1}{2}\)
percent of the maximum model output while for the compen­
sated system it is \(14\frac{3}{4}\) percent. Errors of this magnitude
are undesirable but previous analysis and results have
shown that increasing the adaptive loop gains will decrease
the peak error. Figure 24 shows the error for both original
and compensated systems and indicates the improved control
of oscillations.
It is worthwhile to note that $b$, the dynamic plant parameter, for both sinusoidal and pseudo-random variation is allowed to vary quite severely and to even be negative. Without the adaptive control, compensated or uncompensated, a negative value of $b$ would indicate positive feedback in the plant. This is the same as operating in the right half plane on a root locus diagram which means the plant has become unstable. The model-referenced adaptive control has thus brought stability to an otherwise unstable plant.
III. SUMMARY AND CONCLUSIONS

The first-order model-referenced system was selected for study and the necessary adaptive control was developed using Lyapunov's second method. Several first-order systems were simulated on an analogue computer. This revealed an oscillatory mode in the system. In the light of past and present usage of model-referenced systems there was a need to obtain more information about the oscillatory mode and the possibilities of controlling it.

The model-referenced system as shown in Figure 3 is nonlinear. By linearizing the system about a known equilibrium point the eigenvalues and eigenvectors were found. The oscillatory mode was indicated by the complex pair of eigenvalues. An example was selected and system error was calculated from the linearized system and also found from a simulation of the system shown in Figure 3. A comparison of the error from these two methods using eigenvector excitation of the oscillatory mode was made, and it is concluded that the linearized system is a very good approximation of the nonlinear adaptive system in the neighborhood of the equilibrium point.
A root locus involving the complex eigenvalues of the linearized system, and also simulation studies, showed that an increase in adaptive loop gains, $\mu_1$ and $\mu_2$, caused the frequency of oscillations to increase. However it was found from simulations that large adaptive loop gains are necessary for good adaptation of the plant parameters to the model parameters. The cost of good adaptation qualities is then a substantial increase in the frequency of the oscillatory mode.

Having studied the root locus and knowing that the linearized system was a valid representation, linear compensation techniques were thought to be a possible solution to the oscillatory problem. A general lead compensation network was added to the original adaptive system, and in order to study the new compensated system linearization techniques were applied. The eigenvalues were found and again an oscillatory mode was indicated. The same example used for a study of the original system was chosen for a study of the compensated system. A root locus of the compensated system was then used to determine values for the parameters of the compensator which would give good damping of the oscillations. Initial conditions were chosen that would excite only the oscillatory mode, and once again good correlation was found for results obtained by simulation of the compensated system of Figure 12 and by calculation from the linearized compensated system.
A comparison of the system error for both original and compensated systems showed that an outstanding improvement in the reduction of oscillations was made by the addition of compensation. This was also shown from the study of two systems for which plant parameter and input variations were more general.

It is concluded that model-referenced adaptive control systems designed by Lyapunov's second method should begin with an assumption of large adaptive loop gains which are necessary for good tracking. This assumption implies highly oscillatory modes. It is further concluded then that compensation which can be designed using linearization and root locus techniques should be added to the system for the purpose of controlling the oscillations.
The following recommendations of future work represent problems and items of interest which were not covered in this paper.

1. The problem of the adaptive control drifting and/or not reaching ideal values for constant inputs warrants further study. It is suggested that a small amplitude noise signal at the input might be helpful.

2. It should be shown whether or not the compensated system is stable in the large.

3. Compensation has been added to both adaptive loops of the first-order system. For the example studied in section II-C if the compensation is added only to the adaptive loop involving $K$ the following equation is obtained for plotting a root locus.

$$0 = 1 + \frac{80(d - 1)\lambda}{(\lambda + c)(\lambda^2 + \lambda + 100)}$$

This is identical to equation (42) except that the number 100 in the numerator of (42) is now 80. The two loci are identical then except for values of $d$ at each point on the loci. Any advantages, disadvantages, or redundancies that
occur from compensating various combinations or individual adaptive loops would be a worthwhile and interesting study.

4. If the compensated model-referenced adaptive system is to be fully exploited, work should be done on optimization of the compensator.

5. It is not necessary that $N$ on page 10 be a diagonal matrix. Satisfactory performance might be achieved by a different choice of $N$ and at the same time eliminate the need for a compensator.
BIBLIOGRAPHY


A-1. Analogue Simulation Drawing of the Original System

Any scaling necessary for actual use with an analogue computer has not been shown.
A-2. Analogue Simulation Drawing of the Compensated System

Any scaling necessary for actual use with an analogue computer has not been shown.

\[ p(0) = z(0) - de(0) \]
APPENDIX B
Detailed Computations - Linearized System

B-1. Solution of $e(t)$ for the Linearized System from Equation (25)

$$e(t) = z_1(t) - z_2(t)$$  \hspace{1cm} (25)

From equation (24) \hspace{1cm} z_1(t) = 0

Therefore

$$e(t) = -z_2(t) = -0.01(\frac{-5 - j10}{-5 + j10})(\frac{-.625 - j1.03}{-.625 + j1.03})e^{-(.5 + j10)t}$$

$$= .121e^{-0.5t}[e^{j(10t + 33.8^\circ)} + e^{-j(10t + 33.8^\circ)}]$$

$$= .242e^{-0.5t}\cos(10t + 33.8^\circ)$$

B-2. Inverse Modal Matrix of Linearized System and Solution of $y(0)$

$$M^{-1} = 0.01$$

$$\begin{bmatrix}
0 & 0 & 20 & 40 \\
100 & 0 & 0 & 0 \\
-j5 & -j5 & 1 + j0.05 & -5 - j0.025 \\
-j5 & j5 & -1 + j0.05 & .5 - j0.025
\end{bmatrix}$$

$$y(0) = M^{-1}z(0) = M^{-1}\begin{bmatrix}
0 \\
-.2 \\
-.5 \\
.25
\end{bmatrix} = 0.01\begin{bmatrix}
0 \\
0 \\
-.625 - j1.03 \\
-.625 + j1.03
\end{bmatrix}$$
B-3. Adjoint of $\lambda I - J_{eq}$ for the Compensated System

\[
\text{Adj}[\lambda I - J_{eq}] = \begin{bmatrix}
\lambda (\lambda^3 + 21\lambda^2 + 310\lambda + 2000) & 0 & 0 & 0 \\
1710\lambda & \lambda (\lambda+1)(\lambda^2 + 20\lambda + 290) & -2\lambda (\lambda+1)(\lambda+20) & \lambda (\lambda+1)(\lambda+20) & 100\lambda (\lambda+1) \\
-684\lambda (\lambda+1) & 684\lambda (\lambda+1) & (\lambda+1)(\lambda^3 + 21\lambda^2 + 18\lambda + 400) & 40(\lambda+1)(2.9\lambda+20) & -40\lambda (\lambda+1)^2 \\
342\lambda (\lambda+1) & -342\lambda (\lambda+1) & (\lambda+1)(116\lambda+800) & (\lambda+1)(\lambda^3 + 21\lambda^2 + 252\lambda + 1600) & 20\lambda (\lambda+1)^2 \\
17.1\lambda^2 (\lambda+1) & -17.1\lambda^2 (\lambda+1) & \lambda (\lambda+1)(5.8\lambda+40) & -\lambda (\lambda+1)(2.9\lambda+20) & \lambda^2 (\lambda+1)^2
\end{bmatrix}
\]
B-4. **Inverse Modal Matrix for the Example Compensated System**

\[
\begin{bmatrix}
0 & 0 & .2 & .4 & 0 \\
1 & 0 & 0 & 0 & 0 \\
-.00526 & .00526 & .00056 & .00028 & .00277 \\
.00263 - j.00091 & -.00263 + j.00091 & .00086 + j.00015 & -.00043 - j.000076 & -.00138 - j.00177 \\
.00263 + j.00091 & -.00263 - j.00091 & .00086 - j.00015 & -.00043 + j.000076 & -.00138 + j.00177
\end{bmatrix}
\]
B-5. Solution of \( e(t) \) for the Compensated System from Equations (44), (45), and (51)

\[
\mathbf{w}(t) = M\mathbf{v}(t)
\]  \hspace{1cm} (45)

Using \( M \), equation (44), and \( \mathbf{v}(t) \), equation (51), it is found that

\[
w_1(t) = 0
\]

Therefore from equation (52)

\[
e(t) = -w_2(t)
\]

Then

\[
e(t) = -\left[ (40.5 - j117.3)(-.00037 - j.000736)e^{-(5.5 + j13.03)t} \right.
\]
\[
\quad + (40.5 + j117.3)(-.00037 + j.000736)e^{-(5.5 - j13.03)t}
\]
\[
\quad + 190(.000018)e^{-10t} \bigg]
\]
\[
= -.103e^{-5.5t}\left[ e^{j(13.03t + 187.7^\circ)} + e^{-j(13.03t + 187.7^\circ)} \right]
\]
\[
\quad - .00342e^{-10t}
\]
\[
= -.206e^{-5.5t}\cos(13.03t + 187.7^\circ) - .00342e^{-10t}
\]
\[
= .206e^{-5.5t}\cos(13.03t + 7.7^\circ) - .00342e^{-10t}
\]

The error should contain only the oscillatory mode since eigenvector excitation was used. The term \(-.00342e^{-10t}\) is negligible and has entered the analytic solution due to round-off error in the calculations.
Development of Pseudo-Random Signals

The pseudo-random signals used for r and b were obtained by passing digital signals from a Scientific Control Corporation 650 computer across D-A lines and then through a series of low-pass filters on the analogue computer. The digital computer was used to generate pseudo-random numbers by the following scheme. In a twelve bit register there are \(2^{12}\) different binary numbers that can be represented. All of these combinations except zero can be obtained in a random manner by an exclusive-or of the last two bits, a right shift of the register, and then a transfer of the result of the exclusive-or into the first bit position. This is shown schematically as

![Schematic Diagram]

The maximum output on the D-A lines of the SCC 650 is \(\pm 10\) volts, therefore \(2^{12} - 1\) different voltage levels between \(+10\) and \(-10\) can be transferred to analogue low-pass filters.

The program used to generate pseudo-random numbers from the SCC 650 by the scheme mentioned above is shown on the following page.
All numbers used in the SCC 650 program are in octal or base eight form. Initial values used were

\[
\begin{align*}
A &= 0000 \\
B &= 0000 \\
C &= 7772 \\
D &= 1000 \\
E &= 1000 \\
F &= 7772
\end{align*}
\]

SWITCH SETTING = 0020

The following drawing represents the analogue low-pass filters. The outputs of the filters, shown as \( r \) and \( b \), were then used in the model-referenced system.

![Diagram of the analogue low-pass filters](https://example.com/diagram.png)
VITA

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