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A time and frequency domain approach to the optimization of linear multivariable regulators

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A TIME AND FREQUENCY DOMAIN APPROACH TO THE OPTIMIZATION
OF LINEAR MULTIVARIABLE REGULATORS

BY

EUGENE CHARLES MACHACEK, 1941-

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ABSTRACT

Optimal design and analysis of a multivariable regulator may be achieved in either the frequency or time domain. This paper describes the formulation of the matrix Riccati equation in the time domain and the Wiener-Hopf equation and the root-square-locus in the frequency domain. The necessary requirements which must be satisfied in order to achieve an optimal control vector when using a quadratic performance index are presented for both domains. The resultant optimal control vector is shown to be a linear function of the system state vector. The effect of the quadratic performance index weighting matrices on the optimal system closed-loop poles, as well as the importance of picking "good" weighting matrices, is shown in this paper. A computer cost comparison of the two techniques of obtaining the optimal closed-loop roots indicates a marked advantage of the time domain approach over the frequency domain approach for high order systems.
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I. INTRODUCTION

All problems in control system optimization can be initially described as having:

1. A dynamic system which is to be controlled
2. A desired system output response
3. A set of allowable controls
4. A performance criteria which measures the effectiveness of the controls on the system.

Mathematically modern control theory has been developed through the use of state space formulation so that complex multivariable control systems can be more readily evaluated and optimized for a specific performance criteria. Control system optimization is in fact what has made modern control theory of such importance to control engineers. Because many of the sophisticated modern-day control problems require quality performance as well as efficiency, economy, reliability, and stability, the techniques of modern control theory have been developed to allow these design constraints to be taken into account during the initial design and evaluation of the control system.

Classical control theory is best suited to handle the design of single input and single output linear time-invariant systems. Even then the techniques are mainly pertinent to absolute stability and transient response such as rise time, time constant, maximum overshoot, settling time, phase lag, and steady state accuracy.
Modern control theory, which is more applicable to multivariable control systems than classical control theory, has experienced a tremendous growth during the past few years in the development of computational algorithms to handle the design and analysis of control systems. This along with the availability of digital computers and familiarity with the use of these computers by a growing number of control engineers is making modern control theory a very practical part of control system design and analysis.

In this paper the necessary mathematical state space theory and procedures required to optimize a linear multivariable output regulator will be presented. An output regulator is a controller designed to keep the output of a control system within an acceptable deviation from a reference signal.

Since it is felt that most control engineers are familiar with computer programming and in many cases have at their disposal specific computer programs for matrix manipulation and evaluation, this paper will not delve into computer techniques for solving the equations developed in this presentation. By specifically reviewing the regulator controller, a frame of reference is hopefully maintained throughout this paper and a more in depth presentation of the techniques of control theory is possible. The regulator problem is analysed in both the time domain and frequency domain and the interrelation of the two domains is presented.

In the time domain the development of a quadratic performance index and system description is presented along with the necessary
conditions and procedures required to obtain the optimal control vector using the Riccati matrix equation. The frequency domain analysis of the output regulator is achieved by using Parseval's theorem to convert the quadratic performance index from a time domain representation to a frequency domain representation and by using the Laplace transform to transfer the system matrix equation from the time to the frequency domain. The root-square-locus equation and the relationship of the time domain performance index weighting matrices to the optimal system closed-loop roots is developed in the frequency domain.

A comparison of the computer cost associated with solving a typical third and fourth order problem is made to establish the most economical approach for obtaining the system closed-loop roots.
II. REVIEW OF LITERATURE

Many of the most important concepts of optimal control theory in the time domain such as: performance index, controllability, observability, description of dynamical systems, and the solution for the optimal control vector using the matrix Riccati equation has been documented in papers by R. E. Kalman. Such papers as Kalman (1963), Kalman (1964), and Kalman (1965), to only reference a few, are examples of his prolific contribution to the science of optimal control theory. The work of Gilbert (1963) expands upon that of Kalman on controllability and observability. The work of Athans and Falb (1966) gives an in-depth presentation of the mathematical concepts involved with time domain optimal control theory.

The frequency domain approach to optimal control theory was introduced for the single variable systems by Chang (1961) and then developed for multivariable systems by Whitbeck (1965); Rynaski and Whitbeck (1966); Rynaski, Whitbeck, and Wierwille (1966); and Whitbeck (1968). Wonham (1967) shows the relationship between the time and frequency domain by presenting a proof of the equivalence of system controllability in the time domain with pole assignability in the frequency domain. The work of Tyler and Tuteur (1966) describes the relationship of the quadratic performance index weighting matrix parameters in the frequency domain to the optimal
system dynamics through the use of root locus plots. The frequency
domain solution of a regulator problem and the equivalence between
frequency domain spectral factorization and the solution of the
time domain matrix Riccati equation is described by Willis and
Brockett (1965).
III. DISCUSSION

A. THE MULTIVARIABLE CONTROL SYSTEM AND QUADRATIC PERFORMANCE INDEX

The type of control system discussed in this paper is a linear time-invariant multivariable control system. This type of control system can be mathematically described by linear differential equations with constant coefficients. Using the state variable technique, the control system can then be described as a matrix set of first order equations having the form:

\[ \dot{x}(t) = F \cdot x(t) + G \cdot u(t) \]  \hspace{1cm} (1)

\[ y(t) = H \cdot x(t) \]  \hspace{1cm} (2)

where

\( \dot{x}(t) \) is the time derivative of the m column vector representing the state variables of the differential equation

\( x(t) \) is the m column vector whose components are the state variables of the differential equation

\( u(t) \) is the n column vector whose components are the control inputs to the system

\( y(t) \) is the p column vector whose components are the outputs of the system

\( F \) is an \( m \times m \) constant differential transition matrix representing the relationship between the state variables and their time differential
\( G \) is an \( m \times n \) constant input matrix representing the relationship between the control inputs and the time derivative of the state.

\( H \) is a \( p \times m \) constant output matrix which defines the relationship between the state variables and the output variables.

Equations 1 and 2 represent the state space open-loop description of a dynamic system, often called the plant, where the state of the system \( \mathbf{x}(t) \) is determined by the input \( \mathbf{u}(t) \). The basic requirement of linear optimal control theory is to define the control input vector \( \mathbf{u}(t) \) which forces the system output \( \mathbf{y}(t) \) to respond in a desired manner, under specified constraints. Normally an integral function which includes both the control vector and the output vector is defined to measure the effectiveness and cost of the control vector in producing a desired system response. The optimal control is the control that minimizes this integral function, called the cost function or the performance index. The optimal control for a particular system will therefore depend upon the choice of performance index to be minimized.

The performance index considered in this paper is the quadratic performance index

\[
J = \frac{1}{2} \int_0^\infty \left[ \mathbf{y}'(t) \mathbf{Q} \mathbf{y}(t) + \mathbf{u}'(t) \mathbf{R} \mathbf{u}(t) \right] \, dt \tag{3}
\]

where

\( J \) is the quadratic performance index.
\( y'(t) \) is the transpose of the output vector \( y(t) \), that is if

\[
\begin{bmatrix}
  y_1(t) \\
  y_2(t) \\
  y_3(t) \\
  \vdots \\
  y_p(t)
\end{bmatrix}
\]

then

\[
y'(t) = [y_1(t) \, y_2(t) \, y_3(t) \, \ldots \, y_p(t)].
\]

\( u'(t) \) is the transpose of the control vector \( u(t) \).

\( Q \) is a \( p \times p \) positive semidefinite symmetric matrix. The elements of this matrix weight the contribution that each output makes to the performance index. A symmetric \( p \times p \) matrix is positive semidefinite if and only if all the eigenvalues \( \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_p \) of the matrix are nonnegative and at least one of the eigenvalues is zero.

\( R \) is an \( n \times n \) positive definite symmetric matrix. The elements of this matrix weight the contribution that each control input makes to the performance index. A symmetric \( n \times n \) matrix is positive definite if and only if all its eigenvalues are positive.
The quadratic performance index is used in this paper because it is an often used generalized form in optimal control theory which has the particular advantages, as stated by Tyler and Tuteur (1966) that: 1) it results in a closed form solution so that the properties of the control vector as well as the optimal system can be determined, 2) under reasonable restrictions on the weighting matrices it produces a stable system, 3) the optimal feedback gains can be determined once the numerical elements of the performance index weighting matrices are specified, 4) it results in a class of multivariable systems that satisfy a number of well-known design criteria. This particular performance index for a multivariable regulator takes into account the requirement for keeping the system output small and at the same time keeping the required control inputs no larger than is necessary to achieve a desired dynamic response.

The control system of interest in this paper is a multivariable regulator which is a feedback controller designed to keep the outputs of a dynamic system within an acceptable minimum deviation from a given reference signal. This reference signal $r$ is a constant vector, which is usually zero for a regulator system. The regulator controller can be distinguished from a terminal controller which operates to bring a system output to a desired condition, through an acceptable trajectory, in a specified time interval.
A description of the performance index for a multivariable system may more easily be described if the least-square optimization, similar to that found in Chang (1961) for a single-input single-output linear system and shown in Figure 1, is first considered.

In Figure 1

\[ r \text{ is the constant reference signal} \]
\[ e(t) \text{ is the deviation of the plant output from the desired reference} \]
\[ u(t) \text{ is the control input to the plant} \]
\[ y(t) \text{ is the plant output.} \]

By using the least-square optimization technique the design of an optimal system can be defined as the minimization of the sum of the integral-square error

\[ J_1 = \int_0^\infty [e(t)]^2 dt \quad (4) \]
and the integral-square of the control input

\[ J_2 = \int_0^\infty [u(t)]^2 \, dt. \]  (5)

Considering the transient response of a dynamic system, such things as poor rise time, settling time, overshoot, and steady state error contribute to the error \( e(t) \). By requiring a minimization of \( J_1 \), a restriction is effectively being placed on these dynamic response parameters. That is, a poorly responding system is penalized by this integral function because the integral function is a measure of the amount of actual output deviation from the reference signal. The same is true of \( J_2 \), by minimizing \( J_2 \), a controller is being specified which will require only the necessary control input to the plant which will give the desired output.

Since the reference signal is zero, \( J_1 \) can be rewritten as

\[ J_1 = \int_0^\infty [y(t)]^2 \, dt. \]  (6)

If the performance index \( J \) is defined as

\[ J = J_1 + J_2 \]  (7)

then

\[ J = \int_0^\infty \left\{ [y(t)]^2 + [u(t)]^2 \right\} \, dt. \]  (8)
By requiring a minimum $J$ the control engineer is effectively, within the design requirements and constraints, striving to obtain the best responding system with the least expenditure of control energy.

Consider now the case of the multivariable feedback control system as shown in Figure 2, where the double line represents multivariable signals.

![Multivariable Feedback Control System Diagram](image)

Figure 2. Multivariable Feedback Control System

The quadratic performance index of Equation 3, for a multivariable feedback control system, has the same form as the performance index represented by Equation 8. The difference between the performance index for a multivariable system and that of a single-input and
single-output system is that \( Q \) and \( R \) are used to weight the individual \( p \) outputs and the \( n \) control inputs of the multivariable system.

With a knowledgeable selection of \( Q \) and \( R \) a trade-off study between response cost and control cost of the output and control variables can be made by weighting certain outputs heavier than other outputs and by weighting certain controls heavier than other controls in the performance index. Once this is done the advantage of optimal control theory is that it allows one to obtain a unique optimal control vector \( \mathbf{u}^*(t) \) which will minimize the specified performance index. Optimal control theory, through the use of matrix and vector notation, is capable of making tractable multiple input and multiple output problems.

B. OPTIMIZATION OF A STATE REGULATOR

Consider the linear time-varying system represented by the equation

\[
\dot{\mathbf{x}}(t) = \mathbf{F}(t) \mathbf{x}(t) + \mathbf{G}(t) \mathbf{u}(t)
\]  

(9)

where

\( \dot{\mathbf{x}}(t) \) is the time derivative of the \( m \) column state vector
\( \mathbf{x}(t) \) is the \( m \) column state vector
\( \mathbf{u}(t) \) is the \( n \) column control vector
\( \mathbf{F}(t) \) is an \( m \times m \) time-varying differential transition matrix
\( \mathbf{G}(t) \) is an \( m \times n \) time-varying input matrix.
Also consider the performance index

\[
J = \frac{1}{2} \left[ x'(T) L x(T) \right] + \frac{1}{2} \int_{t_0}^{T} \left[ x'(t) q(t) x(t) + u'(t) r(t) u(t) \right] dt
\] (10)

where

- \( t_0 \) is the initial time at which the response of the system is considered
- \( T \) is the terminal time of system consideration
- \( L \) is an \( m \times m \) constant positive semidefinite matrix whose elements weight the contribution the state makes to the performance index at the terminal time \( T \)
- \( q(t) \) is an \( m \times m \) positive semidefinite time-varying matrix whose elements weight the contribution the state makes to the performance index from \( t_0 \) to \( T \)
- \( r(t) \) is an \( n \times n \) positive definite time-varying matrix whose elements weight the contribution the control vector makes to the performance index from \( t_0 \) to \( T \).

This performance index differs from the one presented earlier because for a state regulator it is desirable to keep the state of the system near zero without excessive use of control energy and so the state vector \( x(t) \) appears in the performance index rather than the output vector \( y(t) \). Also the term \( \frac{1}{2} \left[ x'(T) L x(T) \right] \) is included in the performance index to assure that the state error at
the terminal time $T$ is kept small. During the time interval $t_0$ to $T$, for the situation where the reference signal is equal to zero, the performance index $J$ of Equation 10 measures the relationship between how well the state is kept near zero and how much control input is required.

The problem of optimal control theory is one of determining the control vector $u(t)$ which minimizes the performance index of Equation 10 subject to the $m$ equality constraints of Equation 9. This is essentially a problem of minimization of the functional $J$, where $J$ is a function of a function. Looking first at the problem of determining the extremum of a function, one can find in DeRusso, Roy, and Close (1967) that if $\theta(g)$ is a function of $m$ independent variables the stationary points of this function can be found by differentiating $\theta(g)$ with respect to $g$ and solving the equation obtained when the derivative is set equal to zero. A stationary point is then called an extremum and it may be either a minimum or a maximum. If, however, the variables of the function are not independent but rather constrained by an equation of the form $\beta(g) = 0$ then the necessary conditions for an extremum can be determined by Lagrange's method of multipliers where a number of new parameters, called Lagrange multipliers, equivalent to the number of constraint equations are introduced. These multipliers $P_1, P_2, P_3, \ldots, P_m$ can be represented as components of a vector $p$ which is called the Lagrange multiplier or the costate vector. Then a function, usually called the Lagrange function, can be formed
by adding $\theta(g)$ and $[\beta(g)]' \ p$ as

$$\theta_c = \theta(g) + [\beta(g)]' \ p.$$  \tag{11}

The necessary conditions for an extremum can then be obtained from the Lagrange function as

$$\text{grad}_g \theta_c = 0$$ \tag{12}

and

$$\text{grad}_p \theta_c = 0$$ \tag{13}

where $\text{grad}_g$ and $\text{grad}_p$ are the vector operators

$$\text{grad}_g = \frac{\partial}{\partial g_1} = \begin{bmatrix} \frac{\partial}{\partial g_1} \\ \frac{\partial}{\partial g_2} \\ \vdots \\ \frac{\partial}{\partial g_m} \end{bmatrix} \tag{14}$$
Lagrange's method simplifies the process of obtaining an extremum of a functional by eliminating the process of solving the constraint equation for \( g \) and then substituting \( g \) into \( \theta(g) \) before differentiating. Rather, the conditions for an extremum are obtained directly by solving Equations 12 and 13.

The extremum of the quadratic performance index of Equation 10, subject to the functional constraints of Equation 9, can be found by forming a functional \( H_1 \), similar to the Lagrange function, from the time-varying portion of the performance index and the constraint equation. This functional \( H_1 \) is called the Hamiltonian and is defined as

\[
H_1 = \frac{1}{2} \left[ x'(t) Q(t) x(t) + u^*(t) R(t) u(t) \right]
\]

\[
+ \left[ P(t) x(t) \right]' p(t) + G(t) u(t) \right] p(t)
\]

(16)
where \( p(t) \) is the real Lagrange multiplier vector or the \( m \) costate vector which is the solution of the vector differential equation

\[
\dot{p}(t) = -\frac{\partial H_1}{\partial x(t)}.
\]  

(17)

Performing this differentiation on Equation 16 gives

\[
\dot{p}(t) = -G(t) x(t) - F'(t) p(t).
\]  

(18)

Assuming an optimal control does exist for the system and performance index of Equations 9 and 10 respectively then this optimal control must minimize the Hamiltonian. That is

\[
\frac{\partial H_1}{\partial u(t)} = R(t) u(t) + G'(t) p(t) = 0.
\]  

(19)

Solving Equation 19 for \( u(t) \) gives the control vector

\[
u(t) = -R^{-1}(t) G'(t) p(t).
\]  

(20)

If \( R^{-1}(t) \) exists and if Equation 20 defines a value of \( u(t) \) which gives a minimum value of \( H_1 \), then Equation 20 defines the optimal control vector \( u^*(t) \) which is

\[
u^*(t) = -R^{-1}(t) G'(t) p(t).
\]  

(21)

One of the original requirements of the weighting matrix \( R(t) \) was that it be positive definite, therefore \( R^{-1}(t) \) does exist and Equation 20 defines the value of \( u(t) \) which gives either a minimum or a maximum \( H_1 \). Equation 20 will define a minimum if the second
derivative of $H_1$ with respect to $u(t)$ is positive definite. This is similar to the requirement that the second derivative of a function be positive in order that the first derivative, when set equal to zero, defines a minimum of the function. Taking the second derivative of $H_1$ with respect to $u(t)$ results in

$$\frac{\partial^2 H_1}{\partial u^2 (t)} = R(t). \quad (22)$$

Thus, since $R(t)$ is a positive definite matrix the control vector $u^*(t)$ defined in Equation 21 does in fact minimize $H_1$.

The optimal control vector, as described in this section, has been defined in terms of the control weighting matrix, the system input matrix, and the costate vector. The costate vector is the only term in Equation 21 which has yet to be determined. Substituting Equation 21 into Equation 9 results in

$$\dot{x}(t) = F(t) x(t) - G(t) \cdot R^{-1}(t) \cdot G'(t) \cdot p(t). \quad (23)$$

Equations 18 and 23 can then be written in the partitioned matrix form

$$\begin{bmatrix} \dot{x}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} F(t) & -G(t) \cdot R^{-1}(t) \cdot G'(t) \\ -G(t) & -F'(t) \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} \quad (24)$$

where Equation 24 describes a state space representation of a system described by a $2m$ linear time-varying homogeneous differential equation. A unique solution of $x(t)$ and $p(t)$ for this system can
only be obtained if 2m boundary conditions are known. The initial conditions of the state vector at \( t = t_0 \) provides \( m \) of the boundary conditions. The other \( m \) boundary conditions are found by using the transversality condition which requires that, at the terminal time \( T \), the costate vector \( p(t) \) must satisfy the relationship

\[
p(T) = \frac{\partial}{\partial x(T)} \frac{1}{2} [x'(T) L x(T)].
\]  

(25)

Performing this differentiation results in

\[
p(T) = \Gamma(T) \dot{y}(T)
\]

(26)

which provides the remaining \( m \) boundary conditions required to solve Equation 24.

Ogata (1967) shows that a linear homogeneous vector-matrix differential equation can be represented in the form

\[
\dot{y}(t) = \Gamma(t) y(t)
\]

(27)

where

- \( y(t) \) is an \( m \) column vector
- \( \Gamma(t) \) is an \( m \times m \) differential transition matrix whose elements are assumed to be absolutely integrable as functions of \( t \) in the interval \( t_0 \leq t \leq t_1 \).

There are \( m \) linearly independent solutions to Equation 27 and they can be represented as \( \gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_m \). Any other solutions are linear combinations of \( \gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_m \). An \( m \times m \) matrix \( \gamma(t) \) can be formed such that its columns consist of the
m linearly independent solutions of Equation 27. This matrix is referred to as a fundamental matrix or matrizant. The fundamental matrix will satisfy the differential equation

\[ \dot{\mathbf{y}}(t) = \Gamma(t) \mathbf{y}(t). \]  

(28)

A fundamental matrix may differ from another matrix solution by a multiplicative constant matrix, however, for given initial conditions the fundamental matrix is uniquely determined. That is, if an \( m \times m \) matrix \( \Phi(t, t_0) \) is a solution of the matrix differential equation

\[ \dot{\Phi}(t, t_0) = \Gamma(t) \Phi(t, t_0) \]  

(29)

where

\[ \Phi(t_0, t_0) = \mathbf{I}. \]  

(30)

\( \mathbf{I} \) is the identity matrix. Any fundamental matrix \( \mathbf{y}(t) \) can be written as

\[ \mathbf{y}(t) = \Phi(t, t_0) \mathbf{C} \]  

(31)

where \( \mathbf{C} \) is a nonsingular constant matrix, that is the determinant of \( \mathbf{C} \) does not equal zero. Considering the homogeneous linear vector matrix differential equation of Equation 27 with initial condition

\[ \mathbf{y}(t_0) = \mathbf{y}_0. \]  

(32)
The solution of Equation 27 is given by

$$y(t) = \Phi(t, t_0) \, v(t_0)$$  \hspace{1cm} (33)

where $\Phi(t, t_0)$ is the unique solution of Equations 29 and 30.

Equation 33 can be verified by the following equations. At $t = t_0$ we see that

$$y(t_0) = \Phi(t_0, t_0) \, v(t_0) = I \, v(t_0) = v_0.$$

(34)

Also

$$\dot{y}(t) = \frac{d}{dt} \left[ \Phi(t, t_0) \, v(t_0) \right] = \frac{d}{dt} \left[ \Phi(t, t_0) \right] \, v(t_0)$$

$$= \dot{\Phi}(t, t_0) \, v(t_0) = \Gamma(t) \left[ \Phi(t, t_0) \, v(t_0) \right]$$

$$= \Gamma(t) \, y(t).$$

(35)

Thus one can see that Equation 33 is the solution of Equation 27.

Therefore, the homogeneous solution of Equation 27 is simply a transformation of the initial condition vector. This is the reason the unique fundamental matrix $\Phi(t, t_0)$ is often called the state transition matrix.

Going back to Equation 24, the $2m \times 2m$ fundamental matrix of this equation can be defined by $\Omega(t, t_0)$. The solution of Equation 24 then has the form

$$\left[ \begin{array}{c} x(t) \\ p(t) \end{array} \right] = \Omega(t, t_0) \left[ \begin{array}{c} x(t_0) \\ p(t_0) \end{array} \right]$$

(36)
where $p(t_o)$ is the unknown initial costate vector. Then at $t = T$ the relationship

$$
\begin{bmatrix}
\dot{x}(T) \\
\dot{p}(T)
\end{bmatrix}
= \Omega(T,t)
\begin{bmatrix}
x(t) \\
p(t)
\end{bmatrix}
$$

(37)

will exist. Partitioning the $2m \times 2m$ matrix $\Omega(T,t)$ into four $m \times m$ submatrices gives

$$
\Omega(T,t) = \begin{bmatrix}
\Omega_{11}(T,t) & \Omega_{12}(T,t) \\
\Omega_{21}(T,t) & \Omega_{22}(T,t)
\end{bmatrix}
$$

(38)

Equation 37 can then be rewritten, using Equations 26 and 38, as

$$
\dot{x}(T) = \Omega_{11}(T,t) \dot{x}(t) + \Omega_{12}(T,t) \dot{p}(t)
$$

(39)

$$
\dot{p}(T) = \Omega_{21}(T,t) \dot{x}(t) + \Omega_{22}(T,t) \dot{p}(t) = \Omega \dot{x}(T).
$$

(40)

Solving for $\dot{p}(t)$ by substituting Equation 39 into Equation 40 gives

$$
\dot{p}(t) = \left[\Omega_{22}(T,t) - \Omega \Omega_{12}(T,t)\right]^{-1} \left[\Omega \Omega_{11}(T,t) - \Omega_{21}(T,t)\right] \dot{x}(t).
$$

(41)

Equation 41 indicates that the costate vector $\dot{p}(t)$ and the state vector $\dot{x}(t)$ are related by an equation of the form

$$
\dot{p}(t) = \bar{E}(t) \dot{x}(t)
$$

(42)

where $\bar{E}(t)$ is an $m \times m$ time-varying matrix which depends only on the terminal time $T$ and the terminal weighting matrix $\Omega$ but not on
the state initial conditions. Now that the form of \( p(t) \) has been defined Equation 42 can be substituted into Equation 21 to give

\[
u^*(t) = -R^{-1}(t) G'(t) E(t) x(t).
\]

(43)

This is the equation for the optimal control vector and it can be reduced to the form

\[
u^*(t) = -K(t) x(t)
\]

(44)

which is the same equation as that derived by Greensite (1970) where \( K(t) \) is defined as the state feedback matrix. Thus the optimal control vector \( u^*(t) \) for the state regulator, when using the quadratic performance index, is a linear function of the state vector.

A method to evaluate the matrix \( E(t) \), which is the only term in Equation 43 that is not known, is now required in order to completely specify the optimal control vector. If Equation 42 is differentiated with respect to time then the resultant equation is

\[
\dot{p}(t) = \dot{E}(t) x(t) + E(t) \dot{x}(t).
\]

(45)

Expanding Equation 24 gives

\[
\dot{x}(t) = F(t) x(t) - G(t) R^{-1}(t) G'(t) p(t)
\]

(46)

and

\[
\dot{p}(t) = -G(t) x(t) - F'(t) p(t).
\]

(47)
Substituting Equation 42 into Equation 46 gives
\[ \dot{x}(t) = F(t)x(t) - G(t)R^{-1}(t)G'(t)F(t)x(t) \] (48)
or
\[ \dot{x}(t) = \left[ F(t) - G(t)R^{-1}(t)G'(t)F(t) \right] x(t). \] (49)

Substituting Equation 49 into Equation 45 results in
\[ \dot{p}(t) = \left[ \dot{E}(t) + E(t)F(t) - E(t)G(t)R^{-1}(t)G'(t)F(t) \right] x(t). \] (50)

Substituting Equation 42 into Equation 47 gives
\[ \dot{p}(t) = \left[ -Q(t) - F'(t)E(t) \right] x(t). \] (51)

Subtracting Equation 51 from Equation 50 results in
\[ \left[ \dot{E}(t) + E(t)F(t) - E(t)G(t)R^{-1}(t)G'(t)F(t) \right. \\
\left. + Q(t) + F'(t)E(t) \right] x(t) = 0. \] (52)

Equation 52 must hold for any choice of initial state. Also, since 
\( E(t) \) was found earlier to not depend upon the initial state and \( x(t) \) is a solution of Equation 49 it follows that Equation 52 must hold for any value of \( x(t) \). Then dividing Equation 52 by \( x(t) \) and rearranging terms gives
\[ \dot{P}(t) = -P(t) F(t) + P(t) G(t) R^{-1}(t) G'(t) P(t) \]
\[ -Q(t) - F'(t) P(t). \]

Equation 53 is the matrix differential Riccati equation. The solution of this equation for \( \dot{E}(t) \) gives the last term needed to describe the optimal control vector of Equation 43. The boundary conditions needed to solve Equation 53 can be found by considering Equations 26 and 42 at time equal to \( T \). That is

\[ p(T) = L x(T) \]  \hspace{1cm} (54)

and

\[ P(T) = E(T) x(T). \]  \hspace{1cm} (55)

Subtracting Equation 54 from Equation 55 gives

\[ \left[ E(T) - L \right] x(T) = 0 \]  \hspace{1cm} (56)

or

\[ E(T) = L \]  \hspace{1cm} (57)

which is the required boundary condition to solve the matrix differential Riccati equation.

Another characteristic of \( E(t) \) which will be derived before proceeding is that \( E(t) \) is a symmetric matrix. Since \( E(t) \) is a solution of the matrix Riccati equation and equal to \( L \) at \( t = T \), the transpose of both sides of Equation 53 is
\[
\left[ \frac{d}{dt} E(t) \right]' = -F'(t) E'(t) + E'(t) G(t) R^{-1}(t) G'(t) E'(t) - Q(t) - F'(t) F(t) \tag{58}
\]

Since \(Q(t)\) and \(G(t) R^{-1}(t) G'(t)\) are symmetric matrices their transpose is also a symmetric matrix which is equal to the matrix itself. For any matrix \(E(t)\) the equation

\[
\left[ \frac{d}{dt} E(t) \right]' = \frac{d}{dt} \left[ E'(t) \right] \tag{59}
\]

is true. Substituting Equation 59 into Equation 58 and rearranging terms gives

\[
\frac{d}{dt} \left[ E'(t) \right] = -F'(t) E(t) + E'(t) G(t) R^{-1}(t) G'(t) E'(t) - F'(t) E'(t) - Q(t) \tag{60}
\]

A comparison of Equation 53 and 60 shows that \(E(t)\) and \(E'(t)\) are solutions of the same matrix differential equation. The boundary condition at time \(T\) is given by Equation 57 where \(L\) is symmetric, that is \(L\) equals \(L'\). Then taking the transpose of both sides of Equation 57 gives

\[
E'(T) = L' = L \tag{61}
\]

Since \(E(t)\) and \(E'(t)\) are solutions of the same differential equation with the same boundary conditions, they are equal. That is

\[
E(t) = E'(t). \tag{62}
\]
Therefore $E(t)$, the solution of the matrix differential Riccati equation, is a symmetric matrix.

A more rigorous development of the linear time-varying multivariable state regulator problem can be found in Athans (1966), but what has been described thus far should be sufficient to show that the optimal control vector is specified by state variable feedback and that the solution of the matrix differential Riccati equation is required in order to specify the optimal control vector. Figure 3 shows the structure of the optimal feedback linear time-varying multivariable state regulator based on the equations developed thus far.

The preceding discussion was concerned with the linear time-varying multivariable state regulator. The case of a state regulator for a linear time-invariant multivariable system is now introduced. The equation which describes this system is

$$\dot{x}(t) = Fx(t) + Gu(t) \quad (63)$$

and the performance index is

$$J = \frac{1}{2} \int_0^\infty [x'(t) Q x(t) + u'(t) R u(t)] dt \quad (64)$$

Equations 63 and 64 are the same as Equations 9 and 10 based on the situation where $F$, $G$, $R$, and $Q$ are constant matrices having the same mathematical characteristics as described earlier and that $L = 0$ and $T = \infty$. For this problem one does not have to worry
Figure 3. Multivariable Time-Varying Optimal State Regulator
about the terminal cost \( \frac{1}{2} \begin{bmatrix} x'(T) \end{bmatrix} \mathbf{L} \begin{bmatrix} x(T) \end{bmatrix} \) in the performance index because terminal cost at \( T = \infty \) has no meaning. \( T \) is allowed to go to \( \infty \) in order to guarantee that the state stays near zero after an initial transient interval and then the arbitrary specification of a large terminal time \( T \) is avoided. For this system and performance index the optimal control vector is given by

\[
 u^*(t) = -R^{-1} G' E x(t)
\]

which is equivalent to Equation 43 where \( E \) is the constant \( m \times m \) positive definite matrix solution of the Riccati equation

\[
 -E F + E G R^{-1} G' E - Q - F' E = 0
\]

which is analogous to Equation 53 for the time-varying system.

One condition that is now required of the system which was not necessary when using a performance index with a finite terminal time is that the system be completely controllable. This means, according to Elgerd (1967), that it is possible to find a control vector which, in a specified finite time \( t_f \), will transfer the system between two arbitrarily specified states \( x_a \) and \( x_b \).

Controllability is a necessary requirement since, if the system were uncontrollable and also unstable, the performance index would be infinite for all controls because the time interval of the performance index is infinite. If this were the situation there would be no way to distinguish the optimal control from any control.

The mathematical definition for system controllability, as described by Elgerd (1967), is that the \( m \times mn \) matrix \( \Lambda \), called the
controllability matrix,

\[ \Lambda = \begin{bmatrix} G, \ F \ G, \ F^2 \ G, \ \cdots, \ F^{m-1} \ G \end{bmatrix} \]  

(67)

has rank m. This means the matrix must contain m linearly independent column vectors. \( \Lambda \) has rank m, as shown in Kreyszig (1967) if it contains at least one \( m \times m \) submatrix with a nonvanishing determinant, while the determinant of any square \( (m + 1) \times (m + 1) \) submatrix possibly contained in \( \Lambda \) is zero.

C. OPTIMIZATION OF AN OUTPUT REGULATOR

Previously the problem of finding the optimal control vector which would return the state of the plant back to zero after it had been displaced from zero by an external disturbance was considered. The case of returning the output of the plant, rather than the state, back to zero after an external disturbance is now investigated.

Considering the linear time-varying multivariable system described by the state variable equations

\[ \dot{x}(t) = F(t) x(t) + G(t) u(t) \]  

(68)

\[ y(t) = H(t) x(t) \]  

(69)

where

\( H(t) \) is a \( p \times m \) time-varying matrix describing the relationship between the system state vector and the output vector.
and the performance index given by

\[
J = \frac{1}{2} \left[ y'(T) L y(T) \right] + \frac{1}{2} \int_{t_0}^{T} \left[ y'(t) Q(t) y(t) \right]
+ u'(t) R(t) u(t) \, dt.
\]  

(70)

This performance index measures how well the output \( y(t) \) is kept near zero without excessive use of control energy. The terminal cost \( \frac{1}{2} \left[ y'(T) L y(T) \right] \) is included to take into account the requirement of keeping the output small at the terminal time \( T \). Substituting Equation 69 into Equation 70 results in

\[
J = \frac{1}{2} \left[ x'(T) H'(T) L H(T) x(T) \right] + \frac{1}{2} \int_{t_0}^{T} \left[ x'(t) H'(t) Q(t) H(t) x(t) \right]
+ u'(t) R(t) u(t) \, dt.
\]  

(71)

Comparing this performance index with that of Equation 10 for a time-varying state regulator indicates the only difference is that the matrices \( L \) and \( Q(t) \) of Equation 10 are replaced by \( H'(T) L H(T) \) and \( H'(t) Q(t) H(t) \) in Equation 71. By definition \( L \) and \( Q(t) \) are symmetric, so then \( H'(T) L H(T) \) and \( H'(t) Q(t) H(t) \) are symmetric. If the system described by Equations 68 and 69 is observable, which according to Luenberger (1966), who states a succinct mathematical discription based on the work of Kalman (1963) and similar work by Gilbert (1963), means the matrix
\[
\Psi = \left[ H'(t), \ F'(t) \ H'(t), \ F'(t)^2 \ H'(t), \ldots, \ F'(t)^{p-1} \ H'(t) \right] \tag{72}
\]
has rank \( p \). Then \( H'(t) \) must not be zero for time over the interval \( t_0 \) to \( T \). Since \( Q(t) \) is positive semidefinite, \( y'(t) \quad Q(t) \quad y(t) \geq 0 \) for all \( y(t) \) and \( x'(t) \quad H'(t) \quad Q(t) \quad H(t) \quad x(t) \geq 0 \) for all \( H(t) \quad x(t) \). Observability implies that it is possible to reconstruct the state vector \( x(t) \) from observations of the output vector \( y(t) \). Therefore \( x'(t) \quad H'(t) \quad Q(t) \quad H(t) \quad x(t) \geq 0 \) for all \( x(t) \) and \( H'(t) \quad Q(t) \quad H(t) \) is positive semidefinite. Using this same logic, one can see that since \( L \) is a positive semidefinite matrix

\[
y'(T) \quad L \quad y(T) \geq 0 \tag{73}
\]
for all \( y(T) \) and

\[
y(T) = H(T) \quad x(T). \tag{74}
\]
Taking the transpose of Equation 74 and substituting it into Equation 73 gives

\[
x'(T) \quad H'(T) \quad L \quad H(T) \quad x(T) \geq 0 \tag{75}
\]
for all \( H(T) \quad x(T) \). Observability implies that \( x(T) \) can be generated from \( y(T) \), therefore

\[
x'(T) \quad H'(T) \quad L \quad H(T) \quad x(T) \geq 0 \tag{76}
\]
for all \( x(T) \) and \( H'(T) \quad L \quad H(T) \) is also positive semidefinite.

With these points established the same technique used on the state regulator problem can now be applied to the output regulator.
problem. Given a linear observable time-variant multivariable system described by the equations

\[ \dot{x}(t) = F(t) x(t) + G(t) u(t) \quad (77) \]

\[ y(t) = H(t) x(t) \quad (78) \]

and the performance index

\[ J = \frac{1}{2} \left[ y'(T) L y(T) \right] + \frac{1}{2} \int_{t_0}^{T} \left[ y'(t) Q(t) y(t) \right. \\
\left. + u'(t) R(t) u(t) \right] dt. \quad (79) \]

The optimal control vector can be determined, by using the same technique as was used on the state regulator, to be

\[ u^*(t) = - R^{-1}(t) G'(t) E(t) x(t) \quad (80) \]

where the \( m \times m \) symmetric and positive definite matrix \( E(t) \) is the solution of the Riccati equation

\[ \dot{E}(t) = - E(t) F(t) - F'(t) E(t) - H'(t) Q(t) H(t) \]

\[ + E(t) G(t) R^{-1}(t) G'(t) E(t) \quad (81) \]

with the boundary conditions to this matrix equation

\[ E(T) = H'(T) L H(T). \quad (82) \]

A block diagram of this optimal output multivariable regulator system is shown in Figure 4.
Figure 4. Multivariable Time-Varying Optimal Output Regulator
We now consider the type of control system which the remainder of this paper will be concerned, that is the linear time-invariant output regulator which is controllable and observable and described by the equations

\[
\dot{x}(t) = F x(t) + G u(t) \tag{83}
\]

\[
y(t) = H x(t) \tag{84}
\]

with the performance index given by

\[
J = \frac{1}{2} \int_0^\infty \left[ v'(t) Q v(t) + u'(t) R u(t) \right] dt. \tag{85}
\]

An analysis similar to that carried out on the time-invariant state regulator problem results in the optimal control given by

\[
u^*(t) = - R^{-1} G' E x(t) \tag{86}
\]

where \( E \) is the solution of the time-invariant matrix Riccati equation

\[
- E F - F' E + E G R^{-1} G' E - H' Q H = 0. \tag{87}
\]

Substituting the optimal control vector of Equation 86 into the system Equation 83 gives

\[
\dot{x}(t) = F x(t) - G R^{-1} G' E x(t) \tag{88}
\]

or

\[
\dot{x}(t) = \left[ F - G R^{-1} G' E \right] x(t). \tag{89}
\]
The solution of Equation 89 gives the optimal state of the system. Substitution of this optimal state into Equation 84 will then define the optimal system output. A block diagram of this optimal multivariable time-invariant output regulator is shown in Figure 5.

D. A FREQUENCY DOMAIN APPROACH TO LINEAR MULTIVARIABLE REGULATOR OPTIMIZATION

The frequency domain approach to optimal control theory makes use of a root plotting technique, similar to the root locus plots of classical control theory, called the root-square-locus method. A description of this method can be found in Chang (1961), for the single-input single-output system. Chang's method has been expanded, as shown in Rynaski and Whitbeck (1966), through the use of matrix and vector mathematics to include multivariable input and output control systems.

In the preceding section the optimal control vector \( u^*(t) \) was determined, in the time domain, for a linear multivariable time-invariant output regulator as a linear function of the current state of the system. That is

\[
   u^*(t) = -Kx(t) \tag{90}
\]

where

\[
   K = R^{-1}G^TF \tag{91}
\]

The state feedback constant matrix \( K \) is a function of \( R \), \( G \), and \( F \), where \( F \) is the solution of the matrix Riccati equation, \( G \) is the
Figure 5. Multivariable Time-Invariant Optimal Output Regulator
system input matrix, and $R$ is the control weighting matrix of the quadratic performance index. If $u^*(t)$ is substituted into the system matrix equations

$$
\dot{x}(t) = A x(t) + B u(t) \quad (92)
$$

$$
y(t) = C x(t) \quad (93)
$$

then

$$
\dot{x}(t) = A x(t) - G K x(t) \quad (94)
$$

$$
y(t) = H x(t). \quad (95)
$$

Taking the Laplace transform of Equations 94 and 95 results in

$$
I_s X(s) - X(0) = F X(s) - G K X(s) \quad (96)
$$

$$
Y(s) = H X(s). \quad (97)
$$

Combining Equations 96 and 97 and then rearranging terms gives

$$
\begin{bmatrix} I_s - F + G K \end{bmatrix} X(s) = X(0) \quad (98)
$$

or

$$
X(s) = \begin{bmatrix} I_s - F + G K \end{bmatrix}^{-1} X(0) \quad (99)
$$

and

$$
Y(s) = H \begin{bmatrix} I_s - F + G K \end{bmatrix}^{-1} X(0). \quad (100)
$$
If the input disturbance to the system is considered to be the initial condition vector \( x(0) \), then the system transfer function of output to input is

\[
\frac{Y(s)}{x(0)} = H \left[ I_s - F + G K \right]^{-1}.
\]

(101)

This is the linear multivariable time-invariant output regulator closed-loop optimal transfer function. The right-hand side of Equation 101 can be expanded as

\[
H \left[ I_s - F + G K \right]^{-1} = \frac{H \text{adj} \left[ I_s - F + G K \right]}{\det \left[ I_s - F + G K \right]}
\]

(102)

where \( \text{adj} \left[ I_s - F + G K \right] \), the adjoint of the square matrix \( [I_s - F + G K] \), is the transpose of \( [I_s - F + G K] \) obtained when each term of this matrix is replaced by its cofactor. The closed-loop characteristic equation of the optimal system is simply the denominator of Equation 102 set equal to zero. That is

\[
\det \left[ I_s - F + G K \right] = \left| I_s - F + G K \right| = 0.
\]

(103)

The closed-loop characteristic equation is therefore a function of the system transition and input matrices \( F \) and \( G \) as well as the constant feedback gain matrix \( K \). \( K \) is a function of \( \Sigma \), the solution of the matrix Riccati equation, which in turn is a function of \( Q \) and \( R \) the weighting matrices of the quadratic performance index. Therefore, once the selection of the performance index and weighting
matrices has been made, the system closed-loop roots are automatically specified and the dynamic response of the system has been determined.

In the preceding time domain approach to optimal control theory several restrictions concerning the mathematical form of $Q$ and $R$ were made. One can now see that the dynamics of the optimal multivariable regulator are actually specified by the choice of the performance index weighting matrices. The importance of this point should not be treated lightly since the choice of these parameters is the very basis of optimal control theory. It should be apparent at this time that obtaining a desired closed-loop system response can be a very involved trial-and-error process in search of the necessary performance index weighting matrices. The need for a plotting technique similar to the root locus plots for a scalar system is apparent, so that some feel for the effect of variations in the parameters of $Q$ and $R$ may be achieved. The development of the frequency domain approach to optimal control theory leads to such a plotting technique which is called the root-square-locus.

The work of Whitbeck (1965) shows that the quadratic performance index may be converted from the time domain to the frequency domain by applying Parserval's Theorem to the quadratic performance index. Parserval's Theorem can be developed by considering the functions $x(t)$ and $y(t)$ which are bounded functions, where the absolute value of these functions integrated over the limits of $-\infty$ to $+\infty$ is less than $\infty$. The function $y(t)$ can then be written as
\[ y(t) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} Y(s)e^{st} \, ds \quad (104) \]

where \( Y(s) \) is the Fourier transform of \( y(t) \). Then, given the integral

\[ I = \int_{-\infty}^{\infty} x(t) y(t) \, dt \quad (105) \]

by substituting Equation 104 into Equation 105 one can get

\[ I = \frac{1}{2\pi j} \int_{-\infty}^{\infty} \int_{-j\infty}^{j\infty} x(t) Y(s)e^{st} \, ds \, dt. \quad (106) \]

Interchanging the order of integration gives

\[ I = \frac{1}{2\pi j} \int_{-\infty}^{j\infty} Y(s) \int_{-\infty}^{\infty} x(t)e^{st} \, dt \, ds. \quad (107) \]

Now the integral \( \int_{0}^{\infty} x(t)e^{st} \, dt \) can be rewritten, based on the definition for the Fourier transform as

\[ \int_{-\infty}^{\infty} x(t)e^{st} \, dt = X(-s). \quad (108) \]

Then

\[ I = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} Y(s) X(-s). \quad (109) \]

If \( y(t) = x(t) \) then Equation 104 becomes
Performing the same mathematical operations on Equation 110 as was performed on Equation 105 results in

\[ I = \int_{-\infty}^{\infty} x(t) \, x(t) \, dt = \int_{-\infty}^{\infty} \left[ x(t) \right]^2 \, dt. \] (110)

Equation 111 can then be used to convert a time domain quadratic integral function to a frequency domain integral function.

Considering the quadratic performance index for the time-invariant multivariable output regulator as defined by Equation 85, it is possible to define

\[ 2J_1 = \int_{0}^{\infty} \left[ y'(t) \, Q \, y(t) \right] \, dt \] (112)

and

\[ 2J_2 = \int_{0}^{\infty} \left[ u'(t) \, R \, u(t) \right] \, dt \] (113)

where \( 2J_1 \) is defined as the weighted integral-square error and \( 2J_2 \) is the weighted integral square value of the control input.

Applying Parseval's Theorem to \( 2J_1 \) plus \( 2J_2 \) gives the frequency domain representation of the quadratic performance index of Equation 85.
\[ 2J = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left[ Y(s) \ast G Y(s) + U(s) \ast R U(s) \right] ds \quad (114) \]

where

\[ Y(s) \ast \triangleq Y'(-s). \quad (115) \]

Taking the Laplace transform of Equations 92 and 93 results in

\[ \begin{bmatrix} Is - F \end{bmatrix} X(s) = G U(s) + x(0) \quad (116) \]

\[ Y(s) = H X(s). \quad (117) \]

Rearranging terms and combining Equations 116 and 117 results in

\[ X(s) = \left[ Is - F \right]^{-1} G U(s) + \left[ Is - F \right]^{-1} x(0) \quad (118) \]

and

\[ Y(s) = H \left[ Is - F \right]^{-1} G U(s) + H \left[ Is - F \right]^{-1} x(0). \quad (119) \]

If

\[ W \triangleq H \left[ Is - F \right]^{-1} G \quad (120) \]

\[ B_1 \triangleq H \left[ Is - F \right]^{-1} \quad (121) \]

\[ B \triangleq B_1 x(0) \quad (122) \]

then

\[ Y(s) = W U(s) + B. \quad (123) \]
Taking the transpose of Equation 123 when \( s \mapsto -s \), results in

\[
Y(s)^* = U(s)^* \, W^* + B^* \tag{124}
\]

where

\[
W^* = G' \left[ -Is - F' \right]^{-1} H'. \tag{125}
\]

Substituting Equations 123 and 124 into the frequency domain representation of the quadratic performance index, Equation 114, gives

\[
2J = \frac{1}{2\pi j} \int_{-\infty}^{\infty} \left\{ \left[ U(s)^* \, W^* + B^* \right] Q \left[ W \, U(s) + B \right] \right. \\
+ \left. U(s)^* \, R \, U(s) \right\} ds \tag{126}
\]

or

\[
2J = \frac{1}{2\pi j} \int_{-\infty}^{\infty} \left\{ \left[ U(s)^* \, W^* \, Q \, W \, U(s) + U(s)^* \, W^* \, Q \, B \right] \\
+ B^* \, Q \, W \, U(s) + B^* \, Q \, B + U(s)^* \, R \, U(s) \right\} ds. \tag{127}
\]

What is now required, as was in the time domain approach, is the optimum control vector \( \bar{U}^*(s) \) which will minimize this quadratic performance index. Since the only system of interest will be a stable system, a requirement of \( U(s) \) in Equation 127 is that it be analytic in the right-half plane. This means the poles of all the terms of \( U(s) \) must be in the left-half plane.
The frequency domain representation of the plant control input $\mathbf{u}(s)$ can be written as

$$
\mathbf{u}(s) = \mathbf{u}^*(s) + \lambda \mathbf{u}_1(s) 
$$

(128)

where

- $\mathbf{u}^*(s)$ is the optimum control vector which is analytic in the right-half plane
- $\lambda$ is a constant multiplier
- $\mathbf{u}_1(s)$ is an arbitrary column control vector which must be analytic in the right-half plane because $\mathbf{u}(s)$ and $\mathbf{u}^*(s)$ are analytic in the right-half plane.

Substituting Equation 128 into Equation 127 results in

$$
2J = \frac{1}{2\pi j} \int_{-\infty}^{\infty} \left\{ \mathbf{u}^*(s) \mathbf{u}^* + \lambda \mathbf{u}_1(s) \mathbf{u}_1 \right\} \mathbf{w}^* \mathbf{w} \left[ \mathbf{u}^*(s) + \lambda \mathbf{u}_1(s) \right] ds 
$$

$$
+ \left[ \mathbf{u}^*(s) \mathbf{u}^* + \lambda \mathbf{u}_1(s) \mathbf{u}_1 \right] \mathbf{w}^* \mathbf{w} \mathbf{b}^* \mathbf{b} \left[ \mathbf{u}^*(s) + \lambda \mathbf{u}_1(s) \right] 
$$

$$
+ \left[ \mathbf{u}^*(s) \mathbf{u}^* + \lambda \mathbf{u}_1(s) \mathbf{u}_1 \right] \mathbf{r} \left[ \mathbf{u}^*(s) + \lambda \mathbf{u}_1(s) \right] 
$$

$$
+ \mathbf{b}^* \mathbf{b} \right\} ds 
$$

(129)

and upon expanding Equation 129 the resulting equation is
2J = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left\{ U^*(s) \ast W^* Q W U^*(s) + U^*(s) \ast W^* Q W \lambda U_1(s) \\
+ \lambda U_1(s) \ast W^* Q W U^*(s) + \lambda^2 U_1(s) \ast W^* Q W U_1(s) \\
+ U^*(s) \ast W^* Q B + \lambda U_1(s) \ast W^* Q B + B^* Q W U^*(s) \\
+ B^* Q W U_1(s) + U^*(s) \ast R U^*(s) + U^*(s) \ast R U_1(s) \\
+ \lambda U_1(s) \ast R U^*(s) + \lambda^2 U_1(s) \ast R U_1(s) \\
+ B^* Q B \right\} ds \tag{130}

Rearranging the terms of Equation 130 gives

\begin{align*}
2J &= \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left[ U^*(s) \ast W^* Q W U^*(s) + U^*(s) \ast W^* Q B + B^* Q B \\
&\quad + B^* Q W U^*(s) + U^*(s) \ast R U^*(s) \right] ds \\
&\quad + \frac{\lambda}{2\pi j} \int_{-j\infty}^{j\infty} \left[ U^*(s) \ast W^* Q W U_1(s) + B^* Q W U_1(s) \\
&\quad + U^*(s) \ast R U_1(s) \right] ds \\
&\quad + \frac{\lambda}{2\pi j} \int_{-j\infty}^{j\infty} \left[ U_1(s) \ast W^* Q W U^*(s) + U_1(s) \ast W^* Q B \\
&\quad + U_1(s) \ast R U^*(s) \right] ds \\
&\quad + \frac{\lambda^2}{2\pi j} \int_{-j\infty}^{j\infty} \left[ U_1(s) \ast W^* Q W U_1(s) + U_1(s) \ast R U_1(s) \right] ds \tag{131}
\end{align*}
Equation 131 can be written as

$$2J = J_a + \lambda (J_b + J_c) + \lambda^2 J_d$$ (132)

where

$$J_a = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left[ \mathbf{U}^*(s)^{*} \mathbf{W}^* \mathbf{Q} \mathbf{W} \mathbf{U}^*(s) + \mathbf{U}^*(s)^{*} \mathbf{W}^* \mathbf{Q} \mathbf{B} + \mathbf{B}^* \mathbf{Q} \mathbf{B} \right] ds$$ (133)

$$J_b = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left[ \mathbf{U}^*(s)^{*} \mathbf{W}^* \mathbf{Q} \mathbf{W} \mathbf{U}_1(s) + \mathbf{B}^* \mathbf{Q} \mathbf{W} \mathbf{U}_1(s) \right. \left. + \mathbf{U}^*(s)^{*} \mathbf{R} \mathbf{U}_1(s) \right] ds$$ (134)

$$J_c = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left[ \mathbf{U}_1(s)^{*} \mathbf{W}^* \mathbf{Q} \mathbf{W} \mathbf{U}^*(s) + \mathbf{U}_1(s)^{*} \mathbf{W}^* \mathbf{Q} \mathbf{B} \right. \left. + \mathbf{U}_1(s)^{*} \mathbf{R} \mathbf{U}^*(s) \right] ds$$ (135)

$$J_d = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left[ \mathbf{U}_1(s)^{*} \mathbf{W}^* \mathbf{Q} \mathbf{W} \mathbf{U}_1(s) + \mathbf{U}_1(s)^{*} \mathbf{R} \mathbf{U}_1(s) \right] ds$$ (136)

Considering Equations 133, 134, 135, and 136 one can see that $J_a$ is the optimum component of the quadratic performance index, that $J_d$ is always positive and that $J_b$ is the transpose of $J_c$ when $s \rightarrow -s$. Therefore, the necessary condition for a minimum value of the performance index is that $J_c$ equal zero. This will mean that $J_b$ is also equal to zero and a minimum value of the performance index
will exist. Rearranging terms in Equation 135 and setting $J_c$ equal to zero gives

$$J_c = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left\{ U_1(s) \left[ R + W_\star Q W \right] U^\star(s) + W_\star Q B \right\} ds = 0.$$  \hspace{1cm} (137)

If the term in brackets in Equation 137 is defined as

$$\left( R + W_\star Q W \right) U^\star(s) + W_\star Q B = Z(s)$$  \hspace{1cm} (138)

then, since $U_1(s)$ was defined as an arbitrary column control vector which is analytic in the right-half plane, $U_1(s)_\star$ must be a row vector which is analytic in the left-half plane and over the path of integration the contour integral of Equation 137 will vanish if $Z(s)$ is a vector with all its poles in the right-half plane.

Equation 138 is known as the matrix Wiener-Hopf equation.

Whitbeck (1965) presents two methods of solution of Equation 138, they are spectral factorization and the direct method. The direct method, the most applicable to regulator problems will be the only method described in this paper.

Assuming the vector $Z(s)$ is known, then solving for $U^\star(s)$ gives

$$U^\star(s) = \left[ R + W_\star Q W \right]^{-1} \left[ Z(s) - W_\star Q B \right]$$  \hspace{1cm} (139)

or

$$U^\star(s) = \frac{\text{adj} \left[ R + W_\star Q W \right] Z(s) - W_\star Q B}{\det \left[ R + W_\star Q W \right]}.$$  \hspace{1cm} (140)
The matrix \[ R + W_\ast Q W \] can be written as the ratio of a matrix which contains polynomial terms divided by the plant open-loop poles and the conjugate of the open-loop poles. This can be verified by recalling that the definition of \( W \), according to Equation 120, is

\[ W = H \left[ I_s - F \right]^{-1} G \]  

(141)

which can be rewritten as

\[ W = \frac{H \text{adj} \left[ I_s - F \right] G}{\text{det} \left[ I_s - F \right]} . \]

(142)

Setting the denominator of Equation 142 to zero gives the plant open-loop poles. The denominator of \( W_\ast \) set equal to zero defines the conjugate of the open-loop poles. If \( D \) represents the open-loop poles, \( \overline{D} \) the conjugate of the open-loop poles, and \( k \) is the order of the matrix \( \left[ R + W_\ast Q W \right] \), then

\[ \text{adj} \left[ R + W_\ast Q W \right] = \frac{M}{(D \overline{D})^{k-1}} \]

(143)

where \( M \) is a matrix with polynomial terms. The denominator of \( W_\ast(s) \) can be represented as

\[ \text{det} \left[ R + W_\ast Q W \right] = \frac{N \overline{N}}{D \overline{D}} \]

(144)

where \( N \) is the left-half plane zeros and \( \overline{N} \) is the right-half plane zeros. Setting the left side of Equation 144 equal to zero gives
\[
\det \left[ R + W_{\alpha} Q W \right] = 0
\]  

(145)

which is the root-square-locus for multivariable optimal systems.

This scalar equation is the closed-loop characteristic equation of a multivariable system. By solving Equation 145 for the left-half plane roots, the closed-loop poles of the optimal system are defined as a function of the performance index weighting parameters.

Solving Equation 145 gives the same results as solving Equation 103, except by using Equation 145 it is unnecessary to solve the Riccati matrix equation for each change in \( Q \) and \( R \). By plotting a standard root locus diagram of \( N \), for variations of \( Q \) and \( R \), a trade-off between system speed of response and control magnitudes can be made and the desired weighting matrices can be specified.

The vector \( \left[ Z(s) - W_{\alpha} Q B \right] \) can be redefined so that

\[
\left[ Z(s) - W_{\alpha} Q B \right] = \frac{T_1(s)}{T_2(s) T_3(s)}
\]

(146)

where

- \( T_1(s) \) is a vector which describes the numerator polynomial of \( \left[ Z(s) - W_{\alpha} Q B \right] \)
- \( T_2(s) \) is a polynomial containing all the least common left-half plane poles of \( \left[ Z(s) - W_{\alpha} Q B \right] \)
- \( T_3(s) \) is a polynomial containing all the least common right-half plane poles of \( \left[ Z(s) - W_{\alpha} Q B \right] \).

Since \( Z(s) \) is analytic in the left-half plane \( T_2(s) \) must be a function of the vector \( W_{\alpha} Q B \) only. Substituting Equations 143, 145,
and 146 into Equation 140 results in

\[ U^*(s) = \frac{M T_1(s)}{N \overline{N} T_3(s) [D \overline{D}]^{k-1}}. \] (147)

In order to have a stable system the terms of \( U^*(s) \) must have their poles in the left-half plane. That is, \( U^*(s) \) must be analytic in the right-half plane. \( M T_1(s) \), which is a vector with polynomial terms must contain the factor \( \overline{N} T_3(s) \) because they are the terms which make up the right-half plane roots. The open-loop poles must also cancel into \( M T_1(s) \) since the open-loop poles cannot be closed-loop poles. Equation 147 then reduces to

\[ U^*(s) = \frac{V(s)}{N}. \] (148)

where

\( V(s) \) is an \( n \) column vector containing polynomial terms with coefficients which have yet to be determined.

The roots of the denominator of Equation 148, which are the closed-loop poles of the optimum system, can be found by evaluating the multivariable root-square-locus equation for \( N \). To solve for the undetermined coefficients of \( V(s) \), Equation 148 can be substituted into Equation 138 resulting in

\[ \left[ R + W_s \sigma W \right] \left[ \frac{V(s)}{N} \right] + W_s \sigma B = Z(s). \] (149)
Equation 149 can then be expanded so that there is a common denominator consisting of all the left-half plane poles. Since \( Z(s) \) is analytic in the left-half plane, it must have zeros which cancel the left-half plane poles. Once the desired left-half plane poles, the closed-loop poles of the optimal system, are specified then \( s \) can be allowed to take on values of these poles. This will force the denominator to be zero. Since it is required that the numerator also equal zero, the values of the undetermined coefficients of \( Y(s) \) can be found by equating the numerator to zero for those values of \( s \) equal to the closed-loop poles. This will generate \( m \) simultaneous equations which, when solved, give the coefficients of \( Y(s) \). Once these coefficients are determined the optimal control vector, as defined by Equation 148, is specified.

The optimal control vector can be substituted into Equation 118 which results in

\[
X(s) = \left[ I_s - F \right]^{-1} G U^*(s) + \left[ I_s - F \right]^{-1} x(0). \tag{150}
\]

Equation 150 can be solved for \( X(s) \) and then substituted into the Laplace transformed optimal control law of Equation 90, which is

\[
U^*(s) = -K X(s). \tag{151}
\]

Equation 151 can then be solved for the unknown feedback gain matrix \( K \) which will force the desired closed-loop poles of the system. As it turns out this feedback gain matrix is the same matrix as is defined by Equation 91 in the time domain.
E. COST COMPARISON OF TIME VERSUS FREQUENCY DOMAIN APPROACH

Common usage of the term "optimal control vector" to describe the control vector given in Equation 86 is an unfortunate description of what this vector actually represents. Equation 86 defines a control vector which may well be optimal only from a mathematical point of view. That is, it is the "best" control vector for the multivariable system described by Equations 83 and 84, based on the quadratic performance index described by Equation 85. However, this may have little or no meaning from an engineering point of view. In the final analysis it is usually the damping and response of the outputs of the multivariable regulator system, following an external initial condition disturbance, which must be acceptable for the particular control system. Therefore, a truly optimal control vector will be one which results in acceptable regulator response at a minimum cost of control energy, not one which is the best solution of some mathematical equation. This is the reason then that the approach of determining the performance index weighting matrices that will result in acceptable regulator response and then solving for the optimal control vector, based on these weighting matrices, is the only meaningful approach from an engineering point of view. Two questions that must then be answered are:

1. What is the best method of determining the system closed-loop roots as the performance index weighting matrices are varied?
2. What is the best method for determining the feedback gains which will result in an optimally controlled system once the desired performance matrices have been defined?
Two methods are available to obtain the system closed-loop characteristic equation from which the system roots can be determined. The first method as presented earlier in this paper can be described by considering the time-invariant matrix Riccati equation

\[-E F - F' E + E G R^{-1} G' E - H' Q H = 0.\]  

(152)

Solving this equation defines the matrix \(E\). Once \(E\) is known, the system optimal control vector can be found. That is

\[u^*(t) = -R^{-1} G' E x(t)\]  

(153)

or

\[u^*(t) = -K x(t)\]  

(154)

where

\[K = R^{-1} G' E\]  

(155)

Substituting \(u^*(t)\) into the system equations

\[\dot{x}(t) = F x(t) + G u(t)\]  

(156)

\[y(t) = H x(t)\]  

(157)

gives

\[\dot{x}(t) = F x(t) - G K x(t)\]  

(158)

\[y(t) = H x(t)\]  

(159)
Taking the Laplace transform of these equations gives

\[ Is \mathbf{X}(s) - \mathbf{X}(0) = \mathbf{F} \mathbf{X}(s) - \mathbf{G} \mathbf{K} \mathbf{X}(s) \]  \hspace{1cm} (160)

\[ Y(s) = H \mathbf{X}(s). \]  \hspace{1cm} (161)

Rearranging terms and combining Equations 160 and 161 gives

\[ Y(s) = H [Is - F + GK]^{-1} X(0). \]  \hspace{1cm} (162)

The system transfer function of output to input is then

\[ \frac{Y(s)}{X(0)} = H [Is - F + GK]^{-1} \]  \hspace{1cm} (163)

or

\[ \frac{Y(s)}{X(0)} = \frac{H \text{adj} [Is - F + GK]}{\text{det} [Is - F + GK]}. \]  \hspace{1cm} (164)

The closed-loop characteristic equation is then

\[ |Is - F + GK| = 0 \]  \hspace{1cm} (165)

or

\[ |Is - F + GR^{-1} G'E| = 0. \]  \hspace{1cm} (166)

Therefore, a root locus plot can be generated by solving this equation for variations of \( Q \) and \( R \). This method requires that the Riccati matrix equation be solved each time a change is made in \( Q \) or \( R \).

However, once the values of \( Q \) and \( R \) that assure acceptable dynamic
system response are found, all terms which are required to solve for the feedback gains, described by Equation 155, are known. The optimal control vector can then be specified.

The second method of obtaining the system closed-loop roots makes use of scalar Equation 145. Solving this equation for the left-half plane roots gives the closed-loop optimal roots of the system directly. However, when using this technique it is still necessary, once the desired weighting matrices have been obtained, to determine the optimal feedback gains. Expanding Equation 145 gives

\[
\begin{vmatrix}
R + [H(I_s - F)^{-1} G] \ast Q H (I_s - F)^{-1} G
\end{vmatrix} = 0
\] (167)

which, as shown in Rynaski and Whitbeck (1966), is equal to

\[
\begin{vmatrix}
I_s - F & G R^{-1} G^t \\
-H^t Q H & -I_s - F^t
\end{vmatrix} = 0.
\] (168)

The solution of Equation 168 gives both the desired roots and the adjoint of the system. This technique therefore gives \(2m\) roots for an \(m\) order system and so it requires that a \(2m\) order determinant be solved for an \(m\) order system.

Several papers on the frequency domain approach to optimal control theory helped the author of this paper form an initial premise that solving for the closed-loop roots could best be performed in the frequency domain and thus avoid, as Whitbeck (1968) said, "the sometimes painful chore of waiting for a digital computer to converge to the steady-state solution of a Riccati equation". It was further
theorized that a "good" method would be to use the frequency domain approach while making the repetitive computer runs in search of the weighting matrices which provide desired system dynamic response, then once the weighting matrices had been defined, making a one-time run to solve the matrix Riccati equation and then solve for the required optimal feedback gains which would generate the optimal control vector.

In order to substantiate this premise, several check case problems were considered to determine if there is a significant difference in computer cost between the two techniques. The following third and fourth order systems, with the indicated performance index weighting matrices, were considered in this comparison.

**THIRD ORDER SYSTEM**

\[
\begin{align*}
\dot{x}_1 &= \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1.0 \end{bmatrix} u_1 \\
\dot{x}_2 &= \begin{bmatrix} 10.0 & -1.0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix}
\end{align*}
\quad (169)
\]

\[
\begin{align*}
y_1 &= \begin{bmatrix} 1.0 & 0 & 0 \end{bmatrix} x_1 \\
y_2 &= \begin{bmatrix} 0 & 1.0 & 0 \end{bmatrix} x_2 \\
y_3 &= \begin{bmatrix} 0 & 0 & 1.0 \end{bmatrix} x_3
\end{align*}
\quad (170)
\]

\[
Q = \begin{bmatrix} 14.61 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 3286.1 \end{bmatrix}
\]

\[
R = [.101] \]
Two Fortran programs were written during the course of this study. The time domain program used the Runge-Kutta method of solution for the constant coefficient matrix Riccati differential equation as described in Athans and Levine (1966). Once a steady state solution of the Riccati equation was obtained the program was written to use this solution as the matrix $E$ in Equation 166. Equation 166 was then solved for the system closed-loop roots. The feedback gains, as described in Equation 155 were also calculated in this program.
The frequency domain computer program was written to solve Equation 168. This equation determines the system closed-loop roots.

Both programs used the same SSP routines of MINV, to invert the control vector weighting matrix \( R \), and POLRT to solve for the roots of the polynomial which results upon expansion of the characteristic equation and the root-square-locus equation. Each program was run on the CDC 6600 under the same computer service cost accounting method. The cost accounting method not only indicated the central process time expended on each program but also a unit value which was directly proportional to a dollar cost. This unit charge was determined by considering the central process time used as well as machine core and input-output expenses.

The results of this study provided the following information:

<table>
<thead>
<tr>
<th>THIRD ORDER SYSTEM</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Time Domain</strong></td>
</tr>
<tr>
<td>Closed-Loop Roots</td>
</tr>
<tr>
<td>-14.6704</td>
</tr>
<tr>
<td>-6.63953 ± j8.88079</td>
</tr>
<tr>
<td>Cost</td>
</tr>
<tr>
<td>4.670 seconds</td>
</tr>
<tr>
<td>.244 units</td>
</tr>
</tbody>
</table>
FOURTH ORDER SYSTEM

<table>
<thead>
<tr>
<th></th>
<th>Time Domain</th>
<th>Frequency Domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Closed-Loop Roots</td>
<td>-3.76368 ± j0.84875</td>
<td>-3.76368 ± j0.84875</td>
</tr>
<tr>
<td></td>
<td>-38.0789</td>
<td>-38.0789</td>
</tr>
<tr>
<td>Cost</td>
<td>.241 units</td>
<td>.715 units</td>
</tr>
<tr>
<td></td>
<td>5.257 seconds</td>
<td>63.763 seconds</td>
</tr>
</tbody>
</table>

In addition, a fifth order system was run. Using the time domain program, closed-loop roots were obtained with an expenditure of 7.658 seconds of central process time and .302 units while the frequency domain program was run for 393.241 seconds of central process time and 3.810 units and was finally terminated, due to a lack of time, without ever providing an answer.

It should also be noted that the time domain program did provide the desired feedback gains as well as the closed-loop roots. Therefore in comparison of these two programs, all the necessary information was available upon solving for the roots in the time domain while in the frequency domain it would be necessary to further increase the complexity and cost of the program in order to find the feedback gains.

The results of this cost comparison clearly indicates that the initial premise was wrong, and that unless the system under study is of low order the frequency domain approach quickly becomes much more expensive than the time domain approach and should therefore be avoided.
IV. CONCLUSION

The optimal control vector for a noise-free linear multivariable regulator, when optimized for a quadratic performance index, is derived in both the time and frequency domain in this paper. The reason for describing both domains is to present, from an engineering systems analysis point of view, derivations of the optimal control and root-locus equations as well as the system and weighting matrices requirements which must be satisfied to maintain system stability. It is shown that the optimal control vector for an output regulator is a linear function of the state vector.

The root-square-locus equation, obtained in the frequency domain, is a scalar equation which, when solved, provides the closed-loop poles of the optimal system as a function of the performance index weighting matrices. By plotting these poles for parameter variations of $Q$ and $R$, it is possible to define the performance index weighting matrices which will give desirable system dynamic response. The performance index weighting matrices determine the optimal closed-loop system dynamics, therefore, once these matrices are specified the optimization is basically finished and all that the mathematical procedures of optimal control theory does is define the control vector, in terms of the state vector, which optimizes the system. This is the reason then that a mathematical technique to help choose "good" weighting matrices, those that will give the desired system dynamic response, is desirable. Once the performance index weighting
matrices have been defined in the frequency domain it is shown that it is then possible to solve for the optimal control vector using Equation 148.

Solving for the optimal closed-loop roots in the time domain requires that the matrix Riccati equation, described for the output regulator by Equation 87, be solved for its steady-state value each time a change is made in $Q$ or $R$. Once this steady-state value is found, however, the optimal control vector can be easily defined as shown in Equation 86.

During the initial investigation of the two optimization approaches, looking for the best method of finding the system closed-loop roots, it appeared that the frequency domain approach was the best choice because it avoided, as Whitbeck (1968) said, "the sometimes painful chore of waiting for a digital computer to converge to the steady-state solution of a Riccati equation".

The computer cost comparison of the two approaches was initiated to determine if the cost of obtaining a solution of the matrix Riccati equation was in fact a significant reason for doing the root-locus study in the frequency domain as Whitbeck recommended rather than the time domain. The results of this comparison clearly indicated that for systems of fourth order or greater there is a marked advantage of the time domain approach over the frequency domain approach in computer running time and cost.

Achieving a multivariable optimal control system does involve extensive computational work in order to determine the optimum
control vector. Recently however, some study has been directed at reducing the magnitude of this problem by attempting to derive a method of judiciously picking the most important state variables and thereby defining a suboptimal controller with incomplete state variable feedback. It is hopeful that the material presented in this paper will help to clarify the work that has been done in optimal control theory as well as the work that is being done to reduce the computational load of achieving an optimal system.
BIBLIOGRAPHY


VITA

Eugene Charles Machacek was born on July 23, 1941, in Morton, Minnesota. He received his primary education in Bechyn, Minnesota and his secondary education in Danube, Minnesota. He has received his college education from Saint John's University, in Collegeville, Minnesota and the University of Minnesota, in Minneapolis, Minnesota. He received a Bachelor of Electrical Engineering degree from the University of Minnesota, in Minneapolis, Minnesota, in June 1965.

He has been enrolled in the Graduate School of the University of Missouri-Rolla since September 1966 and is currently employed in the Guidance and Control Mechanics Department of McDonnell Aircraft Company in St. Louis, Missouri.
APPENDIX

LIST OF SYMBOLS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B$</td>
<td>$B_1 x(0)$</td>
</tr>
<tr>
<td>$B_1$</td>
<td>$H [I_s - F]^{-1}$</td>
</tr>
<tr>
<td>$C$</td>
<td>Nonsingular constant matrix</td>
</tr>
<tr>
<td>$D$</td>
<td>Control system open-loop poles</td>
</tr>
<tr>
<td>$\overline{D}$</td>
<td>Conjugate of the open-loop poles</td>
</tr>
<tr>
<td>$E, E(t)$</td>
<td>Solution of the matrix Riccati equation</td>
</tr>
<tr>
<td>$F, F(t)$</td>
<td>State transition matrix</td>
</tr>
<tr>
<td>$G, G(t)$</td>
<td>Input matrix</td>
</tr>
<tr>
<td>$g$</td>
<td>Vector quantity</td>
</tr>
<tr>
<td>$H, H(t)$</td>
<td>Output matrix</td>
</tr>
<tr>
<td>$H_1$</td>
<td>Hamiltonian</td>
</tr>
<tr>
<td>$I$</td>
<td>Integral function</td>
</tr>
<tr>
<td>$I$</td>
<td>Identity matrix</td>
</tr>
<tr>
<td>$J$</td>
<td>Quadratic performance index</td>
</tr>
<tr>
<td>$J_1$</td>
<td>Integral-square error</td>
</tr>
<tr>
<td>$J_2$</td>
<td>Integral-square of the control input</td>
</tr>
<tr>
<td>$K, K(t)$</td>
<td>State feedback matrix</td>
</tr>
<tr>
<td>$L$</td>
<td>Terminal time weighting matrix</td>
</tr>
<tr>
<td>$M$</td>
<td>Matrix with polynomial terms</td>
</tr>
<tr>
<td>$N$</td>
<td>Left-half plane zeros of $\det [R + w^*Qw]$</td>
</tr>
<tr>
<td>$\overline{N}$</td>
<td>Right-half plane zeros of $\det [R + w^*Qw]$</td>
</tr>
</tbody>
</table>
\( p, p(t) \quad \text{Lagrange multiplier vector} \\
Q, Q(t) \quad \text{Output weighting matrix} \\
R, R(t) \quad \text{Control input weighting matrix} \\
r, r \quad \text{Reference signal} \\
T \quad \text{Terminal time} \\
t_0 \quad \text{Initial time} \\
T_1(s) \quad \text{Numerator of } [Z(s) - W_\ast Q B] \\
T_2(s) \quad \text{Left-half plane poles of } [Z(s) - W_\ast Q B] \\
T_3(s) \quad \text{Right-half plane poles of } [Z(s) - W_\ast Q B] \\
u(t) \quad \text{Scalar plant control input} \\
u(t) \quad \text{Control input vector} \\
u^*(t) \quad \text{Optimal control vector} \\
U(s) \quad \text{Laplace transformed plant control vector} \\
U^*(s) \quad \text{Laplace transformed optimal control vector} \\
U_1(s) \quad \text{Arbitrary column vector} \\
V(t) \quad \text{Fundamental matrix} \\
v(t) \quad \text{State vector of a homogeneous differential equation} \\
V(s) \quad \text{Vector containing polynomial terms} \\
W \quad \text{Matrix of transfer functions, } H [I_s - F]^{-1} G \\
x(t) \quad \text{State vector} \\
X(s) \quad \text{Laplace transform of the state vector} \\
y(t) \quad \text{Scalar plant output} \\
y(t) \quad \text{Plant output vector} \\
Y(s) \quad \text{Laplace transform of the plant output vector} \\
Z(s) \quad \text{Wiener-Hopf vector} \)
\( \theta(g) \)  Constraining function of \( \theta(g) \)
\( \Gamma(t) \)  Transition matrix of a homogeneous differential equation
\( \theta_c \)  Lagrange function
\( \theta(g) \)  Arbitrary function of the vector \( g \)
\( \Delta \)  The controllability matrix
\( \lambda \)  Constant multiplier
\( \lambda_1 \)  Matrix eigenvalues
\( \Psi \)  The observability matrix
\( \Phi(t,t_0) \)  Solution of a homogeneous vector matrix equation with initial conditions at \( t_0 \)
\( \Omega(t,t_0) \)  Fundamental matrix of the state and costate differential equation
\( \Omega_{jk}(T,t) \)  Submatrices of \( \Omega(T,t) \)