1971

Frequency division using cascaded phase-locked loops

Curtis B. Abshier

Follow this and additional works at: http://scholarsmine.mst.edu/masters_theses

Part of the Electrical and Computer Engineering Commons

Department:

Recommended Citation

This Thesis - Open Access is brought to you for free and open access by the Student Research & Creative Works at Scholars' Mine. It has been accepted for inclusion in Masters Theses by an authorized administrator of Scholars' Mine. For more information, please contact weaverjr@mst.edu.
FREQUENCY DIVISION
USING CASCADED PHASE-LOCKED LOOPS

By
CURTIS B. ABSHIER, 1945-

A THESIS
Presented to the Faculty of the Graduate School of the
UNIVERSITY OF MISSOURI - ROLLA

In Partial Fulfillment of the Requirements for the Degree
MASTER OF SCIENCE IN ELECTRICAL ENGINEERING

Approved by
William H. Smith (Advisor)  R. E. Farris

Gerald B. Regier
ABSTRACT

In many practical communication systems it is necessary to generate phase coherent subharmonics of an input tone. A useful system for accomplishing this task is a system of phase-locked loops in cascade. It is shown that when the feed-forward function is matched, the phase error of the second loop is zero. Analytical analysis and computer simulation prove that the minimum time to achieve phase-lock is obtained when the feed-forward function is matched. Under this condition, necessary design parameters are determined with ease.
ACKNOWLEDGMENTS

I am particularly indebted to Professor W. H. Tranter for his suggestion of this thesis topic and his guidance and assistance throughout the course of this work. The value of his critical assistance cannot be overemphasized. Without his kindly criticisms, this thesis would have been much less acceptable. However, I alone am responsible for any errors of fact or judgements.

I am also indebted to Dr. G. B. Clark, Director of the Rock Mechanics and Explosives Research Center at the University of Missouri-Rolla, for his support during the course of this study.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>i</td>
</tr>
<tr>
<td>ACKNOWLEDGMENTS</td>
<td>ii</td>
</tr>
<tr>
<td>TABLE OF CONTENTS</td>
<td>iii</td>
</tr>
<tr>
<td>LIST OF ILLUSTRATIONS</td>
<td>iv</td>
</tr>
<tr>
<td>I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II. SUMMARY OF PREVIOUS WORK</td>
<td>2</td>
</tr>
<tr>
<td>III. NATURE OF DEVICE</td>
<td>8</td>
</tr>
<tr>
<td>IV. THEORETICAL ANALYSIS OF CASCADED PHASE-LOCKED LOOPS WITH FEED-FORWARD</td>
<td></td>
</tr>
<tr>
<td>A. Theoretical Analysis to Show that Phase Error of Second Loop is Zero for &quot;Matched&quot; Feed-Forward</td>
<td>16</td>
</tr>
<tr>
<td>B. Linear Analysis of the Device in the Absence of Noise, When Both Loops are First Order.</td>
<td>18</td>
</tr>
<tr>
<td>C. Linear Analysis of the Device in the Presence of Noise, When Both Loops are First Order.</td>
<td>21</td>
</tr>
<tr>
<td>V. A PRACTICAL SYSTEM</td>
<td></td>
</tr>
<tr>
<td>A. Choosing Design Parameters</td>
<td>24</td>
</tr>
<tr>
<td>B. Time Required to Achieve Phase Lock</td>
<td>26</td>
</tr>
<tr>
<td>VI. RECOMMENDATION FOR FURTHER WORK</td>
<td>35</td>
</tr>
<tr>
<td>VII. SUMMARY AND CONCLUSIONS</td>
<td>37</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>39</td>
</tr>
<tr>
<td>APPENDIX</td>
<td></td>
</tr>
<tr>
<td>A. A Program to Obtain the Data Necessary for the Plots of Time-To-Lock vs $\beta$</td>
<td>40</td>
</tr>
<tr>
<td>VITA</td>
<td>43</td>
</tr>
</tbody>
</table>
### LIST OF ILLUSTRATIONS

#### FIGURES

1. Phase-locked loop models
   a. Block diagram of a phase-locked loop  
   b. Nonlinear model for a phase-locked loop
2. Block diagram of two phase-locked loops in cascade
3. Mathematical model for two phase-locked loops in cascade
4. Block diagram of two phase-locked loops in cascade with a feed-forward function
5. Mathematical model for the block diagram shown in Fig. 4
6. Time-to-Lock vs $\beta$ when $c = 1/2$ and $n/m = 1/2$
7. Time-to-Lock vs $\beta$ when $c = 1$ and $n/m = 1/2$
8. Time-to-Lock vs $\beta$ when $c = 2$ and $n/m = 1/2$
9. Time-to-Lock vs $\beta$ when $c = 1/2$ and $n/m = 1/4$
10. Time-to-Lock vs $\beta$ when $c = 1$ and $n/m = 1/4$
11. Time-to-Lock vs $\beta$ when $c = 2$ and $n/m = 1/4$
12. Fig. (5) redrawn in form advantageous for simulation
I. INTRODUCTION

In many practical communication systems, it is necessary to generate phase coherent subharmonics of an input carrier or tone. For example, AM-baseband communications systems often have harmonically related subcarriers. This allows all necessary demodulation subcarriers to be derived from a single tone and results in excellent bandwidth utilization. Another application is found in multichannel digital systems having harmonically related bit rates. Thus, a simple system to accomplish these tasks has widespread application.

The purpose of this thesis is to propose and analyze a device that will develop phase coherent subharmonics of an input tone or carrier. The proposed device uses phase-locked loops in cascade and is analyzed for both the absence and presence of additive noise.
II. SUMMARY OF PREVIOUS WORK

The study of phase-locked loops, whether employed in cascade or individually, is complicated by the relationship between the input signal and the resulting response waveform. This is given by a non-linear, stochastic integro-differential equation for which general solutions are difficult to obtain. Since phase locked loops are extremely practical devices that have a variety of applications, they have consequently been the subject of a great deal of study.

The essential components of a phase-locked loop, a phase detector (which can be modeled by a multiplier), a loop filter, and voltage controlled oscillator (VCO), form a communication receiver. This operates as a phase coherent tracking device by continuously correcting its local oscillator frequency and phase according to a measurement of the error between the frequency and phase of the input signal with that of its local oscillator. A block diagram of a phase-locked loop along with its mathematical model is shown in Fig. 1. The received signal, $e_i(t)$, is assumed a sinusoid of power $A^2$ watts which may be phase or frequency modulated and noise, $n(t)$, which is assumed to be white and Gaussian with one sided spectral density $N_0$ watts/Hz. Thus it may be represented by

$$e_i(t) = \sqrt{2} A \sin[\omega_0 t + \theta_i(t)] + n(t), \quad (2.1)$$

where $\theta_i(t)$ represents the phase and could be phase or frequency modulation. If the received signal were a pure
A sinusoid with constant frequency \( \omega \) and an initial phase \( \theta \), then

\[
\Phi(t) = (\omega - \omega_0)t + \theta .
\]

(2.2)

The output of the voltage-controlled oscillator is a sinusoid whose frequency is controlled by the input voltage, \( e(t) \). It can be expressed as

\[
e(t) = \sqrt{2} K \cos[\omega_0 t + \theta_0(t)].
\]

(2.3)

The output of the multiplier, \( e_d(t) \), is then

\[
e_d(t) = e_i(t) \cdot e(t) = \left\{ \sqrt{2} A \sin[\omega_0 t + \theta_i(t)] + n(t) \right\} \cdot \left\{ \sqrt{2} K \cos[\omega_0 t + \theta_0(t)] \right\}.
\]

(2.4)

The noise process, \( n(t) \), can be represented by

\[
n(t) = \sqrt{2} n_1(t) \sin \omega_0 t + \sqrt{2} n_2(t) \cos \omega_0 t,
\]

(2.5)

where \( n_1(t) \) and \( n_2(t) \) are white Gaussian processes of one-sided spectral density \( N_0 \) watts/Hertz. The voltage, \( e(t) \), in operational form is

\[
E(S) = E_d(S) \cdot F(S),
\]

(2.6)

where \( F(S) \) is the loop filter's transfer function.

Solving for \( E(S) \) and taking the inverse transform, \( e(t) \) is found to be
(a) Block diagram of a phase-locked loop

(b) Nonlinear model for a phase-locked loop

Fig. 1 Phase-locked loop models
\[
e(t) = K \int_{-\infty}^{\infty} \left\{ A \sin [\theta_0(t) - \theta_o(t)] + n'(\tau) \right\} \cdot f(t-\tau) \, d\tau + \text{high frequency terms},
\]

where

\[
n'(t) = -n_1(t) \sin \theta_0(t) + n_2(t) \cos \theta_0(t)
\]

and \( f(t) \) is the impulse response of the loop filter, \( F(S) \).

The loop low-pass filter and the VCO effectively discards the higher-frequency terms. The filter operates on \( \left\{ AK \sin[\theta_1(t) - \theta_o(t)] + K n'(t) \right\} \) to produce the voltage \( e(t) \). The instantaneous frequency of the VCO output is related to its input by

\[
\frac{d\theta_o(t)}{dt} = K_{vco} e(t),
\]

so that when \( e(t) = 0 \), the oscillator frequency is \( \omega_0 \). The constant \( K_{VCO} \) is the VCO gain in radians per second per volt.

The block diagram in Fig. 1(a) can be redrawn to emphasize loop operation in terms of phase error, \( \phi(t) \). Which is

\[
\phi(t) = \theta_1(t) - \theta_o(t)
\]

and which is driven toward zero by loop action. This is the mathematical model shown in operational notation in Fig. 1(b). The sinusoidal nonlinearity gives rise to the fundamental difficulties in describing the action of the loop.
In the absence of noise the model of Fig. 1(b) has been known since the early 1950's and was first studied by Gruen [5]. The phase-locked loop has been analyzed in the absence of noise for a number of filter-transfer functions and for received sinusoids with linearly time-varying frequency [1]. Jaffe and Rechtin [6] were the first to treat the case where additive noise is present. This was accomplished by replacing the sinusoidal nonlinearity of Fig. 1(b) by a linear amplifier of gain $A$, which is valid only so long as $\phi$ is small. The model of Fig. 1(b) first appeared in a paper by Develet [8] where he approximates loop operation by both replacing the sinusoidal nonlinearity by a linear amplifier whose gain is the expected gain of the device and assuming that the input signal is approximately Gaussian. Develet's approach was an approximation using Booton's quasi-linearization technique. Van Trees [12] replaces the sinusoidal nonlinearity by a series expansion and a series of functional equations are solved by the Volterra functional calculus. Viterbi [1, 11] using the Fokker-Planck diffusion equation obtains an exact expression for the stationary probability density for a first order loop and an approximate expression for the second order loop. Weber and Stein [4] obtain an approximate expression for the steady-state joint probability density function of the phase errors of two phase-locked loops in cascade using Fokker-Planck techniques. In the absence of noise, Tranter [3] analyzes two phase-locked loops in cascade using the techniques developed by Jaffe and Rechtin.
Tranter's work was unique in that he considered the input voltage to the VCO of the second loop to be the sum of the input voltage to the VCO of the first loop operated on by a transfer function (feed-forward function) plus the voltage at the output of the loop filter of the second loop.
III. NATURE OF DEVICE

Phase coherent harmonics and subharmonics of an input carrier or tone are easily generated by using phase-locked loops in cascade with the output of the voltage controlled oscillator (VCO) of the first loop being a rectangular pulse train of frequency $\omega_0$. The input, $e_i(t)$, to the first phase-locked loop is assumed to be the sum of the carrier or tone, $e_p(t)$, plus additive noise, $n(t)$. Further, the carrier or tone is assumed to be

$$e_p(t) = \sqrt{2} A \cos (m\omega_0 t + \phi_0), \quad (3.1)$$

where $A$ is the rms value of $e_p(t)$, $m$ is an integer, and $\phi_0$ is some arbitrary phase. The first loop will respond only to the $m^{th}$ harmonic generated by its VCO while each of the other loops respond only to the harmonic to which it is tuned. Note that the total number of phase-locked loops required is one plus the number of harmonics and subharmonics required.

Since any particular harmonic or subharmonic to be generated is to be kept arbitrary, the analysis of the device can be reduced to an analysis of two phase-locked loops in cascade. The first loop is used to track the input carrier and a second loop is used to track a particular harmonic generated by the VCO of the first loop.

A block diagram of two phase-locked loops in cascade is shown in Fig. 2, where the input signal, $e_i(t)$, is the carrier, $e_p(t)$, plus noise, $n(t)$, and is expressed as
Fig. 2 Block diagram of two phase-locked loops in cascade
The output, \( e_0(t) \), of the VCO of the first loop is given by

\[
e_0(t) = \sum_{n=1}^{\infty} \sqrt{2} K_n \sin(n\omega_0 t + n\theta_0),
\]

where \( K_n \) is the rms amplitude of the \( n^{th} \) harmonic, \( \omega_0 \) is the frequency of the pulse train, and \( \theta_0 \) is the reference phase of the fundamental. Since the first loop will respond only to the \( m^{th} \) harmonic and the second loop only to the \( n^{th} \) harmonic, the feedback voltage to the first loop can be expressed as

\[
e_m(t) = \sqrt{2} K_m \sin(m\omega_0 + m\theta_0),
\]

and the input to the second loop as

\[
e_n(t) = \sqrt{2} K_n \sin(n\omega_0 + n\theta),
\]

where \( m \) is a particular value of \( n \). The output, \( e_{n0}(t) \), of the VCO of the second loop is a sinusoid of frequency \( n\omega_0 \) and can be expressed by

\[
e_{n0}(t) = \sqrt{2} K^n \cos(n\omega_0 + n\psi),
\]

where \( \psi \) is some phase. The voltages \( e_1(t) \) and \( e_2(t) \) are the input voltages to the VCO of the first loop and the VCO of the second loop respectfully. The two VCO outputs are given by

\[
\frac{d}{dt} (m\theta_0) = K_1 e_1(t)
\]
and

\[ \frac{d}{dt} (n\Psi) = K_2 e_2(t), \]  

(3.8)

with \( K_1 \) and \( K_2 \) being the VCO gain constants for the first loop and second loop, respectively. Note that two band-pass filters (center frequency at \( m\omega_0 \) for filter in first loop and \( n\omega_0 \) for filter at the input to the second loop) have been added to indicate that a particular loop will respond only to a particular harmonic, and in actual practice the filters would not be required. Therefore, the output, \( e_{d1}(t) \), of the phase detector of the first loop is

\[ e_{d1}(t) = e_i(t) \cdot e_m(t), \]  

(3.9)

and the output, \( e_{d2}(t) \), of the phase detector of the second loop is

\[ e_{d2}(t) = e_n(t) \cdot e_{n0}(t). \]  

(3.10)

The two loops can be separated and following the procedure outlined in Chapter II, the mathematical model for each loop is obtained. The mathematical models are then combined to give the mathematical model for the device. The result is illustrated in Fig. 3, where \( \phi_1 \) and \( \phi_2 \) represent the phase error of the first loop and second loop respectively.

When the phase error of a phase-locked loop is near zero, the loop is said to be in lock. If lock is to occur for
Fig. 3 Mathematical model for two phase-locked loops in cascade
the device shown in Fig. 3, then the term \( m\theta(S) \) multiplied by \( n/m \) must equal \( n\psi(S) \). Therefore, the term, \( \Phi(S) \), used to describe overall system performance is defined by

\[
\Phi(S) = n\theta(S) - n\psi(S),
\]

(3.11)

and is given the name total phase error. From Fig. 3, the phase error of the first loop is

\[
\phi_1(S) = m\theta(S) - m\theta_0(S),
\]

(3.12)

and that of the second is

\[
\phi_2(S) = n\theta_0(S) - n\psi(S).
\]

(3.13)

Therefore, total phase error, expressed as a function of \( \phi_1 \) and \( \phi_2 \), is

\[
\Phi(S) = \frac{n}{m} \phi_1(S) + \phi_2(S).
\]

(3.14)

From (2.14), it is seen that if the phase error of the second loop were always zero, then, the total phase error is merely a constant times the phase error of the first loop. The phase error of the second loop may be adjusted with a feedforward function which was proposed by Tranter [3]. The feedforward function is shown added to the circuit in Fig. 4, and the mathematical model for the device is shown in Fig. 5, where \( G(S) \) represents the function.
Fig. 4  Block diagram of two phase-locked loops in cascade with a feed-forward function
Mathematical model for the block diagram shown in Fig. 4.
IV. THEORETICAL ANALYSIS OF CASCADED PHASE-LOCKED LOOPS WITH FEED-FORWARD

A. Theoretical Analysis to Show that Phase Error of Second Loop is Zero for "Matched" Feed-Forward

The term matched, as used in this thesis, will refer to that feed-forward function which makes the phase error of the second loop, $\phi_2$, zero. The initial conditions on $\phi_2$ are

$$\phi_2(0) = 0$$ (4.1)

and

$$\dot{\phi}_2(0) = 0,$$ (4.2)

which implies that the device is in lock at $t = 0$. Note that $\phi_2(0)$ is zero because of a time delay in the first loop.

The following analysis used to derive the matched feed-forward is based on the mathematical model shown in Fig. 5, where the symbol $\mathcal{L}$ denotes the Laplace operator.

The control voltage to the VCO of the first loop is given by

$$E_1(S) = \mathcal{L}\left\{ A \sin \phi_1(t) \right\} + \mathcal{N}(S) \cdot \left[ K_m F_1(S) \right].$$ (4.3)

Then, $m\Theta_o(S)$ written in terms of $E_1(S)$ is

$$m\Theta_o(S) = E_1(S) \cdot \left( \frac{K_1}{S} \right).$$ (4.4)

The control voltage to the VCO of the second loop can now be expressed as

$$E_2(S) = \left[ \mathcal{L}\left\{ K_n \sin \phi_2(t) \right\} K_F F_2(S) \right] + \left[ \mathcal{G}(S) \cdot E_1(S) \right],$$ (4.5)
and $n\Psi(S)$ written in terms of $E_2(S)$ as

$$n\Psi(S) = E_2(S) \cdot \frac{K_2}{S}. \quad (4.6)$$

Using (4.3), (4.4), (4.5), and (4.6), $n\Psi(S)$ can be expressed as

$$n\Psi(S) = \left[ k'K_2 \mathcal{L} \left\{ K_n \sin \phi_2(t) \right\} \cdot \frac{F_2(S)}{S} \right] +$$

$$\frac{K_2G(S)}{S} \left[ \mathcal{L} \left\{ A \sin \phi_1(t) \right\} + N'(S) \right] \cdot \left[ K_mF_1(S) \right]. \quad (4.7)$$

From 4.4

$$\mathcal{L} \left\{ A \sin \phi_1(t) \right\} = \frac{S_m \phi_0(S) - N'(S)K_1K_mF_1(S)}{K_1K_mF_1(S)}. \quad (4.8)$$

Inserting (4.8) into (4.7) yields

$$S_n\Psi(S) = \mathcal{L} \left\{ K_n \sin \phi_2(t) \right\} \cdot k'K_2F_2(S) + \frac{K_2}{K_1} SG(S)m\phi_0(S). \quad (4.9)$$

From (3.13)

$$S\phi_2(S) = S_n \phi_0(S) - S_n \Psi(S) = S_n \phi_0(S) \left[ 1 - \frac{K_2m}{K_1n} G(S) \right] -$$

$$\mathcal{L} \left\{ K_n \sin \phi_2(t) \right\} \cdot k'K_2F_2(S). \quad (4.10)$$

Note that if $G(S) = (K_1n)/(K_2m)$, then

$$S\phi_2(S) + K'K_2F_2(S) \mathcal{L} \left\{ K_n \sin \phi_2(t) \right\} = 0. \quad (4.11)$$

In the study of phase-locked loops there are two loop filters most commonly used. These are $F_2(S) = 1$ and $F_2(S) = 1 + b/S$, where $b$ is a constant. For $F_2(S) = 1$
\[ S\phi_2(S) + K'K_2 \left\{ K_n \sin \phi_2(t) \right\} = 0, \]  

and the only solution with \( \phi_2(0) = 0 \) is \( \phi_2(t) = 0 \)  

for all time. Additionally for \( F_2(S) = 1 + b/S \) (4.11) becomes

\[ S^2\phi_2(S) + S \left\{ K_n K'K_2 \sin \phi_2(t) \right\} + \left\{ K_n K'K_2 b \sin \phi_2(t) \right\} = 0, \]

and, once again, the only solution for \( \phi_2(0) = \phi_2(0) = 0 \) is (4.13). Therefore, \( \phi_2(t) = 0 \) when \( G(S) = \frac{K_n}{K_2 m} \) and \( F_2(S) \) is either 1 or \( (1 + b/S) \).

B. Linear Analysis of the Device in the Absence of Noise, when Both Loops are First Order

When \( \Phi(t) \) is at all times small compared with one radian, the device is near lock and the sinusoidal nonlinearities of Fig. 5 may be disregarded. Neglecting noise, the closed-loop transfer function is obtained using linear servo theory and, for this case, is

\[ H(S) = \frac{n\Psi(S)}{m\Theta(S)}. \]

The closed-loop transfer function was computed and found to be

\[ H(S) = \frac{\left[ AK_m K_1 K_n K'K_2 \left( \frac{nF_1(S)F_2(S)}{m S^2} \right) + AK_m K_2 \left( \frac{F_1(S)G(S)}{S} \right) \right]}{\left[ 1 + \frac{AK_m K_1 F_1(S)}{S} \right] \left[ 1 + \frac{K_n K'K_2 F_2(S)}{S} \right]} \]
Therefore, from (4.15) and (3.11), total phase error, expressed in terms of $H(S)$, is given by

$$\Phi(S) = m\theta(S) \left[ \frac{n}{m} - H(S) \right]. \quad (4.17)$$

The analysis is primarily concerned with the case where both loops are first order, that is $F_1(S) = F_2(S) = 1$. The input to the device for this section is considered to be a step in frequency occurring at $t = 0$, therefore $m\theta(S)$ is given by the transform of

$$m\theta(t) = m(\omega - \omega_0)t + m\theta_0. \quad (4.18)$$

where $m$ is a constant, $\theta_0$ is some initial phase angle, $m(\omega - \omega_0)$ is the value of the step, and $t$ is time. Using the final-value theorem for Laplace transforms, the steady-state value for $\Phi(t)$ is given by

$$\lim_{t \to \infty} \Phi(t) = \lim_{S \to 0} \left[ s\Phi(S) \right] = \Phi_{ss}. \quad (4.19)$$

For the given choice of $F_1(S)$ and $F_2(S)$,

$$\Phi(S) = Sm\theta(S) \left[ \frac{mS + \frac{n}{m}AK_mK_1 + \frac{n}{m}K_mK'K_2 - AK_mK_2G(S)}{(S + AK_mK_1)(S + K_nK'K_2)} \right]. \quad (4.20)$$

Therefore,

$$\Phi_{ss} = \lim_{S \to 0} S^2m \left\{ \frac{(\omega - \omega_0)}{S^2} + \frac{\theta_0}{S} \frac{nS + \frac{n}{m}(AK_mK_1 + K_nK'K_2)}{(S + AK_mK_1)(S + K_nK'K_2)} \right\} - \frac{AK_mK_2G(S)}{(S + AK_mK_1)(S + K_nK'K_2)} \right\}, \quad (4.21)$$
which yields
\[
\Phi_{ss} = (\omega - \omega_0) \left\{ \frac{n}{m}(A K_m K_1 + K_n K' K_2) - A K_m K_2 \left[ \lim_{S \to 0} G(S) \right] } \right. \\
\left. \frac{1}{A K_m K_1 K_n K' K_2} \right\}.
\] (4.22)

Note that if
\[
\frac{n}{m}(A K_m K_1 + K_n K' K_2) - A K_m K_2 \left[ \lim_{S \to 0} G(S) \right] = 0,
\] (4.23)
then \( \Phi_{ss} \) is equal to zero. From (4.21),
\[
\lim_{S \to 0} G(S) = \frac{n K_1}{m K_2} + \frac{n K_n K'}{m A K_m}.
\] (4.24)

It is interesting to note that so long as the initial frequency offset is small, the steady-state value for the total phase error can be made zero if the feed-forward function obeys (4.24). This shows that two first order phase-locked loops in cascade with feed-forward perform as a single second order loop when the preceding conditions are met.

The steady-state value of \( \Phi(t) \) for the particular feed-forward function derived in section A of this chapter is
\[
\Phi_{ss} = (\omega - \omega_0) \left[ \frac{n}{m}(A K_m K_1 + K_n K' K_2) - A K_m K_2 \frac{n K_1}{m K_2} \right] \left( \frac{A K_m}{A K_m K_1 K_n K' K_2} \right),
\] (4.25)
or
\[
\Phi_{ss} = \frac{n}{m}(\omega - \omega_0) \left( \frac{1}{A K_m K_1} \right),
\] (4.26)

The steady-state value of total phase error given in (4.26) differs from the steady-state phase error for a single first order phase-locked loop only by \( n/m \). The value of the
steady-state phase error for the first loop shown in Fig. 5 is

$$\phi_{1ss} = \frac{\omega - \omega_0}{AK K_1}, \quad (4.27)$$

which says that

$$\Phi_{ss} = \frac{n}{m} \phi_{1ss}. \quad (4.28)$$

Equation (4.28) agrees with (3.14) which says that the total phase error is \(n/m\) multiplied by the phase error of the first loop since the phase error of the second loop is always zero for the matched feed-forward function.

C. Linear Analysis of the Device in the Presence of Noise, when Both Loops are First Order

In order to justify the use of the linear approximations under noisy input conditions, it must be assumed that both the propagation of signal and that of random noise through the device are independent. This assumes that the superposition principle holds for linear systems. Signal and noise will then be additive if the linear approximations are valid. Therefore, the operation of the device can be examined when only noise is applied to the input of the linear model.

Since for this section only noise is considered to be present at the input, the term \(m\theta(S)\) equals zero and from (3.11), \(\Phi(S)\) is \(-n\Psi(S)\). The function, \(\Phi(S)\), depends only on \(N'(S)\), where \(N'(S)\) enters the model for the device after the sinusoidal nonlinearity of the first loop (Fig. 5).
It can be shown, using linear servo theory, that

$$n\Psi(S) = \frac{1}{A} H(S) \cdot N'(S), \quad (4.29)$$

where $H(S)$ is the closed-loop transfer function. Therefore, the power spectral density of $\Phi(t)$ is

$$S_{\Phi}(\omega) = S_{n\Psi}(\omega) = \frac{1}{A^2} |H(j\omega)|^2 \cdot S_n(\omega), \quad (4.30)$$

where $S = j\omega$, and $S_n(\omega)$ is power spectral density of the noise which is considered to be zero mean and white with one-sided power spectral density $N_0$. Then, $S_n(\omega) = N_0 / 2$ and

$$S_{\Phi}(\omega) = \frac{N_0}{2A^2} |H(j\omega)|^2. \quad (4.31)$$

The variance of the total phase error due to noise is

$$\sigma^2_{\phi} = \frac{N_0}{2A^2} \int_{-\infty}^{\infty} |H(j\omega)|^2 \frac{d\omega}{2\pi} = \frac{N_0}{A^2} \int_{0}^{\infty} |H(j\omega)|^2 \frac{d\omega}{2\pi}. \quad (4.32)$$

The equivalent noise bandwidth is defined as

$$B_L = \int_{0}^{\infty} |H(j\omega)|^2 \frac{d\omega}{2\pi}, \quad (4.33)$$

because an ideal low-pass filter with bandwidth $B_L$ Hertz will have the same value of noise power at its output as the variance of $\Phi(t)$ when the input is a white process of one-sided power spectral density of $N_0/A^2$ watts/Hertz.

The noise bandwidth was computed for the case where $F_1(S) = F_2(S) = 1$ and the feed-forward function restricted to insure that $\lim_{S \to 0} = G(S) = G$, a constant, and was found to be
The value, when the feed-forward function is the matched feed-forward function, is

\[ B_L = \frac{AK_mK_1K_nK_2K_2}{4(K_nK_2 + AK_mK_1)} \]  \hspace{1cm} (4.34)

It is well known that the loop-noise bandwidth for the first loop alone is

\[ B_{L1} = \frac{AK_mK_1}{4} \]  \hspace{1cm} (4.36)

Then, for the case of the matched feed-forward function, the noise bandwidth is

\[ B_L = (\frac{n}{m})^2 B_{L1} \]  \hspace{1cm} (4.37)

Note that in (4.34) and (4.35) that the noise bandwidth is a function of \((\frac{n}{m})^2\) and the smaller the ratio of the desired frequency to the carrier frequency the smaller the device noise bandwidth. Therefore, if the linear model is valid, the performance of the device in the presence of noise may be improved by making \(m\) large and \(n\) small. Thus, the device should be used to translate down in frequency and not vice versa.
V. A PRACTICAL SYSTEM

A. Choosing Design Parameters

The analysis involved in the preceding Chapter was for the most part restricted to loops which were first order. For good lock-in and noise performance, both loops should be second order, (i.e. $F_1(S) = 1 + a/S$ and $F_2(S) = 1 + b/S$). However, this yields a fourth order system which is difficult to analyze and optimize.

As in all engineering work, the basic problem reduces to that of picking optimum parameters with respect to some criterion. The criterion usually chosen for phase-locked loops, which are in lock, is minimum mean square phase error. Using Fig. 5 and assuming the linear model, the equations for the phase error of the first and second loops respectively are easily computed as

\[ S^2\phi_1(S) + SAK_m K_1\phi_1(S) + \alpha AK_m K_1\phi_1(S) = S^2m\theta(S) \]

+ Noise terms \hspace{1cm} (5.1)

and

\[ S^2\phi_2(S) + SK_n K'_2\phi_2(S) + bK_n K'_2\phi_2(S) = S^2n\theta_0(S) - SK_2E_2(S) \] \hspace{1cm} (5.2)

These equations may be rewritten in standard form as

\[ S^2\phi_1(S) + 2\tau_1\omega_n S\phi_1(S) + \omega_n^2\phi_1(S) = S^2m\theta(S) + \text{Noise terms} \hspace{1cm} (5.3) \]
and

\[ S^2 \phi_2 (S) + 2 \zeta_2 \omega_n S \phi_2 (S) + \omega_n^2 \phi_2 (S) = S^2 n \theta_0 (S) - SK_2 E_2 (S) \]  \hspace{1cm} (5.4)

where the subscripts 1 and 2 devote values pertaining to the first and second loops respectfully. It is well known [1] that the value of \( \zeta \) which yields minimum mean square phase error for the above system is \( \left( \sqrt{2} \right) / 2 \). Therefore, in a practical system, the design engineer would want to choose \( \zeta_1 \) and \( \zeta_2 = 0.707 \). However, the value chosen for \( \zeta_1 \) and \( \zeta_2 \) may not optimize overall system performance, because \( \phi_1 \) and \( \phi_2 \) have not been proved statistically independent. The value of \( \zeta_1 \) and \( \zeta_2 = 0.707 \) does however yield optimum performance for each individual loop, and since it was impossible for the author to obtain the correlation coefficient for \( \phi_1 \) and \( \phi_2 \), the above value for \( \zeta_1 \) and \( \zeta_2 \) is chosen. This value will at least yield good performance, if not optimum. Further, since the second loop is operating at a frequency of \( n/m \) times the frequency of the first loop, it follows that the resonant frequency of the second loop be \( n/m \) times the resonant frequency of the first loop (i.e., \( \omega_n^2 = \frac{n}{m} \omega_n^1 \)).

Using the values \( \zeta_1 \) and \( \zeta_2 = 0.707 \) and \( \omega_n^2 = \frac{n}{m} \omega_n^1 \), the following relationships between the parameters of the first and second loops are easily obtained as:

\[ AK_m K_1 = 2a, \]  \hspace{1cm} (5.5)

\[ K_2 K_n K' = 2b, \]  \hspace{1cm} (5.6)

\[ \frac{n}{m} a = b, \]  \hspace{1cm} (5.7)

and

\[ \frac{n}{m} A K_m K_1 = K_2 K_n K'. \]  \hspace{1cm} (5.8)
The engineer can design the entire system from a knowledge of the first loop's parameters and the type of feed-forward function desired.

It should be noted that if the feed-forward function is chosen to be matched, then the phase error of the second loop is always zero and $\phi_1$ and $\phi_2$ are necessarily orthogonal. Thus, a $\xi_1 = \xi_2 = 0.707$ yields an optimum system when the feed-forward function is matched.

B. Time Required to Achieve Phase Lock

With the in-lock performance of the system determined, the most important analysis becomes the determination of the time required for the system to achieve phase lock and the effect of a deviation of the feed-forward function from the matched feed-forward function. With the system out of lock and the feed-forward different from the matched value, the system differential equation is fourth order and nonlinear. Thus, computer simulation was chosen as the tool for analysis.

The time required for the cascaded combination with feed-forward to achieve phase lock is a function of the value of the initial frequency offset, the magnitude of the gain terms in each individual loop, and the frequency ratio $n/m$. Consequently, it is difficult to generalize concerning the time required for the device to reach lock, but for the case of the matched feed-forward, the time required for the device to achieve phase lock is the same as the time required for first loop to achieve lock. This is so because
\[ \Phi(t) = \frac{n}{m} \phi_1(t) \]  \hspace{1cm} (5.9)

for a matched feed-forward function.

The matched feed-forward function as derived in Chapter III is

\[ \text{Feed Forward} = \frac{nK_1}{mK_2} \] \hspace{1cm} (5.10)

By defining an arbitrary feed-forward function as

\[ \text{Feed Forward} = \beta \frac{nK_1}{mK_2} \] \hspace{1cm} (5.11)

the effect on the time to achieve phase lock by a deviation of the feed-forward from the matched case can then be analyzed. Thus, if \( \beta = 1 \), then the feed-forward is matched. Using the arbitrary feed-forward function, the relationships given by (5.5), (5.6), (5.7), and (5.8), and defining the ratio of the initial frequency offset to \( AK_mK_1 \) to be \( C \), the time to achieve phase lock can be expressed as a function of \( \beta \), \( C \), and \( n/m \).

The results of the computer simulation are shown in Figs. 6 thru 11, where time-to-lock is plotted against \( \beta \) for various values of \( C \) and \( n/m \). The simulation was accomplished using the IBM System/360 Continuous System Modeling Program, which is discussed in the appendix.

It can be noted from these curves that minimum time-to-lock occurs for \( \beta = 1 \) for all cases considered. Additionally, minimum time-to-lock is very sensitive to \( \beta \) in the region of \( \beta = 1 \). In many cases an error of only 10% in the value of
§ can more than double the required time-to-lock. In conclusion, a feed-forward function which is matched yields the minimum time-to-lock and the implementation of the feed-forward function should be very stable.

It should be noted that the following curves have not been normalized; consequently, time-to-lock depends on the particular numbers chosen for the various parameters in the loops. The numerical values for the parameters in this thesis are given in the appendix.
Fig. 6 Time-to-Lock vs $\beta$ when $c = 1/2$
and $\frac{n}{m} = \frac{1}{2}$
Fig. 7 Time-to-Lock vs $\beta$ when $c = 1$ and

$$\frac{n}{m} = \frac{1}{2}$$
Fig. 8 Time-to-Lock vs $\beta$ when $c = 2$
and $\frac{n}{m} = \frac{1}{2}$
Fig. 9 Time-to-Lock vs $\beta$ when $c = \frac{1}{2}$

and $\frac{n}{m} = \frac{1}{4}$
Fig. 10 Time-to-Lock vs $\beta$ when $c = 1$ and $\frac{n}{m} = \frac{1}{4}$
Fig. 11 Time-to-Lock vs $\beta$ when $c = 2$

and $\frac{n}{m} = \frac{1}{4}$
VI. RECOMMENDATION FOR FURTHER WORK

This thesis proposes a system which uses phase-locked loops in cascade. It is shown that when the feed-forward function is matched that minimum time-to-lock results. Additionally, if it were proved that the same feed-forward which yields minimum time to lock also yields minimum mean square phase error, then a system is obtained that is optimum in both lock and out-of-lock.

This would probably involve obtaining the cross-correlation function for \( \phi_1 \) and \( \phi_2 \), \( R_{\phi_1 \phi_2}(\tau) \), and the auto-correlation function for \( \phi_2 \), \( R_{\phi_2 \phi_2}(\tau) \). These functions would be desirable, since to minimize mean-square phase error for the system the expression

\[
\Phi^2 = \frac{n^2}{m^2} \phi_1^2 + 2 \frac{n}{m} \bar{\phi_1 \phi_2} + \phi_2^2
\]  

should be a minimum, where the bar above the variables denote the expectation operator. Equation (6.1) can be written in terms of \( R_{\phi_1 \phi_2}(\tau) \) and \( R_{\phi_2 \phi_2}(\tau) \) as

\[
\Phi^2 = \frac{n^2}{m^2} \phi_1^2 + 2 \frac{n}{m} R_{\phi_1 \phi_2}(0) + R_{\phi_2 \phi_2}(0).
\]  

The matched feed-forward yields a value of mean-square phase error for the system to be

\[
\Phi^2 = \frac{n^2}{m^2} \phi_1^2.
\]  

(6.3)
Therefore, to prove that the matched feed-forward yields minimum mean square phase error for the system, it would be necessary to prove the expression

\[ 0 \leq 2 \frac{n}{m} R_{\phi_1\phi_2}(0) + R_{\phi_2\phi_2}(0). \]  

(6.4)

In Chapter IV, using the linear model, a feed-forward function was derived which made two first order phase-locked loops in cascade perform as a single, second order phase-locked loop. While this particular feed-forward function provided nothing spectacular, it does not preclude the possibility that some other feed-forward function could provide a better system for some design criteria.

It should be pointed out that the above problems are not trivial in nature and in the author's estimation would be worthy topics for study.
VII. SUMMARY AND CONCLUSIONS

In many practical communication systems, it is necessary to generate phase coherent subharmonics of an input carrier or tone. For example, AM-baseband communications systems often have harmonically related subcarriers. This allows all necessary demodulation subcarriers to be derived from a single tone and results in excellent bandwidth utilization. Another application is found in multichannel digital systems having harmonically related bit rates. Thus, a simple system to accomplish these tasks has widespread application.

Phase coherent subharmonics of an input tone or carrier are easily generated by using phase-locked loops in cascade. For example, suppose the input frequency is to be $n/m$ times the input frequency. The voltage controlled oscillator of the first loop can then be a pulse train of frequency $m\omega_0$ and the phase-locked loop tracks the $m$-th harmonic of this voltage controlled oscillator. This yields an output frequency $n/m$ times the input frequency.

In general when the system is out of lock, the system differential equation is nonlinear and second order when both loops are first order, and is nonlinear and fourth order when both loops are second order. It was shown that when the feed-forward function is matched, that the phase error of the second loop is always zero for both first and second order loops. Therefore, it is possible to divide the order of the system by two when the matched feed-forward
is used. Using the linear approximations and considering both loops to be first order, the closed-loop transfer function was computed, an expression for the steady-state value of total phase error was derived, and the system noise bandwidth was computed.

Finally, relationships are provided by which the design engineer may construct a practical system from a knowledge of the first loop's parameters and the type of feed-forward desired. Further, it is shown that if a matched feed-forward function is used then minimum time to achieve phase lock is obtained.

From the work done in this thesis it is seen that for the case of the matched feed-forward, the analysis of the system reduces basically to an analysis of a single phase-locked loop, and the techniques developed by the many authors of papers on phase-locked loops may be used to analyze the system proposed in this thesis. But, for any case where the feed-forward function is not matched the analysis would most likely involve analog or digital simulation.
BIBLIOGRAPHY


APPENDIX

A. A Program to Obtain the Data Necessary for the Plots of Time-To-Lock vs β.

The program presented in the appendix is not intended to be a complete description but is, rather, a worked example showing how the features of the IBM/360 Continuous System Modeling Program may be used to simulate the proposed system.

To make the worked program convey as much information as possible, the model of the proposed system shown in Fig. 5 is redrawn in Fig. 12 in a form advantageous for simulation.

The following program is a simulation where the input is considered to be a step in frequency, of value R, occurring at t = 0. Both loops are second order and the feed-forward function is restricted to being a gain term only. The numerical values for the constants in this example are: β=(0.5, 1.0, 1.5), R=100, A=100, Km=1.0, a=50.0, K1=1.0, n/m=1/2, Kn=50.0, K'=1.0, b=25.0, and K2=1.0.

INPUT STATEMENTS

```plaintext
JOB CARD
// * LIMITS=(T=5,P=200,R=200)
//S1 EXEC CSMP
//G.DATA DD *
PARAMETER BETA=(0.5,1.0,1.5)
 X1=R*RAMP(0.0)
 R=100.0
 X2=X1-Y3
 X3=SIN(X2)
```
Fig. 12  Fig. (5) redrawn in form advantageous for simulation
Note that the program is allowed to run long enough for the system to reach steady-state. Time required to achieve phase lock may then be defined as the time required for the absolute value of the phase error to always be less than $10^{-6}$, which is an arbitrary value chosen for this thesis.
Curtis B. Abshier was born on June 11, 1945, in Tallapoosa, Missouri. He received his primary and secondary education in Gideon, Missouri. He has received his college education from the University of Missouri-Rolla. He received a Bachelor of Science degree in Electrical Engineering from the University of Missouri-Rolla, in Rolla, Missouri, in January 1970.

He has been enrolled in the Graduate School of the University of Missouri-Rolla since January 1970 and has worked for the Rock Mechanics and Explosives Research Center, University of Missouri-Rolla, for the period of January 1970 to the present.

The author is a student member of the IEEE and an associate member of the Society of the Sigma Xi.