1972

Application of transmission matrices to describe transverse vibrations of non-uniform Bernoulli-Euler beams

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APPLICATION OF TRANSMISSION MATRICES TO DESCRIBE
TRANSVERSE VIBRATIONS OF NON-UNIFORM BERNOULLI-EULER BEAMS

BY

DEAN IRLE PARKER, 1948–

A THESIS

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Approved by

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David Dean Keith
ABSTRACT

This report investigates the application of transmission matrices to determine natural frequencies of non-uniform Bernoulli-Euler beams. The classes of non-uniform beams considered includes: truncated wedges, truncated cones, and truncated rectangular pyramids. Two transmission matrices were derived from solutions to the Bernoulli-Euler equation. One was an exact closed form solution which was applicable for the above classes of beams excluding uniform and nearly uniform beams. The second solution was an approximate one limited to the use for nearly uniform beams, but which does give the correct solution to the uniform beam in the limiting case.

The transmission matrix has two advantages:

(a) It allows for the consideration of multi-segmented beams where the cross-sectional parameters are discontinuous at each segment boundary.

(b) Once the 16 transmission matrix elements are calculated, natural frequencies for any set of boundary conditions can be directly obtained.

The formulated transmission matrices were verified by comparing calculated natural frequencies for one and two segment beams to those previously reported in the literature.

In concluding this work the first three natural frequencies were calculated for three segment beams. The two set of boundary conditions considered were fixed-fixed and pinned-pinned. The beam geometry was composed of non-uniform first and third segments which were symmetric about a uniform mid-segment.
ACKNOWLEDGMENTS

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He is also greatly indebted to his wife for her understanding during his graduate studies and her patience in typing the manuscript.

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NOMENCLATURE

Basic Notation

\( x, y, z \) = right hand cartesian coordinate system

\( A(x) \) = cross-sectional area in the y-z plane

\( b(x) \) = base dimension of a beam in the z direction

\( h(x) \) = height dimension of a beam in the y direction

\( d(x) \) = diameter of a circular beam in the y-z plane

\( \ell \) = length of a beam in the x direction

\( I(x) \) = moment of inertia in the y-z plane

\( E \) = modulus of elasticity

\( \rho \) = mass density

\( \omega_i \) = \( i \)th natural frequency (rad/sec)

\( W \) = transverse displacement

\( \phi \) = slope

\( V \) = shear force

\( M \) = moment

\( \cdot \) = denotes \( \frac{d}{dx} \)

\( j \) = \( \sqrt{-1} \)

Matrix Notation

\( \{ \} \) = column matrix

\( [ ] \) = general matrix

\( [ ]^T \) = transpose of a matrix

\( [ ]^{-1} \) = inverse of a matrix

\( [ -U - ] \) = diagonal \( U \) matrix
Nomenclature (continued)

\[ [\mathbf{T}] \] = diagonal identity matrix
\[ [\mathbf{T}] \] = general transmission matrix
\[ [\mathbf{T}] \] = forward form transmission matrix
\[ [\mathbf{R}] \] = rearward form transmission matrix
\[ \{\mathbf{\psi}\} \] = state vector

\[ [C] \] = compatibility matrix relating the \( i \)th state vector to the \( j \)th state vector
\[ [C] \] = matrix of constants for transmission matrix containing Bessel functions
\[ [v] \] = initial condition matrix for transmission matrix containing Bessel functions
\[ [\beta] \] = Beta matrix for transmission matrix containing Bessel functions
\[ [\beta] \] = Beta matrix for nearly uniform transmission matrix
\[ [v] \] = initial condition matrix for nearly uniform transmission matrix

Definitions

\[ I_\ell \] = moment of inertia at \( x = \ell \)
\[ A_0 \] = cross-sectional area at \( x = 0 \)
\[ A_\ell \] = cross-sectional area at \( x = \ell \)
\[ I_0 \] = moment of inertia at \( x = 0 \)
\[ \phi_1 \] = dimensionless frequency parameter \( = \frac{\ell^2 \omega_1}{\sqrt{\rho A_\ell / EI_\ell}} \)
\[ \xi \] = slope parameter \( = (A_\ell / A_0)^{1/n} - 1 \)
\[ n \] = parameter defined by the exponents of \( A(x) \) and \( I(x) \)

\[ A(x) = A_0 \left(1 + \xi x / \ell \right)^n \]
\[ I(x) = I_0 \left(1 + \xi x / \ell \right)^{n+2} \]
Nomenclature (continued)

\[ \begin{align*}
Z &= 1 + \varepsilon x / \ell \\
\tilde{Y}(x) &= 1/\text{EI}(x) \\
\bar{Z}(x) &= A(x) \cdot \omega^2 \\
\mu &= \left(2\varepsilon / \xi \right) \left(\rho A_o / \text{EI}_o\right)^{1/4} \sqrt{\omega} \\
\Omega &= \mu \sqrt{Z} \\
\tilde{\Omega} &= \Omega \Big|_{x=\ell} = \mu \sqrt{1 + \xi} \\
\text{CON1} &= \left(\rho A_o \right)^{-3/8} \left(\text{EI}_o\right)^{-1/8} \omega^{-3/4} \\
\text{CON2} &= \sqrt{\omega} \left(\rho A_o / \text{EI}_o\right)^{1/4} 2\xi / \xi \\
\Lambda &= \text{CON1} Z^{-2n^{-1/4}} \\
M &= \text{CON2} (\sqrt{Z} - 1) \\
\text{DET}(\omega) &= \text{remainder term for frequency determinates} \\
\text{Cr} &= \text{constant for beams with rectangular cross-sections} \\
&= 2(12 \rho/E)^{1/4} \\
\text{Ce} &= \text{constant for beams with elliptical/circular cross-sections} \\
&= 2(16 \rho/E)^{1/4} \\
\gamma &= (\xi / \xi) \sqrt{(1 + \xi) / h_o} \\
\varepsilon &= \text{frequency root error} \\
\omega / \omega_o &= \text{frequency ratio for three segment beams} \\
\omega_o &= \text{natural frequency for a uniform beam of length} L \text{ and area} \ A
\end{align*} \]
INTRODUCTION

Problems related to vibrating continuous systems are frequently encountered in the engineering profession. Such systems have continuously distributed mass and elasticity and, consequently, possess an infinite number of degrees of freedom. The theory governing continuous systems is well established; however, for many problems the systems are so complex that the governing equations cannot be easily solved. One technique for solving these more complex problems is to employ a lumped parameter model where the system is subdivided into \( N \) finite degrees of freedom. A second technique which is applied most easily for systems with segments in a chainlike array is to describe the system by transmission matrices.

This report features the application of transmission matrices to certain classes of non-uniform, transversely vibrating Bernoulli-Euler beam systems. The classes considered will consist of special geometries of transverse beams. To more clearly define these beams reference is made to Fig. (1) and the following list of assumptions:

![Fig. 1 General Beam Orientation](image-url)
Assumptions

1) Transverse Vibrations: The beam is constrained at the boundaries $x = 0$ and $x = L$ permitting vibrations only in the $x$-$y$ plane.

2) Isotropic Homogeneous Material: The material is assumed to be isotropic homogeneous, thus, $E$ and $\rho$ are constant in all coordinate directions.

3) Bernoulli-Euler Beam: The beam segments are governed by Bernoulli-Euler Theory whereby the effects due to rotatory inertia and shear deformations are neglected.

4) Non-uniform Beams: The beams are considered to have non-uniform cross sections in the $z$-$y$ plane; hence, the moment of inertia $I$ and the cross-sectional area $A$ have a variation with respect to the spatial coordinate $x$.

To provide a background on the subject of non-uniform Bernoulli-Euler beams chapter I reviews the literature in this area. Various exact solutions and approximate solutions to the Bernoulli-Euler equations are presented along with some of the numerical schemes.

Chapter II is a basic review of general transmission matrix concepts. The relationships necessary to compute normal modes and corresponding natural frequencies are formulated for various boundary conditions.

It is shown in chapter III that the transmission matrix element $T_{il}$ satisfies the same differential equation as that of the mode shape function of the Bernoulli-Euler beam normally obtained by separation of variables. Consequently, the solutions of Cranc and Adler, $^{[11]}$ and Mok and Murray, $^{[14]}$ and others are applicable in solving for the trans-
mission matrix elements. The advantage of the transmission matrix form is that the beam can be subdivided into segments and be treated in a piecewise linear manner. These segments can be of increasing or decreasing cross-sectional areas. Also, the transmission matrix is independent of the boundary conditions.

Chapter IV validifies the transmission matrix formulated in chapter III by comparison of results for one and two segment beams whose natural frequencies and mode shapes are found in the literature.

Finally, in chapter V the transmission matrix is utilized in a study of the natural frequencies of three segment beams.
CHAPTER I
BERNOULLI-EULER BEAM THEORY

The "Bernoulli-Euler Beam Theory" is an elementary theory in mechanics dating back to the early eighteenth century. Daniel Bernoulli, around 1740, was the first to obtain a differential equation which describes the transverse vibration of a bar. Soon thereafter, Euler independently presented a similar equation; hence, the title Bernoulli-Euler is generally given to this equation.[1]

In its modern nomenclature the Bernoulli-Euler equation which governs the free transverse vibrations of a beam is a fourth order partial differential equation given as:[2]

$$\frac{\partial^2}{\partial x^2} \left[ EI(x) \frac{\partial^2}{\partial x^2} \ddot{W}(x,t) \right] + \rho A(x) \frac{\partial^2}{\partial t^2} \ddot{W}(x,t) = 0 \quad (1.1)$$

In order to solve for the principal modes the form of $\ddot{W}(x,t)$ is assumed, by separation of variables, to be:

$$\ddot{W}(x,t) = W(x)e^{i\omega t} \quad (1.2)$$

Equation (1.2) when substituted in Eq. (1.1) yields the differential equation for the principal mode shapes as:

$$\frac{d^2}{dx^2} \left[ EI(x) \frac{d^2}{dx^2} W(x) \right] - \rho A(x) \omega^2 V(x) = 0 \quad (1.3)$$

1.1 Uniform Beam Solution

Solutions to Eq. (1.3) for a uniform beam are of simple form
containing trigonometric and hyperbolic functions. By a uniform beam it is implied that $I(x)$ and $A(x)$ are constants with respect to the spatial variable, $x$; consequently, Eq. (1.3) can be written as:

$$\frac{d^4 W(x)}{dx^4} = \frac{\partial A}{EI} \omega^2 W(x) = 0$$

(1.4)

for the uniform case and the solution is given by:

$$W(x) = c_1 \sin px + c_2 \cos px + c_3 \sinh px + c_4 \cosh px$$

(1.5)

$$p = \omega^{1/2} \left(\frac{\partial A}{EI}\right)^{1/4}$$

In order to evaluate the constants in Eq. (1.4), four boundary conditions must be specified at corresponding values of $x$. These boundary conditions determine the four constants in terms of one constant and provide an expression for the infinite number of natural frequencies.

1.2 Non-uniform Beams

For non-uniform beams the solution is more complex and exact solutions exist only for special situations. Consequently, for many vibration problems involving non-uniform beams approximate solutions or numerical schemes must be used.

Kirchoff\textsuperscript{[3]} in 1879 was the first investigator to solve Eq. (1.3) exactly for the two cases: (a) a very thin wedge, and (b) a very sharp cone. The boundary conditions used considered the sharp end free and the large end fixed. In both cases, the solution is expressed in terms of Bessel functions. Ward\textsuperscript{[4]} later in 1913 extended Kirchoff's solution to include three different boundary conditions. The sharp end was
free and the other end was either fixed, free, or pinned.

One of the earliest numerical methods was presented by Lord Rayleigh.\textsuperscript{[5]} His method equates the maximum kinetic and potential energies and solves for a frequency equation. In order to calculate the frequency a deflection curve must be assumed. Rayleigh showed that the first natural frequency could be determined with good accuracy by assuming any reasonable deflection curve. If the true, fundamental mode deflection curve is used the frequency calculated will be exact. For all other curves the frequency calculated would be an upper bound. Morrow\textsuperscript{[6]} extended the method to a process of continuous approximation where the first deflection curve, predicted a more accurate second deflection curve, etc.

Numerical schemes were developed by Myklestad\textsuperscript{[7]} in 1944 and Prohl\textsuperscript{[8]} in 1945. These methods, which are nearly the same, are generally termed the Myklestad-Prohl method or MP method. It is a step-by-step numerical method of integration for estimating frequencies and mode shapes. The actual beam is replaced by an equivalent system consisting of discrete masses connected by sections of massless, elastic elements.

Another numerical scheme known as the Stodala method was first presented in 1940. It is described and applied by Anderson.\textsuperscript{[9]} The procedure consists of solving the differential equation of equilibrium for vibration by a method of successive approximations. Higher modes are readily found by using the orthogonality relation between normal modes. The numerical methods of Myklestad-Prohl and Stodala have been used by Housner and Keightly\textsuperscript{[10]} to develop mode shapes and natural frequencies for cantilever beams, with linearly tapered width and height, considering constant modulus of elasticity and density.
More recently, Cranch and Adler\cite{11} in 1956 have presented exact solutions for a group of non-uniform beams. By assuming the functional form of $\rho A(x)$ and $EI(x)$, Eq. (1.3) can be separated into the product of two second order differential operators. The sum of the solutions to the two second order equations yields the total solution which is expressed in Bessel functions. Their functional form and beam orientation is given by:

$$\rho A(x) = \rho_0 A_0 \left(\frac{x}{L}\right)^n \quad (1.6)$$
$$EI(x) = E_0 I_0 \left(\frac{x}{L}\right)^{n+2} \quad (1.7)$$

![Diagram of Cranch and Adler Beam Orientation](image)

Fig. 2 Cranch and Adler Beam Orientation

For geometries satisfying Eqs. (1.6) and (1.7) the solution was applied to two groups: (a) cantilever one segment beams, and (b) free-free two segment beams.

Conway, Becker, and Dubil\cite{12,13} give an exact solution for truncated wedge and cone beams for nine possible combinations of free, fixed,
and pinned boundary conditions. Their solution uses a variable transformation to solve Eq. (1.3) yielding a solution again in terms of Bessel functions.

Mok and Murray [14] present an approximate solution for the case of slender beams where characteristics slightly differ from those of a uniform beam. The solution assumes weak dependencies of $I$, $A$, and $H$ with respect to $x$. By applying order of magnitude approximation to various dimensionless quantities the displacement solution has been obtained as:

$$W(x) = H (D_1 \cosh m + D_2 \sinh m + D_3 \cos m + D_4 \sin m)$$

$$H = (\pi^3 \omega^3 E^2 I^2)^{-1/4}$$

$$m = \omega^{1/2} \int_0^{1/2} Q^{1/2} dx$$

$$D_1, D_2, D_3, D_4$$ arbitrary constants

The solution is basically a perturbation of the uniform case and becomes increasingly more accurate in approaching a uniform beam. When $A$, $I$, $E$ and $\rho$ are constant, the solution converges to the uniform solution Eq. (1.3).

Gaines and Volterra [15, 16, 17] present approximate formulas for the upper and lower bound values of the first three natural frequencies for cantilever bars of variable cross section. For cones, truncated cones, wedges and truncated wedges the effects of rotatory inertia and shear are taken into account and compared to elementary theory. In a later paper their technique is applied to free-free truncated double wedges and free-free double cones.

Wang [18] solves Eq. (1.3) in terms of generalized hypergeometric functions by the method of Frobenius. For the beams considered, the height and base vary along the beam according to any two arbitrary powers
of the spatial coordinate \( x \). In a NASA report by Wang and Worley,\cite{19} the natural frequencies and mode shapes are tabulated for three cases: (a) one segment cantilever beam (b) one segment truncated cantilever beams, and (c) two segment free-free beams.

Mabie and Rogers\cite{20} extend the work on non-uniform cantilever beams to include the boundary conditions of pinned-fixed. They consider two geometries (a) constant height and linearly varying base and (b) constant base and linearly varying height.

The application of transmission matrices to non-uniform beams has occurred only recently. Rocke\cite{21} in 1966 presented exact transmission matrices for a one-dimensional longitudinal vibrating non-uniform rod. He suggested the application of approximate and exact transmission matrices to describe transverse vibrations of non-uniform Bernoulli-Euler beams. In a later report by Rocke and Roy\cite{22} the application of approximate transmission matrices to describe transverse beam vibration was developed.
2.1 General Transmission Matrix Concepts

The concept of transmission matrices is a well established concept in vibration theory. In its most general definition a transmission matrix describes the manner in which sinusoidal forces and displacements are transmitted through a linear elastic element during steady state. Pestel and Leckie \[23\] have presented various transfer matrices for many common engineering problems. The terminology transfer matrix and transmission matrix appear to be used interchangeably. Rubin\[24\] has given the transmission matrix a general discussion.

A general linear elastic element operating in a steady state is schematically represented in Fig. (3). The arrows indicate the positive sense of the state vectors.

\[
\begin{bmatrix}
\{\psi_1\} \\
T \\
\{\psi_2\}
\end{bmatrix}
\]

Fig. 3 General Transmission Element

By definition:

\(\{\psi_1\}\) is the input state vector

\(\{\psi_2\}\) is the output state vector
\[ [T] \text{ is the forward form transmission matrix} \]
\[ [R] \text{ is the rearward form transmission matrix} \]

The input and output state vectors are related by:

\[
\begin{align*}
\{ \psi_1 \} &= [T]_{12} \{ \psi_2 \} \\ 
\{ \psi_2 \} &= [R]_{21} \{ \psi_1 \}
\end{align*}
\]

(2.1) \quad (2.2)

The elements in the forward and rearward transmission matrices are not independent. They are interrelated by:

\[
[T] = \begin{bmatrix}
\tilde{A} & \tilde{B} \\
\tilde{C} & \tilde{D}
\end{bmatrix}
\]

(2.3)

\[
[R] = \begin{bmatrix}
\tilde{D}^T & -\tilde{B}^T \\
-\tilde{C}^T & -\tilde{A}^T
\end{bmatrix}
\]

(2.4)

Where \( \tilde{A}, \tilde{B}, \tilde{C}, \) and \( \tilde{D} \) are square submatrices of equal order.

If a system is composed of \( n \) segments the total transmission matrix can be obtained by combining the segment transmission matrices. The transmission approach is most useful when \( n \) segments are in a chainlike arrangement, as shown in Fig. (4).

![Segment Transmission Matrices](Image)
Depending on directional definitions, the state vectors at the output of one segment may not be the same as the input for the next segment and, in general, are related by:

\[
\{\psi_2\} = [^C.][23] \{\psi_3\} \\
\{\psi_4\} = [^C.][45] \{\psi_5\} \\
\vdots \\
\{\psi_{2i}\} = [^C.][2i(2i+1)] \{\psi_{2i+1}\} \\
\]

The \([^C.][ij]\) are compatibility matrices taking into account differences in sign of the state vector elements.

The \([\overline{T}][ij]\) are either the rearward or forward transmission matrices relating the \(i\)th state to the \(i+1\) state. That is:

\[
\{\psi_1\} = [\overline{T}][12] \{\psi_2\} \\
\{\psi_3\} = [\overline{T}][34] \{\psi_4\} \\
\vdots \\
\{\psi_j\} = [\overline{T}][j(j+1)] \{\psi_{j+1}\} \\
\vdots \\
\{\psi_{2n-1}\} = [\overline{T}][2n-1(2n)] \{\psi_{2n}\} \\
\]

The total transmission matrix can be obtained by combining Eqs. (2.5) – (2.11) as follows:

\[
\{\psi_1\} = [\overline{T}][12] \{\psi_2\} 
\]
\[ \{ \psi_1 \} = [\mathbf{T}]_{12} [^\ast \mathbf{C}]_{23} \{ \psi_3 \} \]

\[ \{ \psi_1 \} = [\mathbf{T}]_{12} [^\ast \mathbf{C}]_{23} [\mathbf{T}]_{34} \{ \psi_4 \} \]

\[ \vdots \]

\[ \{ \psi_1 \} = [\mathbf{T}]_{12} [^\ast \mathbf{C}]_{23} [\mathbf{T}]_{34} \ldots \]

\[ \ldots [^\ast \mathbf{C}]_{(2n-2)(2n-1)} [\mathbf{T}]_{(2n-1)(2n)} \{ \psi_{2n} \} \]

\[ (2.12) \]

Therefore, the transmission matrix from point 1 to point 2n of an n segment beam can be written as:

\[ [\mathbf{T}]_{(1)(2n)} = \prod_{i=1,3,5}^{2n-3} [\mathbf{T}]_{(1)(i+1)} [^\ast \mathbf{C}]_{(i+1)(i+2)} [\mathbf{T}]_{(2n-1)(2n)} \]

\[ (2.13) \]

Where:

\[ \{ \psi_1 \} = [\mathbf{T}] \{ \psi \} \]

\[ 1 \quad (1)(-n) \quad 2n \]

\[ (2.14) \]

It should be noted that the sign convention of \( \{ \psi_1 \} \) and \( \{ \psi_{2n} \} \) is not necessarily the same. If it is necessary to have \( \{ \psi_1 \} \) and \( \{ \psi_{2n} \} \) measured in the same sense another compatibility matrix may be necessary.

\[ \{ \psi_2n \} = [^\ast \mathbf{C}]_{(2n)} \{ \psi_2n \} \]

\[ (2.15) \]

\[ \vdots \]

\[ \{ \psi_1 \} = [\mathbf{T}]_{(1)(2n)} [^\ast \mathbf{C}]_{(2n)} \{ \psi_2n \} \]

\[ (2.16) \]

Where \( \{ \psi_1 \} \) and \( \{ \psi_2n \} \) have the same positive sign convention.
The usual multiplication situation is when the state vectors all have the same positive sign convention as shown in Fig. (5).

\[
\begin{align*}
\{\psi_1\} & \rightarrow [T]_{12} \rightarrow \{\psi_2\} \rightarrow [T]_{34} \rightarrow \{\psi_3\} \rightarrow \cdots \rightarrow \{\psi_{2n-1}\} \rightarrow [T]_{(2n-1)(2n)} \rightarrow \{\psi_{2n}\}
\end{align*}
\]

**Fig. 5 Compatible Sign Conventions**

For this situation all the compatibility matrices reduce to identity matrices and the transmission matrix relating the \(i^{th}\) state to the \((i+1)^{th}\) state is the forward form.

\[
[T]_{i(i+1)} = [I]_{i(i+1)}
\]  
(2.17)

\[
[T]_{j(j+1)} = [T]_{j(j+1)}
\]  
(2.18)

Therefore, the total transmission matrix is given as:

\[
[T] = \prod_{i=1,3,5}^{2n-1} [T]_{i(i+1)}
\]  
(2.19)

For the compatible situation the total transmission matrix is simply the ordered product of the segment forward form transmission matrices. In most references Eq. (2.19) is the only multiplication rule discussed. However, in some cases, as will be shown in chapter III, it becomes necessary to orient some of the state vectors' sign conventions in an opposite sense to the system overall positive sense. This
orientation constitutes an incompatible situation. Consider, as an example, the system shown in Fig. (6).

\[
\begin{pmatrix}
\psi_1 \\
\end{pmatrix} \rightarrow [T]_{12} \begin{pmatrix}
\psi_2 \\
\end{pmatrix} \rightarrow [T]_{34} \begin{pmatrix}
\psi_3 \\
\psi_4 \\
\psi_5 \\
\end{pmatrix} \rightarrow \cdots \begin{pmatrix}
\psi_{2n} \\
\end{pmatrix}
\]

**Fig. 6 Incompatible Transmission Segments**

Examining the second element in Fig. (6) and using the rearward form one obtains:

\[
\{\psi_2\} = [\mathbf{\cdot C.}]_{23} \{\psi_3\} \tag{2.20}
\]

\[
\{\psi_3\} = [R]_{34} \{\psi_4\} \tag{2.21}
\]

\[
\{\psi_4\} = [\mathbf{\cdot C.}]_{45} \{\psi_5\} \tag{2.22}
\]

\[
\therefore \{\psi_2\} = [\mathbf{\cdot C.}]_{23} [R]_{34} [\mathbf{\cdot C.}]_{45} \{\psi_5\} \tag{2.23}
\]

For this incompatible situation the transmission matrix for segment 2 to be used in Eq. (2.13) is the rearward form,

\[
[T]_{34} = [R]_{34}
\]

and the compatibility matrices $[\mathbf{\cdot C.}]_{23}$ and $[\mathbf{\cdot C.}]_{45}$ do not reduce to the identity matrix. The exact nature of the compatibility matrices for a transverse beam segment is discussed in Appendix A.
For transverse beam segments the state vector contains four elements: shear force, bending moment, transverse displacement, and slope. In matrix form the state vector for point $j$ is written as:

$$\{\psi\} = \begin{bmatrix} V_j \\ M_j \\ W_j \\ \phi_j \end{bmatrix}$$

(2.24)

These four quantities completely describe the state of a Bernoulli-Euler beam at any particular point.

The state vector for any given point in a continuous system satisfies a matrix differential equation of the form:

$$\frac{d}{dx} \{\psi(x)\} = [a(x)] \{\psi(x)\}$$

(2.25)

For a Bernoulli-Euler beam, consider a $dx$ segment as shown in Fig. (7).
Assuming sinusoidal displacements, (basic assumption of a transmission matrix):

\[ w = w_0 e^{j\omega t} \]  
\[ \ddot{w} = d^2 w = -\omega^2 w \]  

(2.26)  
(2.27)

Summing the forces on the dx segment in Fig. (7) gives:

\[ v - (v + \frac{\partial v}{\partial x} \, dx) = - \rho A(x) \, dx (\omega^2 \, v) \]

\[ \frac{\partial v}{\partial x} = \rho A(x) \omega^2 \, v, \text{ or} \]

\[ \frac{dv}{dx} = \rho A(x) \omega^2 \, v \]  

(2.28)

Summing moments on the segment neglecting rotatory inertia gives:

\[ m - (m + \frac{\partial m}{\partial x} \, dx) - (v + \frac{\partial v}{\partial x} \, dx) \, dx = 0 \]

\[ - \frac{\partial m}{\partial x} \, dx - vdx - \frac{\partial v}{\partial x} (dx)^2 = 0 \]

\[ \frac{\partial m}{\partial x} = -v, \text{ or} \]

\[ \frac{dm}{dx} = -v \]  

(2.29)

By definition the deflection and slope are related by:

\[ \frac{dw}{dx} = \phi \]  

(2.30)

From basic strength of materials (neglecting shear deformations) the moment and slope are related as:

\[ \frac{d^2 w}{dx^2} = \frac{-m}{EI(x)}, \text{ or} \]

\[ \frac{d^2 w}{dx^2} = \frac{-m}{EI(x)} \]
\[
\frac{d\phi}{dx} = \frac{-m}{EI(x)} \tag{2.31}
\]

Combining Eqs. (2.28) - (2.31) the matrix differential equation is obtained as:

\[
\begin{bmatrix}
0 & 0 & \rho A(x) \omega^2 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & -1/EI(x) & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\phi \\
M \\
W \\
\end{bmatrix}
\]

(2.32)

and by defining,

\[
Z(x) = \rho A(x) \omega^2 
\tag{2.33}
\]

\[
Y(x) = 1/EI(x) 
\tag{2.34}
\]

the governing alpha matrix becomes:

\[
\begin{bmatrix}
0 & 0 & Z(x) & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & -Y(x) & 0 & 0 \\
\end{bmatrix} 
\tag{2.35}
\]

The transmission matrix also satisfies a basic matrix differential equation, Eq. (2.36). The derivation of Eq. (2.36) is well documented in references (21–24).

\[
\frac{d}{dx} [T(x)] = - [T(x)][\alpha(x)] 
\tag{2.36}
\]

In order to solve Eq. (2.36) it is necessary to develop matrix initial conditions. The first such initial condition is formulated by writing the transmission matrix from \(x = 0\) to some arbitrary point \(x\) and taking the limit as \(x \to 0\).
The derivative matrix initial conditions are given as:

\[
\begin{align*}
\frac{d}{dx} [T(o)] &= -[I_\times] [\alpha(o)] \\
\frac{d^2}{dx^2} [T(o)] &= [I_\times] [\alpha(o)]^2 - [I_\times] \frac{d}{dx} [\alpha(o)] \\
\frac{d^3}{dx^3} [T(o)] &= -[\alpha(o)]^3 + 2[\alpha(o)] \frac{d}{dx} [\alpha(o)] + \left[ \frac{d}{dx} [\alpha(o)] \right] [\alpha(o)] + \frac{d^2}{dx^2} [\alpha(o)]
\end{align*}
\]

(2.39) \hspace{1cm} (2.40) \hspace{1cm} (2.41)

2.3 Principal Modes

The transmission matrix is easily used to determine natural frequencies. The method involves partitioning the transmission matrix and expanding. Then the boundary conditions are used to determine a characteristic determinate which must be zero. Take for example a cantilever beam with boundary conditions free at \(x = 0\) and fixed at \(x = l\). Expanding the partitioned transmission matrix yields:

\[
\begin{bmatrix}
V_o \\
M_o \\
W_o \\
\phi_o
\end{bmatrix} = 
\begin{bmatrix}
T & T & T & T \\
11 & 12 & 13 & 14 \\
T & T & T & T \\
21 & 22 & 23 & 24 \\
T & T & T & T \\
31 & 32 & 33 & 34 \\
T & T & T & T \\
41 & 42 & 43 & 44
\end{bmatrix}
\begin{bmatrix}
V_x \\
M_x \\
W_x \\
\phi_x
\end{bmatrix}
\]

(2.42)
The free-fixed boundary condition reduces Eq. (2.42a) to:

\[
\begin{bmatrix}
V \\
\omega
\end{bmatrix}
= \begin{bmatrix}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{bmatrix}
\begin{bmatrix}
V_i \\
W_i
\end{bmatrix}
+ \begin{bmatrix}
T_{13} & T_{14} \\
T_{23} & T_{24}
\end{bmatrix}
\begin{bmatrix}
V_e \\
W_e
\end{bmatrix}
\tag{2.42a}
\]

The free-fixed boundary condition reduces Eq. (2.42a) to:

\[
\begin{bmatrix}
V \\
\omega
\end{bmatrix}
= \begin{bmatrix}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{bmatrix}
\begin{bmatrix}
V_i \\
W_i
\end{bmatrix}
\tag{2.43}
\]

\[
\text{DET} (w) = T_{11} T_{22} - T_{21} T_{12}
\tag{2.43a}
\]

The right hand side of Eq. (2.43a) is the frequency determinate for free-fixed boundary conditions. Similar frequency determinates for other boundary conditions are listed in Appendix B.

The frequency determinates are zero only for specific values of \( \omega \), i.e., the natural frequencies, because transmission elements for a specific system are frequency dependent. The general frequency determinates are expressed as:

\[
\text{DET} (\omega) = T_{ij} T_{km} - T_{im} T_{kj}
\tag{2.44}
\]

In order to calculate natural frequencies the value for \( \omega \) is incremented until the determinate is zero i.e.,:

\[
\text{DET} (\omega_n) = 0
\tag{2.45}
\]

Mode shapes are also easily obtained. The method involves using the rearward form transmission matrix as a function of \( x \). An example is presented for the same free-fixed beam where the rearward form transmission matrix is written from the point \( x = 0 \) to an arbitrary point \( x \) as:

\[
\{ \psi(x) \} = [R(x)] \{ \psi(0) \}
\tag{2.46}
\]
The boundary condition of fixed at \( x = 0 \) eliminates two terms of the state vector.

\[
\begin{bmatrix}
V(x) \\
M(x) \\
W(x) \\
\phi(x)
\end{bmatrix} = [R(x)]
\begin{bmatrix}
V_o \\
M_o \\
W_o \\
\phi_o
\end{bmatrix}
\] (2.47)

Expand by partitioning the matrix:

\[
\begin{bmatrix}
V(x) \\
\phi(x)
\end{bmatrix} =
\begin{bmatrix}
R_{33}(x) & R_{34}(x) \\
R_{43}(x) & R_{44}(x)
\end{bmatrix}
\begin{bmatrix}
W_o \\
\phi_o
\end{bmatrix}
\] (2.48)

Where:

\[
W(x) = R_{33}(x) \, W_o + R_{34}(x) \, \phi_o
\] (2.49)

In order to relate \( W_o \) to \( \phi_o \) Eq. (2.48) is evaluated for the boundary condition at \( x = \ell \).

\[
\begin{bmatrix}
W_\ell \\
\phi_\ell
\end{bmatrix} =
\begin{bmatrix}
R_{33}(\ell) & R_{34}(\ell) \\
R_{43}(\ell) & R_{44}(\ell)
\end{bmatrix}
\begin{bmatrix}
W_o \\
\phi_o
\end{bmatrix}
\] (2.50)

\[
\therefore \phi_o = - \frac{R_{33}(\ell)}{R_{34}(\ell)} \, W_o
\] (2.51)

Combining Eq. (2.49) and (2.51) yields:

\[
\frac{W(x)}{W_o} = R_{33}(x) - R_{34}(x) \, \frac{R_{33}(\ell)}{R_{34}(\ell)}
\] (2.52)

The rearward transmission elements are also frequency dependent, consequently, Eq. (2.52) is valid only for a principal mode i.e., \( \omega = \omega_n \). Thus, the mode shape function at a principal mode can be determined within a constant \( W_o \). For other boundary conditions the procedure is similar.
3.1 Basic Differential Equation

The elements of a transmission matrix for a transverse beam are interrelated by various differential equations. Writing Eq. (2.36) in expanded form, the relationships are obtained as:

\[
\frac{d}{dx} \begin{bmatrix}
T_{11} & T_{12} & T_{13} & T_{14} \\
T_{21} & T_{22} & T_{23} & T_{24} \\
T_{31} & T_{32} & T_{33} & T_{34} \\
T_{41} & T_{42} & T_{43} & T_{44}
\end{bmatrix} = \begin{bmatrix}
T_{11} & T_{12} & T_{13} & T_{14} \\
T_{21} & T_{22} & T_{23} & T_{24} \\
T_{31} & T_{32} & T_{33} & T_{34} \\
T_{41} & T_{42} & T_{43} & T_{44}
\end{bmatrix} \begin{bmatrix}
0 & 0 & \bar{Z}(x) & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & -\bar{Y}(x) & 0 & 0
\end{bmatrix}
\]

Expanding the first column of the right hand matrix yields:

\[
\frac{d}{dx} T_{i1} = +T_{i2}
\]

\[
\frac{d}{dx} T_{i1} = +T_{i2}
\]

\[
\frac{d}{dx} T_{i3} = +T_{i4}
\]

\[
\frac{d}{dx} T_{i4} = +T_{i2}
\]

\[\therefore \frac{d}{dx} T_{i1} = T_{i2} \quad \text{for } i = 1, 2, 3, 4 \ldots \quad (3.1)\]

By similar expansion of the second, third, and fourth column, one obtains:

\[\frac{d}{dx} T_{i2} = \bar{Y}(x) \quad (3.2)\]

\[\frac{d}{dx} T_{i3} = -\bar{Z}(x) \quad (3.3)\]

\[\frac{d}{dx} T_{i4} = -T_{i3} \quad (3.4)\]
Equations (3.1) - (3.4) can be combined into a fourth order differential equation of the form:

\[ 0 = \frac{d^4}{dx^4} T_{11} - 2 \frac{Y'(x)}{Y(x)} \frac{d^3}{dx^3} T_{11} + \left[ 2 \left( \frac{Y'(x)}{Y(x)} \right)^2 - \frac{Y''(x)}{Y(x)} \right] \frac{d^2}{dx^2} T_{11} - Z(x) \bar{Y}(x) T_{11} \]  

(3.5)

By substituting the values of \( \bar{Y}(x) \) and \( \bar{Z}(x) \):

\[ \frac{d^4}{dx^4} T_{11} + 2 \frac{EI'(x)}{EI(x)} \frac{d^3}{dx^3} T_{11} + \frac{EI''(x)}{EI(x)} \frac{d^2}{dx^2} T_{11} - \frac{\rho A(x)}{EI(x)} \omega^2 T_{11} = 0 \]  

(3.6)

Combining the derivatives of \( EI(x) \) Eq. (3.6) can be rewritten as:

\[ \frac{d^2}{dx^2} \left[ EI(x) \frac{d^2}{dx^2} T_{11} \right] - \frac{\rho A(x)}{EI(x)} \omega^2 T_{11} = 0 \]  

(3.7)

The form of Eq. (3.7) is identical to that of Eq. (1.3). Thus, the terms \( T_{11} \) are governed by the same differential equation as the mode shape function in the Bernoulli-Euler solution. Consequently, solutions to the Bernoulli-Euler equation for uniform or non-uniform beams are also applicable solutions for \( T_{11} \).

3.2 Transmission Matrix Solution

Cranch and Adler [11] have shown that differential equations of the form of Eq. (3.7) can be separated into the product of two second-order linear differential operators written as:

\[ \left[ \frac{1}{R} \frac{d}{dx} S \frac{d}{dx} + 0 \right] \left[ \frac{1}{R} \frac{d}{dx} S \frac{d}{dx} - 0 \right] T_{11} = 0 \]  

(3.8)

Expanding Eq. (3.8) and equating coefficients in Eq. (3.6) gives the conditions:

\[ \bar{R} = \rho A(x) \]  

(3.9)
where \( \overline{C} \) is a constant.

Equations (3.9) and (3.10) when substituted into Eq. (3.12) and (3.13) yield:

\[
\begin{align*}
\text{EI}(x) &= \left\{ \overline{C} \int \frac{\df}{\df x} \right\}^2 \text{ when } \frac{dS}{dx} \neq 0 \quad (3.14) \\
\text{EI}(x) &= \overline{C}^2 \left\{ \frac{\df}{\df x} \right\} \text{ when } \frac{dS}{dx} = 0 \quad (3.15)
\end{align*}
\]

Cranch and Adler state that when \( \overline{R} \cdot x^n \) and \( S \cdot x^m \) and where \( m \neq n + 2 \), a solution to Eq. (3.8) exists in terms of Bessel functions.

They define \( \overline{R} \) by:

\[
\overline{R} = \frac{\df}{\df x} = \frac{\df_{A_0}}{\df_{A_0}} \left( x/\ell \right)^n, \text{ or} \\
\overline{R} = x^n
\]

and \( \text{EI}(x) \) is determined by Eq. (3.14) as:

\[
\text{EI}(x) = \text{EI}_0 \left( x/\ell \right)^{n+2} \quad (3.17)
\]

The resulting \( S \) term is:

\[
S = + \sqrt{\frac{\df_{A_0}}{\df_{E_0}}} \left( x/\ell \right)^{n+1}, \text{ or} \\
S = x^m \text{ where } m = n + 1
\]

Substituting into Eq. (3.8):

\[
0 = \left[ \frac{1}{x^n} \frac{\df}{\df x} \left( x^{n+1} \frac{\df}{\df x} \right) + \sqrt{\frac{\df_{A_0}}{\df_{E_0}}} \ell \omega \right] \left[ \frac{1}{x^n} \frac{\df}{\df x} \left( x^{n+1} \frac{\df}{\df x} \right) - \sqrt{\frac{\df_{A_0}}{\df_{E_0}}} \ell \omega \right] T_{i_1}
\]

(3.18)
The solution to Eq. (3.18) for \( n = \text{integer} \) is given by:

\[
T_{i1}(x) = x^{-n/2} \left[ C_{1i} J_n(\eta) + C_{2i} Y_n(\eta) + C_{3i} I_n(\eta) + C_{4i} K_n(\eta) \right]
\]

(3.19)

and where \( n \neq \text{an integer} \) the form is:

\[
T_{i1}(x) = x^{-n/2} \left[ C_{1i} J_n(\eta) + C_{2i} J_{-n}(\eta) + C_{3i} I_n(\eta) + C_{4i} I_{-n}(\eta) \right]
\]

(3.20)

where:

\[
\eta = 2\sqrt{xEI} \left( \frac{\rho A_o}{EI_o} \right)^{1/4}
\]

(3.21)

The solutions given in Eqs. (3.19) - (3.21) cannot easily define truncated beams. The length measurement as defined in Eqs. (3.16) - (3.17) requires that the \( x \) coordinate origin coincide with the focal point of the beam (refer to Fig. (2)). For truncated beam segments cascaded in a chainlike array, it becomes cumbersome to determine the focal point for each segment. Rocke has shown that for a truncated beam a coordinate transformation allows the \( x \) axis to originate at a point where the beam has a finite area.\[[21]\] He defines:

\[
\bar{R} = \rho A(x) = \rho A_o (1 + \xi x/L)^n
\]

(3.22)

where:

\[
\xi = \left( \frac{A_e}{A_o} \right)^{1/n} - 1
\]

(3.23)

and the transformed variable is defined:

\[
Z = 1 + (\xi x/L)
\]

(3.24)
From the definition of $R$ in Eq. (3.22) the corresponding function for $EI(Z)$ is determined from Eq. (3.14) as:

$$EI(Z) = \left[ C \int \frac{\rho A(Z)}{\rho A(Z)} \frac{g}{\xi} dZ \right]^2 = \frac{C^2}{Z^n} (\xi/\xi)^2 \left[ \frac{z^{n+1}}{n+1} + K \right]^2$$

(3.25)

Taking the integration constant $K = 0$, then evaluating the boundary conditions at $x = 0$ or $Z = 1$ yields:

$$EI(Z) \bigg|_{Z=1} = EI_0$$

$$c^2 = EI_0/\rho A \ (\xi/\xi)^2 (n + 1)^2$$

(3.26)

$$EI(Z) = EI_0 Z^{n+2}, \text{ or}$$

$$EI(x) = EI_0 \ (1 + \xi x/\xi)^n + 2$$

(3.27)

To insure that we have satisfied the necessary conditions in order to use the separation presented by Cranch and Adler Eq. (3.12) must be examined.

$$\frac{dS}{dx} = (\xi/\xi) \sqrt{\rho A_0 EI_0} \ (n+1) Z^n$$

$$\frac{dS}{dZ} = (\xi/\xi) \sqrt{\rho A_0 EI_0}$$
\[
\frac{dS}{dx} = \sqrt{\frac{\rho A_o \cdot EI_o}{\rho A_o}} \frac{d}{dx} (\xi/\ell)^2 (n+1)^2 \cdot \frac{\rho A_o}{\rho A_o} \cdot Z^n
\]

\[
\frac{dS}{dx} = \frac{\rho A_o \cdot EI_o}{\rho A_o} (n+1) \cdot Z^n = \sqrt{\frac{\rho A_o \cdot EI_o}{\rho A_o}} \frac{d}{dx} (\xi/\ell)^2 (n+1)^2 \cdot \frac{\rho A_o}{\rho A_o} \cdot Z^n
\]

\[
(\xi/\ell) \cdot (n+1)^2 = \sqrt{(\xi/\ell)^2} \cdot (n+1)^2
\]

(3.28)

Therefore, for \( \xi \geq 0 \) and \( n \geq 1 \) Eq. (3.28) is satisfied and consequently, Eq. (3.12) is satisfied. Thus, the separation is valid for beams with positive slope.

Applying the separation in Eq. (3.28) and using the transformation \( Z \) the fourth order operator is separated as:

\[
0 = \left[ \frac{1}{Z^n} \frac{d}{dZ} \left( Z^{n+1} \frac{d}{dZ} \right) + \sqrt{\frac{\rho A_o \cdot EI_o}{\rho A_o}} \frac{d}{dx} (\xi/\ell)^2 \cdot \omega \right] \left[ \frac{1}{Z^n} \frac{d}{dZ} \left( Z^{n+1} \frac{d}{dZ} \right) - \sqrt{\frac{\rho A_o \cdot EI_o}{\rho A_o}} \frac{d}{dx} (\xi/\ell)^2 \cdot \omega \right] \cdot T_{11} \quad (3.29)
\]

By comparison to Eq. (3.18) - (3.21) the solution to Eq. (3.29) for \( n \) an integer is:

\[
T_{11} = Z^{-n/2} \left\{ C_{1i} \text{J}_n (\Omega) + C_{2i} \text{Y}_n (\Omega) + C_{3i} \text{I}_n (\Omega) + C_{4i} \text{K}_n (\Omega) \right\} \quad (3.30)
\]

For \( n \neq \) an integer:

\[
T_{11} = Z^{-n/2} \left\{ C_{1i} \text{J}_n (\Omega) + C_{2i} \text{J}_n (\Omega) + C_{3i} \text{I}_n (\Omega) + C_{4i} \text{I}_n (\Omega) \right\} \quad (3.31)
\]

Where:

\[
\Omega = 2 \sqrt{Z \cdot \frac{\rho \cdot \ell^2}{\rho \cdot \ell^2} \cdot \omega \cdot \left( \frac{\rho A_o \cdot EI_o}{\rho A_o} \right)^{1/\mu}} \quad (3.32)
\]
3.3 Transmission Matrix (n = an integer)

Having the solution for the \(T_{41}\) transmission elements, the next two steps will consist of (a) determining the values for the \(C_{ji}\) constants and (b) calculating the solutions for the three remaining columns in the transmission matrix.

The method of solving for the \(C_{ji}\) constants involves applying the matrix initial conditions from Eq. (2.38) - (2.41), when \(x = 0\) or \(z = 1\). 

The matrix initial conditions for \(T_{41}\) are rearranged into the form:

\[
\begin{bmatrix}
T_{11}(x) & T_{21}(x) & T_{31}(x) & T_{41}(x) \\
\frac{dT_{11}}{dx}(x) & \frac{dT_{21}}{dx}(x) & \frac{dT_{31}}{dx}(x) & \frac{dT_{41}}{dx}(x) \\
\frac{d^2T_{11}}{dx^2}(x) & \frac{d^2T_{21}}{dx^2}(x) & \frac{d^2T_{31}}{dx^2}(x) & \frac{d^2T_{41}}{dx^2}(x) \\
\frac{d^3T_{11}}{dx^3}(x) & \frac{d^3T_{21}}{dx^3}(x) & \frac{d^3T_{31}}{dx^3}(x) & \frac{d^3T_{41}}{dx^3}(x)
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & \overline{Y}(x) \\
0 & 0 & -Y(x) & \frac{d\overline{Y}(x)}{dx}
\end{bmatrix}
\begin{bmatrix}
x = 0
\end{bmatrix}
\]

(3.33)

The right hand side of Eq. (3.33) is referred to as the \([v]\) matrix. Calculating the derivatives and substituting for \(\overline{Y}(x)\) the \([v]\) matrix becomes:

\[
[v] = 
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{EI_o} \\
0 & 0 & -\frac{1}{EI_o} & -\frac{r}{k} (n+2)
\end{bmatrix}
\]

(3.33a)

In order to facilitate taking the derivatives of the Bessel functions, we define:

\[
\Omega = \nu r
\]

(3.34)
Therefore, we can write the $T_{11}$ solution for $n = 1$ integer as:

$$T_{11} = r^{-n} \{ C_{11} J_n(\mu r) + C_{21} Y_n(\mu r) + C_{31} I_n(\mu r) + C_{41} K_n(\mu r) \}$$

(3.37)

Evaluating $T_{11}$ at $x = 0$:

$$T_{11} \bigg|_{x=0} = C_{11} J_n(\mu) + C_{21} Y_n(\mu) + C_{31} I_n(\mu) + C_{41} K_n(\mu)$$

(3.38)

$$\frac{d}{dx} T_{11} = \frac{d}{d\xi} T_{11} = \frac{\xi}{\xi^2} \frac{1}{2 \xi} \frac{d}{dr} T_{11}$$

$$\frac{d}{dx} T_{11} \bigg|_{x=0} = \frac{\xi \mu}{2 \xi}$$

(3.39)

$$\frac{d^2}{dx^2} T_{11} \bigg|_{x=0} = \left( \frac{\xi \mu}{2 \xi} \right)^2 \{ C_{11} J_{n+1}(\mu) + C_{21} Y_{n+1}(\mu) + C_{31} I_{n+1}(\mu) + C_{41} K_{n+1}(\mu) \}$$

(3.40)

$$\frac{d^3}{dx^3} T_{11} \bigg|_{x=0} = \left( \frac{\xi \mu}{2 \xi} \right)^3 \{ -C_{11} J_{n+3}(\mu) - C_{21} Y_{n+3}(\mu) - C_{31} I_{n+3}(\mu) - C_{41} K_{n+3}(\mu) \}$$

(3.41)

Putting Eqs. (3.38) - (3.41) into the matrix form of Eq. (3.33) and simplifying:

$$\begin{bmatrix} J_n(\mu) & Y_n(\mu) & I_n(\mu) & K_n(\mu) \\ -J_{n+1}(\mu) & -Y_{n+1}(\mu) & I_{n+1}(\mu) & -K_{n+1}(\mu) \\ J_{n+2}(\mu) & Y_{n+2}(\mu) & I_{n+2}(\mu) & K_{n+2}(\mu) \\ -J_{n+3}(\mu) & -Y_{n+3}(\mu) & I_{n+3}(\mu) & -K_{n+3}(\mu) \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix} = \begin{bmatrix} \nu \end{bmatrix}$$

(3.42)
Where the \( \nu \) matrix is given by:

\[
[\nu] =
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{2\xi}{\mu \xi} & 0 & 0 \\
0 & 0 & 0 & \nu_{34} \\
0 & 0 & \nu_{34} & \nu_{44}
\end{bmatrix}
\]

(3.43)

\( \nu_{34} = (2\xi/\mu \xi)^2 \left(1/EI_0\right) \)  
(3.43a)

\( \nu_{43} = -(2\xi/\mu \xi)^3 \left(1/EI_0\right) \)  
(3.43b)

\( \nu_{44} = -(2\xi/\mu \xi)^3 \left(\xi/\xi\right) (n+2) \left(1/EI_0\right) \)  
(3.43c)

The 4x4 matrix composed of Bessel functions is referred to as the Beta matrix. Therefore, the \( C_{ji} \) constants can be solved by matrix inversion as:

\[
[\beta][C] = [\nu]
\]

(3.44)

or

\[
[C] = [\beta]^{-1} [\nu]
\]

(3.45)

In the second step we must determine the relationships for \( T_{12} \), \( T_{13} \), and \( T_{14} \). This is accomplished by applying the derivative relationships of Eqs. (3.1), (3.2), and (3.4):

\[
T_{12} = \frac{d}{dx} T_{i1} = \frac{\xi}{\xi} \frac{d}{dZ} T_{i1} = \frac{\xi}{\xi} \frac{1}{2r} \frac{d}{dr} T_{i1} (r)
\]

\[
T_{11} = \frac{\xi u}{2\xi} \frac{-(n+1)}{\xi} \{-C_{11} J_{n+1}(\mu r) - C_{21} Y_{n+1}(\mu r) + C_{31} I_{n+1}(\mu r) - C_{41} K_{n+1}(\mu r)\}
\]

(3.46)
\[ T_{14} = \frac{1}{Y(x)} \frac{d}{dx} \quad T_{12} = \frac{1}{Y(Z)} \frac{d}{dZ} \quad T_{12} = \frac{1}{Y(r)} \frac{d}{dr} \quad T_{12} \]

\[ T_{11} = \frac{1}{Y(r)} \left( \frac{\xi u}{2\ell} \right)^2 r^{-(n+2)} \left\{ C_{11} J_{n+2}(\mu r) + C_{31} J_{n+2}(\mu r) + C_{41} K_{n+2}(\mu r) \right\} \]

\[ T_{13} = \frac{d}{dx} \quad T_{14} = -\frac{\xi}{\ell} \frac{1}{2r} \frac{d}{dr} \quad T_{14} \]

\[ T_{11} = -EI_0 \left( \frac{\xi u}{2\ell} \right)^3 r^{n+1} A_1 - (n+2) \left( \frac{\xi u}{2\ell} \right)^2 \frac{\xi}{\ell} EI_0 r^n A_2 \]

Where:

\[ A_1 = -C_{11} J_{n+3}(\mu r) - C_{21} Y_{n+3}(\mu r) + C_{31} J_{n+3}(\mu r) - C_{41} K_{n+3}(\mu r) \]

\[ A_2 = C_{11} J_{n+2}(\mu r) + C_{21} Y_{n+2}(\mu r) + C_{31} J_{n+2}(\mu r) + C_{41} K_{n+2}(\mu r) \]

By applying Eq. (3.45) the \( C_{ji} \) constant can be determined. From the definitions in Eqs. (3.31) and Eqs. (3.46) - (3.50) expressions have been developed which determine the entire transmission matrix. The final step is to evaluate the transmission matrix for a beam segment of length \( \ell \):

\[ x = \ell \quad Z = 1 + \xi \]

\[ r = \sqrt{1 + \xi} \]

\[ \tilde{\Omega} = \Omega \quad | \quad x = \ell \]

Thus, we have developed the transmission matrix for a Bernoulli-Euler beam segment of length \( \ell \) for \( n \) an integer. For \( n \) not an integer, the transmission matrix is derived in Appendix C.

### 3.4 Nearly Uniform Solution

By examining the Bessel function transmission matrix presented in
section 3.3 the question arises as to the permissible range of the slope parameter $\xi$. If one investigates a beam which is approaching a uniform beam it can be observed,

$$\lim_{A_2^+A_0} \xi = \lim_{A_4^+A_0} \left[ (A_4/A_0)^{1/n} - 1 \right] = 0$$

As $\xi$ approaches zero the arguments of the Bessel function for a beam segment behave as:

$$\lim_{\xi \to 0} \frac{2\xi}{\xi} (\rho A_0/EI_0)^{1/4} \omega^{1/2} \sqrt{1 + \xi} = \infty$$

For $\Omega$ approaching infinity some of the Bessel functions also approach infinity. Consequently, it is anticipated that for a numerical scheme the range of the slope parameter would be limited. The exact nature of the permissible range of $\xi$ will be presented in detail in chapter IV.

In order to treat beams with small values of $\xi$, a second transmission solution was developed. Mok and Murray have presented an approximate solution for nearly uniform Bernoulli-Euler beams,\footnote{14} i.e., it is implied that $A(x)$ and $I(x)$ deviate only slightly from that of a uniform beam. Since solutions to the Bernoulli-Euler equation are also applicable solutions to $T_{t1}$ as shown in section 3.1, Mok and Murray's solution can be applied to the transmission elements $T_{t1}$.

$$T_{t1} (x) = H \{ D_{11} \cosh m + D_{21} \sinh m + D_{31} \cos m + D_{41} \sin m \} \tag{3.51}$$

$$H = \{ \rho^{3/2} \omega^3 E^2 I(x)^2 \}^{-1/4} \tag{3.52}$$

$$Q = \sqrt{\rho A(x)/EI(x)} \tag{3.53}$$
Applying the basic definitions for \( Z, \xi, \rho A(Z) \), and \( EI(Z) \) stated in Eqs. (3.22) - (3.27) yields:

\[
H = (\rho A_o)^{-3/6} (EI_o)^{-1/8} \omega^{-3/4} Z^{-2n-1}/4
\]  

or

\[
H = \text{CON1} \, Z^{-2n-1}/4
\]

where:

\[
\text{CON1} = (\rho A_o)^{-3/6} (EI_o)^{-1/8} \omega^{-3/4}
\]

and the \( m \) term becomes:

\[
m = \sqrt{\omega} \left(\frac{\rho A_o}{EI_o}\right)^{1/4} \left(2\ell/\xi\right) (\sqrt{Z} - 1)
\]  

or

\[
m = \text{CON2} \, (\sqrt{Z} - 1)
\]

where:

\[
\text{CON2} = \sqrt{\omega} \left(\frac{\rho A_o}{EI_o}\right)^{1/4} \left(2\ell/\xi\right)
\]

Now that the form of \( T_{11} \) is defined one must again complete the two steps: (a) determine the \( D_{ji} \) constants and (b) calculate the solutions for \( T_{12}, T_{13}, \) and \( T_{14} \).

In order to evaluate the constants the matrix initial conditions must be examined. The formulation is identical to that presented in Eq. (3.33). First a \([B]\) matrix is formed from the derivatives of \( T_{11} \) as:

\[
\begin{bmatrix}
T_{11}' & T_{21}' & T_{31}' & T_{41}' \\
T_{11}'' & T_{21}'' & T_{31}'' & T_{41}'' \\
T_{11}''' & T_{21}''' & T_{31}''' & T_{41}'''
\end{bmatrix} = [B] [D]
\]
The first row of the [B] matrix is obtained from:

\[
T_{11} = H_0 \begin{bmatrix}
D_{11} \cosh m_0 + D_{12} \sinh m_0 + D_{13} \cos m_0 + D_{14} \sin m_0 \\
\end{bmatrix}
\]

\[x=0\]  

(3.62)

The \(H_0\) and \(m_0\) terms indicate that the \(H\) and \(m\) are evaluated at \(x = 0\).

 Casting Eq. (3.62) into the [B] matrix form yields:

\[
\begin{align*}
B_{11} &= H_0 \cosh m_0 \\
B_{12} &= H_0 \sinh m_0 \\
B_{13} &= H_0 \cos m_0 \\
B_{14} &= H_0 \sin m_0
\end{align*}
\]

(3.63) (3.64) (3.65) (3.66)

The second row of the [B] matrix is determined by differentiating \(T_{11}\) and evaluating at \(x = 0\),

\[
\begin{align*}
B_{21} &= H'_0 \cosh m_0 + H_0 m'_0 \sinh m_0 \\
B_{22} &= H'_0 \sinh m_0 + H_0 m'_0 \cosh m_0 \\
B_{23} &= H'_0 \cos m_0 - H_0 m'_0 \sin m_0 \\
B_{24} &= H'_0 \sin m_0 + H m'_0 \cos m_0
\end{align*}
\]

(3.67) (3.68) (3.69) (3.70)

Similarly, the third row is obtained by differentiating \(T_{11}\) twice and evaluating at \(x = 0\),

\[
\begin{align*}
B_{31} &= \{H''_0 + H_0 (m'_0)^2\} \cosh m_0 + \text{CON}3 \sinh m_0 \\
B_{32} &= \{H''_0 + H_0 (m'_0)^2\} \sinh m_0 + \text{CON}3 \cos m_0 \\
B_{33} &= \{H''_0 + H_0 (m'_0)^2\} \cos m_0 - \text{CON}3 \sin m_0 \\
B_{34} &= \{H''_0 + H_0 (m'_0)^2\} \sin m_0 + \text{CON}3 \cos m_0
\end{align*}
\]

(3.71) (3.72) (3.73) (3.74)

where:

\[
\text{CON}3 = 2 H'_0 m'_0 + H_0 m''_0
\]

(3.75)
The fourth row of the [B] matrix is obtained by evaluating the third derivative of $T_{11}$ at $x=0$,

$$
B_{41} = \{H'''' + \text{CON5}\} \cosh m_o + \{\text{CON4} + H_o(m_o')^3\} \sinh m_o
$$

$$
B_{42} = \{H'''' + \text{CON5}\} \sinh m_o + \{\text{CON4} + H_o(m_o')^3\} \cosh m_o
$$

$$
B_{43} = \{H'''' - \text{CON5}\} \cos m_o + \{-\text{CON4} + H_o(m_o')^3\} \sin m_o
$$

$$
B_{44} = \{H'''' - \text{CON5}\} \sin m_o + \{\text{CON4} - H_o(m_o')^3\} \cos m_o
$$

where:

$$
\text{CON4} = 3H''''(m_o')^2 + 3H' m_o' + H m''
$$

$$
\text{CON5} = 3H'(m_o')^2 + 3H m_o'
$$

After the [B] matrix is determined the constants $D_{ji}$ are solved by equating Eq. (3.33) to (3.61) where:

$$
[B] [D] = [v]
$$

or

$$
[D] = [B]^{-1} [v]
$$

The [v] is defined in Eq. (3.33a).

In order to calculate the remaining terms of the transmission matrix, $T_{i2}$, $T_{i3}$, and $T_{i4}$, we utilize the differential relationships in Eq. (3.1), (3.2) and (3.4) where the $T_{i2}$ terms are related to $T_{i1}$ by:

$$
T_{i2} = \frac{d}{dx} T_{i1}
$$

$$
T_{i2} = H'(D_{i1} \cosh m + D_{2i} \sinh m + D_{3i} \cos m + D_{4i} \sin m) + H m' (D_{i1} \sinh m + D_{2i} \cosh m - D_{3i} \sin m + D_{4i} \cos m)
$$

For convenience in writing define:
Therefore, the $T_{12}$ can be written as:

$$T_{12} = H' P_0 + H m' P_1$$  (3.88)

The $T_{14}$ elements are determined as:

$$T_{14} = \{1/\sqrt{V(x)} \} \frac{d}{dx} T_{12}$$

and finally the $T_{13}$ elements are:

$$T_{13} = - \frac{d}{dx} T_{14}$$

$$T_{13} = - \{1/\sqrt{V(x)} \} \{H'' P_0 + [2H' m' + H m''] P_1 + H (m')^2 D_2 \}$$

$$- \{1/\sqrt{V(x)} \} \{H''' P_0 + [3H'' m' + 3H' m'' + H m''' P_1 + H(m')^3 P_3 \}$$

In order to determine the transmission matrix for a Bernoulli-Euler beam segment, all one must do is to evaluate $[T]$ at $x=t$. The derivatives of $H$ and $M$ are calculated for $x=0$ and $x=L$ in Appendix D.

In summary, section 3.3 sets up the transmission matrix solution for positive integer $n$. Since that solution cannot describe uniform or nearly uniform beams, section 3.4 derives a second solution. This solution for nearly uniform beams is valid for arbitrary $n$. 
CHAPTER IV
APPLICATION TO ONE AND TWO SEGMENT BEAMS

In chapter III it was briefly stated that the Bessel function solution was not valid for beam segments where the slope parameter \( \xi \) approaches zero, i.e., when a beam is approaching a uniform beam. A second solution based on Mok and Murray's approximate method was developed to handle cases for small values of \( \xi \). However, the exact nature of the cut-off point where the Bessel function solution ceases to be useable and the second solution applies must be examined. In chapter IV, the purpose is to first describe this cut-off point and secondly check the validity of the solutions.

4.1 Determination of the Cut-off Point

In order to describe the cut-off point one must examine the arguments of the Bessel functions used in the solution. From Eq. (3.32) the argument, evaluated for a beam segment of length \( \xi \), in the Bessel functions is:

\[
\tilde{\Omega} = (2\xi/\varepsilon) (pA_o/EI_o)^{1/4} \left( \omega \sqrt{1 + \frac{\xi}{\varepsilon}} \right)
\]  

(4.1)

Taking the limit of Eq. (4.1) we have:

\[
\lim_{\xi \to 0} \tilde{\Omega} = \infty
\]  

(4.2)

The large argument expansions for the Bessel function and modified Bessel functions are given as:

\[
J_n (\tilde{\Omega}) = \sqrt{2/\pi} \tilde{\Omega} \cos (\tilde{\Omega} - \pi/4 - n\pi/2)
\]  

(4.3)

\[
Y_n (\tilde{\Omega}) = \sqrt{2/\pi} \tilde{\Omega} \sin (\tilde{\Omega} - \pi/4 - n\pi/2)
\]  

(4.4)

\[
I_n (\tilde{\Omega}) = e^{\tilde{\Omega}/\sqrt{2\pi}} \tilde{\Omega}
\]  

(4.5)
Upon inspection of Eq. (4.3), (4.4), and (4.6) it is observed that in the limit as \( \xi \) approaches infinity that \( J_n, Y_n, \) and \( K_n \) approach zero. On the other hand the nature of the modified Bessel function \( I_n \) is given by:

\[
\lim_{\xi \to 0} I_n(\tilde{\Omega}) = \lim_{\tilde{\Omega} \to \infty} e^{\tilde{\Omega} \sqrt{2\pi \tilde{\Omega}}}
\]

Applying l'Hôpital's rule the limit becomes:

\[
\lim_{\xi \to 0} I_n(\tilde{\Omega}) = \lim_{\tilde{\Omega} \to \infty} 2 e^{\tilde{\Omega} \sqrt{2\pi \tilde{\Omega}}}
\]

\[
\lim_{\xi \to 0} I_n(\tilde{\Omega}) \to \infty
\]

(4.7)

When using the IBM 360-50 computer the maximum allowable argument value for an exponential function is 174.63. Thus, for \( \tilde{\Omega} \geq 174.63 \) the computer cannot execute the exponential function in Eq. (4.5) and consequently, cannot return values for \( I_n(\tilde{\Omega}) \). Thus, the first solution ceases to be useable when \( \tilde{\Omega} \geq 174.63 \).

In order to display the nature of the cut-off point we must examine \( \tilde{\Omega} \). First, consider beams with rectangular cross-sections where:

\[
A_o = b_o h_o
\]

\[
I_o = b_o h_o^3 / 12
\]

Substituting into Eq. (4.1) yields:

\[
\tilde{\Omega} = 2 \left( 12 \rho / E \right)^{1/4} \sqrt{\omega} \left( \ell / \xi \right) \sqrt{(1 + \xi)} / h_o
\]

(4.8)

and by defining,
\[ Cr = 2 \left( \frac{12\rho}{E} \right)^{1/4} \]  
(4.9)

\[ \gamma = \left( \frac{\ell}{\xi} \right) \frac{\sqrt{(1+\xi)}}{h_o} \]  
(4.10)

\[ \tilde{\Omega} = C_r \gamma \sqrt{\omega} \]  
(4.11)

For elliptical cross-sections we have:

\[ A_o = \pi b_0 h_o / 4 \]

\[ I_o = \pi b_0 h_o^3 / 64 \]

Which upon substitution into Eq. (4.1) yields:

\[ \tilde{\Omega} = 2 \left( \frac{16\rho}{E} \right)^{1/4} \sqrt{\omega} \left( \frac{\ell}{\xi} \right) \frac{\sqrt{(1+\xi)}}{h_o} \]  
(4.12)

and by defining:

\[ C_e = 2 \left( \frac{16\rho}{E} \right)^{1/4} \]  
(4.13)

\[ \tilde{\Omega} = C_e \gamma \sqrt{\omega} \]  
(4.14)

Figures (9) and (10) present a graphical representation of Eq. (4.11) and (4.14) for the physical properties of aluminum. The fact that the properties of aluminum were used does not lessen their generality because \( \rho/E \) value is nearly constant for many other common metals.

<table>
<thead>
<tr>
<th>Material</th>
<th>( \rho/E )</th>
<th>( \text{in}^2/\text{sec}^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminum</td>
<td>0.255 x 10^{-10}</td>
<td></td>
</tr>
<tr>
<td>Carbon Steel</td>
<td>0.256 x 10^{-10}</td>
<td></td>
</tr>
<tr>
<td>Stainless Steel</td>
<td>0.262 x 10^{-10}</td>
<td></td>
</tr>
</tbody>
</table>

In examining Figs. (9) and (10) the curve for the case of \( \gamma = \infty \) corresponds to a uniform beam and the curve for \( \gamma = 0 \) corresponds to a pointed or sharp beam. A pointed or sharp beam has one end with zero area. Since the \( \gamma = \infty \) curve intersects the level specified by the \( \tilde{\Omega} = 174.63 \) at \( \omega = 0 \), the Bessel function solution cannot be applied to obtain natural frequencies for a uniform beam. On the other hand, since the
Fig. 9 Cut-off Point for Beams of Rectangular Cross-Sections
Fig. 10 Cut-off Point for Beams of Elliptical Cross-Sections
\[ Y = 0 \] curve never intersects the \( \Omega = 174.63 \), the Bessel function solution can be used to calculate an infinite number of natural frequencies for a pointed or sharp beam. The majority of the beams fall within these two limits where the Bessel function solution can calculate only a finite number of natural frequencies before the value of \( \Omega \) exceeds \( 174.63 \). The available number of natural frequencies for most problems is well over five, which is about the maximum number of frequencies of interest. However, a small number of beam problems lie close to the \( Y \rightarrow \infty \) curves. For these curves the Bessel function solution cannot calculate all of the natural frequencies of interest. Consequently, the second or nearly uniform solution must be used.

The \( \gamma \) parameter is a function of \( \xi, \xi, \) and \( h_0 \). It is difficult to gain a physical feeling for its values. A rough approximation predicted by the \( \gamma \) term for normal height to length ratios is:

1) \( 0.111 < \xi < \infty \) \( \quad \rightarrow \quad \text{Bessel Function Transmission Matrix} \)
   or \( 0 < A_o/A_{\xi} < 0.9 \)

2) \( 0 < \xi \leq 0.111 \) \( \quad \rightarrow \quad \text{Nearly Uniform Transmission Matrix} \)
   or \( 0.9 < A_o/A_{\xi} < 1.0 \)

### 4.2 Programming Technique

The two transmission matrix solutions presented in chapter III are too complex for manual calculations. Consequently, a computing scheme was developed and its basic block diagram is shown in Fig. (11). This scheme was designed to treat beams with up to five segments where each segment can have a different slope.

In the block diagram after the input data for the beam has been stored, step 1 orients the positive sense of the segment such that \( \xi \) is positive. The positive sense of slope is taken as increasing area as \( x \)
Input beam data for each segment

1

Initialize \( \omega \)

Frequency loop

Segment loop

Calculate \( \Omega \) for particular segment

\( \tilde{\Omega} < 174.63 \)

Calculate \( T \) from Bessel function solution

\( \tilde{\Omega} \geq 174.63 \)

Calculate \( T \) from nearly uniform solution

\( \text{NEQ} \)

\( \text{NEQ} = 1 \)

Calculate \( R \) from \( T \)

\( \text{NEQ} = 0 \)

Store segment \( T \) matrices

Replace \( T \) by \( [T] = [C][R][C] \)

Calculate total \( T \) matrix by multiplication of segment \( T \) matrices

\( \text{DET}(\omega) \)

2

Output nat. freq.

Fig. 11 Block Diagram of Computing Scheme
Fig. 12 Block Diagram For Step 1

Fig. 13 Method of Successive Bisection
increases. The NEQ value is later used to employ the compatibility matrices and rearward form transmission matrix for negative slopes.

The segment loop calculates and stores all the segment transmission matrices. Then the total transmission matrix is calculated by the matrix multiplication of the segments' matrices.

The frequency loop iterates on $\omega$ until the $\text{DET}(\omega)$ is zero i.e., a natural frequency is determined. The terms involved in $\text{DET}(\omega)$ are dependent on the boundary conditions desired. In this loop step 2 is an iterative method based on a successive bisection method outlined in reference 25. As shown in Fig. (13) the $\omega$ value is incremented until a sign change occurs in the value of $\text{DET}(\omega)$. When a sign change occurs then the increment between the last two frequencies is halved successively for 14 iterations. After 14 iterations the last two iterated frequencies for the first mode differ in the fifth digit by $= 1$. For higher modes the difference is less.

Care must be taken as to the initial value of $\omega$ to insure that it is smaller than the first natural frequency. Also the initial increment must be small enough to insure that an entire sign change is not passed over.

4.3 Application to One Segment Beams

To show that the transmission matrix obtained is valid natural frequencies were calculated and compared to those for several one segment beams given in the literature. Six different boundary conditions were used in order to involve all of the 16 transmission matrix elements in at least one frequency determinate. For the transmission elements involved for each set of boundary conditions refer to Appendix B. As an example, free-fixed boundary conditions specify a determinate term given
as:

\[ \text{DET}(\omega) = T_{11}T_{22} - T_{21}T_{12} \] (4.15)

After calculating natural frequencies correctly for the six boundary conditions it is still possible that an entire row or column may be incorrect by a multiplicity factor. In order to insure that this did not happen a one segment beam was subdivided into several segments and the total transmission matrix was obtained using the multiplication rule in Eq. (2.13). By multiplying the segment matrices to form the total transmission matrix, the elements in the total matrix are computed from nearly all the rows and columns of the segment matrices. Consequently, if one row or column of the segment transmission matrix is incorrect it would show up in the natural frequency values calculated from a multi-segment beam. In the initial programming such a mistake was detected.

The groups of one segment beams considered were: (a) truncated wedge (b) truncated cone and (c) truncated rectangular pyramid.

4.4 Truncated Wedges

The truncated wedge beam has a linear variation in the height dimension and a constant base dimension. The cross-section in the y-z plane is rectangular. The fact that the area need not necessarily be zero at \( x=0 \) implies the beam is truncated.

![Truncated Wedge Diagram](image-url)
For the truncated wedge in Fig. (14) the following relationships apply:

\[ n = 1 \]  \quad (4.16)

\[ A(x) = A_0 (1 + \frac{\xi x}{\xi}) \]  \quad (4.17)

\[ I(x) = I_0 (1 + \frac{\xi x}{\xi})^3 \]  \quad (4.18)

For a rectangular cross-section z-y plane.

\[ A_0 = b_0 \frac{h_0}{2} \]  \quad (4.19)

\[ I_0 = \frac{b_0 h_0^3}{12} \]  \quad (4.20)

Substituting Eqs. (4.4) and (4.5) into Eqs. (4.2) and (4.3) yields:

\[ b = b_0 \]  \quad (4.21)

\[ h = h_0 \frac{z}{2} \]  \quad (4.22)

\[ \xi = \left( \frac{h_0}{h_2} \right) - 1 \]  \quad (4.23)

The first wedge calculation using the Bessel solution was made on a beam of \( \xi = 1.5 \). Values in the literature were present for free-fixed, fixed-fixed, pinned-pinned, and pinned-fixed. A representative comparison is shown in Table I where the term L.B. indicates lower bound and U.B. indicates upper bound. The numbers in parentheses refer to references in the bibliography from which the comparisons in that column were based. The \( e \) term is computed by:

\[ e = \frac{\phi_{\text{comp}} - \phi_{\text{reference}}}{\phi_{\text{reference}}} \times 100\% \]  \quad (4.24)

Thus, a positive \( e \) indicated that the computed value is larger than the reference value. In the tables the line indicates that the reference does not present values for that particular mode.
TABLE I  Free-Fixed Truncated Wedge ( $\xi = 1.5$)

<table>
<thead>
<tr>
<th>Mode</th>
<th>$\Phi_{\text{comp.}}$</th>
<th>(17)</th>
<th>(13)U.B.</th>
<th>(13)L.B.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.9344</td>
<td>0.0025%</td>
<td>0.0025%</td>
<td>0.046%</td>
</tr>
<tr>
<td>2</td>
<td>17.4886</td>
<td>-0.60%</td>
<td>0.00023%</td>
<td>0.82%</td>
</tr>
<tr>
<td>3</td>
<td>44.0267</td>
<td>—</td>
<td>-0.065%</td>
<td>5.00%</td>
</tr>
</tbody>
</table>

Further frequencies were calculated using the Bessel function solution for $\xi = 1.0$. The boundary conditions considered were, free-fixed, free-free, and fixed-free. The $\xi$ value for the free-free boundary condition is given in Table II.

TABLE II  Free-Free Truncated Wedge ($\xi = 1.0$)

<table>
<thead>
<tr>
<th>Mode</th>
<th>$\Phi_{\text{comp.}}$</th>
<th>(11)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>16.724</td>
<td>-0.006%</td>
</tr>
<tr>
<td>2</td>
<td>45.503</td>
<td>0.0066%</td>
</tr>
<tr>
<td>3</td>
<td>88.711</td>
<td>-0.445%</td>
</tr>
<tr>
<td>4</td>
<td>146.27</td>
<td>0.0014%</td>
</tr>
</tbody>
</table>

From the computations for $\xi = 1.0$ and $\xi = 1.5$ all 16 of the Bessel function transmission elements were used. The remaining step consisted of taking a one segment beam and subdividing it into several segments to compute the natural frequencies. For an aluminum free-fixed beam Table III was prepared with the following dimensions.

$$
\begin{align*}
    b_o &= 2.0'' \\
    b_\xi &= 2.0'' \\
    h_o &= 1.0'' \\
    h_\xi &= 2.0'' \\
    \xi &= 35''
\end{align*}
$$
TABLE III  Natural Frequency Comparison Using Multiple Segments
($\xi = 1.0$)

<table>
<thead>
<tr>
<th>Mode</th>
<th>1 Seg.</th>
<th>5 Seg.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>357.6664</td>
<td>357.6664</td>
</tr>
<tr>
<td>2</td>
<td>1713.344</td>
<td>1713.344</td>
</tr>
<tr>
<td>3</td>
<td>4421.016</td>
<td>4421.016</td>
</tr>
</tbody>
</table>

From Table III it can be inferred that segmenting a beam only adds very small roundoff error which in this case is negligible. In general, the results displayed in Table I - III indicate that the Bessel function transmission matrix has been formulated correctly for truncated wedges and provides accurate results with minimal numerical error.

In the foregoing discussion only general truncated wedges have been considered. The two limiting cases of (a) sharp wedges and (b) uniform or nearly uniform rectangular beams were not mentioned. The numerical scheme cannot solve exactly for a sharp beam ($\xi = \infty$) because it is impossible to use the value of infinity in a computing system. However, a sharp wedge can be approximated by making one end very much smaller than the other end i.e., $\xi \to \infty$. The values used in Fig. (15) are for free-fixed boundary conditions and display the magnitude of $\xi$ necessary in order to accurately approximate a sharp wedge. The $e$ values calculated for $\xi = 3999$ are shown in Table IV.

TABLE IV  Sharp Wedge Approximation ($\xi = 3999$)

<table>
<thead>
<tr>
<th>Mode</th>
<th>(\phi_{\text{comp.}})</th>
<th>(19)</th>
<th>(13)</th>
<th>(11)</th>
<th>(15)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.3124</td>
<td>-0.049%</td>
<td>-0.14%</td>
<td>-0.060%</td>
<td>-0.051%</td>
</tr>
<tr>
<td>2</td>
<td>15.200</td>
<td>-0.047%</td>
<td>0.0%</td>
<td>-0.035%</td>
<td>-0.013%</td>
</tr>
<tr>
<td>3</td>
<td>30.005</td>
<td>-0.049%</td>
<td>-0.17%</td>
<td>-0.096%</td>
<td>-0.047%</td>
</tr>
</tbody>
</table>
Fig. 15 Sharp Wedge Approximation
For the other limiting case of nearly uniform or uniform rectangular beams, the perturbation solution based on Mok and Murray must be used. This transmission matrix solution must be checked in a fashion similar to that used for the Bessel function solution. For this purpose a uniform beam was examined for 5 boundary conditions. The comparisons for three of the boundary conditions are listed in Table V. Since the value is very small the actual dimensionless frequency parameters are listed instead of \( \xi \).

**TABLE IV** Sharp Wedge Approximation (\( \xi = 3999 \)) (continued)

<table>
<thead>
<tr>
<th>Mode</th>
<th>( \Phi_{\text{comp.}} )</th>
<th>(19)</th>
<th>(13)</th>
<th>(11)</th>
<th>(15)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>49.739</td>
<td>-0.049%</td>
<td>-0.12%</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>5</td>
<td>74.405</td>
<td>-0.047%</td>
<td>0.007%</td>
<td>---</td>
<td>---</td>
</tr>
</tbody>
</table>

**TABLE V** Uniform Beam For Various Boundary Conditions (\( \xi \to 0 \))

<table>
<thead>
<tr>
<th>Boundary Condition</th>
<th>Mode</th>
<th>( \Phi_{\text{comp.}} )</th>
<th>( \Phi_{\text{Ref. 2,5,19}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Free-Fix</td>
<td>1</td>
<td>3.51602</td>
<td>3.51601 (19)</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>22.0345</td>
<td>22.03449 (19)</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>61.6972</td>
<td>61.69721 (19)</td>
</tr>
<tr>
<td>Pin-Pin</td>
<td>1</td>
<td>9.8696</td>
<td>9.869604 (2)</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>39.4784</td>
<td>39.478416 (2)</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>88.8264</td>
<td>88.826436 (2)</td>
</tr>
<tr>
<td>Fix-Fix</td>
<td>1</td>
<td>22.3732</td>
<td>22.3733 (5)</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>61.6727</td>
<td>61.6728 (5)</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>120.903</td>
<td>120.9034 (5)</td>
</tr>
</tbody>
</table>
The two boundary conditions of pinned-fixed and free-free were also calculated and compared similarly. From the calculations for a uniform beam for the five boundary conditions, the perturbation solution appears to be correct.

Again, to be sure that no column or row has been inadvertently multiplied by a factor, a wedge was subdivided into several segments and natural frequencies were computed. The material properties of aluminum and the dimensions below were used in these calculations,

\[ b_0 = 2.0'' \quad b_2 = 2.0'' \quad l = 37.5'' \]
\[ h_0 = 2.0'' \quad h_2 = 2.00001'' \]

and the results given in Table VI indicate that the perturbation solution is a valid transmission matrix solution.

<p>| TABLE VI Natural Frequency Comparison of Multi-Segmenting of One Segment Beams (( \xi = 0 )) |
|-------|-----|-----|</p>
<table>
<thead>
<tr>
<th>Mode</th>
<th>1 Seg.</th>
<th>3 Seg.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>804.191</td>
<td>804.190</td>
</tr>
<tr>
<td>2</td>
<td>3216.77</td>
<td>3216.76</td>
</tr>
<tr>
<td>3</td>
<td>7237.72</td>
<td>7237.70</td>
</tr>
</tbody>
</table>

4.5 Truncated Cones

The truncated cone is a solid cone of circular cross-section (solid cylinder with linear variation of the diameter).

Fig. 16 Truncated Cone
For the truncated cone in Fig. (16) we have the following relationships.

\[ n = 2 \]  \hspace{2cm} (4.25)

\[ b_0 = h_o = d_o \]  \hspace{2cm} (4.26)

\[ A_o = \left(\frac{\pi}{4}\right) d_o^2 \]  \hspace{2cm} (4.27)

\[ I_o = \frac{\pi}{64} d_o^4 \]  \hspace{2cm} (4.28)

\[ A(x) = A_o (1 + \xi x/L)^2 \]  \hspace{2cm} (4.29)

\[ I(x) = I_o (1 + \xi x/L)^4 \]  \hspace{2cm} (4.30)

\[ \xi = \sqrt{\frac{A_L}{A_o}} - 1 \]  \hspace{2cm} (4.31)

Since the two transmission solutions were checked for truncated wedges the basic approach is assumed to be correct. For truncated cones basically we want to show that the two solutions are also valid for \( n = 2 \).

**TABLE VII** Free-Fixed Truncated Cone (\( \xi = 1.0 \))

<table>
<thead>
<tr>
<th>Mode</th>
<th>( \phi ) comp.</th>
<th>(13) U.B.</th>
<th>(13) L.B.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.62515</td>
<td>-0.0011%</td>
<td>0.049%</td>
</tr>
<tr>
<td>2</td>
<td>19.5476</td>
<td>0%</td>
<td>0.086%</td>
</tr>
<tr>
<td>3</td>
<td>48.5789</td>
<td>-0.04%</td>
<td>5.17%</td>
</tr>
</tbody>
</table>

**TABLE VIII** Nearly Uniform Free-Free Cylinder (\( \xi = 0.001 \))

<table>
<thead>
<tr>
<th>Mode</th>
<th>( \phi ) comp.</th>
<th>(19)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>22.362</td>
<td>-0.0504%</td>
</tr>
<tr>
<td>2</td>
<td>61.6417</td>
<td>-0.0505%</td>
</tr>
<tr>
<td>3</td>
<td>120.842</td>
<td>-0.0508%</td>
</tr>
<tr>
<td>4</td>
<td>199.759</td>
<td>-0.0503%</td>
</tr>
<tr>
<td>5</td>
<td>298.405</td>
<td>-0.0504%</td>
</tr>
</tbody>
</table>
As shown in Table VII and VIII the two transmission matrices can be applied when \( n = 2 \).

Again in order to approximate a pointed or sharp beam the \( \xi \) value must be made very large. For free-fixed boundary conditions, Fig. (18) shows the magnitude of \( \xi \) necessary to accurately approximate a sharp cone. The \( \xi \) value monotonically decreases as \( \xi \) increases. For \( \xi = 3999 \) the \( \xi \) term listed in Table IX is negligible.

<table>
<thead>
<tr>
<th>Mode</th>
<th>( \phi_{\text{comp.}} )</th>
<th>(16)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8.7149</td>
<td>-0.05%</td>
</tr>
<tr>
<td>2</td>
<td>21.1351</td>
<td>-0.05%</td>
</tr>
<tr>
<td>3</td>
<td>38.4345</td>
<td>-0.05%</td>
</tr>
</tbody>
</table>

4.6 Truncated Rectangular Pyramids

The rectangular pyramid is similar to a wedge except that both the base and height vary linearly.

Fig. 17 Truncated Rectangular Pyramid
Fig. 18 Sharp Cone Approximation
\( n = 2 \) \hspace{1cm} (4.32)
\[
\zeta = \sqrt{\frac{A_0}{A_x}} - 1 \hspace{1cm} (4.33)
\]
\[
I(x) = I_0 x^n \hspace{1cm} (4.34)
\]
\[
A(x) = A_0 x^2 \hspace{1cm} (4.35)
\]

Since for a truncated rectangular pyramid \( I(x) \) and \( A(x) \) have the same mathematical variation as a truncated cone \((n = 2)\), its application is similar. One calculation is presented in Table X.

### TABLE X Free-Fixed Truncated Rectangular Pyramid \((\xi = 4)\)

<table>
<thead>
<tr>
<th>Mode</th>
<th>( \phi_{\text{comp.}} )</th>
<th>( (19) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.1964</td>
<td>-0.013%</td>
</tr>
<tr>
<td>2</td>
<td>18.386</td>
<td>0.011%</td>
</tr>
<tr>
<td>3</td>
<td>39.834</td>
<td></td>
</tr>
</tbody>
</table>

The limiting case for \( \xi \to 0 \) is the rectangular uniform beam treated in section 4.3. The approximation of a sharp pyramid follows the same trends as that reported for a truncated cone.

#### 4.7 Two Segment Free-Free Beams

In the literature natural frequencies are present for two segment free-free beams. The area is first increasing then decreasing. Since the Bessel solution is valid only for a positive slope the coordinate systems must be oriented as shown in Fig. (19).
This situation represents the incompatible sign convention system discussed in chapter II where:

\[
\{\psi_1\} = [T]_{12} \left[ \%C_{-}\right]_{23} [R]_{34} \left[ \%C_{+}\right]_{45} \{\psi_5\} \quad (4.36)
\]

\[
[T]_{15} = [T]_{12} \left[ \%C_{-}\right]_{23} [R]_{34} \left[ \%C_{+}\right]_{45} \quad (4.37)
\]

The form of the \([ \%C_{\pm}]\) matrices is given in Appendix A. In reference (17) only the symmetrical modes are presented, which are modes 2 and 4 in Table XI, and the results compare very well for these two cases.

**TABLE XI Double Cones (\(\xi = 2/3\))**

<table>
<thead>
<tr>
<th>Mode</th>
<th>(\phi_{\text{comp.}})</th>
<th>(\phi_{\text{comp.}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.9917</td>
<td>0%</td>
</tr>
<tr>
<td>2</td>
<td>13.606</td>
<td>0.006%</td>
</tr>
<tr>
<td>3</td>
<td>25.651</td>
<td>0.006%</td>
</tr>
</tbody>
</table>
TABLE XI  Double Cones ($\xi = 2/3$) (continued)

<table>
<thead>
<tr>
<th>Mode</th>
<th>$\Phi_{\text{comp.}}$</th>
<th>(17)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>40.958</td>
<td>---</td>
</tr>
<tr>
<td>5</td>
<td>60.690</td>
<td>0.0%</td>
</tr>
</tbody>
</table>

Similar data was computed for double wedge beams.

In summary chapter IV the cut-off point for the Bessel function was examined and the computing scheme discussed. In the validity computations the correctness of the entire computing and theoretical scheme was methodically proven. From its application to truncated wedges where $n = 1$ it was shown that:

1. all 16 elements of the two transmission matrices are correct.
2. a beam with increasing area (positive slope) could accurately be subdivided into multi-segments.
3. a pointed wedge cannot be treated exactly, but it can be accurately approximated by making $\xi > 3999$.

For the case when $n = 2$ it was shown that truncated cones and rectangular pyramids can also be accurately described. Finally, the calculations on two segment beams shows that:

1. the form of the compatibility matrices are correct.
2. beams where adjoining segments have different slopes, i.e., the area is increasing and then decreasing, can be handled.

Thus, in general the technique presented in chapter III and developed into a computing scheme in chapter IV was concluded to be a valid technique for multi-segmented Bernoulli-Euler beams.
CHAPTER V

NATURAL FREQUENCIES FOR THREE SEGMENT BERNOULLI-EULER BEAMS

5.1 Three Segment Beam Geometry and Dimensions

In chapter IV the transmission matrix solutions were shown to accurately describe principal mode frequencies of one and two segment Bernoulli-Euler beams. It is our purpose in chapter V to apply these solutions to a study of the natural frequencies of a class of three segment beams with the geometry shown in Fig. (20).

Fig. 20 Three Segment Beams

The base dimension b is the same for the entire beam. The mid section is a rectangular uniform beam and the first and third segments are truncated wedges. The dimensionless numbers that are pertinent to this geometry are:

\[ \text{area ratio} = \frac{A_0}{A_1} = \frac{h_0}{h_1} \]  \hspace{1cm} (5.1)
\[ \text{length ratio} = \frac{L_1}{L_2} \]  \hspace{1cm} (5.2)

The two boundary conditions that are investigated are pinned-pinned
and fixed-fixed. The natural frequency values are cast into a dimensionless form by dividing the natural frequency of the three segment beam by the frequency of a uniform beam with length $L$ and area $A_1$.

$$\text{Frequency Ratio} = \frac{\omega}{\omega_0}$$  \hspace{1cm} (5.3)

Where $\omega_0$ is calculated from:

$$\omega_0 = \frac{\phi_i \cdot EI}{L^2 \cdot \rho A_1}$$  \hspace{1cm} (5.4)

The term $\omega_0$ for a uniform beam is well documented as (2,5) and the first three principal mode roots are:

- **pinned-pinned** $\phi_i = (n \pi)^2$
  $\phi_i = 9.869604, 39.478416, 88.826436$  \hspace{1cm} (5.5)

- **fixed-fixed** $\phi_i = 22.3733, 61.6728, 120.9034$  \hspace{1cm} (5.6)

In order to calculate numerical values for the frequency ratios $\rho$ and $E$ for aluminum and the following dimensions were used:

$$\rho = 0.2539 \times \text{lb}-\text{sec}^2/\text{in}^4$$  \hspace{1cm} (5.7)

$$E = 10 \times 10^6 \text{ psi}$$  \hspace{1cm} (5.8)

$$h_1 = 2.0''$$  \hspace{1cm} (5.9)

$$b_1 = 2.0''$$  \hspace{1cm} (5.10)

$$L = 37.5''$$  \hspace{1cm} (5.11)

For the parameters used the $\omega_0$ values for the first three modes are:

- **pinned-pinned** $\omega_0 = 804.2, 3216.8, 7237.7 \text{ rad/sec}$  \hspace{1cm} (5.12)

- **fixed-fixed** $\omega_0 = 1823.0, 5025.2, 9851.4 \text{ rad/sec}$  \hspace{1cm} (5.13)

For the three segment beams described, the first three calculated frequency ratios were tabulated in Tables XII–XVII by varying the length.
and area ratios. The data was plotted into graphical form in Figs. (21) - (26) where the \( \omega/\omega_0 \) term was the ordinate and the \( A_0/A_1 \) term was the abscissa. Separate curves were drawn for the length ratios of \( L_1/L_\infty = \infty, 2, 1, 0.5, 0.25, 0. \)

The length ratio of zero corresponds to a uniform beam, while the length ratio of infinity corresponds to a double wedge with no uniform section. The area ratio of 1 also corresponds to a uniform beam regardless of the length ratio. Consequently, all the curves pass through the point characterized by \( A_0/A_1 = 1 \) and \( \omega/\omega_0 = 1 \).
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TABLE XIV Third Mode Frequency Ratio for Three Segment Pinned-Pinned Beams

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TABLE XVI Second Mode Frequency Ratio for Three Segment Fixed-Fixed Beams

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TABLE XVII Third Mode Frequency Ratio for Three Segment Fixed-Fixed Beams

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Fig. 21 First Mode Frequency Ratio for Three Segment Pinned-Pinned Beams
Fig. 22 Second Mode Frequency Ratio for Three Segment Pinned-Pinned Beams
Fig. 23 Third Mode Frequency Ratio for Three Segment Pinned-Pinned Beams
Fig. 24 First Mode Frequency Ratio for Three Segment Fixed-Fixed Beams
Fig. 24a First Mode Frequency Ratio for Three Segment Fixed-Fixed Beams
Fig. 25 Second Mode Frequency Ratio for Three Segment Fixed-Fixed Beams
Fig. 26 Third Mode Frequency Ratio for Three Segment Fixed-Fixed Beams
5.2 Discussion of Tables

For the pinned-pinned beams, Figs. (21) - (23), the first mode has the smallest variation of frequency ratio for the region \( A_0/A_1 > 1 \). In fact, the curves for values of \( L_1/L_2 = 1, 0.5, 0.25, 0 \) in this region are nearly coincident. For the second and third modes the curves are more distinct. Consequently, for \( A_0/A_1 > 1 \) the length ratio variations have minor affect on the first natural frequencies but they have a more pronounced affect for the second and third modes.

For pinned-pinned beams where \( A_0/A_1 < 1 \) the opposite trend is observed. The first mode has the largest variation of frequency ratio, while the second and third modes have a somewhat smaller variation in frequency ratio. Thus, for \( A_0/A_1 < 1 \) the length ratio has a major affect on the first natural frequency with a less affect on the second and third modes.

For fixed-fixed beams Figs. (24) - (26) the length ratio has the least affect on the first mode frequency ratio. In expanded view Fig. (24a) the curves for the first mode in the region \( A_0/A_1 < 1 \) are very close. Also in the region \( A_0/A_1 > 1 \) the curves lie close together. In fact, for \( L_1/L_2 = \infty, 2, 1 \) the curves are nearly coincident. For the second and third modes the curves have a wider variation of frequency ratio for the various length ratios.
CONCLUSIONS

This paper has investigated the application of two transmission matrix solutions to several groups of non-uniform Bernoulli-Euler beams. On the basis of a natural frequency comparison for one and two segment beams, the following conclusions are drawn:

1) All 16 elements of the two transmission matrices presented are valid for the truncated wedge, truncated cone, and truncated rectangular pyramid segments.

2) Non-uniform beams can be subdivided into multi-segments and treated in a piecewise linear basis.

3) The compatibility matrix formulation is correct and allows for the arbitrary choice of sign conventions at the segment level.

4) The transmission matrix can be directly used to determine natural frequencies for any set boundary conditions.

In the application of the transmission matrices to the group of three segment beams it has been concluded that:

1) In general, the pinned-pinned beams have a smaller range of frequency variation than the fixed-fixed beams. Consequently, the natural frequencies of three segment pinned-pinned beams are less sensitive to changes in length and area ratio.

2) For the first mode the frequency ratio curves for pinned-pinned beams lie close to the \( L_1/L_2 = 0 \) curve, i.e., uniform beam, as the length and area ratios change. However, the curves for the fixed-fixed beams lie closer to the \( L_1/L_2 = \infty \) curve, i.e., two segment double wedge, as the length and area
ratios vary.

3) For the second and third modes the frequency ratio curves for the fixed-fixed and pinned-pinned beams do not lie close to either of the limits. Instead they are more or less evenly distributed between the two curves for $L_1/L_2 = \infty$ and $L_1/L_2 = 0$. 
APPENDIX A

COMPATIBILITY MATRICES FOR BERNOULLI-EULER BEAM SEGMENTS

Due to the fact that the Bessel function solution is valid only for positive values of slope, the coordinate system must be oriented at the small end of a beam segment. If in a chain arrangement of the beam segments the area is increasing and then decreasing, at least one of the beam segments will have its positive sense opposite to the overall positive sense for the system. Moreover, in this case, the compatibility matrices as stated in section 2.1 will not reduce to the identity matrix.

Consider as an example the three segment Bernoulli-Euler beam in Fig. (27) where the second segment has the positive sense of its state vector opposite to the overall positive sense.

Now examining the state vectors \( \{\psi_2\} \) and \( \{\psi_3\} \) one observes:

\[
V_2 = -V_3 \quad W_2 = W_3 \\
M_2 = M_3 \quad \phi_2 = -\phi_3
\]
In matrix form:

\[
\begin{bmatrix}
V_1 \\
M_1 \\
W_1 \\
\phi_1
\end{bmatrix} =
\begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
V_2 \\
M_2 \\
W_2 \\
\phi_2
\end{bmatrix}
\]

Therefore:

\[
[\mathbf{C}_2]_{23} =
\begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
\]

(A.1)

For the state vectors \{\psi_4\} and \{\psi_5\} we observe the same relationships:

\[
\begin{align*}
V_4 &= -V_5 \\
M_4 &= M_5 \\
W_4 &= W_5 \\
\phi_4 &= -\phi_5
\end{align*}
\]

Consequently:

\[
[\mathbf{C}_5]_{45} =
\begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
\]
APPENDIX B

FREQUENCY DETERMINATES

In chapter II the method was defined to determine the $2 \times 2$ frequency determinates. For some of the commonly encountered boundary conditions refer to the following list.

1) Free (x=0) - Fixed (x=l)  \[ \text{DET}(\omega) = T_{11} T_{22} - T_{12} T_{21} \]

2) Pinned (x=0) - Pinned (x=l)  \[ \text{DET}(\omega) = T_{21} T_{34} - T_{31} T_{24} \]

3) Fixed (x=0) - Fixed (x=l)  \[ \text{DET}(\omega) = T_{31} T_{42} - T_{41} T_{32} \]

4) Pinned (x=0) - Fixed (x=l)  \[ \text{DET}(\omega) = T_{21} T_{42} - T_{41} T_{32} \]

5) Free (x=0) - Free (x=l)  \[ \text{DET}(\omega) = T_{13} T_{24} - T_{23} T_{14} \]

6) Pinned (x=0) - Sliding (x =l)  \[ \text{DET}(\omega) = T_{22} T_{33} - T_{32} T_{23} \]

7) Fixed (x=0) - Sliding (x=l)  \[ \text{DET}(\omega) = T_{32} T_{43} - T_{42} T_{33} \]
APPENDIX C

TRANSMISSION MATRIX (n \neq \text{integer})

For \( n \neq \text{integer} \) the solution is given in Eq. (3.31) as:

\[
T_{11}(r) = r^{-n} \left[ C_{1i} J_n(\mu r) + C_{2i} J_{-n}(\mu r) + C_{3i} I_n(\mu r) + C_{4i} I_{-n}(\mu r) \right] \quad \text{(C.1)}
\]

Again in order to evaluate the \( C_{ij} \) constants we must evaluate the matrix initial conditions at \( x = 0, \ z = 1 \). The \( [v] \) matrix will be the same as in Eq. (3.33a) however, the derivatives of \( T_{ij} \) will be different.

\[
T_{11} \bigg|_{x=0} = C_{1i} J_n(\mu) + C_{2i} J_{-n}(\mu) + C_{3i} I_n(\mu) + C_{4i} I_{-n}(\mu) \quad \text{(C.2)}
\]

\[
\frac{2}{\mu} \frac{d}{dz} T_{11} \bigg|_{x=0} = -C_{1i} J_{n+1}(\mu) + C_{2i} J_{-n-1}(\mu) + C_{3i} I_{n+1}(\mu) + C_{4i} I_{-n-1}(\mu) \quad \text{(C.3)}
\]

\[
\left( \frac{2}{\mu} \right)^2 \frac{d^2}{dz^2} T_{11} \bigg|_{x=0} = C_{1i} J_{n+2}(\mu) + C_{2i} J_{-n-2}(\mu) + C_{3i} I_{n+2}(\mu) + C_{4i} I_{-n-2}(\mu) \quad \text{(C.4)}
\]

\[
\left( \frac{2}{\mu} \right)^3 \frac{d^3}{dz^3} T_{11} \bigg|_{x=0} = -C_{1i} J_{n+3}(\mu) + C_{2i} J_{-n-3}(\mu) + C_{3i} I_{n+3}(\mu) + C_{4i} I_{-n-3}(\mu) \quad \text{(C.5)}
\]

Thus, casting into \( [\beta] \) matrix notation:

\[
[\beta] = \begin{bmatrix}
J_n(\mu) & J_{-n}(\mu) & I_n(\mu) & I_{-n}(\mu) \\
-J_{n+1}(\mu) & J_{-n-1}(\mu) & I_{n+1}(\mu) & I_{-n-1}(\mu) \\
J_{n+2}(\mu) & J_{-n-2}(\mu) & I_{n+2}(\mu) & I_{-n-2}(\mu) \\
-J_{n+3}(\mu) & J_{-n-3}(\mu) & I_{n+3}(\mu) & I_{-n-3}(\mu)
\end{bmatrix} \quad \text{(C.6)}
\]

\[
[\beta] [C] = [v] \quad \text{(C.7)}
\]

or

\[
[C] = [\beta]^{-1} [v] \quad \text{(C.8)}
\]
Thus, Eq. (3.8) can be solved for the constants.

In order to calculate the remaining elements of the transmission matrix we utilize Eq. (3.1) - (3.4).

\[
T_{i2} = \frac{d}{dx} T_{i1}
\]

\[
T_{i2} = \frac{\xi \mu}{2 \ell} n^{-1} (-C_{1i}J_{n+1}(\mu r) + C_{2i}J_{n-1}(\mu r) + C_{3i}I_{n+1}(\mu r) + C_{4i}I_{n-1}(\mu r))
\]

\[
T_{i4} = \frac{1}{\gamma(x)} \frac{d}{dx} T_{i2}
\]

\[
T_{i4} = (\xi \mu / 2 \ell)^2 EI_o n^{n+2} \{C_{1i}J_{n+2}(\mu r) + C_{2i}J_{n-2}(\mu r) + C_{3i}I_{n+2}(\mu r)
\]
\[+ C_{4i}I_{n-2}(\mu r))\]

\[
T_{i3} = \frac{-d}{dx} T_{i4}
\]

\[
T_{i3} = -(\xi \mu / 2 \ell)^3 EI_o y^{n+1} A_3 - (\xi \mu / 2 \ell)^2 EI_o (n+2) y^n A_4
\]

\[
A_3 = -C_{1i}J_{n+3}(\mu r) + C_{2i}J_{n-3}(\mu r) + C_{3i}I_{n+3}(\mu r) + C_{4i}I_{n-3}(\mu r)
\]

\[
A_4 = C_{1i}J_{n+2}(\mu r) + C_{2i}J_{n-2}(\mu r) + C_{3i}I_{n+2}(\mu r) + C_{4i}I_{n-2}(\mu r)
\]
APPENDIX D
DERIVATIVES OF PERTURBATION SOLUTION QUANTITIES

In the evaluation of the \([B]\) matrix in Eqs. (3.60) - (3.82) it is necessary to calculate the first three derivatives of \(H\) and \(m\) at \(x=0\).

First consider the \(H\) term defined as:

\[
H = \text{CON}1 \frac{(-2n-1)}{4}
\]

Recall that:

\[
Z = 1 + \frac{\xi x}{\ell}
\]

\[
\frac{d}{dx} = \frac{\xi}{\ell} \frac{d}{dZ}
\]

Then the \(H\) terms are given as:

\[
H_0 = H \bigg|_{x=0} = \text{CON}1
\]

\[
H'_0 = \left. \frac{d}{dx} H \right|_{x=0} = \frac{\xi}{\ell} \left. \frac{d}{dZ} H \right|_{Z=1} = -\text{CON}1 \frac{(2n+1)}{4} \left(\frac{\xi}{\ell}\right)
\]

\[
H''_0 = (\frac{\xi}{\ell})^2 \left. \frac{d^2}{dZ^2} H \right|_{Z=1} = \text{CON}1 \frac{(2n+5)}{4} \left(\frac{\xi}{\ell}\right)^2
\]

\[
H'''_0 = (\frac{\xi}{\ell})^3 \left. \frac{d^3}{dZ^3} H \right|_{Z=1} = H''_0 \frac{(2n+9)}{4} \left(\frac{\xi}{\ell}\right)
\]

Secondly consider the \(m\) term defined as:

\[
m = \text{CON}2 \left(\sqrt{Z} - 1\right)
\]

\[
m_o = m \bigg|_{x=0} = m \bigg|_{Z=1} = 0
\]

\[
m'_o = (\frac{\xi}{\ell}) \left. \frac{d}{dZ} m \right|_{Z=1} = \sqrt{\frac{\xi}{\ell}} \space \text{CON}2
\]

\[
m''_o = (\frac{\xi}{\ell})^2 \left. \frac{d^2}{dZ^2} m \right|_{Z=1} = -\frac{1}{4} (\frac{\xi}{\ell})^2 \space \text{CON}2
\]

\[
m'''_o = (\frac{\xi}{\ell})^3 \left. \frac{d^3}{dZ^3} m \right|_{Z=1} = \frac{3}{8} (\frac{\xi}{\ell})^3 \space \text{CON}2
\]
In order to evaluate the \( [T] \) for a Bernoulli-Euler beam segment of length \( L \), the first three derivatives of \( H \) and \( m \) must be calculated at \( x = \ell \).

For the \( \tilde{H} \) terms we obtain:

\[
\tilde{H} = H \bigg|_{x = \ell} = H(1 + \varepsilon)^{(-2n-1)/4} = \text{CON1} \quad (D.11)
\]

\[
\tilde{H}' = \frac{d}{dx} H \bigg|_{x = \ell} = H_0(1 + \varepsilon)^{(-2n-5)/4} = \text{CON2} \quad (D.12)
\]

\[
\tilde{H}'' = \frac{d^2}{dx^2} H \bigg|_{x = \ell} = H_0(1 + \varepsilon)^{(-2n-9)/4} = \text{CON3} \quad (D.13)
\]

\[
\tilde{H}''' = \frac{d^3}{dx^3} H \bigg|_{x = \ell} = H_0(1 + \varepsilon)^{(-2n-13)/4} = \text{CON4} \quad (D.14)
\]

The \( \tilde{m} \) terms are:

\[
\tilde{m} = m \bigg|_{x = \ell} = \text{CON2} \{ \sqrt{1 + \varepsilon} - 1 \} \quad (D.15)
\]

\[
\tilde{m}' = \frac{d}{dx} m \bigg|_{x = \ell} = m_0(1 + \varepsilon)^{-1/2} \quad (D.16)
\]

\[
\tilde{m}'' = \frac{d^2}{dx^2} m \bigg|_{x = \ell} = m_0^{1/2} (1 + \varepsilon)^{-3/2} \quad (D.17)
\]

\[
\tilde{m}''' = \frac{d^3}{dx^3} m \bigg|_{x = \ell} = m_0^{3/2} (1 + \varepsilon)^{-5/2} \quad (D.18)
\]
BIBLIOGRAPHY


VITA

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