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Analysis of cascade of four-terminal networks

Ernest Randolph Roehl

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ANALYSIS OF CASCADE OF FOUR-TERMINAL NETWORKS

BY

ERNEST RANDOLPH ROEHL

A
THESIS

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>List of Illustrations</td>
<td>ii</td>
</tr>
<tr>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>Review of Literature</td>
<td>3</td>
</tr>
<tr>
<td>Elementary Discussion of Networks</td>
<td>9</td>
</tr>
<tr>
<td>Network Equilibrium Conditions</td>
<td>19</td>
</tr>
<tr>
<td>Mathematical Solution of the System of Simultaneous Linear Differential</td>
<td></td>
</tr>
<tr>
<td>Equations with Constant Coefficients for the General Network</td>
<td>23</td>
</tr>
<tr>
<td>The Determinantal Method of Solution</td>
<td>28</td>
</tr>
<tr>
<td>Derivation of Fundamental Relations</td>
<td>34</td>
</tr>
<tr>
<td>Solution for Cascade of Identical Dissymmetrical Networks on the Iterative Basis</td>
<td>39</td>
</tr>
<tr>
<td>Solution for Cascade of Identical Dissymmetrical Networks on the Image Basis</td>
<td>47</td>
</tr>
<tr>
<td>A Partial Experimental Verification of Results</td>
<td>62</td>
</tr>
<tr>
<td>Conclusions</td>
<td>66</td>
</tr>
<tr>
<td>Summary</td>
<td>67</td>
</tr>
<tr>
<td>Appendix</td>
<td>68</td>
</tr>
<tr>
<td>Bibliography</td>
<td>73</td>
</tr>
</tbody>
</table>
LIST OF ILLUSTRATIONS

Figures

1. Elementary network .......................... 9
2. Circuit illustration of mesh currents ....... 10
3. Illustration of simple two-mesh network .... 11
4. Schematic of a four-terminal network
   showing reference directions for voltage
   and current .................................. 24
5. Cascade of identical dissymmetrical networks
   on the iterative basis ........................ 33
6. Internal portion of the cascaded structure
   of Fig. 5 .................................... 41
7. Cascade of identical dissymmetrical
   networks on the image basis ............... 47
8. Internal portion of the cascaded
   structure of Fig. 7 ............................ 48
9. Laboratory circuit used to obtain
   experimental results .......................... 62
10. Diagram of individual four-terminal
    network used in cascade connection
    in laboratory circuit ....................... 63
11. Cascade of identical dissymmetrical
    networks with networks combined
    as two four-terminal networks / ........... 71
INTRODUCTION

The object of this thesis is to find analytic expressions for the voltage and current variations along a structure composed of a cascade of identical dissymmetrical four-terminal networks driven harmonically at the sending end and working into an impedance at the receiving end and an impedance that can be placed at any of the junctions along the cascade. The method of analysis that is applied is analogous to that used in solving the long-line problem.

A transmission system usually consists of the line or cable itself plus numerous other links whose purpose it is to correct defects in the line performance or to supply selective or other characteristics to the system that may be required by the class of service that is expected of the structure as a whole. Each of these single links can be considered to be a four-terminal network. Then the complete line is composed of a number of these four-terminal networks connected in cascade. The transmission lines or cables are known as distributed-constant networks while the remaining portions of the line are known as lumped-constant networks. Since the performance of the distributed-constant four-terminal networks can be reduced to the same basis as that of the lumped-constant networks, the system as a whole becomes homogeneous and a consistent method of analysis for the determination of the overall performance can be developed.
A solution for the voltages and currents along the line has been obtained for a communication line of this type loaded with an impedance at the end of the line, but so far no solution has been published for the case considered in this paper. Since it is sometimes of advantage to place an additional load at different junctions along a communication line, it seems appropriate to determine the equations for the voltages and currents at the various junctions along the line so that these can be calculated if desired.

It is the purpose of this paper to develop these equations.
Transmission lines were used in electrical communications for many years before their theoretical behavior was clearly understood. The commercial development of the electric telegraph began in the early 1840's.

In 1855, Sir William Thomson (Lord Kelvin) made the first attempt at an analysis of the propagation of electrical energy through a long uniform circuit. His analysis was applicable only to the single wire conductor with concentric return such as was used in undersea cable circuits. The return could be through the sea water itself, but electrically this would amount to practically the same thing. The land circuits in those days were all of the single conductor with earth-return type, to which Thomson's analysis did not apply because he did not consider the inductive effect, which in such a line affects the resulting behavior considerably. His treatment takes into account only the capacitance and resistance, which is sufficient in cable circuits over a limited frequency range but insufficient in land circuits.

G. Kirchhoff was the first to introduce the inductive effect into the analysis of the long-line behavior. The
results of his investigation are given in two papers published in 1857. It appeared later that W. Weber had


independently solved the same problem but had withheld publication until he could include the results of certain experimental verifications which he was undertaking in collaboration with R. Kohlrousch. This duplication of effort indicates the problem must have attracted considerable attention at that time. Although the results of these investigations agreed in every essential detail, they were of little practical value because of their highly idealized nature. The inductive effect does not appear in its true light so far as practical circuits are concerned because the single wire was assumed to be so far removed from other conductors that it was considered isolated. For any reasonable degree of rigor, the single conductor with earth return (the type of line used at that time) cannot be considered as a single isolated conductor, but must be treated
by reducing it to an equivalent two-conductor problem, for which the inductive effect is more easily and clearly expressible in engineering terms. The two-conductor circuit came into practical use in the early 30's when it was found that the ground return introduced too much extraneous noise for satisfactory telephonic communica-

With the advent of the telephone into the communications field a better understanding of the theoretical behavior of long lines came into being because of the fact that good telephonic communication requires a much more carefully designed transmission circuit than is needed for satisfactory telegraphic communication. This statement of course applies to the speeds that were in use for telegraphic circuits in that early period, and not to the high-speed telegraphic circuits of today, which require as careful a consideration of details as do any of the other facilities. At any rate, this fact partially accounts for the almost thirty-year gap of relative inactivity in connection with the communications aspect of the long-line problem from the time of Kirchhoff's and Weber's work until the more detailed investigations of Oliver Heaviside

were made in 1886-87. Heaviside also made the only

(5) O. Heaviside, Formulation of Inductance Concept,
Phil. Mag., Vol 1, p. 53, August 1876.

other significant contribution of an earlier date when
he formulated the inductance concept in a more rigorous
and practical fashion than was done by either Kirchhoff
or Weber.

Kirchhoff and Weber did not view the problem from
the field standpoint at all as the Maxwellian concept of
the electro-magnetic field was not yet available when they
carried out their investigations. They considered the
major seat of the phenomenon of propagation of electricity
to lie in the conductor itself rather than in the medium
immediately surrounding it. Maxwell's concise formulation
of the field theory enabled a much more convenient and
accurate introduction of the capacitance and inductance
parameters as derived quantities in terms of which an
approximate representation of the problem could be
carried through.

While the field theory made possible a far better
appreciation of the significant factors involved in the
long-line problem, it also indicated that the rigorous
situation was extremely more involved than had been
supposed. A partially rigorous treatment for the single-
circular conductor was first made by J. J. Thomson in


1886 and for the concentric cable by him in 1889.


A fuller treatment of the single-conductor problem was given by A. Sommerfeld in 1899. The first successful attempt at an exact treatment of two infinitely long parallel wires was made by G. Mie in 1900.


Although there have been numerous other contributions to this subject both within and following the period
sketched above, those mentioned are probably the most important.
A general network consists of any finite number of meshes linked together in the most general manner conceivable and excited by means of forces of arbitrary form. The networks dealt with in this thesis will be of the type known as linear networks. A linear network is one in which the network elements or parameters, i.e., the resistances, inductances, and capacitances are constant.

A clearer picture of what is meant by the term mesh and its relation to the network can be obtained by inspecting figure 1. The openings in the Network, A and B, are called meshes because of their similarity to meshes in a fish net. The boundary of any mesh is called the mesh-contour, and consists of the continuous line which a pencil would have to follow in travelling completely around the mesh. For example, the mesh contour of A is the circuit composed of $L_1$, $R_1$, $R_2$, $L_2$, $S_2$, and $S_1$. However, what has here been defined as mesh-contour is generally also referred to as mesh in engineering literature. The term mesh will be used for mesh-contour in this paper.

The common portion of meshes A and B, i.e., the
branch composed of $R_2$, $L_2$, and $S_2$, is termed a common branch or mutual branch. Points a and b are called branch-points.

Currents $i_1$ and $i_2$ are defined as mesh-currents. They are those currents which are assumed to circulate exclusively along the contours of their respective meshes as illustrated in Fig. 1 and Fig. 2. Their assumed positive direction is indicated by an arrow in each mesh as also shown. As this direction may be arbitrarily chosen, clockwise directions will be chosen here. From the figures it is clear that the total currents in the mutual branches are then always the difference between two mesh-currents.

Next we will discuss the network parameters. In general each mesh in the network will be considered to consist of all three kinds of elements, i.e., inductance, resistance, and capacitance. Furthermore it will be assumed that each mutual branch contains all three kinds of elements, i.e., all the coupling branches between any two meshes will be of the most general type. Although mutual inductance has no effect in the particular circuit used to verify the theoretical calculations of this paper, it will be included in this discussion for the sake of generality. In the two-
mesh network of Fig. 3, the inductance common to the two meshes consists not only of the self-inductance $L_4$, but also includes the mutual inductance $M$.

In Fig. 3 as in Fig 1, the condensers are designated by the symbols $S_1$, $S_2$, and $S_3$, which stand for elastance, which is the reciprocal of capacitance. The reason for using elastance instead of capacitance will become clear from the following discussion.

The best way to introduce the method of notation used is to use the simple two-mesh network of Fig. 3 as an illustration. First the equilibrium condition for this system will be written. In order to do this all the counter-voltages that appear in each mesh must be added up and these sums must be equated to zero because there are no impressed forces in the meshes. Mesh # 1 will be considered first. This mesh contains various resistances, inductances, and elastances. The mesh-current $i_1$ flowing through these elements produces part of the counter-voltages that are to be added up. The counter-voltages of self-induction are:

$$\left( L_1 + L_2 + L_4 \right) \frac{di_1}{dt}$$

The counter-voltages of resistance are:
\[(R_1 + R_2) i_1 \]  
(2)

The condenser-counter-voltages are:
\[(S_1 + S_2) \int i_1 \, dt \]  
(3)

Now it can be seen why it is better to use elastance rather than capacitance. Elastances add up in series just like resistances or inductances.

The total counter-voltage induced in Mesh \# 1 by mesh-current \(i_1\) is the sum of the above three expressions (1), (2), (3). Here it is of advantage to introduce the following notation:
\[
\begin{align*}
L_{11} &= L_1 + L_2 + L_3 \\
R_{11} &= R_1 + R_2 \\
S_{11} &= S_1 + S_2
\end{align*}
\]  
(4)

The quantities \(L_{11}\), \(R_{11}\), \(S_{11}\) are respectively all the inductance, resistance and elastance in Mesh \# 1. They are called the mesh parameters. In terms of these the total counter-voltage induced in mesh \# 1 by \(i_1\) may be written:
\[
L_{11} \frac{di_1}{dt} + R_{11} i_1 + S_{11} \int i_1 \, dt .
\]  
(5)

This is not all of the counter-voltage induced in Mesh \# 1. The common branch between the meshes carries not only \(i_1\) but \(i_2\) also. The counter-voltage induced there by \(i_1\) is already contained in the expression (5). The mesh-current \(i_2\) also induces a counter-voltage in this common branch which adds linearly to that given by (5) to form the total counter-voltage appearing in Mesh \# 1. This
component of counter-voltage induced into mesh # 1 will now be determined. That due to the common self-inductance \( L_4 \) is:

\[
-L_4 \frac{dI_2}{dt}
\]  

(6)

The minus sign appears due to the fact that the assumed positive direction for \( I_2 \) is opposite to that of \( I_1 \). Hence since the counter-voltage:

\[
L_4 \frac{dI_1}{dt}
\]

was designated as positive, the counter-voltage:

\[
L_4 \frac{dI_2}{dt}
\]

which is fed from mesh # 2 into mesh # 1 by \( I_2 \), must be considered negative. Next the counter-voltage fed from mesh # 2 into mesh #1 by \( I_2 \) due to the mutual inductance will be considered. This is given by:

\[
M \frac{dI_2}{dt}
\]  

(7)

The question of the algebraic sign of this term will be considered later. The common resistance \( R_2 \) causes the introduction of the counter-voltage:

\[
-R_2 I_2
\]  

(8)

into mesh #1. Finally, the common conductance \( S_2 \) causes:

\[
-S_2 \int I_2 dt
\]  

(9)

to appear in mesh #1. The total counter-voltage fed from mesh #2 into mesh #1 by virtue of the common branch is given by the sum of the expressions (6), (7), (8), and (9). Before adding these it is convenient to introduce the notation:
These quantities $L_{12}$, $R_{12}$, and $S_{12}$ are called the mutual parameters. Their subscripts indicate to which common branch they belong. The introduction of the minus sign is a perfectly arbitrary procedure. It is done merely to avoid writing minus signs later. The only rule that must be remembered to apply this convention is that if the assumed positive mesh directions, as indicated by the arrows, are opposite in the common branch under consideration, then the common elements are prefixed with negative signs in forming the mutual parameters as in (10). If the arrows coincide in the common branch, then the counter-voltages fed across this branch are positive, and hence the common elements involved remain positive in forming the mutual parameters. With the notation indicated by (10), the total counter-voltage induced into mesh #1 by the current $i_2$ becomes:

$$L_{12} \frac{di_2}{dt} + R_{12}i_2 + S_{12}\int i_2 dt.$$  

(11)

A few words will now be said regarding the algebraic sign of the mutual inductance $M$. From the preceding discussion it is clear that the purely mutual inductance $M$ plays the same part as a common self inductance except for the question of algebraic sign. In forming the mutual parameters according to the expression (10) the common
inductance, resistance, or elastance is prefixed with a negative or positive sign according to whether the mesh-arrows in this common branch are opposite or coincident respectively, as pointed out above. This same idea cannot be applied to the purely mutual inductance however, because it does not form a physical part of the common branches as do the other parameters. Therefore we must be guided entirely by the direction in which voltages are induced. The expression (7) represents a counter-voltage fed from mesh #2 into mesh #1. Its algebraic sign has been assumed positive. This means that a positive mutual inductance must be defined as one which will cause a positive induced counter-voltage in mesh #1 for a positive time rate of change of $i_2$. This, of course, depends upon the positive assumed directions of both $i_1$ and $i_2$, because these fix the positive directions for counter-voltages in the respective meshes. In general, if the mutual inductance between any two meshes causes the introduction of a positive rate of change of current in the other according to the positive assumed directions, then that mutual inductance is numerically positive with regard to the mutual parameter as expressed by \((10)^{10}\). The question

\[(10)\] R. M. Kershner and G. P. Corcoran, Alternating Current Circuits, p. 195.
regarding the sign of the purely mutual inductance doesn't arise of course until numerical values are substituted in the above equations.

The total counter-voltage appearing in mesh #1 is then the sum of the expressions (5) and (11) so the condition of dynamic equilibrium for mesh #1 becomes:

\[
L_{11} \frac{di_1}{dt} + R_{11} i_1 + S_{11} \int i_1 dt + L_{12} \frac{di_2}{dt} + R_{12} i_2 + S_{12} \int i_2 dt = 0.
\]  

(12)

In order to do the same thing for mesh #2, we define:

\[
\begin{align*}
L_{22} &= L_3 + L_4 + L_5 \\
R_{22} &= R_2 + R_3 \\
S_{22} &= S_2 + S_3
\end{align*}
\]

Then the counter-voltage induced in mesh #2 by the mesh-current \( i_2 \) is given by:

\[
L_{22} \frac{di_2}{dt} + R_{22} i_2 + S_{22} \int i_2 dt.
\]  

(14)

The counter-voltage induced in mesh #2 by the current \( i_1 \) due to the common branch between meshes #1 and #2 is given by:

\[
L_{21} \frac{di_1}{dt} + R_{21} i_1 + S_{21} \int i_1 dt
\]

(15)

where obviously:

\[
\begin{align*}
L_{21} &= L_{12} \\
R_{21} &= R_{12} \\
S_{21} &= S_{12}
\end{align*}
\]

(16)

The relations (16) should be kept in mind as they are referred to frequently later on.

The total counter-voltage appearing in mesh #2 is
therefore the sum of the expressions (14) and (15), so
that the condition of dynamic equilibrium for mesh #2
becomes:

\[ \frac{d}{dt}i_1 + R_{21}i_1 + L_{21}\int i_1 dt + \frac{d}{dt}i_2 + R_{22}i_2 + S_{22}\int i_2 dt = 0. \tag{17} \]

Equations (12) and (17) together fully describe the
force-free equilibrium condition of the two-mesh network
illustrated in Fig. 3. The preceding discussion organizes
the method of procedure for setting up the equilibrium
conditions for a network. The object is now to put the
method in such a form that it is almost impossible to get
confused as to algebraic signs and network parameters
and currents, no matter how complicated the situation may be.
We have actually accomplished this already as will be seen
from what follows.

The two-mesh network of Fig. 3 was used merely as a
vehicle for the introduction of the defining equations (10),
(13), and (16), regarding the mesh and mutual parameters.
In the general case it is supposed that there are n meshes
in the network, where n be any finite number. The defini-
tions that were used for the two-mesh case above can be
extended to the general case.

If the \( K \)th mesh of the \( n \)-mesh network, where \( K \) is
any integer between 1 and \( n \), is considered the sum of all
of the inductance in this mesh is denoted by \( L_{kk} \), the sum
of all the resistances by \( R_{kk} \), and the sum of all the
elastances by \( S_{kk} \). These are called the mesh-parameters
of the \( K \)th mesh.
In this general case that is being considered, each mesh in the network is supposed to be coupled to every other mesh in the net by means of all three kinds of coupling, i.e., inductive, resistive, and elastive coupling, just as in the two-mesh case above. In the mutual branch between mesh K and some other mesh i, different from K, the mutual parameters involved will be denoted by $L_{ik}$, $R_{ik}$, and $S_{ik}$ respectively. Here $L_{ik}$ or $L_{ki}$ denotes the sum of all the inductance in the common branch, i.e., common self plus mutual inductance, with the same convention regarding signs as was applied to the two-mesh case above.

Now that the meshes are fixed and numbered, the positive assumed directions denoted by arrows and the mesh and mutual parameters determined by means of the above notation, the equations expressing the dynamic equilibrium between impressed and counter-forces in the various meshes of the network will be set up.
NETWORK EQUILIBRIUM CONDITIONS

It will be assumed that only one driving force exists in the network, and that this one is located in mesh #1 since this is the type of structure dealt with in this paper. Now all the counter-voltages must be added up in each mesh and this sum must be equated to the impressed voltage in that mesh. Starting with mesh #1, we have the counter-voltage induced in it due to its own mesh-current given by

\[ L_{11} \frac{dI_1}{dt} + R_{11} I_1 + S_{11} \int I_1 \, dt \]  

(18)

The counter-voltage induced into it by \( I_2 \) is:

\[ L_{12} \frac{dI_2}{dt} + R_{12} I_2 + S_{12} \int I_2 \, dt \]  

(19)

The counter-voltage induced into it by \( I_3 \) is:

\[ L_{13} \frac{dI_3}{dt} + R_{13} I_3 + S_{13} \int I_3 \, dt \]  

(20)

Similarly, that induced into mesh #1 by \( I_4 \) is:

\[ L_{14} \frac{dI_4}{dt} + R_{14} I_4 + S_{14} \int I_4 \, dt \]  

(21)

We continue in this way until we come to the counter-voltage which is induced into mesh #1 by the current in the nth (last) mesh. This is:

\[ L_n \frac{dI_n}{dt} + R_n I_n + S \int I_n \, dt \]  

(22)

The sum of all the expressions (18) to (22) then represents the total counter-voltage that appears in mesh #1. To obtain the equation expressing the condition of dynamic equilibrium of forces in mesh #1, this sum must be equated to the impressed voltage.

In order to conserve space we introduce a new method of notation rather than writing this equation out in full.
It can be seen that the form of the equations (18) to (22) are alike. If \( i_1 \) is factored out of each term of (18) we have:

\[
(L_{11} \frac{di}{dt} + R_{11} + s_{11} \int dt)i_1.
\]

(18a)

The three terms in the parenthesis together are termed a differential-integral operator. They express a certain operation that is to be performed upon \( i_1 \), namely that given by (23). This operator will be denoted by a single letter:

\[
a_{11} = (L_{11} \frac{di}{dt} + R_{11} + s_{11} \int dt)
\]

(23)

where the subscripts on the operator correspond to those on the mesh parameters. Now the term (18) may be written

\[
a_{11}i_1
\]

(18b)

which is much shorter.

Similarly the term (19) may be written in the form:

\[
a_{12}i_2
\]

(19a)

where:

\[
a_{12} = (L_{12} \frac{di}{dt} + R_{12} + s_{12} \int dt).
\]

(24)

This can be continued down to the last expression (22) which becomes:

\[
a_{1n}i_n
\]

(22a)

where:

\[
a_{1n} = (L_{1n} \frac{di}{dt} + R_{1n} + s_{1n} \int dt).
\]

(25)

The equilibrium condition for mesh #1 in this compressed form therefore becomes:

\[
a_{11}i_1 + a_{12}i_2 + a_{13}i_2 + \cdots + a_{1n}i_n = e_1
\]

(26)

where \( e_1 \) is the impressed voltage in mesh #1.
The above procedure must be repeated identically for mesh #2. Thus it can be seen that the equilibrium condition for this mesh becomes:

\[ e_{21}^{-1}i_1 + e_{22}^{-1}i_2 + e_{23}^{-1}i_3 + \cdots + e_{2n}^{-1}i_n = 0 \]  

(27)

because there are no impressed voltages in any mesh except the first. The operators \( a \) have the same significance as for mesh #1 with proper allowance for subscripts.

For each mesh in the network an equilibrium equation like the above must be written. In order to bring this whole set of \( a \)-operators under a single roof so to speak, we write for them the one defining equation:

\[ a_{ik} = (L_{ik} \frac{di}{dt} + R_{ik} + S_{ik}) \int dt \]  

(28)

where the indices \( i \) and \( k \) may take on any integral values from 1 to \( n \). For \( i = k \), the operator \( a_{ii} \) or \( a_{kk} \) involves the mesh parameters \( L_{ii} \), \( R_{ii} \), and \( S_{ii} \); whereas for \( i \neq k \), the operator \( a_{ik} \) involves the mutual parameters \( L_{ik} \), \( R_{ik} \), and \( S_{ik} \). Remembering that \( L_{ik} = L_{ki} \), \( R_{ik} = R_{ki} \), and \( S_{ik} = S_{ki} \), we recognize that:

\[ a_{ik} = a_{ki} \]  

(29)

which will later prove a most useful relation.

Now the complete set of equilibrium conditions for the \( n \)-mesh network of general form can be written. These are:

\[
\begin{align*}
\left\{ a_{11}i_1 + a_{12}i_2 + a_{13}i_3 + \cdots + a_{1n}i_n = e_1 \\
\left. \begin{array}{l}
\quad a_{21}i_1 + a_{22}i_2 + a_{23}i_3 + \cdots + a_{2n}i_n = 0 \\
\quad \quad \vdots \\
\quad \quad a_{ni}i_1 + a_{n1}i_2 + a_{n2}i_3 + \cdots + a_{nn}i_n = 0 \\
\end{array} \right. \\
\end{align*}
\]

(30)
\[ a_1 + a_2 + a_1 + \cdots + a_1 = 0 \]

This gives a straightforward method. It can be seen that the physical makeup of the entire network is contained in the operators \( a_{ik} \). If a network is given, the set of \( a_{ik} \)'s may be written down by inspection; and conversely, the network can be constructed by inspection if the set of coefficients are given.

Observing the general structure of the system (30), it can be seen that the terms on the diagonal from the upper left to the lower right hand corner (called the principal diagonal) represent counter-voltages induced in meshes by their own mesh-currents. All the other terms represent counter-voltages which are fed or induced from one mesh into another by virtue of the common branch between these two.

An important structural feature of the system (30) is due to the symmetry condition (29). This causes the elements to be symmetrical about the principal diagonal.

Mathematically (30) is a system of simultaneous linear differential equations with constant coefficients. The method of obtaining their solution will be given next.
MATHEMATICAL SOLUTION OF THE SYSTEM OF SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS FOR THE GENERAL NETWORK

For the general case we wish to know the value of all \( n \) currents in order to specify the condition of the network at all times.

Let us first consider a two-mesh case, where \( n=2 \).

Then (30) would be:

\[
\begin{align*}
   a_{11}i_1 + a_{12}i_2 &= d_1 \\
   a_{21}i_1 + a_{22}i_2 &= 0.
\end{align*}
\]

(31)

Each equation involves both \( i_1 \) and \( i_2 \) as unknowns. The elementary way of solving this pair of equations would be to eliminate one of the currents between the two, and thus obtain a linear differential equation in one variable alone. This will now be illustrated. Eliminating \( i_2 \) from (31) by multiplying the first equation by \( a_{22} \) and the second by \(-a_{12}\) and then adding we have:

\[
(a_{11}a_{22} - a_{12}a_{21})i_1 = a_{22}d_1.
\]

(32)

From the previous discussion it can be noted that:

\[
\begin{align*}
a_{11}a_{22} &= (L_{11}\frac{d}{dt} + R_{11} + S_{11}\int dt)(L_{22}\frac{d}{dt} + R_{22} + S_{22}\int dt) \\
&= L_{11}L_{22}\frac{d^2}{dt^2} + (L_{11}R_{22} + L_{22}R_{11})\frac{d}{dt} \\
&+ (L_{11}S_{22} + L_{22}S_{11} + R_{11}R_{22} + R_{11}S_{22} + R_{22}S_{11})\int dt \\
&+ S_{11}S_{22}\int dt dt.
\end{align*}
\]

(33)

Furthermore:

\[
a_{12}a_{21} = (L_{12}\frac{d}{dt} + R_{12} + S_{12}\int dt)^2
\]

(34)

on account of the symmetry condition. Hence:
Thus eq. (32) actually becomes.

\[
\begin{align*}
\left( L_{11}L_{22} - L_{12}^2 \right) \frac{\Delta^2 I_1}{2} + \left( L_{11}R_{22} + L_{22}R_{11} - 2L_{12}R_{12} \right) \frac{dI_1}{dt} \\
+ \left( L_{11}S_{22} + L_{22}S_{11} - 2L_{12}S_{12} \right) \frac{d^2 I_1}{dt^2} \\
+ \left( S_{11}S_{22} - S_{12}^2 \right) \int I_1 dt = L_{22} \frac{d^2 I_2}{dt^2} + R_{22} \frac{dI_2}{dt} + S_{22} \int I_2 dt.
\end{align*}
\]

(35)

In order to eliminate the integral signs from this equation, it is necessary to differentiate it twice with respect to time. The result then is a linear differential equation with constant coefficients involving only \( I_1 \) and could then be solved in the ordinary manner. (11)

(11) Lyman M. Kells, Elementary Differential Equations, pp. 92-114.

It is obvious that the elementary process of solution by elimination is very laborious. The same process of elimination has to be gone through in connection with \( I_2 \). This of course would involve mounting effort if the network involved more than two meshes. From this discussion it can be seen that elementary methods of attack are inadequate for this situation.

Therefore a direct method of solution will be given, the two mesh examples for which the pair of equations (31) express the equilibrium conditions will be considered again.
The driving force in the first mesh will be assumed to be given by:
\[ e_1 = E_1 e^{j\omega t} \]  
(36)
where the real portion is understood, and \( E_1 \) may be considered as the complex voltage amplitude.

The first problem is to find the particular integrals or the steady-state solutions for this case. Since the driving force is harmonic of angular frequency \( \omega \), the steady-state mesh currents must be of the same form.

Hence we assume
\[ \begin{align*}
    i_1 &= I_1 e^{j\omega t} \\
    i_2 &= I_2 e^{j\omega t}
\end{align*} \]
(37)
where the real portions are understood and \( I_1 \) and \( I_2 \) are the complex current amplitudes. Substituting values from (36) and (37) into (31) we have:
\[ \begin{align*}
    a_{11} I_1 e^{j\omega t} + a_{12} I_2 e^{j\omega t} &= E_1 e^{j\omega t} \\
    a_{21} I_1 e^{j\omega t} + a_{22} I_2 e^{j\omega t} &= 0
\end{align*} \]
(38)
In terms of the network parameters, the first term of the first equation is:
\[ a_{11} I_1 e^{j\omega t} = I_1 \left( L_{11} \frac{d}{dt} + R_{11} + S_{11} \int dt \right) e^{j\omega t} \]
\[ = I_1 \left( L_{11} j\omega + R_{11} + S_{11} \right) e^{j\omega t} \]
(39)
Similarly the second term of this equation is:
\[ a_{12} I_2 e^{j\omega t} = I_2 \left( L_{12} \frac{d}{dt} + R_{12} + S_{12} \right) e^{j\omega t} \]
(40)
Now it is seen that \( \frac{d}{dt} \) is replaced by \( j\omega \) and \( dt \) is replaced by the reciprocal of \( j\omega \). The three terms in the parentheses are no longer operators but definitely known functions of \( \omega \). However, they still have the same uniform appearance.
This suggests the introduction of a new symbol whereby they may be denoted in order to save time and space in writing equations. So the following symbols will be used:

\[
\begin{align*}
b_{11} &= (L_{11}j\omega + R_{11} + \frac{G_{11}}{j\omega} \\
b_{12} &= (L_{12}j\omega + R_{12} + \frac{G_{12}}{j\omega})
\end{align*}
\]  

(41)

Then the first equation (38) becomes:

\[
b_{11}I_1e^{j\omega t} + b_{12}I_2e^{j\omega t} = E_1e^{j\omega t}.
\]  

(42)

In exactly the same manner we may obtain for the second equation (38):

\[
b_{21}I_1e^{j\omega t} + b_{22}I_2e^{j\omega t} = 0
\]  

(43)

where:

\[
\begin{align*}
b_{21} &= (L_{21}j\omega + R_{21} + \frac{G_{21}}{j\omega} \\
b_{22} &= (L_{22}j\omega + R_{22} + \frac{G_{22}}{j\omega})
\end{align*}
\]  

(44)

Observing equations (42) and (43) it can be seen that since the factor \(e^{j\omega t}\) can never be zero, it can be cancelled out and the pair of condition equations (38) become:

\[
\begin{align*}
b_{11}I_1 + b_{12}I_2 &= E_1 \\
b_{21}I_1 + b_{22}I_2 &= 0.
\end{align*}
\]  

(45)

If this pair of condition equations can be satisfied, then the assumed steady-state solutions (37) are valid and useful. It should be noted that this is a pair of simultaneous algebraic equations with the steady-state current amplitudes \(I_1\) and \(I_2\) as unknowns. It may be solved by any of the usual methods.

The coefficients of this pair of equations given by
(41) and (44) are in the form of impedances. For instance, $b_{11}$ may be written as:

$$b_{11} = R_{11} + j (L_{11} w - S_{11})$$

which is of the form:

$$R + jX.$$ 

Thus it is seen that $b_{11}$ is the impedance of mesh #1 for the angular frequency $w$. $b_{22}$ is that for mesh #2, while $b_{12} = b_{21}$ is the impedance of the common branch between meshes 1 and 2. These coefficients will be called the mesh and mutual impedances of the network.

Now the general case as expressed by the n-dimensional system will be considered. In the same way as for the two mesh case the steady-state currents will be assumed to have the following form:

$$\begin{cases} 
I_1 = I_1 e^{jwt} \\
I_2 = I_2 e^{jwt} \\
\vdots \\
I_n = I_n e^{jwt}
\end{cases}$$

(46)

Substitution into (30) gives the system of condition equations:

$$\begin{cases} 
b_{11} I_1 + b_{12} I_2 + \cdots + b_{1n} I_n = E_1 \\
b_{21} I_1 + b_{22} I_2 + \cdots + b_{2n} I_n = 0 \\
\vdots \\
b_{n1} I_1 + b_{n2} I_2 + \cdots + b_{nn} I_n = 0
\end{cases}$$

(47)
where:

\[ b_{ik} = b_{ki} = R_{ik} + j(L_{ik}w - \frac{S_{ik}}{w}). \]  

(48)

Here again the terms on the principal diagonal are mesh voltages and the balance of the terms represent feed-over voltages which are due to the coupling between each mesh and every other mesh in the network. The condition (48) is again responsible for the symmetry of the system (47) about the principal diagonal.

In carrying out the solution of the system (47) it is almost necessary to make use of the determinantal method. Ordinary elimination methods can be used but these involve the expenditure of much more time and energy than the determinantal method.

THE DETERMINANTAL METHOD OF SOLUTION

A brief outline of the method of solving linear simultaneous algebraic equations by means of determinants will be given. First, the determinant of the system (47) is defined by the following arrangement of the coefficients:

\[ D = \begin{vmatrix}
  b_{11} & b_{12} & \cdots & b_{1n} \\
  b_{21} & b_{22} & \cdots & b_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{n1} & b_{n2} & \cdots & b_{nn}
\end{vmatrix}. \]  

(49)

The coefficients \( b_{ik} \) are called the elements of the determinant \( D \). It should be noted that the determinant is merely an arrangement of the coefficients in the same order as to row and column as they appear in the system of equations for
which it is written. The method of determining the value of the determinant will be taken up a little later. The first index or subscript of any element denotes the row in which it appears, while the second index denotes the column. Thus $b_{49}$ would be the element at the intersection of the fourth row and ninth column. On account of the symmetry condition:

$$b_{ik} = b_{ki}$$

the network determinant (49) is symmetrical about its principal diagonal. This is always the case for linear networks.

Next it will be explained what is meant by principal minors of a determinant. A principal minor is also a determinant, but one having one row and column less than the original determinant. It is formed from the latter by cancelling one row and one column, and allowing the remainder to move together so as to fill in the voids left by the cancellation. It is generally denoted by the capital letter corresponding to that used to denote the elements. It is further supplied with the same subscripts or indices as those corresponding to the element located at the intersection of the cancelled row and column. Finally, the principal minor is prefixed by an algebraic sign which is positive if the sum of the indices is even and negative if this sum is odd and this combination is known as a cofactor. (12) Thus the minor of (49) which is
obtained by cancelling the \( i \)th row and \( k \)th column would be:

\[
B_{ik} = \begin{vmatrix}
\ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\end{vmatrix}
\]

The heavy dotted lines indicate the positions of the cancelled row and column. The algebraic sign of the cofactor is controlled by the factor:

\((-1)^{i+k}\)

which is obviously positive for \( i+k \) even and negative for \( i+k \) odd. Minors may also be formed by cancelling two or more rows and columns simultaneously. These are called second, third, etc. minors to distinguish them from the principal or first minor described above. They are useful in affording short-cuts in the process of evaluating determinants.

Now a method of evaluating a determinant such as (49) will be described. This may be written:

\[
D = b_{11}B_{11} + b_{12}B_{12} + \cdots + b_{nn}B_{nn}
\]
The form of this expansion is clear from the above. It follows successively along the terms of the first row. This is called an expansion of $D$ by principal minors along the first row. $D$ may be expanded in this way along any row or column. Thus expanding along the second column, for example, we have:

$$D = b_{12}b_{12} + b_{22}b_{22} + b_{32}b_{32} + \cdots + b_{n2}b_{n2}. \quad (51a)$$

The numerical result in any case is the same no matter which row or column is chosen to expand along. In a practical example there may be definite preferences in this respect from the standpoint of the simplicity of the result. Thus, if a certain row or column contains a number of zeroes, this one would undoubtedly be chosen for the expansion.

Such an expansion reduces the given determinant to a sum of other determinants having one row and column less. Each of these must again be expanded in the above manner. This again yields a sum of determinants which have two rows and columns less than the original. This process is repeated until the result involves nothing but elements. The last step obviously involves the expansion of determinants having only two rows and columns. Such a one is, for example:

$$\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$$

which yields by expansion:

$$b_{11}b_{22} - b_{12}b_{21}$$

so that finally there are no determinants left. A single
element alone may be regarded as a determinant having only one row and column.

As an illustration let us consider the numerical example:

\[
D = \begin{vmatrix}
5 & 3 & 2 \\
1 & 0 & 7 \\
6 & 4 & 5
\end{vmatrix}
\]

The second row or column is the one to choose. Take the second row for example, then:

\[
D = -1 \begin{vmatrix}
3 & 2 \\
4 & 5
\end{vmatrix} + 0 - 7 
\begin{vmatrix}
5 & 3 \\
6 & 4
\end{vmatrix}
\]

\[
= -1(15-8) -7(20-18) = -7 -14 = -21.
\]

Returning now to the solution of the system of simultaneous equations (47), the so-called Cramer rule for finding the solutions will be stated. We have, for example:

\[
I = \frac{1}{D} \begin{vmatrix}
E \quad b_1 \quad \ldots \quad b_{1n} \\
1 \quad 1_2 \quad \ldots \quad b_{2n} \\
0 \quad b_{22} \quad \ldots \quad b_{2n} \\
0 \quad b_{n2} \quad \ldots \quad b_{nn}
\end{vmatrix}
\]

(52)

Where \( D \) is given by (49). The formation of this result is a quotient whose denominator is the determinant of the system and whose numerator is a determinant which is

(13) Ibid, pp. 132-133
obtained from the original one by replacing the elements of the first column by the quantities that appear on the right hand side of the system (47). The replacement is made for the first column of D because the unknown solved for is the current number one. For current number K the Kth column is replaced, thus:

\[
I_k = \frac{1}{D} \begin{vmatrix}
            b_{11} \cdots b_{1(K-1)} & E_1 b_{1(K+1)} \cdots b_{1n} \\
            b_{21} \cdots b_{2(K-1)} & 0 & b_{2(K+1)} \cdots b_{2n} \\
            \vdots & \ddots & \ddots & \ddots \\
            b_{n1} \cdots b_{n(K-1)} & 0 & b_{n(K+1)} \cdots b_{nn}
          \end{vmatrix} \quad (53)
\]

By letting K take on any integer value from 1 to n, the corresponding steady-state mesh-currents are obtained.

By recognizing the following (53) can be put in a more compact form. Suppose the determinant in the numerator of (53) is expanded by principal minors along the Kth column. Obviously this expansion will contain only one term because the Kth column is composed entirely of zeros except for the first element which is \(E_1\). The minor corresponding to this element, however, is the same as that corresponding to the element \(b_{1k}\) in the determinant D. This minor is \(B_{1k}\). Hence (53) may be written:

\[
I_k = (-1)^{1+k} \frac{E_1 b_{1k}}{D} \quad (53a)
\]

which is a compact expression.
DERIVATION OF FUNDAMENTAL RELATIONS

If the voltages $E_1$ and $E_2$ are thought of as forming the closing loops to those meshes of the network which would be left open at the terminals 1-1 and 2-2 respectively, and these meshes are designated as 1 and 2, we may write

$$
\begin{align*}
I_1 &= \frac{B_{11}}{D} E_1 - \frac{B_{12}}{D} E_2 \\
I_2 &= \frac{B_{21}}{D} E_1 + \frac{B_{22}}{D} E_2
\end{align*}
$$

where $D$ is the network determinant and $B_{ik}$ $B_{ki}$ are its first minors. Here it is convenient to introduce the notation

$$
\begin{align*}
Y_{11} &= \frac{B_{11}}{D} \\
Y_{12} &= \frac{B_{12}}{D} \\
Y_{21} &= \frac{B_{21}}{D} \\
Y_{22} &= \frac{B_{22}}{D}
\end{align*}
$$

and then write instead of (54)

$$
\begin{align*}
I_1 &= Y_{11} E_1 + Y_{12} E_2 \\
I_2 &= Y_{21} E_1 + Y_{22} E_2
\end{align*}
$$

If we solve these equations for the voltages we obtain relations of the form:
The Z's may be found in terms of the y's by simply solving the system (56) and comparing coefficients with those in (57). Writing for the determinant of (56)

\[
|y| = \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix} = y_{12} y_{22} - y_{12} \quad (58)
\]

we find

\[
\begin{align*}
Z_{11} &= \frac{y_{12}}{y} \\
Z_{12} &= \frac{y_{22}}{y} \\
Z_{22} &= \frac{y_{11}}{y}
\end{align*}
\]

\quad . \quad (59)

The y's and z's may be determined as admittances and impedances of the network. Namely, if we suppose that the terminals 2-2' are shorted, then obviously \( E_2 \) becomes zero, and the equation (56) give

\[
\begin{align*}
y_{11} &= \frac{I_1}{E_1} \\
y_{21} &= \frac{I_2}{E_1}
\end{align*}
\]

\quad (60)

while if we imagine the end 1 shorted, then \( E_1 \) 0, and we have

\[
\begin{align*}
y_{12} &= \frac{I_1}{E_2} \\
y_{22} &= \frac{I_2}{E_2}
\end{align*}
\]

\quad (61)

The admittances \( y_{11} \) and \( y_{22} \) are obviously those looking into the ends 1 or 2, respectively, with the opposite ends short-circuited. The admittances \( y_{21} = y_{12} \) is also a
short-circuit admittance but involves the ratio of current at the far end to voltage at the driven end. This will be called a transfer admittance; the other two will be spoken of as driving-point admittances. The fact that with alternate ends shorted or driving we have

\[
\frac{I_2}{E_1} = \frac{I_1}{E_2}
\]

is simply an expression of the reciprocity theorem for this network. The system of \( y \)'s is referred to as the short-circuit driving-point and transfer admittances. These three functions uniquely characterize the behavior of the four-terminal network with respect to its two pairs of terminals.

Similarly if for the system (57) we regard the end 2 as open, then \( I_2 = 0 \), and we have

\[
\begin{align*}
Z_{11} &= \frac{E_1}{I_1} \\
Z_{12} &= \frac{E_2}{I_1}
\end{align*}
\]

while if end 1 is considered open, \( I_1 = 0 \), and (57) give

\[
\begin{align*}
Z_{12} &= \frac{E_1}{I_2} \\
Z_{22} &= \frac{E_2}{I_2}
\end{align*}
\]

Here \( Z_{11} \) and \( Z_{22} \) are driving-point impedances, and \( Z_{12} = Z_{21} \) is a transfer impedance. The system \( Z \)'s will, therefore, be called the open-circuit driving-point and transfer impedances. These are three alternative functions whereby
the external behavior of the network may be specified.

The $y$'s or $Z$'s may thus be obtained experimentally for any given network.

If we wish to express $E_1$ and $I_1$ in terms of $E_2$ and $I_2$ we have

$$
\begin{align*}
E_1 &= AE_2 - BI_2 \\
I_1 &= CE_2 - DI_2
\end{align*}
$$

and if we wish to express $E_2$ and $I_2$ in terms of $E_1$ and $I_1$ we have

$$
\begin{align*}
E_2 &= DE_1 - BI_1 \\
I_2 &= CE_1 - DI_1
\end{align*}
$$

where $A$, $B$, $C$, and $D$ are known as the general circuit parameters. These are known as generalized circuit constants in power work. \(14\) It can be seen that $A$ and $D$ are

\[ (14) \quad J. G. Tarboux, Introduction to Electric Power Systems, p. 71. \]

numerics, $B$ has the dimensions of ohms, and $C$ has the dimensions of mhos.

If in addition to (58) we write for the determinant of $Z$

$$
|Z| = \begin{vmatrix} Z & Z \\ 11 & 12 \\ Z_{21} & Z_{22} \end{vmatrix}
$$

the interrelations between the various sets of coefficients are expressed by
\[
\begin{align*}
\begin{aligned}
    y_{11} &= \frac{p_{11}}{d} = \frac{z_{22}}{z} = \frac{d}{b} \\
    y_{12} &= y_{21} = \frac{p_{12}}{d} = \frac{z_{12}}{z} = \frac{1}{b} \\
    y_{22} &= \frac{p_{22}}{d} = \frac{z_{11}}{z} = \frac{a}{b} \\
    z_{11} &= \frac{x_{22}}{y} = \frac{a}{c} \\
    z_{12} &= z_{21} = \frac{x_{12}}{y} = \frac{1}{c} \\
    z_{22} &= \frac{x_{11}}{y} = \frac{d}{c} \\
    a &= -\frac{y_{22}}{y_{12}} = \frac{z_{11}}{z_{12}} \\
    b &= -\frac{1}{y_{12}} = \frac{z}{z_{12}} \\
    c &= -\frac{x}{y_{12}} = \frac{1}{z_{12}} \\
    d &= -\frac{x_{11}}{y_{12}} = \frac{z_{22}}{z_{12}}
\end{aligned}
\end{align*}
\]
SOLUTION FOR CAScade OF IDENTICAL DIssYMMETRICAL NETWORKS ON THE ITERATIVE BASIS

The external behavior of the long transmission line can be completely specified by means of its sending-and receiving-end voltages and currents, and their interrelation in turn can be determined in terms of two functions, the characteristic impedance and propagation functions. (15)

(15) L. A. Ware and H. R. Reed, Communication Circuits, pp. 52-71.

Although the general type of four-terminal network that is being considered here need not necessarily be the analytic representation of a long line, it is desirable to attempt to specify its behavior in similar terms.

The type of network that will be considered here is illustrated in Fig. 5. It consists of a number of identical

Fig. 5. Cascade of identical dissymmetrical networks on the iterative basis.
four-terminal networks connected in cascade working out of an impedance $Z_s$ and into an impedance $Z_k$ at the end of the line and an impedance $Z_L$ that can be varied along the line at the junctions between any of the four-terminal networks. This system is called a uniform recurrent structure. In this arrangement the networks are all oriented in the same direction; i.e., ends 1 are on the left and ends 2 on the right. An important modification of this scheme will be discussed later.

The voltages and currents at the junctions between $Z_s$ and the variable positioned load $Z_L$ are numbered from $E_o$ and $I_0$ to $E_n$ and $I_n$, $n$ being the number of networks between the driving voltage and the variable load. The voltages and currents at the junctions between $Z_L$ and $Z_k$ are numbered from $E_{o1}$ and $I_{o1}$ to $E_{m1}$ and $I_{m1}$, $m$ being the number of networks between the variable positioned load $Z_L$ and the fixed load $Z_k$. The four-terminal network characteristics, the generator voltage $E_g$, and the impedances $Z_s$, $Z_L$, and $Z_k$ are known and it is desired to find analytic expressions for the voltage and current variations along the structure, i.e., from junction to junction. The physical aspects of this problem are similar to those found in the long line. The major difference lies in the fact that the voltages and currents here are discontinuous functions of distance along the structure, the various junctions being referred to by subscripts.
The method of analysis that will be applied is entirely analogous to that used in solving the long line problem. First, the voltage and current equilibrium equations for a typical internal section of the structure will be written, and after finding formal solutions to these, the various arbitrary constants involved will be evaluated by means of the boundary conditions.

\[ I_k = I_{k-1} - y_{21} E_{k-1} - y_{22} E_k + y_{11} E_k + y_{12} E_{k+1}, \]  
\[ \left( E_{k-1} + \frac{y_{11} + y_{22}}{y_{12}} \right) E_k + E_{k+1} = 0. \]  

**Fig. 6. Internal portion of the cascaded structure of Fig. 5.**

An internal portion of the uniform structure is shown in Fig. 6. Applying the \( y \)-system (56) to the junction \( K \), first considering \( I_k \) as the output of the preceding network and then as the input to the succeeding one, we have with due attention to the current directions

\[ I_k = -y_{21} E_{k-1} - y_{22} E_k + y_{11} E_k + y_{12} E_{k+1}, \]  
\[ \left( E_{k-1} + \frac{y_{11} + y_{22}}{y_{12}} \right) E_k + E_{k+1} = 0. \]  

Applying the \( z \)-system (57) to the junction \( K \) in the same manner we get
\[ E_k = Z_{21} I_{k-1} - Z_{22} I_{k} = Z_{11} I_{k} - Z_{12} I_{k+1} \] \tag{72}

so that

\[ I_{k-1} - \left( \frac{Z_{11} + Z_{22}}{Z_{12}} \right) I_k + I_{k+1} = 0. \] \tag{73}

From (67) and (68) we see that

\[ \frac{V_{11} + V_{22}}{V_{12}} = -\frac{Z_{11} + Z_{22}}{Z_{12}} = - (A + D). \] \tag{74}

Hence the voltage and current equilibrium conditions (71) and (73) become

\[
\begin{align*}
E_{k-1} - (A+D)E_k + E_{k+1} &= 0, \\
I_{k-1} - (A+D)I_k + I_{k+1} &= 0.
\end{align*}
\] \tag{75}

By analogy with the uniform long-line we will attempt a solution of (75) by assuming the following forms for the voltages and currents between the generator and the variable positioned load \( Z_k \):

\[
\begin{align*}
E_k &= A_1 e^{-k \gamma} + A_2 e^{k \gamma}, \\
I_k &= B_1 e^{-k \gamma} + B_2 e^{k \gamma},
\end{align*}
\] \tag{76}

where \( \gamma \) plays the same part for the uniform recurrent structure that \( \alpha \), the propagation constant, plays for the uniform long-line, and the index \( k \) takes the place of the continuous variable \( x \). Thus \( \gamma \) becomes the propagation function per network in the recurrent structure just as \( \alpha \) is the propagation function per unit length in the long-line.

Substituting the assumed solutions (76) into (75) we find after some factoring that the former will be valid subject to the conditions...
Non-vanishing solutions of the form (76), therefore, are obtained provided
\[ e^r - (A+D) + e^{-r} = (A+D), \] (77)
which determines the propagation functions \( r \). Namely, (78) gives
\[ \cosh r = \frac{A+D}{2}, \] (79)
so that
\[ \sinh r = \sqrt{\frac{(A+D)^2}{2} - 4}, \] (80)
and
\[ e^{\pm r} = \frac{(A+D) \pm \sqrt{(A+D)^2 - 4}}{2}, \] (81)
from which
\[ r = \ln \frac{(A+D) + \sqrt{(A+D)^2 - 4}}{2}. \] (82)

Returning to the solutions (76) the next step is to evaluate the constants \( A_1, A_2, B_1, \) and \( B_2 \). The interrelation between voltage and current as expressed by (70) or (72) will suffice to determine two of these constants in terms of the other two. Since (70) and (72) are not independent, either one may be used. Thus substituting (76) into (72) we have
\[ A_1 e^{-kr} + A_2 e^{kr} = Z_{11} B_1 e^{-kr} + Z_{12} B_2 e^{kr} - Z_{11} B_1 e^{-(k+1)r} - Z_{12} B_2 e^{(k+1)r}. \] (83)
Equating coefficients of \( e^{kr} \) and \( e^{-kr} \) gives
\[ A_1 = \left( Z_{11} - Z_{12} e^{-k} \right) B_1 \]
\[ A_2 = \left( Z_{11} - Z_{12} e^{k} \right) B_2. \] (84)

Speaking of the behavior of this recurrent lumped
structure in terms of wave propagation, $A_1$ and $A_2$ may be designated as the incident and reflected amplitudes of the net voltage and $B_1$ and $B_2$ as the corresponding current amplitudes. The factors of $B_1$ and $B_2$ in (84) will be regarded as characteristic impedances, with positive and negative signs respectively, of the recurrent lumped structure. Since the general case that is being treated is disymmetric, it is logical to expect that the characteristic impedances should be different in the positive and negative directions, i.e., for the incident and the reflected wave amplitudes. Let us denote these by $Z_{o1}$ and $Z_{o2}$ respectively, and write

$$
B_1 = \frac{A_1}{Z_{o1}} \quad \text{and} \quad B_2 = \frac{A_2}{Z_{o2}}
$$

Then comparing with (84) and using (81) and (68) we find

$$
Z_{o1} = \sqrt{\frac{(A+D)^2 - 4 + (A-D)}{2D}} \
Z_{o2} = \sqrt{\frac{(A+D)^2 - 4 - (A-D)}{2D}}
$$

We will assume the following forms for the voltages and currents between the variable positioned load $Z_L$ and the load $Z_k$

$$
E^1_k = A^1_1 e^{-k_1^1 \gamma} + A^1_2 e^{k_1^1 \gamma} \
I^1_k = B^1_1 e^{-k_1^1 \gamma} + B^1_2 e^{k_1^1 \gamma}
$$

By analogy with equations (84) and (85) it can be seen
that \( A_1^l \) and \( B_1^l \) and also \( A_2^l \) and \( B_2^l \) are related in the
following manner

\[
\begin{align*}
B_1^l &= \frac{A_1^l}{Z_{01}} \\
B_2^l &= \frac{A_2^l}{Z_{02}}
\end{align*}
\]  
(88)

where \( Z_{01} \) and \( Z_{02} \) are the impedances defined by equation (86).

At the boundaries of the structure and at the load \( Z_L \) we may write the following condition equations

\[
\begin{align*}
E_0 + I_0 Z_s &= E_g \\
E_n + I_n Z_L &= 0 \\
E_l - I_l Z_N &= 0 \\
E_0 - (I_n - I_l) Z_L &= 0
\end{align*}
\]  
(89)

Substituting (76), (85), (87) and (88) into these we get

\[
\begin{align*}
\frac{Z_{01} - Z_k}{Z_{01}} e^{-myA_1^l + Ze_{02}^l + Ze_{02}^l} &= 0 \\
\frac{Z_{01} - Z_k}{Z_{01}} A_1^l + Z_{02}^l A_2^l - Z_{01} e^{-nyA_1^l + Ze_{02}^l} &= 0 \\
\frac{Z_{01} - Z_k}{Z_{01}} A_1^l - Z_{02}^l A_2^l - Z_{01} e^{-nyA_1^l + Z_{02}^l e^{nyA_2^l}} &= 0 \\
\frac{Z_{01} - Z_k}{Z_{01}} A_1^l - Z_{02}^l A_2^l &= E_g
\end{align*}
\]  
(90)

Denoting the determinant of this system by \( \Delta \) we have

\[
\begin{align*}
\Delta &= \left[ \frac{Z_{02}^2 (Z_k - Z_{01}) e^{-(m-n)Y} (Z_k + Z_{01})(m+Z_{01} Z_{02}) e^{(m+n)Y}}{(Z_{01} Z_{02})^2} \right] \\
&+ \left[ \frac{Z_{02}^2 (Z_k + Z_{02}) e^{(m-n)Y} (Z_k - Z_{01})(m-Z_{01} Z_{02}) e^{-(m+n)Y}}{(Z_{01} Z_{02})^2} \right]
\]  
(91)
where \( M = Z_L(Z_{01} + Z_{02}) \), and

\[
A_1 = \frac{E_k}{\Delta} \frac{1}{Z_{01}Z_{02}} \left[ (Z_k - Z_{01})Z_{02}^2 e^{(n-m)r} + (Z_k + Z_{02})Z_{01}z_0 e^{(m+n)r} \right]
\]

\[
A_2 = \frac{E_k}{\Delta} \frac{1}{Z_{01}Z_{02}} \left[ (Z_k - Z_{01})Z_{02}^2 e^{(m+n)r} - (Z_k + Z_{02})Z_{01}z_0 e^{(m-n)r} \right]
\]

\[
A_1 = \frac{E_k}{\Delta} \frac{M}{Z_{01}Z_{02}^2} (Z_{02} + Z_k) e^{mr}
\]

\[
A_2 = -\frac{E_k}{\Delta} \frac{Z_{01}Z_{02}}{2} (Z_{01} - Z_k) e^{-mr}
\]

Defining as the reflection coefficients

\[
r_s = \frac{Z_g - Z_{02}}{Z_s + Z_{01}} \quad : \quad r_s = \frac{Z_k - Z_{01}}{Z_k + Z_{01}}
\]

the voltage and current solutions take the final form

\[
E_k = \frac{E_k Z_{01} [Z_{02}^2 e^{-(m-n+k)r} + (M+Z_{01}Z_{02}) e^{(m+n-k)r}]}{\Delta}
\]

\[
I_k = \frac{E_k [Z_{02}^2 e^{-(m-n+k)r} + (M+Z_{01}Z_{02}) e^{(m+n-k)r}]}{\Delta}
\]

\[
E_{k1} = \frac{E_k M [Z_0 e^{(m-k^2)r} + Z_{02} e^{-(m-k^1)r}]}{\Delta}
\]

\[
I_{k1} = \frac{E_k M [e^{(m-k^2)r} - r_k e^{-(m-k^1)r}]}{\Delta}
\]

where:
g = \left( z_0 + z_{01} \right) \left[ Z_{02} e^{-(m-n)R} + \left( M + Z_{01} z_{02} \right) e^{(m+n)R} \right] \\
- r_0 \left( z_0 + z_{01} \right) \left[ R_{01} e^{(m+n)R} - z_{01} e^{(m-n)R} \right]

**SOLUTION FOR CASCADE OF IDENTICAL DISSYMMETRICAL NETWORKS ON THE IMAGE BASIS**

Instead of the structure of Fig. 5, the cascade arrangement shown in Fig. 7 will be considered. Here the networks are alternately reversed so that similar ends are adjacent throughout the structure. The junctions are, therefore, symmetrical although the networks are themselves dissymmetrical. The same functional relations that hold for voltages and currents at junctions of even subscripts will not hold for the voltages and currents at junctions of odd subscripts because these two kinds of junctions are obviously different. According to Fig. 7

\[ \text{Fig. 7. Cascade of identical dissymmetrical networks on the image basis.} \]

the even subscripts refer to the junctions of ends 1 and the odd subscripts refer to the junctions of ends 2.
Furthermore, the end to which $Z_L$ is connected will be an end 1 when $n$ is even and an end 2 when $n$ is odd. Also, the end to which $Z_k$ is connected will be an end 1 when $m$ is even and an end 2 when $m$ is odd. Hence a distinction will have to be made between these various cases.

We will consider the structure of Fig. 7 for the cases $n$ even and $n$ odd. Again, we will begin by writing the voltage and current equilibrium conditions for a typical internal portion of the structure. This is illustrated in Fig. 8. Comparison with Fig. 7 shows that the subscript $r$ must be an even integer. Applying the fundamental relations (56) successively to the junctions $r-2$ to $r+1$, with due regard to the directions of the currents, and considering the currents first as the output from the preceding network and then as the input to the succeeding one, we get the following series of equations.

\[
\begin{align*}
I_{r-2} &= -y_{12}E_{r-3} - y_{11}E_{r-2} = y_{11}E_{r-2} + y_{12}E_{r-1} \\
I_{r-1} &= -y_{21}E_{r-2} - y_{22}E_{r-1} = y_{22}E_{r+1} + y_{21}E_r \\
I_r &= -y_{12}E_{r-1} - y_{11}E_r = y_{11}E_r + y_{12}E_{r+1} \\
I_{r+1} &= -y_{21}E_r - y_{22}E_{r+1} = y_{22}E_{r+1} + y_{21}E_{r+2}
\end{align*}
\]
Using the right-hand sides of these, and substituting for
the \( y \)'s by means of (67), we get
\[
\begin{align*}
E_{r-3} - 2DE_{r-2} + E_{r-1} &= 0 \\
E_{r-2} - 2AE_{r-1} + E_r &= 0 \\
E_{r-1} - 2DE_r + E_{r+1} &= 0 \\
E_{r-2}AE_{r+1} + E_{r+2} &= 0
\end{align*}
\]  
(96)

Applying the relations (57) in a similar manner we
have
\[
\begin{align*}
E_r &= Z_1 I - Z_2 I \\
E_{r-1} &= Z_{21} I_{r-2} - Z_{22} I_{r-1} \\
E_r &= Z_{12} I_{r-2} - Z_{11} I_r \\
E_{r+1} &= Z_{21} I_{r+1} - Z_{22} I_{r+2}
\end{align*}
\]  
(97)

With these and the substitutions (68) we have
\[
\begin{align*}
I_{r-3} - 2AI_{r-2} + I_{r-1} &= 0 \\
I_{r-2} - 2DI_{r-1} + I_r &= 0 \\
I_{r-1} - 2AI_r + I_{r+1} &= 0 \\
I_r - 2DI_{r+1} + I_{r+2} &= 0
\end{align*}
\]  
(98)

The fact that the equations (96) and (98) alternately
contain the parameters \( A \) and \( D \) shows that the voltage and
current relations do not follow a uniform variation from
junction to junction. This is to be expected since the
junctions are alternately those of ends 1 and 2. However,
if we derive relations for the voltage and current which
skip every other junction, i.e., progress by two networks
at one time, then we may expect a uniform result.
If in (96) we multiply the second equation by 2D, and then add the first three or multiply the third equation by 2A and add the last three we get respectively

\[ \begin{align*}
E_{r-3} - 2(2AD-1) E_{r-1} + E_{r+1} &= 0 \\
E_{r-2} - 2(2AD-1) E_{r} + E_{r+2} &= 0
\end{align*} \]  

(99)

Treating the current equations (98) in a similar manner we have

\[ \begin{align*}
I_{r-3} - 2(2AD-1) I_{r-1} + I_{r+1} &= 0 \\
I_{r-2} - 2(2AD-1) I_{r} + I_{r+2} &= 0
\end{align*} \]  

(100)

These results show that a uniform relation can be expected for the variation of either voltage or current along alternate junctions. Since these equations all contain the same parameter 2(2AD-1), they also show that the propagation properties along the odd or even junctions are the same, and furthermore are the same for voltage and current. This makes it possible to assume the following forms for the voltages and currents between the generator and the variable positioned load \( Z_L \)

\[ \begin{align*}
E_k &= A_1 e^{-k\theta} + A_2 e^{k\theta} \text{ for } K \text{ even} \\
E_k &= A_1 e^{-k\theta} + A_2 e^{k\theta} \text{ for } K \text{ odd}
\end{align*} \]  

(101)

and

\[ \begin{align*}
I_k &= B_1 e^{-k\theta} + B_2 e^{k\theta} \text{ for } K \text{ even} \\
I_k &= B_1 e^{-k\theta} + B_2 e^{k\theta} \text{ for } K \text{ odd}
\end{align*} \]  

(102)

i.e., the solutions for the odd and even junctions can differ only by their amplitudes.

For the determination of the propagation function, which is here denoted by \( \Theta \), we may substitute either of
the assumed solutions (101) or (102) into (99) or (100).

Substituting the first equation (101) into the second equation (99) and factoring, we have

\[(A_1 e^{-kθ} + A_2 e^{kθ}) [e^{2θ} - 2(2AD-1) + e^{-2θ}] = 0\]

a non-vanishing solution, therefore, requires

\[2AD-1 = \cosh 2θ,\]

or

\[AD = \frac{1 + \cosh 2θ}{2} = \cosh^2 θ,\]

so that

\[\cosh θ = \sqrt{AD}\]

\[\sinh θ = \sqrt{AD-1} = \sqrt{BC}\]

\[e^{±θ} = (\sqrt{AD} ± \sqrt{BC})\]  \hspace{1cm} (103)

For the determination of \(G_1\) and \(G_2\) in terms of \(A_1\) and \(A_2\), we substitute (101) into any of the equations (96) which relate successive junctions. Here it must be remembered that \(r\) was taken to designate an even integer.

Hence if the third equation (96) is chosen, the second solution (101) must be used for the first and last terms, and the first solution (101) must be used for the second term. Thus we get after some factoring

\[G_1 e^{-rθ}(e^{θ-2D} \frac{A_1}{G_1} + e^{-θ}) + G_2 e^{rθ}(e^{-θ-2D} \frac{A_2}{G_2} + e^{θ}) = 0, \hspace{1cm} (104)\]

which must hold for all even values of \(r\). This requires that the coefficients of \(e^{rθ}\) and \(e^{-rθ}\) vanish separately.

For non-vanishing values of \(G_1\) and \(G_2\) this requires

\[e^{θ-2D} \frac{A_1}{G_1} + e^{-θ} = 0,\]

and
\[ e^{-\theta} - 2D \frac{A_2}{\sigma_2} + e^{\theta} = 0. \]

Making use of the relations (103) this gives

\[ \frac{A_1}{\sigma_1} = \frac{A_2}{\sigma_2} = \sqrt{\frac{A}{D}} \]  

which is the desired relationship.

Similarly if we substitute the solutions (102) into the third equation (98), bearing in mind that \( r \) is an even integer, and demand non-vanishing values for \( H_1 \) and \( H_2 \), we get

\[ e^{-\theta} - 2A \frac{B_1}{H_1} + e^{\theta} = 0, \]

and

\[ e^{-\theta} - 2A \frac{B_2}{H_2} + e^{\theta} = 0, \]

from which we find

\[ \frac{B_1}{H_1} = \frac{B_2}{H_2} = \sqrt{\frac{D}{A}} \]  

The results (105) and (106) express the solutions for odd integers in (101) and (102) in terms of those for even integers.

The relations between the voltage and current amplitudes are found by substituting these solutions into any of the left-hand equations (95) or (97). For example, substituting into the third equation (97) gives

\[ A_1 e^{-k\theta} + A_2 e^{k\theta} = z_{12} \sqrt{\frac{A}{D}} \left( B_1 e^{-(k-1)\theta} + B_2 e^{(k-1)\theta} \right) \]

\[ -z_{11} \left( B_1 e^{-k\theta} + B_2 e^{k\theta} \right) \]

equating coefficients of \( e^{-k} \) and \( e^k \) we have

\[ A_1 = (z_{12} \sqrt{\frac{A}{D}} e^{\theta} - z_{11}) B_1 \]

and
\[ A_2 = (Z_{12} \sqrt{\frac{A}{D}} e^{-\theta} - z_{11}) B_2. \]

Substituting for the Z's from (68) and using the last relation (103) this gives

\[ A_1 = \frac{A B}{\sqrt{C D}} \cdot B_1 \]
\[ A_2 = -\frac{A B}{\sqrt{C D}} \cdot B_2 \]

(107)

The factor of \( B_1 \) or minus the factor is the characteristic impedance of the recurrent structure for this case. Furthermore, since the structure is considered to begin with an end 1, the characteristic impedance must be distinguished by a suitable subscript. It will be written as

\[ Z_{II} = \frac{A B}{\sqrt{C D}}. \]

(108)

The I in the subscript is added to distinguish this impedance from the iterative impedance.

With (105), (106), (107), and (108) we may now rewrite our solutions for the voltages and currents between the generator and the variable positioned load \( Z_L \) in the form

\[
\begin{align*}
E_k &= A_1 e^{-k\theta} + A_2 e^{k\theta} \quad \text{for } K \text{ even} \\
E_k &= \frac{D}{A} (A_1 e^{-k\theta} + A_2 e^{k\theta}) \quad \text{for } K \text{ odd}
\end{align*}
\]

(109)

\[
\begin{align*}
I_k &= A_1 \frac{e^{-k\theta}}{z_{II}} - A_2 \frac{e^{k\theta}}{z_{II}} \quad \text{for } K \text{ even} \\
I_k &= \frac{A}{\sqrt{D}} \left( \frac{A_1 e^{-k\theta} - A_2 e^{k\theta}}{z_{II}} \right) \quad \text{for } K \text{ odd}
\end{align*}
\]

(110)

By analogy to equations (101) and (102) the following forms will be assumed for the voltages and currents between
the variable positioned load $Z_l$ and the load $Z_k$

$$\begin{align*}
E'_k &= A'_1 e^{-k'\theta} + A'_2 e^{k'\theta} \quad \text{for } k^1 \text{ even} \\
E'_k &= G'_1 e^{-k'\theta} + G'_2 e^{k'\theta} \quad \text{for } k^1 \text{ odd}
\end{align*}$$

and

$$\begin{align*}
I'_k &= B'_1 e^{-k'\theta} + B'_2 e^{k'\theta} \quad \text{for } k^1 \text{ even} \\
I'_k &= H'_1 e^{-k'\theta} + H'_2 e^{k'\theta} \quad \text{for } k^1 \text{ odd}
\end{align*}$$

where again the solutions for the odd and even junctions can differ only by their amplitudes.

Obviously the propagation function is that given by (103). By carrying out the procedure for this case as was done for the previous case, i.e., using equations analogous to equations (104) through (107) it can be seen that

$$\begin{align*}
A'_1 &= \frac{AB}{CD} B'_1 \\
A'_2 &= \frac{AB}{CD} B'_2 
\end{align*}$$

The next step is to evaluate $A_1$, $A_2$, $A_1^1$, and $A_2^1$ from boundary conditions. These are the same as for the preceding case and are therefore given by (89). In substituting into these we must distinguish between the following four cases: (1) $n$ even and $m$ even; (2) $n$ odd and $m$ odd; (3) $n$ odd and $m$ even; (4) $n$ even and $m$ odd.

First the case for $m$ even and $n$ even will be considered. Then the boundary conditions give the following equations

$$\begin{align*}
-Z_l e^{-n\delta} A_1 + Z_u e^{n\delta} A_2 + (Z_l + Z_r) A'_1 - (Z_l - Z_r) A'_2 &= 0 \\
-(Z_l - Z_r) e^{-m\delta} A'_1 + (Z_l + Z_r) e^{-m\delta} A'^1_2 &= 0 \\
-(Z_l - Z_r) e^{m\delta} A'_1 + (Z_l + Z_r) e^{m\delta} A'^1_2 &= 0 \\
(Z_s + Z_r) A_1 + (Z_s - Z_r) A_2 &= 0
\end{align*}$$

(114)
Denoting the determinant of this system by \( \Delta \) we have

\[
\Delta = -(Z_s + Z_{II})(Z_s^2(\frac{Z_k}{Z_{II}}) e^{-(m-n)\theta} + Z_{II} P(Z_k + Z_{II}) e^{(m+n)\theta})
\]

\[
+ (Z_s - Z_{II})(Z_{II} Q(Z_k - Z_{II}) e^{-(m+n)\theta} - Z_{II}^2(Z_k + Z_{II}) e^{(m-n)\theta})
\]

(115)

where \( P = 2Z_s + Z_{II} \), \( Q = 2Z_s - Z_{II} \),

and

\[
A_1 = -\frac{Z_{II} E_{g} [Z_{II}^2(Z_k - Z_{II}) e^{-(m-n)\theta} + Z_{II} P(Z_k + Z_{II}) e^{(m+n)\theta}]}{\Delta}
\]

\[
A_2 = -\frac{Z_{II} E_{g} [(Z_k - Z_{II}) Q e^{-(m+n)\theta} - Z_{II}^2(Z_k + Z_{II}) e^{(m-n)\theta}]}{\Delta}
\]

\[
A_1' = -\frac{2Z_s E_{g} [Z_k Z_{II}] e^{m\theta}}{\Delta}
\]

\[
A_2' = -\frac{2Z_s Z_{II}^2 E_{g} (Z_k - Z_{II}) e^{-m\theta}}{\Delta}
\]

(116)

Defining as the reflection coefficients

\[
r_{s1} = \frac{Z_s - Z}{Z_s + Z_{II}}; \quad r_{k1} = \frac{Z - Z_{II}}{Z_k + Z_{II}}
\]

(117)

the voltage and current solutions take the final form

\[
E_k = \begin{cases} 
\frac{Z_{II} E_{g} [Z_{II} n_{k1} e^{(m-n)\theta} + P e^{-(m+n)\theta} + n_{k1} Q e^{(m-n)\theta}}}{\Delta} \\
-\frac{Z_{II} E_{g} e^{(m+n)\theta}}{\Delta} 
\end{cases} \quad \text{for } k \text{ even}
\]

\[
E_k = \sqrt{\frac{P}{\Delta}} \begin{cases} 
\frac{Z_{II} E_{g} [Z_{II} n_{k1} e^{(m-n)\theta} + P e^{-(m+n)\theta} + n_{k1} Q e^{(m-n)\theta}}}{\Delta} \\
\end{cases} \quad \text{for } k \text{ odd}
\]

(118)
\[ I_k = E_k \left[ Z_{kl} e^{-(k-m+n)\theta} + p e^{-(k+m+n)\theta} - n_k e^{(k-m-n)\theta} \right] \]
\[ + Z_{kl} E_k e^{(k+m-n)\theta} \quad \text{for } k \text{ even} \]
\[ I_k = \frac{A}{D} \left[ \right] \quad \text{for } k \text{ odd} \]

\[ E'_k = \frac{Z_k Z_{kl} E_k}{H} \left[ e^{(m-k')\theta} + n_{k'} e^{-(m-k')\theta} \right] \quad \text{for } k' \text{ even} \]
\[ E'_k = \frac{D}{A} \left[ \right] \quad \text{for } k' \text{ odd} \]

\[ I'_k = \frac{2 Z_k Z_{kl} E_k}{H} \left[ e^{(m-k)\theta} - n_{k} e^{-(m-k)\theta} \right] \quad \text{for } k' \text{ even} \]
\[ I'_k = \frac{A}{D} \left[ \right] \quad \text{for } k' \text{ odd} \]

where

\[ H = (Z_s + Z_{kl}) \left[ Z_{kl} n_k e^{-(m-n)\theta} + p e^{(m+n)\theta} \right] \]
\[ - n_{s} (Z_s + Z_{kl}) \left[ Re_{k} e^{-(m+n)\theta} - Z_{kl} e^{(m-n)\theta} \right]. \]

Next the case for \( m \) odd and \( n \) odd will be considered. The boundary conditions (39) give the following equations
\[-(z_i - z_{\text{Ia}}) e^{-\theta A_i} + (z_i + z_{\text{Ia}}) e^{\theta A_i} - \sqrt{A} z_i A_i = 0\]

\[-z_b e^{\theta A_i} + z_b e^{\theta A_i} + \sqrt{A} (z_i + z_{\text{Ia}}) A_i - \sqrt{A} (z_i - z_{\text{Ia}}) A_i = 0\]

\[(z_s + z_{\text{Ia}}) A_i - (z_s - z_{\text{Ia}}) A_i = z_{\text{Ia}} E_b\]  \hspace{1cm} (122)

Denoting the determinant of this system by \( \Delta \) we have

\[\Delta = \frac{\sqrt{D}}{A} \left[ (N - z_{\text{Ia}} z_{\text{Ia}})(z_k - z_{\text{Ia}}) e^{-m-n \theta} - (M + z_{\text{Ia}} z_{\text{Ia}})(z_k + z_{\text{Ia}}) e^{m+n \theta} \right]\]

\[+ \frac{\sqrt{D}}{A} \left[ (N - z_{\text{Ia}} z_{\text{Ia}})(z_k - z_{\text{Ia}}) e^{-m-n \theta} - (N + z_{\text{Ia}} z_{\text{Ia}})(z_k + z_{\text{Ia}}) e^{m-n \theta} \right]\]  \hspace{1cm} (123)

where \( M = z_L (z_{\text{Ia}} + z_{\text{Ia}}) \) and \( N = z_L (z_{\text{Ia}} - z_{\text{Ia}}) \).

and

\[A_1 = \sqrt{D} \frac{z_{\text{Ia}} E_b (z_k - z_{\text{Ia}})(N - z_{\text{Ia}} z_{\text{Ia}}) e^{-m-n \theta} - (z_k + z_{\text{Ia}})(M + z_{\text{Ia}} z_{\text{Ia}}) e^{m+n \theta}}{\Delta}\]

\[A_2 = -\sqrt{D} \frac{z_{\text{Ia}} E_b (N - z_{\text{Ia}} z_{\text{Ia}})(z_k - z_{\text{Ia}}) e^{-m-n \theta} - (z_k + z_{\text{Ia}})(N - z_{\text{Ia}} z_{\text{Ia}}) e^{m-n \theta}}{\Delta}\]

\[A_1' = -\frac{3E_b z_{\text{Ia}} z_{\text{Ia}} z_k (z_k + z_{\text{Ia}}) e^{-m \theta}}{\Delta}\]

\[A_2' = -\frac{3E_b z_{\text{Ia}} z_{\text{Ia}} z_k (z_k - z_{\text{Ia}}) e^{-m \theta}}{\Delta}\]  \hspace{1cm} (124)

In these eqns. we have introduced

\[z_{\text{Ia}} = \sqrt{D} A, \quad z_{\text{I1}} = \sqrt{\frac{D}{A}},\]

which is the same as \( z_{\text{I1}} \), except that \( A \) and \( D \) are inter-
changed. Defining an additional reflection coefficient
\[ r_{k2} = \frac{z_k - z_{I2}}{z_k + z_{I2}} \] (126)

the voltage and current solutions take the final form.

\[
E_k = \frac{z_{I1}E_g r_{k2} (N - z_{I1} z_{I2}) e^{-(k+m-n)\theta} - (M + z_{I1} z_{I2}) e^{-(k+m+n)\theta}}{J} \\
+ \frac{z_{I1}E_g r_{k2} (M - z_{I1} z_{I2}) e^{(k-m-n)\theta} + (N + z_{I1} z_{I2}) e^{(k+m-n)\theta}}{J} \\
E_k = \frac{\sqrt{D}}{\sqrt{A}} \left\{ \begin{array}{l}
\text{for } k \text{ even.} \\
\text{for } k \text{ odd}
\end{array} \right. (128)
\]

\[
I_k = \frac{E_g r_{k2} (N - z_{I1} z_{I2}) e^{-(k+m-n)\theta} - (M + z_{I1} z_{I2}) e^{-(k+m+n)\theta}}{J} \\
+ \frac{E_g r_{k2} (M - z_{I1} z_{I2}) e^{(k-m-n)\theta} - (N + z_{I1} z_{I2}) e^{(k+m-n)\theta}}{J} \\
I_k = \frac{\sqrt{A}}{\sqrt{D}} \left\{ \begin{array}{l}
\text{for } k \text{ even} \\
\text{for } k \text{ odd}
\end{array} \right. (127)
\]

\[
E'_k = \frac{\sqrt{A}}{\sqrt{D}} z_{I1} z_{I2} z_{I1} z_{I2} E_g \left[ e^{(m-k')\theta} + r_{k2} e^{-(m-k')\theta} \right] \text{ for } k' \text{ even} \\
E'_k = \frac{\sqrt{D}}{\sqrt{A}} \left\{ \begin{array}{l}
\text{for } k' \text{ odd}
\end{array} \right. (129)
\]
\[ I'_k = \frac{A}{D} \sum_{\alpha=1}^{J} \left( n^{-\beta} e^{-\frac{(m-n)^2}{m+n}} \right) \text{ for } K' \text{ even} \]
\[ I'_k = \frac{A}{\sqrt{D}} \left( \cdot \cdot \cdot \right) \text{ for } K' \text{ odd} \]  

where 
\[ J = (Z + Z) (N - Z) \sum_{\alpha} x^2 e^{-\frac{(m-n)^2}{m+n}} + (Z + Z) \sum_{\alpha} x^2 e^{-\frac{(m-n)^2}{m+n}} \] 
\[ + (Z + Z) \left\{ (M - Z) \sum_{\alpha} x^2 e^{-\frac{(m-n)^2}{m+n}} \right\}. \]  

(130)

Now the case for \( n \) even and \( m \) odd will be considered.

The boundary conditions (89) give the following equations
\[ -Z e^{-\beta} A_i + Z e^{-\beta} A_i + (Z + Z) A_i - (Z - Z) A_i = 0 \]
\[ -(Z - Z) e^{-\beta} A_i + (Z + Z) e^{-\beta} A_i = 0 \]
\[ -(Z - Z) e^{-\beta} A_i + (Z + Z) e^{-\beta} A_i + Z A_i - Z A_i = 0 \]
\[ (Z + Z) A_i - (Z - Z) A_i = Z e^\beta \]  

(131)

By inspection, i.e., by comparison with the previous cases, the voltage and current equations can be seen to take the final form
\[ E_k = Z e^\beta \sum_{\alpha} e^{-\beta \cdot \cdot \cdot} \text{ for } K' \text{ even} \]
\[ E_k = \sqrt{D} \left( \cdot \cdot \cdot \right) \text{ for } K' \text{ odd} \]  

(132)
\[ I_k = E_k \left[ Z_i \frac{r_{(k-m+n)}}{r_{(k+m-n)}} + P \frac{e^{(n-x)} - r_{k_2}}{r_{k_2}} \right] \]

\[ + Z_i \frac{e^{(k+m-n)}}{L} \quad \text{for } K \text{ even} \]

\[ I_k = \sqrt{A} \left\{ \right\} \quad \text{for } K \text{ odd} \]

(133)

\[ E_k' = \frac{2Z_i Z_k E_g \left[ e^{(m-k)} \theta + r_{k_2} e^{-(m-k) \theta} \right]}{L} \quad \text{for } K' \text{ even} \]

\[ E_k' = \sqrt{A} \left\{ \right\} \quad \text{for } K' \text{ odd} \]

(134)

\[ I_k' = \frac{2Z_i E_g \left[ e^{(m-k) \theta} - r_{k_2} e^{-(m-k) \theta} \right]}{L} \quad \text{for } K' \text{ even} \]

\[ I_k' = \frac{\sqrt{A}}{D} \left\{ \right\} \quad \text{for } K' \text{ odd} \]

(135)

where

\[ L = (Z_s + Z_i) \left[ Z_i \frac{r_{(k-m+n)}}{r_{(k+m-n)}} + P \frac{e^{(n-x)} - r_{k_2}}{r_{k_2}} \right] \]

\[ - r_{s_1} (Z_s + Z_i) \left[ G \frac{e^{(m+n) \theta} - Z_i e^{(m-n) \theta}}{r_{k_2}} \right]. \]

Finally, the case for \( n \) odd and \( m \) even will be considered. The boundary conditions (39) give the following equations
\[-(Z_x - Z_{x_2})e^{-\alpha A} + (Z_x + Z_{x_2})e^{\alpha A} + \frac{\bf{B}}{V_A} Z_x A_x' - \frac{\bf{B}}{V_A} Z_x A_x'' = 0\]

\[-(Z_x - Z_{x_1})e^{-\alpha A} + (Z_x + Z_{x_1})e^{\alpha A} + \frac{\bf{B}}{V_A} (Z_x + Z_{x_1}) A_x' - \frac{\bf{B}}{V_A} (Z_x - Z_{x_1}) A_x'' = 0\]

\[(Z_x + Z_{x_2}) A_x - (Z_x - Z_{x_2}) A_x = Z_x, E_g\]  

(136)

By inspection, the voltage and current equations can be seen to take the final form

\[E_k = Z_x E_g \left[ \frac{\alpha_k (N-Z_xZ_{x_2})e^{(-K-m+n)\theta} - (M+Z_xZ_{x_2})e^{(-K-m+n)\theta}}{V_A} \right] + Z_x E_g \left[ \frac{\alpha_k (M-Z_xZ_{x_2})e^{(K-m-n)\theta} - (N+Z_xZ_{x_2})e^{(K+m-n)\theta}}{V_A} \right] \text{ for } K \text{ even} \]

\[E_k = \sqrt{\frac{B}{A}} \left\{ \begin{array}{c} \text{ for } K \text{ odd} \end{array} \right. \]

\[I_k = E_g \left[ \frac{\alpha_k (N-Z_xZ_{x_2})e^{(-K-m+n)\theta} - (M+Z_xZ_{x_2})e^{(-K-m+n)\theta}}{V_A} \right] + E_g \left[ \frac{\alpha_k (M-Z_xZ_{x_2})e^{(K-m-n)\theta} - (N+Z_xZ_{x_2})e^{(K+m-n)\theta}}{V_A} \right] \text{ for } K \text{ even} \]

\[I_k = \sqrt{\frac{A}{B}} \left\{ \begin{array}{c} \text{ for } K \text{ odd} \end{array} \right. \]

(137)

\[E_k' = -2\sqrt{\frac{A}{B}} Z_x Z_{x_2} E_g \left[ \frac{e^{(m-K)\theta} + \alpha_k e^{(-m-K)\theta}}{V_A} \right] \text{ for } K' \text{ even} \]

\[E_k' = \sqrt{\frac{B}{A}} \left\{ \begin{array}{c} \text{ for } K' \text{ odd} \end{array} \right. \]

(138)
\[ I'_{k} = \begin{cases} \sqrt{\frac{A}{D}} \left[\frac{Z_z E_d}{F} \sum_{m \leq M} c^{(m-k)\theta} e^{-\frac{r_{k} c^{(m-k)\theta}}{l_1}} \right] & \text{for } K' \text{ even} \\ \sqrt{\frac{A}{D}} \left\{ \sum_{m \leq M} c^{(m-k)\theta} e^{-\frac{r_{k} c^{(m-k)\theta}}{l_1}} \right\} & \text{for } K' \text{ odd} \end{cases} \]

where

\[ F = (Z_s + Z_z) \left[ (N-Z_z Z_{z1}) r_{k1} e^{-(m-k)\theta} (M+Z_z Z_{z2}) c^{(m+k)\theta} \right] \]
\[ + r_{s1} (Z_s + Z_z) \left[ (M-Z_z Z_{z1}) r_{k1} e^{-(m+k)\theta} (N+Z_z Z_{z2}) c^{(m-k)\theta} \right]. \]

A PARTIAL EXPERIMENTAL VERIFICATION OF RESULTS

A transmission line composed of ten identical four-terminal networks connected in cascade on the iterative basis and a resistance \( Z_s \) plus load resistances \( Z_L \) and \( Z_K \) was set up in the laboratory and readings were taken of the voltages at the first three junctions. These voltages were also calculated using equation (94) and the values obtained by these two methods were compared as a means of verification.

A diagram of the laboratory circuit follows:

![Laboratory Circuit Diagram](image)

**Fig. 9. Laboratory Circuit Used to Obtain Experimental Results**
Fig. 10. Diagram of individual four-terminal network used in cascade connection in laboratory circuit.

Fig. 10. $C_1$ is a 0.1 microfarad condenser and $C_2$ consists of two 0.1 microfarad condensers connected in parallel which, of course, is equivalent to one 0.2 microfarad condenser. The upper arm of the $\pi$-type four-terminal network is a radio-frequency coil which contains a resistance in addition to the inductance. The inductance of the coils was obtained in the laboratory by use of parallel resonance data and the values of resistance of coils

(16) F. E. Terman, Measurements in Radio Engineering, pp. 67-69

were obtained by making use of the conventional impedance diagram for alternating current circuits. (17) Rand L

(17) Kerchner and Corcoran, op. cit. p. 24

were found to equal 30 ohms and 0.63 millihenries respectively.
Here \( Z_0, Z_4, \) and \( Z_L \) are all 100 ohms resistances. \( n, \) the number of four terminal networks between the generator and variable positioned load, is 5, and \( m, \) number of four-terminal networks between the variable positioned load and the fixed load at the end of the line, is 5.

The frequency at which these readings were taken was 15.924 kilocycles per second, so \( w = 2\pi f = 10^5 \) radians per second.

It can be seen that the first step to take in solving for the expected values of voltage is to determine the general circuit parameters \( A, C, \) and \( D. \) These can be obtained by making use of equation (69) from which it can be seen that \( Z_{11}, Z_{12}, Z_{22}, \) are required. These can be obtained by solving for the input impedance into an individual four-terminal network when first end 1 is open circuited and then when end 2 is open circuited. The following values are obtained for the \( Z's\)

\[
Z_{11} = 35.6 \angle 4.45^\circ \text{ ohms}; \quad Z_{22} = 26 \angle -69.8^\circ \text{ ohms}; \quad Z_{12} = 54.5 \angle -109^\circ \text{ ohms.}
\]

Using equations (109) we obtain

\[
A = 0.651 \angle 113.45^\circ 1 \quad C = 0.0183 \angle 109^\circ \text{ ohms}
\]

\[
D = 0.476 \angle 39.2^\circ
\]

Then using equations (82) and (86) we have

\[
Z_{01} = 68.2 \angle 57.45^\circ \text{ ohms}; \quad Z_{02} = 55.7 \angle -38.3^\circ \text{ ohms}
\]

\[
\gamma = 1.572 \angle 74.6^\circ
\]

Then using equations (93) we have

\[
x_0 = 0.445 \angle 8.7^\circ; \quad x_k = 0.58 \angle -28.85^\circ
\]
Substituting these values into (94) and using $E_g$, the driving voltage, as 3 volts we obtain values of the following magnitudes for the voltages at the first three junctions of the line:

$E_0 = 1.375$ volts; $E_1 = 0.915$ volts; $E_2 = 0.675$ volts

The input voltage $E_g$ was set equal to 3 volts and the frequency was set at 15,924 kilocycles per second on the audio oscillator. The wave shape of this voltage was checked to make certain that it was sinusoidal by placing a cathode ray oscilloscope across the output of the oscillator and viewing the wave on the screen of the oscilloscope. The afore-mentioned voltages were then obtained by means of a vacuum tube voltmeter. The values of these were:

$E_0 = 1.3$ volts; $E_1 = 0.90$ volts; $E_2 = 0.64$ volts

This seems to be a reasonable verification as the largest deviations between the calculated and experimental results was less than six percent.
CONCLUSIONS

The analytic expressions obtained for the voltage and current variations along the structure can be used to determine the voltages and currents at any of the junctions along the line to a high degree of accuracy. This analysis presents a convenient method of determining these values.

The method presented in this paper is not the only method of determining these expressions. A somewhat shorter method has been suggested by Dr. Zaborsky and is outlined in the appendix.
SUMMARY

The method for the determination of the analytic expressions for the voltage and current variations along a structure composed of a cascade of identical dissymmetrical four-terminal networks driven harmonically at the sending-end and working into an impedance at the receiving-end and an impedance that can be placed at any of the junctions along the cascade was presented. The analysis effected is analogous to the method used in solving the long line problem.

The solutions were considered for the cascade of four-terminal networks on two bases. First the solution was considered on the iterative basis and then on the image basis. Furthermore, there are four types of solutions when the cascade is considered on the image basis. Denoting the number of four-terminal networks between the generator and the variable positioned load by \( n \) and the number of four-terminal networks between the variable positioned load and the load at the receiving end of the line by \( m \), solutions for these four cases are presented: (1) \( n \) even and \( m \) even; (2) \( n \) odd and \( m \) odd; (3) \( n \) odd and \( m \) even; (4) \( n \) even and \( m \) odd.
OUTLINE OF ALTERNATE METHOD OF SOLUTION

The case of the cascade of identical dissymmetrical networks on the iterative basis will be considered. Fig. (5), page 39 illustrates this case. For this structure we have the following equations

\[ E_{k-1} = A E_k + B I_k \]  
\[ I_{k-1} = C E_k + D I_k \]  
\[ E_k = A E_{k+1} + B I_{k+1} \]  
\[ I_k = C E_{k+1} + D I_{k+1} \]

where all quantities have been defined previously. Combining equations (1), (3), and (4) we have

\[ E_{k-1} - (A+D) E_k + E_{k+1} = 0, \]  
\[ I_{k-1} - (A+D) I_k + I_{k+1} = 0. \]

Since (5) and (6) are equations of finite differences of the second order with constant coefficients, the general network solutions are

\[ E_k = a \lambda_1^k + b \lambda_2^k, \]  
\[ I_k = c \lambda_1^k + d \lambda_2^k, \]

where \( \lambda_1 \) and \( \lambda_2 \) are roots of the characteristic equation

\[ \lambda^2 - (A+D)\lambda + 1 = 0. \]

Solving for these roots we have

\[ \lambda_1 = \frac{1}{2} (A+D) + \frac{1}{2} \sqrt{(A+D)^2 - 4}, \]  
\[ \lambda_2 = \frac{1}{2} (A+D) - \frac{1}{2} \sqrt{(A+D)^2 - 4}, \]

These two roots are reciprocals, i.e., \( \lambda_2 = \frac{1}{\lambda_1} \), and using
the notation \( \lambda_i = \lambda \), equations (7) and (8) become
\[
E_k = a \lambda^k + b \lambda^{-k},
\]
\[
I_k = c \lambda^k + d \lambda^{-k}.
\]

To determine the constants \( a, b, c, \) and \( d \) (12) and (13) are substituted into (3).

This gives
\[
a \lambda^k + b \lambda^{-k} = A (a \lambda^{k+1} + b \lambda^{-(k+1)}) + B (c \lambda^{k+1} + d \lambda^{-(k+1)})
\]
or
\[
\lambda^k (a - A a \lambda - B c \lambda) = \lambda^{-k} (-b + A b / \lambda + B d / \lambda).
\]

Equation (14) is valid for all values for \( K \), which is possible only if

\[
a - A a \lambda - B c \lambda = 0,
\]
\[
- b + A b / \lambda + B d / \lambda = 0,
\]
or
\[
\frac{c}{a} = \frac{1 - A \lambda}{B \lambda} \quad \text{and} \quad \frac{d}{b} = \frac{\lambda - A}{B}.
\]

Substituting the initial conditions
\( K = 0, E_k = E_s, \) and \( I_k = I_s \)
into (12) and (13), we have
\( E = a + b, \) and \( I = c + d \)
Combining these latter two equations with (17) and (18)
we have
\[
a = \frac{\lambda [(A - \lambda) E_s + B I_s]}{1 - \lambda^2},
\]
\[
\frac{b}{l - \lambda^2} = \frac{(1 - A \lambda) E_s - B \lambda I_s}{1 - \lambda^2},
\]
\[
s = \frac{1 - \lambda^2}{1 - \lambda^2} \left[ \frac{(A - \lambda) E_s + I_s}{B} \right].
\]
\[ d = \left[ \frac{\lambda - A}{\lambda - 1} \right] \left[ \frac{(\lambda - A) \lambda - 1}{\lambda - 1} \right] E_s - I_s \lambda \]  \hfill (22)

Substituting (19), (20), (21), and (22) into (12) and (13) and re-arranging we have

\[ E_k = E_s \left[ (A - \lambda) \lambda^{t-k} - (A - 1) \lambda^{-1} \right] + I_s B \left[ \lambda^{t-k} - \lambda^{-1} \right] \hfill (23) \]

\[ I_k = E_s \left[ (A - \lambda)(A - 1) \lambda^{t-k} + \lambda^{-1} \right] + I_s B \left[ (A - 1) \lambda^{t-k} + (A - \lambda) \lambda^{-1} \right] \hfill (24) \]

For \( k = n \), \( E_n = E_R \) and \( I_n = I_R \)

and with the notations

\[ A_n = \frac{(A - 1) \lambda^{t-n} - (A - \lambda) \lambda^{-n}}{\lambda^{t-n} - \lambda^{-n}} = \frac{(A - 1) \lambda^{t-n} - (A - \lambda) \lambda^{-n}}{\sqrt{(A + D)^2 - 4}} \hfill (25) \]

\[ B_n = \frac{\lambda^{t-n} - \lambda^{-n}}{\lambda^{n-1}} = \frac{B \left[ \lambda^{t-n} - \lambda^{-n} \right]}{\sqrt{(A + D)^2 - 4}} \hfill (26) \]

\[ C_n = \frac{(A - 1) (A - \lambda) (A - 1) \lambda^{t-n}}{B (\lambda^{t-n} - \lambda^{-n})} = \frac{(A - 1) (A - \lambda) B_n}{B^2} \hfill (27) \]

\[ D_n = \frac{-A - (A - 1) \lambda^{t-n} - (A - \lambda) \lambda^{-n}}{\lambda^{t-n} - \lambda^{-n}} = \frac{(A - 1) \lambda^n + (A - 1) \lambda^n}{\sqrt{(A + D)^2 - 4}} \hfill (28) \]

equations (23) and (24) become

\[ E_R = D_n E_s - B_n I_s \hfill (29) \]

\[ I_R = -C_n E + A_n I_s \hfill (30) \]

or since \( A D - B C = 1 \)

\[ E_s = A_n E_R + B_n I_n \hfill (31) \]

\[ I_s = C_n E_R + D_n I_n \hfill (32) \]
Consequently \( A_n, B_n, C_n, \) and \( D_n \) are the general circuit constants of a line composed of a cascade of \( n \) elements with general circuit constants \( A, B, C, \) and \( D \) each.

Now the structure illustrated in Fig. 11 will be considered.

Fig. 11. Cascade of identical dissymmetrical networks with networks combined as two four-terminal networks.

The condition equations are

\[
\begin{align*}
E &= Z \cdot I, \\
\frac{E}{R} &= \frac{I}{R} \\
E &= A \cdot E + B \cdot I, \\
I &= C \cdot E + D \cdot I, \\
I &= A \cdot E + B \cdot I, \\
I &= C \cdot E + D \cdot I, \\
E_s &= E_s + I_s z_s, \\
I_s &= I_s + I_s z_s. \\
\end{align*}
\]

Substituting (33) into (34) and (35) we have

\[
\begin{align*}
E &= (A_m + \frac{B_m}{Z_R}) \cdot E_R, \\
I &= (C + \frac{D_m}{Z_R}) \cdot E_R.
\end{align*}
\]
Then, dividing (34a) by (35a)

\[
\frac{E}{I_L} = \frac{A Z R + B m}{m Z + D m}
\]

Solving for \( I_L \) and substituting into (39) we have

\[
I_L = E \left[ \frac{C Z + D m}{A Z + B m} + \frac{1}{Z L} \right] = \frac{E}{M}
\]

where the value of \( M \) is obvious. Then substituting into (36) and (37) we have

\[
E_s = (A_n + B_n) E \quad \text{(36a)}
\]

\[
I_s = (C_n + D_n) E \quad \text{(37a)}
\]

Dividing (36a) by (37a) we have

\[
\frac{E_s}{I_s} = \frac{E_s - I_s Z_s}{I_s} = \frac{A_n M + B_n}{C M + D_n} = N
\]

Then

\[
I_s = \frac{E_s}{2 + N} \quad \text{(40)}
\]

where all quantities are known.

With \( E_s \) and \( I_s \) known other voltages and currents can be calculated from equations (33) through (39). The voltages and currents at the various junctions may then be calculated from (23) and (24).
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VITA

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