Fractals with arbitrary segment lengths

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FRACTALS WITH ARBITRARY SEGMENT LENGTHS

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ABSTRACT

Work in the area of fractal geometry has generally focused on a specific facet of the discipline at the expense of other interesting features. This approach often generates more questions than answers for the general audience due to the lack of unification across all views. It appears that a common thread to relate all aspects of fractal characteristics is missing. This paper addresses this question and presents some new and fascinating results.

For example, in-depth mathematical analysis often defers to the intriguing and attractive graphical displays produced by mapping the complex plane to the pixel field on a CRT. Both areas, mathematics and graphics, are generally developed or presented independently. The development of common attribute linkages is done separately or perhaps not at all. First, a completely modular survey of the state of the art concerning regular fractal geometry is given.

In addition, a method for calculating the fractal dimension of asymmetric fractals is proposed, where a symmetric fractal is a special case of an asymmetric fractal. This new method includes fractals which have variable segment lengths. Fractals designed by allowing variable length segments depict natural phenomena more clearly than those designed with uniform length segments.
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I. INTRODUCTION

The investigation of fractal geometry and fractal curves spans relatively few years but has deep mature roots and has attracted many practitioners for a variety of reasons. Mathematicians favor the uniqueness of the peculiar and non-Euclidean fractional dimension. Naturalists appreciate the amazing similarity between natural phenomena such as an irregular coastline, and a numerical representation of the same through the use of non-integer dimensionality. Film makers enjoy the fractal geometry concept as an art form for creating very detailed and authentic looking landscapes. As we trace the historical and mathematical background of fractals, some coincidences shall emerge which link the mathematical, artistic, and applied uses of fractal geometry.

First a brief review of existing work is given in Parts I through IV, and the mathematical, aesthetic, and practical facets of fractal geometry are emphasized. Part V proposes a new method for calculating the fractal dimension of fractals with segments of non-uniform length (traditional fractals use uniform length segments). Part VI discusses current and future work being done in multi-dimensional fractal generation. Some figures are photographs, while most were generated on an interactive graphics system and plotted using a pen plotter.
II. HISTORY

The ancient Greek mathematicians were probably first to have any impact, although remote, on the subject of fractal curves. They initially contributed a variety of definitions for a curve. Analytic geometers of the seventeenth century gave more definition to curves by making them the graphical representation of algebraic equations. By the last half of the nineteenth century, "monster" curves emerged, so-called because they strongly countered the definition of a curve. That such "monster" curves exist, was proposed by the Italian logician Giuseppe Peano in 1890. In fact, between the years 1875 and 1925, several mathematicians were espousing beliefs that ran counter to the accepted Euclidean geometry, thereby giving some measure of credibility to these "pathological" constructions. Although Peano, David Hilbert, and Waclaw Sierpinski added to the body of knowledge concerning plane or space filling monster curves, it was not until the early 1900's that Helge von Koch, a Swedish mathematician, introduced the familiar "snowflake" curve which exhibited fractional dimensionality. It is this type of curve that precisely embodies the essence of fractal geometry. This Triadic Koch curve is created as a single dimension entity (even though we must view it in a two dimensional plane). As additional recursive iterations push its overall length toward infinity, its $D_t$, (topological or Euclidean dimension), remains one. Even though the additional
iterations appear to make the curve more dense, the first and last iterations of the curve are dimensionally identical. It was not until the work of Felix Hausdorff in 1919 (and extended in 1934 by A. S. Besicovitch) that the problem of increased density without dimensional increase was addressed. Hausdorff proposed a method of differentiating between the density dependent stages of curves by redefining the dimension of a curve using a real number instead of the Euclidean integer, thus defining a fractional dimension. It is this Hausdorff or fractal dimension, referred to as $D_f$, which allows us to observe that a curve with a fractal dimension of 1.5000 (Koch's Quadric Curve), is slightly more dense than a curve with a fractal dimension of 1.2618 (Koch's Triadic curve), despite the fact that both have a Euclidean (topological) dimension of one (see Figure 1). In this instance, as the fractal dimension ranges between one and two, the higher the fractal dimension, the denser the appearance of the curve.

In the early 1950's Benoit B. Mandelbrot, a Polish-born French mathematician, used this redefined and refined idea of dimensionality to describe natural and physical phenomena. He first generated sets of numbers that when plotted, produce irregular but repetitive images of great detail. His recursive generation of complex numbers and their mapping to the complex plane has not only provided a useful modeling tool, but inspired an art form as well.
FIGURE 1. Initial Two Iterations

1/4 of Koch Quadric Curve
1/3 of Koch Triadic Curve
III. MATHEMATICAL DEFINITION

A. COMPLEX NUMBERS

The use of complex numbers in the generation of several types of fractal images warrants a brief review of some basic operations on complex numbers and the geometry attendant on those operations (Ahlfors, 1979). If the set of real numbers is viewed as being geometrically one dimensional, then the set of complex numbers may be logically extended to require two dimensions. The general form of the complex number \( Z \), is

\[ Z = X + Yi, \]

where \( X \) represents the real component (Re), which may be plotted on the horizontal axis, or abscissa. Likewise, \( Y \) represents the imaginary component (Im), which may be plotted on the vertical axis or ordinate (See Figure 2a).

The basic operations addition, subtraction, and multiplication are simple to perform either numerically or graphically.

Numerically, addition is performed by adding the real parts of two complex numbers to form the real part of the resultant complex number, and then adding the imaginary parts of the two numbers to form the imaginary part of the resultant complex number. For example, adding the complex numbers

\[ Z_1 = X_1 + Y_1i \quad \text{and} \quad Z_2 = X_2 + Y_2i \]

produces

\[ Z_1 + Z_2 = (X_1 + X_2) + (Y_1 + Y_2)i. \]

Subtraction of the two numbers similarly produces

\[ Z_1 - Z_2 = (X_1 - X_2) + (Y_1 - Y_2)i. \]

Multiplication can be accomplished by applying the rules for polynomial multiplication so that

\[ Z_1 \times Z_2 = X_1X_2 + (X_1Y_1 + Y_1X_2)i + (Y_1Y_2)i^2. \]
Plot of a Complex Number

Vector Representation

Addition and Subtraction

Multiplication

FIGURE 2. Complex Number Operations
By defining $i^2 = -1$, therefore

$$Z_1 \times Z_2 = (X_1X_2 - Y_1Y_2) + (X_1Y_2 + X_2Y_1)i.$$  

Graphically, the addition and subtraction operations may be carried out by using the same approach as vector addition and subtraction. For example, using complex numbers $Z_1 = X_1 + Y_1i$ and $Z_2 = X_2 + Y_2i$, both shown graphically as two vectors, which have their tail located at the (0,0) point of the Cartesian plane (see Figure 2b). To add the two vectors, translate the tail of one vector to the head of the other, while maintaining the original direction of both vectors. The resultant vector with tail at (0,0) and head coincident with the head of the translated vector, is the result of the addition, and the coordinates of the head are the result of the numerical addition. To subtract the two vectors, first, reverse direction of one of the vectors and proceed with the technique for addition (see Figure 2c). For graphical multiplication, two more attributes of the original vectors need to be calculated; the modulus and the argument. The modulus is simply the absolute value of the vector. In the case of $Z = X + Yi$, the modulus ($m$), is

$$m(Z) = |Z| = \sqrt{x^2 + y^2}.$$  

The argument is the angle the vector makes with the positive X axis. In the case of $Z = X + Yi$, the argument (arg) is

$$\text{arg}(Z) = \arctan \left( \frac{Y}{X} \right).$$  

To multiply the two vectors graphically, multiply the moduli and add the arguments to get the modulus and argument of the product vector, as in Figure 2d (Silver, 1987).
The definition of Dragon fractals (also called regular fractals or "Monsters") requires some discussion of self-similarity, recursive processes, and Hausdorff dimensions, which are defined later in this section. As a curve is viewed at increasingly greater magnification, the piece of curve being viewed begins to look more like a straight line and bears no resemblance to the original curve. In other words, increased magnification, drastically simplifies the geometry of the object (McDermott, 1983).

This example demonstrates no self-similarity or recursive construction whatsoever. Alternately, if a curve, upon greater magnification, shows an exact pattern repeat or even a statistically similar pattern repeat of the unmagnified view, a fractal is indicated. How then are these Dragon or Monster curves created and why the strange names?

Before Calculus, a "curve" was defined as either the intersection of two surfaces (a plane intersects a cone, making a conic-section curve), or as the locus of a moving point (a point describes a circular curve as it rotates about a center point, or an elliptical curve as it moves about two foci). Calculus brought the additional constraint that a "curve" must be smooth. To complete our curve definition we will now discuss the relationship between curves and tangents along those curves.

Defining a line tangent to a circle at some point is
relatively straightforward (See Figure 3a). After drawing a radius from the center point 0 to a point on the circle $P$, construct a line $L$, perpendicular to the radius. This line $L$ then, is the tangent line. Now look at another type of curve, the parabola defined by the equation $y = x^2 + 1$, shown in Figure 3b. If the tangency is examined at the lowest point on the curve, $P$, it is apparent that the line $y = 1$ is a tangent line. However, at a point such as $Q$, the tangent line is not apparent. A definition of tangency should include the fact that a tangent line "touches" a curve at the point under consideration. So, although a circle tangent line is rather simple to construct, tangency is much more difficult to establish for other types of curves, and requires investigation of the curve with the idea of "limits" in mind (Protter/Morrey, 1964).

A general curve, given by some function $y = f(x)$, is shown in Figure 3c. Also shown is a line through two points on the curve, $P$ and $Q$. This line and any line passing through two points on a curve is called a secant. The coordinates of point $P$ are denoted by $(x, y)$ or $(x, f(x))$, and the coordinates of $Q$ are denoted by $(x + d, f(x + d))$, where $d$ is some positive or negative amount of displacement in $x$ ($d$ is shown positive). Now the slope of the line is found by using the formula for slope when both points are known,

$$\text{slope} = m = (y_2 - y_1)/(x_2 - x_1).$$

In this case,

$$m = (f(x + d) - f(x))/((x + d) - x) = (f(x + h) - f(x))/d.$$
FIGURE 3. Tangency
Geometrically, one can see from this slope statement that as \( d \) tends to 0, the point \( Q \) approaches point \( P \). As shown in Figure 3d, a secant line \( L_n \) rotates about point \( P \) and approaches a limit line. Intuitively, this limiting line is the tangent line to the curve at point \( P \). Additionally, if a function \( f \) possesses a derivative, then as \( d \) tends to 0, the slope, \( m \), is defined as

\[
m = \lim_{d \to 0} \frac{f(x+d)-f(x)}{d} = f'(x).
\]

In other words, the slopes of the lines \( L_n \) tend to a limit slope which is the derivative of the function at point \( P \). By definition then, the tangent to a curve with equation \( y = f(x) \) at point \( P(x, f(x)) \) is the line through point \( P \) that has a slope, \( m \), equal to \( f'(x) \).

Therefore, the tangent line is constructed by applying the point-slope formula for a line, to the derivative and the point. A general point-slope formula is \( y - y_1 = m(x - x_1) \), where the point is \( (x_1, y_1) \), \( y_1 = f(x_1) \), and \( m = f'(x_1) \).

Calculating the tangent for the parabola in Figure 3b, described by \( y = x^2 + 1 \), at the point where \( x_1 = 2 \), since then, \( y_1 = f(x_1) = f(2) = 5 \),

\[
\begin{align*}
y_1 &= f(x_1) = f(2) = 5, \\
n &= f'(x_1) = f'(2) = 4.
\end{align*}
\]

Applying the point-slope formula,

\[
y - y_1 = m(x - x_1)
\]

yields \( y = 4x - 3 \) as the equation of the tangent line. From our discussion and the example of a standard curve, it seems that any continuous curve should possess, at any one
point along the curve, a tangent line that changes continuously as it is moved from point to point along the curve. It also seems that by differentiating the function, the slope of the tangent line can be calculated. Mathematicians, however, have provided us with examples that are counter to this intuitive definition of a curve.

In particular, the Italian mathematician and logician Giuseppe Peano introduced a continuous curve that did not possess a different tangent at all points. This type of curve ran so strongly counter to the intuitive and accepted form of a curve that it was called a "Monster" curve. Peano's Monster is built by moving a point continuously over a square shape. This curve is definitely continuous, but analyzing several points along this rather straight sided curve, shows that many points share identical slopes, and therefore share identical tangent lines. David Hilbert introduced a method for generating a Peano-type curve (see Figure 4).

Note that in the basic form of the curve in Figure 4a we have described three distinct straight line parts. Dividing each of these three parts that make up a "square U", again into three elements, 1, 2, and 3, and applying the "square U" shaped pattern recursively to element 1 of the first part, to element 2 of the second part, and to element 3 of the third part, produces a more complex curve. Closer viewing of Figure 4d retains a "self-similarity" to the curve in Figure 4b. Reapplying this recursive construction to each new straight element two more times produces a much
FIGURE 4. Hilbert's Peano Curve
more complex curve and extends the self-similarity property (Hofstadter, 1986; Hubbard, 1986). Figure 4 presents two facts.

The first fact is that by adding more and more recursive iterations, the curve becomes longer. As the number of iterations grows, the curve length also grows. The relationship between numbers of iterations and total length is shown by examining the curve in Figure 5, which is a simple variation of Figure 4. Iteration 0 shows one segment of length one. The following table shows the segment/length relationships depicted in Figure 4:

<table>
<thead>
<tr>
<th>Iteration Number</th>
<th>Number of Segments</th>
<th>* Segment Length</th>
<th>= Total Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5^0</td>
<td>(1/3)^0</td>
<td>1 = 1.000</td>
</tr>
<tr>
<td>1</td>
<td>5^1</td>
<td>(1/3)^1</td>
<td>5/3 = 1.666</td>
</tr>
<tr>
<td>2</td>
<td>5^2</td>
<td>(1/3)^2</td>
<td>25/9 = 2.777</td>
</tr>
<tr>
<td>3</td>
<td>5^3</td>
<td>(1/3)^3</td>
<td>125/27 = 4.629</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>n</td>
<td>5^n</td>
<td>(1/3)^n</td>
<td>5^n/3^n = (5/3)^n</td>
</tr>
</tbody>
</table>

The table shows that as the number of iterations, n, approaches infinity, the total length of the curve goes to infinity.

The second fact depicted by Figure 4 is that the amount of area that this curve occupies appears to be increasing with increasing iterations. This particular example of a
FIGURE 5. Iteration vs. Length
Peano type curve in Figure 4 actually does increase in area. Discussion of this "plane-filling" type of fractal is treated in detail in section III D.

Traditional Euclidean curves have a topological dimension, $D_t = 1$, and no area associated with them. As a curve becomes more and more complex due to added iterations, the curve becomes longer and the area covered by the curve appears larger. As a curve approaches infinite length, the surface contained by the planar curve is more filled, and the dimension seems to be greater than 1. So, even in cases where a curve has a topological dimension of 1, as recursive iterations grow and the curve tends to "fill in" an area, there is a theoretical dimension that reflects how much "filling" has occurred. This dimension becomes fractionally larger or smaller. This real number is called the Hausdorff dimension and is one of the main tenets of fractal geometry. Euclidean geometry describes a line or curve as one dimensional, a plane as a two dimensional entity, and a volume as three dimensional. Fractional or Hausdorff dimensions, however, allow for a continuous blending from 1 dimension to 2, and from 2 dimensions to 3. Another version of the Peano curve is one introduced by Waclaw Sierpinski (see Figure 6). Compared to Hilbert's curve, this version seems to have more "covering power" in the same number of recursions.
FIGURE 6. Sierpinski's Peano Curve
The initial shape or "initiator curve" used to generate the closed curve of Sierpinski's Peano curve is a square. To construct the first iteration, place a scaled-down copy of the square on each corner of the large square. The dimensions of Sierpinski's scaled-down squares may be calculated by using the following equations. These equations relate the side length of the large square, the side length of the small square, and the diagonal of the small square:

\[
\text{Small square side-length} = \left(\frac{\text{Large square side-length}}{4}\right) \times \sqrt{2},
\]

\[
\text{Small square diagonal-length} = \frac{\text{Large square side-length}}{2}.
\]

For example, if the side length of the first square was 8, the scaled down square would be constructed with side lengths equal to \(8/4\) times the square root of 2, or \(2\sqrt{2}\). Or, the length of the small square diagonal would equal \(8/2 = 4\).

Now rotate the small squares 45 degrees around their own centers. This distorted square, shown as the first iteration in Figure 6a, would be scaled down by the method presented, and four scaled down copies would be placed coincident with the four corners of the previous iteration. The generator function or activity then, is to continue placing the scaled down copies of distorted squares on the corners of the next iteration.
C. CALCULATING FRACTAL DIMENSIONS

So far we have discussed two types of dimensions: Euclidean or topological (integer), and fractal (real). Fractal dimensions depend upon two parameters: the parameter concerning the "magnification" of the original curve, which equals the inverse of the scale factor, and the parameter concerning the number of identical segments in the generator curve.

In general, the fractal dimension is defined as the exponent of the scale factor inverse, and produces the number of segments in the generator curve. Allowing the scale factor to be $1/x$, the inverse of the scale factor to be $x$, and the number of generator segments to be $y$, the general equation for the fractal dimension, $D_f$, may be stated:

$$D_f = \frac{\log e y}{\log e x} = \frac{\ln y}{\ln x}$$

For instance, if a line of length 1 is divided by 3, 3 segments of length $1/3$ are formed. The scale factor then is

$$\text{segment-length/total-length} = (1/3)/1 = 1/3$$

and the magnification factor, $x$, is the inverse of $1/3$. So

$$x = 1/\text{scale factor} = 1/(1/3) = 3.$$  
The case where the length of the original line or curve is other than 1 is valid, since the scale factor is a ratio.
The fractal dimension definition may also be stated

\[ D_f = \frac{\ln y}{\ln (1/r)} \]  \hspace{1cm} [1]

where \( r = 1/b \), and \( b \) equals the number of identical sub-intervals into which the unit line is divided. For the most primitive types of geometry like a line, square, or cube, the \( y \) value (the number of copies created) can be calculated by raising the inverse scale factor (the \( x \) value) to the power of the topological dimension (1, 2, or 3). For example, in the case of the straight generator line which is divided by 3, the \( y \) value or number of copies is simply the \( x \) or inverse scale factor (3), raised to the first power, since it is a one dimensional line. Likewise, in a two dimensional square, if the \( x \) factor is 3, the \( y \) value is \( 3^2 \) or 9 and similarly, for a cubic structure, \( y \) will equal \( 3^3 \) or 27. In this degenerative case the equation becomes

\[ D_f = \frac{\ln b^{(D_t)}}{\ln (1/(1/b))} \]

where \( b \) is the number of segments of length \( 1/b \). Degenerate cases of the three types of geometric entities with increasing dimension, demonstrate the idea (see Figure 7).

The first type is a straight line. Dividing the line by three produces three copies of the original. Then the inverse scale factor, \( x \), is equal to \( 1/(1/b) \). Since \( b = 3 \), this \( x \) becomes \( 1/(1/3) \) or 3. \( D_t \), the topological dimension is 1. Therefore,
FIGURE 7. Fractal Dimension Calculations
\[
D_f = \frac{\ln b^{(D_t)}}{\ln (1/(1/b))} = \frac{\ln 3^1}{\ln (1/(1/3))} = 1.
\]

The second type is a planar square, with \( D_t = 2 \). The square is divided so that its sides are cut into 3 segments and the square can then be divided so as to yield nine copies of the original square. With \( D_t = 2 \) and \( b = 3 \), then

\[
D_f = \frac{\ln b^{(D_t)}}{\ln (1/(1/b))} = \frac{\ln 3^2}{\ln (1/(1/3))} = 2.
\]

The third type is a cube with \( D_t = 3 \). Dividing all sides by three produces a cube which can be segmented into 27 copies of the original cube. With \( D_t \) and \( b \) both equal to 3,

\[
D_f = \frac{\ln b^{(D_t)}}{\ln (1/(1/b))} = \frac{\ln 3^3}{\ln (1/(1/3))} = 3.
\]

When a construction, such as these three simple degenerate cases, yields an integer as its fractal dimension, and the topological dimension is equivalent to the fractal dimension, the entity is normally not a fractal, except in special and rare conditions.

Following is a recursive construction that does qualify as a fractal and cannot use the relationship \( y = b^{(D_t)} \). Figure 8 shows the first four iterations of a construction known as the Triadic Koch Curve, also known as Koch's Island or Koch's Snowflake. It is the work of Helge von Koch, a Swedish mathematician. The actual Koch curve begins with an
FIGURE 8. $1/3$ of Triadic Koch Curve
equilateral triangle as its "initiator" or initial curve. Only 1/3 of the initiator is examined (one of the triangle sides) for the sake of clarity. The "generator", which is the recursive engine, is formed by dividing the initiator into three equal segments and replacing the center segment with two segments, equal in length to the original three. That is to say, that with the original curve length equal to 1, and the original number of divisions, b, equal to 3, the four segments that replaced the original three segments become the number of copies or y. As each straight line segment is replaced by its generator, the curve becomes increasingly intricate. The fractal dimension of this curve, with \( x = 1/(1/b) \), \( b = 3 \), and \( y = 4 \), is

\[
D_f = \frac{\ln y}{\ln (1/(1/b))} = \frac{\ln 4}{\ln (1/(1/3))} = 1.2618.
\]

This fractal dimension is the same for the complete Koch curve. In order to change the dimension, a different recursive generator would be necessary, which would, of course, change the image.

So far, a fractal has been defined as a curve whose Hausdorff dimension, \( D_f \), is non-integer, and is also greater than the topological dimension, \( D_t \). A more correct reference would be the Hausdorff-Besicovitch dimension, since Besicovitch derived the final form (Mandelbrot, 1983). Note, however, that the fractal dimension, \( D_f \), may indeed be an integer, as long as the integer \( D_f \) is not equal to the integer \( D_t \).
D. PLANE-FILLING NON-FRACTALS

A type of curve that is germane to the discussion is the plane-filling or space-filling curve (McWorter/Morrill, 1987) that is classified as a non-fractal, due to the fact that $D_f = D_t$. A good example of a space-filling curve is Hilbert's Peano curve, discussed earlier and depicted in Figure 4. During earlier iterations of the Peano curve its length grows, as well as the area it occupies (Gardner, 1976).

Consequently, as the iterations continue, the dimension tends toward 2. This is counter to our fractal definition because as the curve's dimension approaches 2, we will eventually fill the entire plane and $D_f = D_t$. Instead of iterating about the initiator curve and becoming more complex without claiming additional area as fractals do, this curve fills space until it is fully 2 dimensional. It is this "monstrosity" that disturbed early mathematicians since one dimensional curves should have no ability to fill space. Though the space-filling curve is not a fractal, it points out a disruptive variation on early curve theory; a curve whose path seems to be one dimensional, yet at the curve limit, it occupies a two dimensional area. Remember that although the space-filling curve is not a fractal, it is very useful in modeling natural and biological networks.
E. MANDELBROT, CANTOR, and JULIA SETS

Moving into the second and third dimensions with fractal geometry, highlights the environment responsible for the very popular graphical displays. It is here that the very brief primer on complex number operations will come in handy. First, though, a brief description of "strange" or "fractal" attractors is warranted. The common name for the attractor concept given by Mandelbrot (1983) is the "theory of strange attractors and of chaotic (or stochastic) evolution". He explains the idea of an attractor with the example of a ball in a funnel, spiraling toward the lowest point. The lowest point, of course, is located in the center of the funnel. Assuming that the funnel hole is small enough to not allow the ball to pass, this point of dynamic equilibrium could be called an attractor point. As might be expected, when a stable attractor exists, there is also an unstable repulsion point. Mandelbrot's example of a repulsion point is the same ball poised atop a pencil point, constantly being pushed away from the pencil point by gravity. This concept of points of attraction and points of repulsion is significant to the discussion of fractal geometry because the sets of points that comprise the graphical representations of the geometry are generated and driven by such recursive influences. A more concrete example is given by the simple squaring function \( y = x^2 \). Assume that \( x \) may be any positive real number. Also assume that \( x \) has a value that is greater than 1. Now, if the \( x \) value is recursively replaced with the previously squared value of \( x \), the value of \( y \) becomes larger and
approaches infinity as a limit. (Starting with an initial value of 1.500 for $x$, $y$ becomes 2.250, 5.062, 25.628, etc). However, for any value of $x$ that is less than 1, the $y$ value becomes smaller and approaches zero as a limit. (If $x$ initially equals .750, $y = 0.562, 0.316, 0.100$, etc). So, considering the function $y = x^2$, for all positive real values of $x > 1$, infinity is an attractor. Similarly, for values of $x < 1$, zero is an attractor. The precise value of $x = 1$ is very unstable, like the ball on a pencil point analogy, because if the value of $x$ becomes even slightly larger or smaller than 1, the function value will quickly head off for infinity or zero. Likewise, $x = 1$ is a point less repulsion for values of $x$ that are greater than or less than 1. The realm of a one dimensional positive real number system differs from that of the two dimensional complex plane as defined earlier in the section on complex numbers. Instead of $y = x^2$, use $w = z^2$, where $w$ and $z$ are complex. Since in the complex plane both positive and negative real numbers will be used, the value $|z| = 1$, for the function $w = z^2$, simply becomes a unit circle centered about the origin $(0,0)$, with a radius equal to 1 (see Figure 9).

To summarize the activity produced by the simple function $w = z^2$, three conditions are possible in two dimensions (Hearn/Baker, 1986);

- Values of $z$ with $|z|$ less than 1 drive $z$ to 0, therefore, zero is an attractor of such points.
- Values with $|z|$ greater than 1 drive $z$ towards infinity, therefore, infinity is an attractor of such points.
(0, 0) = Attraction Point for all |Z| < 1

Infinity = Attraction Point for all |Z| > 1

One Attraction Zone
\[ Z = Z_0 + C \]
\[ C = 0 \]

Two Attraction Zones
\[ Z = Z_0 + C \]
\[ C = -1 + 0i \]

Three Attraction Zones
\[ Z = Z_0 + C \]
\[ C = -0.12 + 0.74i \]

Three Attraction Zones
\[ Z = Z_0 + C \]
\[ C = -0.194 + 0.6557i \]

Cantor Set

FIGURE 9. Julia Sets
Values with $|z| = 1$ stay there and graphically form a boundary between the two areas of attraction. These points form a set known as the Julia Set (a circle in this case) named for the French mathematician Gaston Julia (Peitgen and Richter, 1986).

This initial activity used a simple squaring function $w = z^2$. The most general form, $Z = f(Z, C)$, will now be introduced, where both $Z$ and $C$ are complex numbers;

$$Z_1 = Z_0^2 + C$$

As you can see, the simple squaring equation merely used a value of $C$ equal to zero, which produced a single attraction point at the origin $(0,0)$. The recursive nature of this function is now evident in that the new value of $Z$ is computed by squaring the previous complex value of $Z$ and adding to it the complex value of $C$. Adding a non-zero value of $C$ to the equation allows for single and multiple, non-origin, zones of attraction. By changing the real and/or imaginary parts of the $C$ value, the number and complexity of the attraction zones is varied. In Figure 9, a single zone of attraction with $C = 0$ produces a circular Julia Set, the simplest type of Julia Set. The multiple zone Julia Sets have non-zero $C$ values and real/imaginary variations. The last type of Julia Set to be discussed is the type that contains points of attraction, but is not connected as the others were. This non-connection allows points that are inside the Julia Set to be attracted to infinity as well as points outside the Julia Set. This Julia Set subset is called a Cantor Set.
The Cantor set can be very difficult to visualize because of its sparseness. Using the technique developed in section III C, the Cantor set can be discussed in terms of initiator curve, generator curve, and fractal dimension. As shown in Figure 10, the initiator curve consists of a single line which is segmented into three pieces as in previous examples. Each segment is 1/3 the length of the initiator curve. The generator function, however, removes the center segment instead of replacing it with multiple segments as before. Continuing recursively with this generator yields smaller and smaller segment lengths until all that remains is point data of zero dimension. The Cantor set becomes so sparse after a few iterations that it is sometimes referred to as the Cantor Dust (Mandelbrot,1983).

Calculating the fractal dimension (Falconer,1985),

\[
D_f = \frac{\ln y}{\ln (1/(1/b))} = \frac{\ln 2}{\ln (1/(1/3))} = \frac{0.6931}{1.0986} = 0.6308.
\]

Although this calculation introduces a fractal dimension less than 1, the fractal definition remains intact since the topological dimension of the points is \(D_t = 0\), and \(D_f > D_t\). This discussion has touched upon curves where \(D_f = D_t\), like the primitive cases of a 0 dimensional point, a 1 dimensional line, a 2 dimensional planar square, and a 3 dimensional cube. Also presented were curves that are not actually fractals; the Peano plane-filling curves. Finally, fractals where \(D_f > D_t\) were discussed (Koch's curves).
FIGURE 10. Cantor Set

$Df = 0.63089$
A collection of sets of true fractal curves, i.e. those with $C$ not zero, can be depicted graphically by the Mandelbrot Set, named after Benoît Mandelbrot of IBM, who is the founding father of modern fractal research (Dewdney, 1985). The Mandelbrot Set displayed in Figure 11 is the set of all possible values for the complex number $C$ that have not escaped to infinity. The areas around the edges of the darkened set represent an area where Cantor Sets are located, points that are on their way to infinity, and their route is depicted by a lighter color.

To give a feeling for the computational intensity involved in graphically depicting a display of this type, the construction of some initial iterations is examined. Remember that the operating function is $Z_1 = Z_0^2 + C_0$, where $Z_1$, $Z_0$ and $C_0$ are complex numbers, and $C_0$ is not equal to zero. (Even if the initial value for $Z_0$ is chosen to be zero, it will be non-zero after the first iteration). The initial values are $Z_0 = 0.5 + 0.09i$ and $C_0 = 0.5 + 0.09i$. Then,

\[
Z_1 \Rightarrow Z_0^2 + C_0
\]

\[
= (0.5 + 0.09i)(0.5 + 0.09i) + (0.5 + 0.09i)
\]

\[
= 0.25 + 0.09i - 0.0081 + 0.5 + 0.09i
\]

\[
= 0.7419 + 0.18i
\]

\[
Z_2 \Rightarrow Z_1^2 + C_0
\]

\[
= (0.7419 + 0.18i)(0.7419 + 0.18i) + (0.5 + 0.09i)
\]

\[
= 0.5504 + 0.267i - 0.0324 + 0.5 + 0.09i
\]

\[
= 1.018 + 0.357i
\]
FIGURE 11. Mandelbrot Set
\[ Z_3 \Rightarrow Z_2^2 + C_0 \]
\[ = (1.018+.357i)(1.018+.357i) + (.5+.09i) \]
\[ = 1.0363 + .7268i - .1274 + .5 + .09i \]
\[ = 1.4089 + .8168i \]

\[ Z_4 \Rightarrow Z_3^2 + C_0 \]
\[ = (1.4089+.8168i)(1.4089+.8168i) + (.5+.09i) \]
\[ = 1.9849 + 2.3015i - .6671 + .5 + .09i \]
\[ = 1.8178 + 2.3915i \]

\[ Z_5 \Rightarrow Z_4^2 + C_0 \]
\[ = (1.8178+2.3915i)(1.8178+2.3915i) + (.5+.09i) \]
\[ = 3.3043 + 8.6845i - 5.7192 + .5 + .09i \]
\[ = -1.9149 + 8.7845i \]

\[ Z_6 \Rightarrow Z_5^2 + C_0 \]
\[ = (-1.9149+8.7845i)(-1.9149+8.7845i) + (.5+.09i) \]
\[ = 3.6668 - 33.6428i - 77.1674 + .5 + .09i \]
\[ = -73 - 33.5528i \]

\[ Z_7 \Rightarrow Z_6^2 + C_0 \]
\[ = (-73-33.5528i)(-73-33.5528i) + (.5+.09i) \]
\[ = 5329 + 4898.7088i - 1125.7903 + .5 + .09i \]
\[ = 4203.7097 + 4898.7988i \]

As is evident from the quickly increasing values of the real and imaginary parts of our \( Z \), plotting these numbers on the complex plane yields a curve headed for infinity after calculating only the first 7 iterates. Since the Mandelbrot set is comprised of all the complex numbers \( C \) for which
\( z^2 + C \) remains finite after a large number of iterations, the choice of \( C = 0.5 + 0.09i \) is not in the Mandelbrot set. Indeed, one of the hardest parts of generating an interesting graphical display is the judicious selection of the seed value used for \( C \). Also bear in mind that the mapping function \( W = z^2 + C \) is not the only map that may provide interesting graphics. It is, however, one of the simplest functions of a complex variable whose graphics are interesting, and it was the one first studied by Mandelbrot. Examining different areas of the Mandelbrot set under various scales affords much variety without changing the mapping function.

The images of Figures 11 and 12 (Smith/Staller, 1986) are views of the Mandelbrot set at two different magnifications. These views demonstrate the characteristic of self-similarity which was discussed earlier. Figure 12 depicts the area pinpointed by the box in Figure 11, under a higher degree of magnification. The similarity is obvious. Continuing to magnify specific areas indicates that the magnified area does not just become simpler and tend to a straight line. Instead, it maintains the similar shape of the original curve.
FIGURE 12. Mandelbrot Set (5X Magnification)
IV. APPLICATIONS

As may be evident by this point, hardcore, directly useful engineering applications for fractal geometry are not immediately obvious. However, several natural geometric phenomena such as irregular coastlines, cloud formations, the intricate network of the lungs' alveolus, the brain's neural pathways, vegetation patterns, the convoluted surface of the brain, as well as fracture patterns on metal surfaces and dendritic cooling patterns within metal castings, have all been shortchanged when it comes to a mathematical definition which is based on integer-dimensioned Euclidean geometry. Fractal geometry, especially the fractal dimension, will allow modeling of previously termed chaotic systems with a single set of recursive equations. These models can be used in the analysis of system dynamics such as earthquake prediction and pollution prediction, much as Finite Element Analysis predicts mechanical distortion due to applied stresses. Aside from the practical applications, the artistic applications are already well known and accepted. Animating films using fractal generated mountain ranges and rivers as well as Euclidean shapes is a technique making a bid to replace some movie sets and physical modeling techniques.

The list of physiological applications alone is becoming extensive (Gleick, 1987):
- The job of the alveolar lining of the lungs is to provide the greatest possible lung area in the smallest possible
space. The standard description of bronchial branching up until now has been "exponential". Recent investigation has shown this bronchial branching to be fractal.

- The urinary collection system can be shown to have a fractal structure.

- The biliary duct in the liver, as well as the branching of blood vessels from the aorta to the capillaries, has a decidedly fractal nature.

- The His-Purkinje Network, a network of special fibers in the heart that carries pulses of electric current to contracting muscles, can be described in fractal terms.
V. FRACTALS WITH VARIABLE LENGTH SEGMENTS

This section examines the next step of generalization, to develop a method for calculating the fractal dimension, \( D_f \), when the generator curve segments are not identical. To capture the concept of more natural phenomena (shorelines, mountain ranges, clouds, trees, etc.), the use of traditional uniform length segments must be generalized to include non-uniform length segments in the generator curve. Two approaches were investigated, and one of them was rejected.

In the first attempt, using the table of length relationships developed in Section III B, an expression for \( D_f \) was developed which relates to segment length directly. In the discussion of Figure 4, it was established that the total fractal length, \( L \), could be expressed by

\[
L = y^n \times (1/x)^n = \frac{y^n}{x^n} \tag{2}
\]

where

- \( y \) = the number of segments in the generator curve
- \( n \) = the number of iterations
- \( 1/x \) = the initiator segment length as a fraction of the initiator total length.

For example, using the familiar Koch curve with all segments = 1/3,

- \( y = 4 \)
- \( n = 1 \)
- \( 1/x = 1/3 \)
and then,

\[ L = \frac{y^n}{x^n} \]
\[ = 4^1 \times (1/3)^1 = 4/3 = 1.3333 \]

By calculating the log of the total length, the fractal dimension, \( D_f \), can be expressed as follows:

\[ D_f = \frac{\ln y^n}{\ln x^n} = \frac{n \ln y}{n \ln x} = \frac{\ln y}{\ln x}. \quad [3] \]

Using equation [3] for the Koch curve example,

\[ D_f = \frac{\ln 4}{\ln 3} = \frac{\ln 4}{\ln 3} = 1.2618. \]

It is evident from the derivation of equations [2] and [3] that although varying \( n \), the number of iterations, certainly affects the total length of the curve, it does not change the fractal dimension. For example, using equation [2] for the Koch curve with 2 iterations, with \( y = 4, \ n = 2, \) and \( 1/x = 1/3, \)

then,

\[ L = 4^2 \times (1/3)^2 = 16/9 = 1.7777, \]

which is longer than the 1.3333 value achieved for the total length after one iteration. However, using equation [3], the fractal dimension stays the same:

\[ D_f = \frac{\ln y^2}{\ln x^2} = 1.2618. \]
Using equations [2] and [3], the definition of fractals with varying generator segment lengths can now be attempted in the following manner. When enlarging or reducing segment lengths, express the segment lengths in the lowest common denominator. For instance, in the Koch example, the initiator and generator segments are all equal to 1/3. Suppose the length of 2 & 3 is reduced to 1/4. The lowest common denominator for 1/3 and 1/4 is 1/12. So all segment lengths should be expressed in 1/12's.

As an example, Figure 13 shows instances of segment reductions. In the first reduction sequence, segments 2 & 4 are reduced from 1/3 to 1/6. Therefore, the lowest common fraction is 1/6. If the segments in this generator are broken into segments 1/6 long, there are 8 of them as shown in 13b. The initiator curve expressed in 1/6's yields 6 segments. Therefore the total length and fractal dimension can be calculated by setting

\[ y = 8, \quad n = 1, \quad \text{and} \quad \frac{1}{x} = 1/6. \]

Then,

\[ L = 8^1 \times (1/6)^1 = 8/6 = 1.3333, \text{ and} \]

\[ D_f = \frac{\ln 8^1}{\ln 6^1} = 1.1605. \]
FIGURE 13. Two Segment Length Reduction
Continuing, Figure 13c shows a generator curve where the two original segments of length 1/3, have been reduced to a length of 1/12. So, there are 2 segments 1/12 long and 3 segments 1/3 long. Subdividing the segments of length 1/3 into 12 short (1/12) segments, plus the 2 reduced segments, yields a generator with 14 segments 1/12 long and an initiator with 12 segments 1/12 long, or \( y = 14 \) and \( x = 12 \). Therefore

\[
\frac{\ln y}{\ln x} = \frac{\ln 14}{\ln 12} = \frac{2.6390}{2.4849} = 1.0620.
\]

Similarly, in 13d, where 2 segments are reduced to 1/24, which is also the lowest common fraction, \( y = 26 \), \( x = 24 \), and

\[
\frac{\ln y}{\ln x} = \frac{\ln 26}{\ln 24} = \frac{3.2580}{3.1780} = 1.0251.
\]

As the reduced segments become smaller, the \( D_f \) approaches 1.000, which is the case when the generator curve and the initiator curve are both straight lines.

Now consider the case for segment enlargement. Referring again to Figure 13, the fractal dimension for the case where all segments equal 1/3, is 1.4649. As one or more segments is enlarged, an increase in the total length of the curve, along with a larger fractal dimension is expected. Using the latest technique for reducing segment lengths, only this time proceeding to enlarge segments 2 and 4 from 1/3 to 5/12, the lowest common fraction is 1/12. There are three segments equal to
1/3 (or 4/12) and two segments (2 & 4) enlarged to 5/12 each. Therefore

\[ y = 22, \quad n = 1, \quad \text{and} \quad 1/x = 1/12. \]

Then

\[ L = 22^1 \times (1/12)^1 = 1.8333, \]

and

\[ \frac{\ln y}{\ln x} = \frac{\ln 22}{\ln 12} = \frac{3.0910}{2.4849} = 1.2438. \]

As expected, the total length has increased, but, to the contrary, the fractal dimension has decreased from 1.4649 to 1.2438. It is obvious, then, that this first approach is not general enough to handle all cases of segment reduction and enlargement.

To solve the problem of scaling fractal segments both up and down, the author proposes the following equation:

\[ D_f = \frac{\ln ((\Sigma k_i)/s)}{\ln x} \quad \text{[4]} \]

where \( s = 1/x \) is the scale factor, and \( k_i \) is the length of each segment in the generator curve. For example, in Figure 14a, where \( 1/x = 1/3 \), all \( k_i \) values are equal to 1/3 since no segment has been shortened or lengthened.

Therefore,

\[ \Sigma k_i = 1/3 + 1/3 + 1/3 + 1/3 + 1/3 = 5/3, \quad \text{and} \]

\[ \Sigma k_i = 5/3. \]
FIGURE 14. Two Segment Length Reduction
\[
\frac{\ln \left( \sum k_i/s \right)}{\ln x} = \frac{\ln \left( \frac{5}{3}/(1/3) \right)}{\ln 3} = D_f = 1.4649.
\]

Reducing the length of the two vertical segments (2&4) to 1/6 as shown in Figure 14b, yields a generator curve with three segments (1,3,5) that are equivalent in length to the initiator segments, and two segments (2&4) that are half the length of the initiator segments. Therefore, the \( k_i \) values are 1/3, 1/6, 1/3, 1/6, and 1/3. Then

\[
\sum k_i = \frac{1}{3} + \frac{1}{6} + \frac{1}{3} + \frac{1}{6} + \frac{1}{3} = \frac{4}{3}, \quad \text{and}
\]

\[
D_f = \frac{\ln \left( \sum k_i/s \right)}{\ln x} = \frac{\ln \left( \frac{4}{3}/(1/3) \right)}{\ln 3} = 1.2619.
\]

Continuing the use of equation [4] and further reducing the length of segments \( k_2 \) & \( k_4 \) to 1/12, (Figure 14c), the \( k_i \) values are 1/3, 1/12, 1/3, 1/12, 1/3.

\[
\sum k_i = \frac{1}{3} + \frac{1}{12} + \frac{1}{3} + \frac{1}{12} + \frac{1}{3} = \frac{7}{6}, \quad \text{and}
\]

\[
D_f = \frac{\ln \left( \sum k_i/s \right)}{\ln x} = \frac{\ln \left( \frac{7}{6}/(1/3) \right)}{\ln 3} = 1.1403.
\]

Figure 14d shows that if \( k_2 = k_4 = 1/24 \), then

\[
\sum k_i = \frac{1}{3} + \frac{1}{24} + \frac{1}{3} + \frac{1}{24} + \frac{1}{3} = \frac{13}{12}, \quad \text{and}
\]
Examining the extreme case of segment reduction implies that segments \( k_2 = k_4 = 0 \). This is essentially the initiator curve. In this case, 
\[
\Sigma k_i = \frac{1}{3} + 0 + \frac{1}{3} + 0 + \frac{1}{3} = 1.00, \text{ and }
\]
\[
\frac{\ln \left( \frac{\Sigma k_i}{s} \right)}{\ln x} = \frac{\ln \left( \frac{1}{(1/3)} \right)}{\ln 3} = 1.0000,
\]
as expected.

To evaluate equation [4] in describing the \( D_f \) of fractals where segments have been enlarged, relative to the initiator segments, look at Figure 15. As usual, in Figure 15a, the normal "all segments equal in length" configuration is shown, producing \( D_f = 1.4649 \). In Figure 15b, segments 2 & 4 have been lengthened by 25% or 1.25 times the length of the initiator segments. Therefore, segments 2 & 4 have a \( k_i \) value of \( 5/12 \), and
\[
\Sigma k_i = \frac{1}{3} + \frac{5}{12} + \frac{1}{3} + \frac{5}{12} + \frac{1}{3} = \frac{11}{6}, \text{ and }
\]
\[
\frac{\ln \left( \frac{\Sigma k_i}{s} \right)}{\ln x} = \frac{\ln \left( \frac{(11/6)}{(1/3)} \right)}{\ln 3} = 1.5517.
\]
All Segments
$k$ value = 1/3
$D_f = 1.4649$

Segments 2 & 4
$k$ value = 5/12
$D_f = 1.5517$

Segments 2 & 4
$k$ value = 1/2
$D_f = 1.6309$

FIGURE 15. Two Segment Length Enlargement
Figure 15c shows segments 2 & 4 lengthened by 50% or 1.50 times the length of an initiator segment. Therefore, segments 1 to 5 in the Figure 15c generator have \( k_i \) values of 1/3, 1/2, 1/3, 1/2, and 1/3, respectively. Then,

\[
\sum k_i = \frac{1}{3} + \frac{1}{2} + \frac{1}{3} + \frac{1}{2} + \frac{1}{3} \approx 2.0
\]

And so it seems that equation [4] does define the \( D_f \) for segments that vary in length, both longer and shorter than the initiator segment length.

To exercise equation [4], the new definition of \( D_f \), consider a fractal similar to those in Figures 14 and 15. Instead of enlarging or reducing both segment lengths, however, enlarge segment 2 by 50% and reduce segment 4 by 50%, which means \( k_2 = \frac{1}{2} \) and \( k_4 = \frac{1}{6} \). As Figure 16 shows, imposing these length constraints on segments 2 & 4 allows for a variety of geometric interpretations. The interpretation depends upon whether the varying segments are allowed to assume angles other than vertical, and whether segment 3 is allowed to assume a \( k_i \) value other than 1/3. The first case in Figure 16a, holds segments 2 & 4 to be vertical and allows segment 3 to be lengthened. The \( D_f \) calculation for this example is

\[
\sum k_i = \frac{1}{3} + \frac{1}{2} + 0.4708 + \frac{1}{6} + \frac{1}{3} \approx 1.8041
\]
FIGURE 16. Two Segment Length Variations
\[ D_f = \frac{\ln \left( \frac{\sum k_i}{s} \right)}{\ln x} = \frac{\ln \left( \frac{1.8041}{1/3} \right)}{\ln 3} = 1.5371. \]

Figure 16b shows the situation where the angle between segments 1 & 2 is greater than 180 degrees, segment 3 is constrained so that \( k_3 = 1/3 \), and segment 4 is held vertical. The resulting fractal dimension calculations are

\[ \sum k_i = \frac{1}{3} + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} + \frac{1}{3} = \frac{5}{3}, \]

and

\[ D_f = \frac{\ln \left( \frac{\sum k_i}{s} \right)}{\ln x} = \frac{\ln \left( \frac{5/3}{1/3} \right)}{\ln 3} = 1.4649, \]

which, coincidentally, is the same \( D_f \) for the conditions in Figure 15a, although the geometry is very different.

The last example, Figure 16c, of a fractal where segments 2 & 4 are held to \( k_i \) values of \( \frac{1}{2} \) and \( \frac{1}{6} \) respectively, operates under the conditions that segment 2 remain vertical, the \( k_i \) value of segment 3 is allowed to vary, and segment 4 is not required to be vertical. To produce a fractal dimension,

\[ \sum k_i = \frac{1}{3} + \frac{1}{2} + 0.4333 + \frac{1}{6} + \frac{1}{3} = 1.7666, \]

and

\[ D_f = \frac{\ln \left( \frac{\sum k_i}{s} \right)}{\ln x} = \frac{\ln \left( \frac{1.766}{1/3} \right)}{\ln 3} = 1.5174. \]
To further test the general nature of equation [4], the equation is applied to fractal dimension calculations that involve the familiar Koch curve. Figure 17 contains the basic "all segments equal to 1/3, all $k_i$ values equal 1/3" case, as well as several situations with varying segment lengths, expressed as varying $k_i$ values.

The first variation involves reducing the length of segment 3 by 50%, thereby setting $k_3 = 1/6$ while all other segments, 1, 2, and 4, retain a $k_i$ value of 1/3. The calculations for Figure 17b are as follows

$$\sum_{i} k_i = 1/3 + 1/3 + 1/6 + 1/3 = 7/6,$$

$$D_f = \frac{\ln \left( \frac{\sum k_i}{s} \right)}{\ln x} = \frac{\ln \left( \frac{7/6}{1/3} \right)}{\ln 3} = 1.1403.$$ 

The next Figure, 17c, depicts the reduction of both segments 2 & 3 by 25%, for $k_i$ values of 1/4. The sum

$$\sum_{i} k_i = 1/3 + 1/4 + 1/4 + 1/3 = 7/6,$$

$$D_f = \frac{\ln \left( \frac{\sum k_i}{s} \right)}{\ln x} = \frac{\ln \left( \frac{7/6}{1/3} \right)}{\ln 3} = 1.1403.$$ 

Notice that if segments 2 & 3 are allowed to be reduced 50% and have $k_i$ values of 1/6, the graphical representation degenerates to a straight line equal to the initiator curve, and $D_f = 1.0$ as expected.
FIGURE 17. Koch Curve Generators with Arbitrary Segment Lengths

All Segments
k values = 1/3
Df = 1.2618

Segment 3
k value = 1/6
Df = 1.1403

Segments 2 & 3
k values = 1/4
Df = 1.1403

Segment 2
k value = 5/12
Df = 1.3170

Segments 2 & 3
k values = 1/2
Df = 1.4649
The third variation, Figure 17d, lengthens segment 2 by 25% giving a $k_i$ value of $5/12$. Then
\[
\sum_{i} k_i = 1/3 + 5/12 + 1/3 + 1/3 = 17/12, \text{ and}
\]
\[
D_f = \frac{\ln \left( \frac{\sum k_i}{s} \right)}{\ln x} = \frac{\ln \left( \frac{17/12}{1/3} \right)}{\ln 3} = 1.3170.
\]

The last illustration, 17e, addresses the configuration where both segments 2 & 3 are lengthened by 50% yielding $k_2 = k_3 = 1/2$. To calculate $D_f$,
\[
\sum_{i} k_i = 1/3 + 1/2 + 1/2 + 1/3 = 5/3, \text{ and}
\]
\[
D_f = \frac{\ln \left( \frac{\sum k_i}{s} \right)}{\ln x} = \frac{\ln \left( \frac{5/3}{1/3} \right)}{\ln 3} = 1.4649.
\]

In the extreme cases of segment enlargement and reduction, the curve tends to either cross itself or be reduced to Cantor Dust, which was described earlier. Although dependent upon the shape of the particular generator, this "self-crossing" takes place long before the $D_f$ reaches 2.0000. Curves of this extreme segment enlargement prove to be at least uninteresting.

To demonstrate the effect of unequal segment lengths, consider the multiple iterations of the fractal shown in Figure 18. This fractal is generated by the constraints imposed upon the familiar Koch curve as shown in Figure 17b. Compare the iterations in Figure 18 with the standard, symmetrical, iterations in Figure 7.
FIGURE 18. Koch Curve Iterations with Arbitrary Segment Lengths
Notice how the unequal segment fractal looks more like a natural structure at an earlier stage of iteration, compared to the more formal structure of the classic Koch curve. It is in this closer approximation of nature, that the "arbitrary segment length" approach finds value. Figure 19 shows the generator, the $D_f$, and the fourth iterate of several arbitrary segment length fractals. Figure 20 shows the fourth iterate of several more Koch fractals that have arbitrary length segments.
FIGURE 19. Fractals with Arbitrary Segment Lengths
FIGURE 20. Fractals with Arbitrary Segment Lengths
VI. FUTURE WORK AND DISCUSSION

Presented to this point is how fractionally dimensioned objects can be generated, represented by the one-plus dimension of the Dragon or Monster curves, and sets of them concentrated into the Julia, Cantor, and Mandelbrot Sets. In these cases, the topological dimension, \( D_t \), is one, and the Hausdorff or Fractal dimension, \( D_f \), is greater than \( D_t \). A special case of curves are the space-filling curves, where eventually, \( D_f = D_t = 2 \). However, these fractals generally have dimensions ranging from 1 to 2 and are viewed in a 2-D plane.

Now consider the fractal and space-filling entities whose dimensions range from 2 to 3 and are viewed in a 3-D volume. An aid to visualization is a 2-D piece of aluminum foil. As it becomes more and more wrinkled, its Hausdorff dimension approaches 3 without reaching it (except in the case of volume-filling surfaces where, similar to the space-filling curves, \( D_f = D_t = 3 \)). An example of a 2-3D fractal is a landscape which resembles a mountain range (Mandelbrot, 1983).

Extrapolating this idea to include fractals ranging in dimension from 3 to 4 is the work of Alan Norton, also of the IBM Watson Research Center. Since these four dimensional entities cannot be viewed directly, viewing is limited to two or three dimensional projection slices. To extend our number representation, a number system is employed that was developed by Sir William Rowan Hamilton (1805-1865), and known as the Quaternion system. In the same way that complex numbers are pairs of real numbers, Quaternions are quadruples
of real numbers. For example,

\[
\text{Complex Number} = A + Bi
\]
\[
\text{Quaternion} = A + Bi + Cj + Dk
\]

Where \(i\), \(j\), and \(k\) are all unit vectors in three orthogonal directions, normal to the real axis, but not to each other. Addition and multiplication of quaternions is handled as you would polynomials in \(i\), \(j\), and \(k\). However, quaternion algebra has some unique properties for products:

1. \(X \cdot Y \neq Y \cdot X\) (no commutation)
2. \(i^2 = j^2 = k^2 = -1\) (extension of \(i^2\))
3. \(ij = -ji = k,\quad jk = -kj = i,\quad ki = -ik = j\)
4. \(ji = -k,\quad kj = -i,\quad ik = -j\)

Assuming for the moment that instead of plotting numbers in the two dimensional complex plane to produce a two dimensional fractal, the goal is to plot in the four dimensional quaternion realm. Obviously one cannot plot along four axes that are orthogonal. One approach is to use the Einsteinian notion of the fourth dimension being time. This approach yields three orthogonal axes plus time. Then plot a three dimensional point along the \(x\), \(y\), and \(z\) axes and note the time of the plotting as the fourth dimension. Unfortunately, the only way to get a glimpse of this 4-d monster is to review an animation of the time dependent plotting. The computational load imposed by such an animation would be enormous (Pickover, 1987; Sorenson, 1984; Stein, 1983). Future work to be done as an extension of this thesis is to expand the arbitrary segment length idea to 2-3D fractals.


VITA

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