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Ming Xin

S. N. Balakrishnan
Missouri University of Science and Technology, bala@mst.edu

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A NEW FILTERING TECHNIQUE FOR A CLASS OF NONLINEAR SYSTEMS

Ming Xin*, S.N. Balakrishnan**
xin@umr.edu , bala@umr.edu
Department of Mechanical and Aerospace Engineering and Engineering Mechanics
University of Missouri-Rolla, Rolla, MO 65401
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Abstract

In this paper, a new nonlinear filtering technique (θ -D filter) is presented. This filter is derived by constructing the dual of a new nonlinear regulator control technique, $\theta - D$ approximation which involves approximate solution to the Hamilton Jacobi Bellman (HJB) equation. The structure of this filter is similar to the State Dependent Riccati Equation Filter (SDREF). However, this method does not need time-consuming online computation of the algebraic Riccati equation at each sample time compared with the SDREF. By manipulating the perturbation terms both asymptotic stability and optimality properties can be obtained. A simple pendulum problem is investigated to demonstrate the effectiveness of this new technique.

1. INTRODUCTION

Nonlinear filtering theory has attracted considerable attentions in many fields. The most popular method is the Extended Kalman Filter [1]. Although this procedure generally yields satisfactory results, there is documented evidence of erratic filter behavior such as premature collapse of the error covariance [2] and lost of observability [3].

Another recently emerging filtering technique is the State Dependent Riccati Equation Filter (SDREF). The SDREF is formulated by constructing the dual of the State Dependent Riccati Equation (SDRE) based nonlinear regulator control design technique [4] which systematically solves the nonlinear regulator problem. The resulting SDREF has the same structure as the continuous steady-state linear Kalman filter. It is based on a paramerization that brings the nonlinear system to a linear structure having state dependent coefficients. This method creates additional degrees of freedom that can be used to overcome the limitations in a traditional filtering method [3]. However, this filter is difficult to implement particularly in higher order systems since it needs to solve the algebraic Riccati Equation online. In [5], Haessig and Frieland have given an application of SDREF to the simultaneous state and parameter estimation. But using this formulation one still needs to address the online computation load problem.

In [6], Shue and Agarwal et.al. have also derived another extension of the Kalman filter by taking the dual of the linear quadratic regulator (LQR) theory. The difference is that they have formulated the filter from the Hamilton Jacobi Bellman Inequality (HJBI). By adding a positive definite matrix, it converted HJBI into a set of equalities. These equations simplified the computations and can be solved off-line. But finding the positive definite matrix that makes HJBI an equality is not clear.

In this paper, a new nonlinear filter technique (θ -D filter) is proposed. It is a dual version of a new suboptimal nonlinear controller design method, $\theta - D$ approximation (submitted to this conference). This new filter takes the same structure as the linear Kalman filter but the filter gain is state-dependent. Unlike the SDREF, this filter does not need online solution of the Riccati equation. It can obtain an approximately analytical solution to the state dependent Riccati equation. In addition, by manipulating a set of perturbation terms appropriately, both stability and good performance can be achieved. In Section 2, the formulation of the θ -D controller design method is described. The dual $\theta - D$ filter is presented in Section 3. In Section 4, a two dimensional pendulum problem is studied and comparison is made between the SDREF and the $\theta - D$ filter. Conclusions are given in section 5.

2. SUBOPTIMAL CONTROL OF A CLASS OF NONLINEAR SYSTEMS, $\theta - D$ APPROXIMATION TECHNIQUE

Consider the state feedback control problem for the class of nonlinear time-invariant systems described by

$$\dot{x} = f(x) + B(x)u \tag{1}$$

with the cost function:

$$J = \frac{1}{2}\int_0^\infty [x^T Q x + u^T R u]dt \tag{2}$$

where $x \in \Omega \subset R^n, f:\Omega \to R^n, B \in R^{n\times m}, u:\Omega \to R^m, Q \in R^{n\times n}, R \in R^{m\times m}$. Q is semi-definite matrix and R is positive definite matrix; g is a constant matrix and f(0)=0.

To ensure that the control problem is well posed we assume that a solution to the optimal control problem (1), (2) exists.

* Ph.D student, ** Professor, contact person

We also assume that $f(x)$ is locally Lipschitz in x on a set Ω and zero state observable through Q.

The optimal solution of the infinite-horizon nonlinear regulator problem can be obtained by solving the Hamilton-Jacobi-Bellman (HJB) partial differential equation [7]:

$$\frac{\partial V^T}{\partial x} f(x) - \frac{1}{2}\frac{\partial V^T}{\partial x} B(x)R^{-1}B(x)^T \frac{\partial V}{\partial x} + \frac{1}{2}x^T Q x = 0 \quad (3)$$

with $\quad V(0) = 0$

The optimal control is given by

$$u = -R^{-1}B^T \frac{\partial V}{\partial x} \quad (4)$$

and V(x) is the optimal cost , i.e.

$$V(x) = \min_u \frac{1}{2}\int_0^{\infty} (x^T Q x + u^T R u)\, dt \quad (5)$$

The HJB equation is extremely difficult to solve in general, rendering optimal control techniques of limited use for nonlinear systems.

Now consider perturbations added to the cost function:

$$J = \frac{1}{2}\int_0^{\infty} [x^T(Q + \sum_{i=1}^{\infty} D_i\theta^i)x + u^T R u]dt \quad (6)$$

where θ and D_i are chosen such that $\left\|\sum_{i=1}^{\infty} D_i\theta^i\right\|_2$ is a small number compared to $\|Q\|_2$.

Write the original state equation as:

$$\dot{x} = f(x) + B(x)u = [A_0 + \theta(\frac{A(x)}{\theta})]x + (g_0 + \theta\frac{g(x)}{\theta})u \quad (7)$$

where A_0 is a constant matrix such that (A_0, g_0) is a stabilizable pair.

Define $\quad \lambda = \frac{\partial V}{\partial x} \quad (8)$

By using (8) in (3), we have

$$\lambda^T f(x) - \frac{1}{2}\lambda^T B(x)R^{-1}B(x)^T\lambda + \frac{1}{2}x^T(Q + \sum_{i=1}^{\infty} D_i\theta^i)x = 0 \quad (9)$$

Assume a power series expansion of λ as

$$\lambda = \sum_{i=0}^{\infty} T_i(x)\theta^i x \quad (10)$$

Here we assume that T_i s are symmetric.

Substitute (7) and (10) into the HJB equation (3) and equate the coefficients of powers of θ to zero to get the following equations:

$$T_0A_0 + A_0^T T_0 - T_0 g_0 R^{-1} g_0^T T_0 + Q = 0 \quad (11)$$

$$T_1(A_0 - g_0R^{-1}g_0^T T_0) + (A_0^T - T_0 g_0 R^{-1} g_0^T)T_1 = -\frac{T_0A(x)}{\theta} - \frac{A^T(x)T_0}{\theta} + T_0 g_0 R^{-1}\frac{g^T}{\theta}T_0 + T_0\frac{g}{\theta}R^{-1}g_0^T T_0 - D_1 \quad (12)$$

$$T_2(A_0 - g_0R^{-1}g_0^T T_0) + (A_0^T - T_0 g_0 R^{-1} g_0^T)T_2 = -\frac{T_1A(x)}{\theta} - \frac{A^T(x)T_1}{\theta} + T_0 g_0 R^{-1}\frac{g^T}{\theta}T_1 + T_0\frac{g}{\theta}R^{-1}g_0^T T_1 + T_0\frac{g}{\theta}R^{-1}\frac{g^T}{\theta}T_0 + T_1 g_0 R^{-1} g_0^T T_1 + T_1 g_0 R^{-1}\frac{g^T}{\theta}T_0 + T_1\frac{g}{\theta}R^{-1}g_0^T T_0 - D_2 \quad (13)$$

$$\vdots$$

$$T_n(A_0 - g_0R^{-1}g_0^T T_0) + (A_0^T - T_0 g_0 R^{-1} g_0^T)T_n = -\frac{T_{n-1}A(x)}{\theta} - \frac{A^T(x)T_{n-1}}{\theta} + \sum_{j=0}^{n-1} T_j(g_0R^{-1}\frac{g^T}{\theta} + \frac{g}{\theta}R^{-1}g_0^T)T_{n-1-j} + \sum_{j=0}^{n-2} T_j g R^{-1} g^T T_{n-2-j} + \sum_{j=1}^{n-1} T_j g_0 R^{-1} g_0^T T_{n-j} - D_n \quad (14)$$

The expression for control can be obtained in terms of the power series for λ as

$$u = -R^{-1}B(x)^T\lambda = -R^{-1}B(x)^T\sum_{i=0}^{\infty} T_i(x)\theta^i x \quad (15)$$

It is easy to find that the equation (11) is an algebraic Riccati equation. The rest of equations are Lyapunov equations which are *linear* in terms of T_0. In the rest of this paper we will call it the θ-D approximation technique. The algorithm without D_i term is called the θ approximation. The algorithm [8] would result in the θ approximation. One of the problems with θ approximation is that large initial conditions may give rise to large control. In order to deal with this problem, we construct the following expression for D_i:

$$D_1 = k_1 e^{-l_1 t}[-\frac{T_0A(x)}{\theta} - \frac{A^T(x)T_0}{\theta}] \quad (16)$$

$$D_2 = k_2 e^{-l_2 t}[-\frac{T_1A(x)}{\theta} - \frac{A^T(x)T_1}{\theta}] \quad (17)$$

$$\vdots$$

$$D_n = k_n e^{-l_n t}[-\frac{T_{n-1}A(x)}{\theta} - \frac{A^T(x)T_{n-1}}{\theta}] \quad (18)$$

The idea in constructing D_i in this manner is that large control results from the state dependent term A(x) on the right hand side of the equations (11)-(14). So we choose D_i such that

$$-\frac{T_{i-1}A(x)}{\theta} - \frac{A^T(x)T_{i-1}}{\theta} - D_i = \varepsilon_i[-\frac{T_{i-1}A(x)}{\theta} - \frac{A^T(x)T_{i-1}}{\theta}] \quad (19)$$

where ε_i is a small number chosen to satisfy some conditions required in the proof of convergence and stability of the above algorithm. On the other hand, the exponential term $e^{-l_i t}$ is used to let the perturbation terms in the cost function and HJB equation diminish as the time evolves. This will guarantee the HJB equations to be solved asymptotically. Since the solution of HJB equation is the sufficient condition for optimality, the suboptimal control (15) will approximate the optimal control asymptotically. In $\theta - D$ technique, D_i terms play a crucial role in both guaranteeing stability and modulating the performance.

3. FORMULATION OF $\theta - D$ FILTER

Considering that the steady state linear Kalman filter is the dual of the steady state linear regulator, we can obtain the filtering counterpart of the $\theta - D$ controller by taking the dual of the coeffient matrices.

Consider the stochastic nonlinear system

$$\dot{x} = f(x) + \Gamma w \quad (20)$$

$$y = h(x) + v \quad (21)$$

where w is a Gaussian zero-mean white process noise with $E[w(t)w^T(t+\tau)] = W(t)\delta(\tau)$ and v is a Gaussian zero-mean white measurement noise with $E[v(t)v^T(t+\tau)] = V(t)\delta(\tau)$. The W(t) and V(t) are power spectral densities.

In the SDREF formulation, it brings the original system

(20),(21) to the state dependent coefficient form:

$$\dot{x} = F(x)x + \Gamma w \tag{22}$$
$$y = H(x)x + v$$

The SDREF is given by

$$\dot{\hat{x}} = F(\hat{x})\hat{x} + K_f(\hat{x})[y(x) - H(\hat{x})\hat{x}] \tag{23}$$

where $K_f(\hat{x}) = P(\hat{x})H^T(\hat{x})V^{-1}$ (24)

and P is the positive definite solution to

$$F(\hat{x})P(\hat{x}) + P(\hat{x})F^T(\hat{x}) - P(\hat{x})H^T(\hat{x})V^{-1}H(\hat{x})P(\hat{x}) + \Gamma^T W \Gamma = 0 \tag{25}$$

Now in $\theta - D$ method, if we consider (10) and assume

$$\lambda = \sum_{i=0}^{\infty} T_i(x)\theta^i x = P(x)x \tag{26}$$

we can obtain the state dependent Riccati equation by substituting (26) into the HJB equation (3):

$$A^T(x)P(x) + P(x)A(x) - P(x)B(x)R^{-1}B^T(x)P(x) + Q = 0 \tag{27}$$

So the solution to $\theta - D$ method $\sum_{i=0}^{\infty} T_i(x)\theta^i$ can be used as the solution to the state dependent Riccati equation.

Rewrite (20) (21) as:

$$\dot{x} = f(x) + \Gamma w = [A_0 + \theta(\frac{A(x)}{\theta})]x + \Gamma w \tag{28}$$

$$y = (H_0 + \theta\frac{H(x)}{\theta})x + v \tag{29}$$

The $\theta - D$ filter is given by

$$\dot{\hat{x}} = (A_0 + \theta\frac{A(\hat{x})}{\theta})\hat{x} + K_f(\hat{x})[y(x) - (H_0 + \theta\frac{H(\hat{x})}{\theta})\hat{x}] \tag{30}$$

where $K_f(\hat{x}) = P(\hat{x})(H_0^T + H^T(\hat{x}))V^{-1}$ (31)

$$P(\hat{x}) = \sum_{i=0}^{\infty} T_i(\hat{x})\theta^i \tag{32}$$

and $T_i(\hat{x})$ is the solution to the following equations (33)-(36) in which we take the dual of $A_0,\ A(x),\ g_0,\ and\ g(x)$ in (11)-(14), i.e. $A_0^T, A^T(\hat{x}), H_0^T, and\ H^T(\hat{x})$.

$$T_0A_0^T + A_0T_0 - T_0H_0^TR^{-1}H_0T_0 + Q = 0 \tag{33}$$

$$T_1(A_0^T - H_0^TR^{-1}H_0T_0) + (A_0 - T_0H_0^TR^{-1}H_0)T_1 = -\frac{T_0A(\hat{x})}{\theta} - \frac{A^T(\hat{x})T_0}{\theta} + T_0H^TR^{-1}\frac{H}{\theta}T_0 + T_0\frac{H^T}{\theta}R^{-1}H_0T_0 - D_1 \tag{34}$$

$$T_2(A_0^T - H_0^TR^{-1}H_0T_0) + (A_0 - T_0H_0^TR^{-1}H_0)T_2 = -\frac{T_1A(\hat{x})}{\theta} - \frac{A^T(\hat{x})T_1}{\theta} + T_0H_0^TR^{-1}\frac{H}{\theta}T_1 + T_0\frac{H^T}{\theta}R^{-1}H_0T_1 + T_0\frac{H^T}{\theta}R^{-1}\frac{H}{\theta}T_0 + T_1H_0^TR^{-1}H_0T_1 + T_1H_0^TR^{-1}\frac{H}{\theta}T_0 + T_1\frac{H^T}{\theta}R^{-1}H_0T_0 - D_2 \tag{35}$$

$$\vdots$$

$$T_n(A_0^T - H_0^TR^{-1}H_0T_0) + (A_0 - T_0H_0^TR^{-1}H_0)T_n = -\frac{T_{n-1}A(\hat{x})}{\theta} - \frac{A^T(\hat{x})T_{n-1}}{\theta} + \sum_{j=0}^{n-1}T_j(H_0^TR^{-1}\frac{H}{\theta} + \frac{H^T}{\theta}R^{-1}H_0)T_{n-1-j} + \sum_{j=0}^{n-2}T_jH^TR^{-1}HT_{n-2-j} + \sum_{j=1}^{n-1}T_jH_0^TR^{-1}H_0T_{n-j} - D_n \tag{36}$$

4. ILLUSTRATIVE EXAMPLE [3]

In order to demonstrate the potential of the $\theta - D$ filter, we applied it to a two dimensional pendulum problem and compare it with the SDREF approach. The pendulum model has two state variables: $x = [\theta, \dot{\theta}]^T$, where θ and $\dot{\theta}$ are the angular position and angular rate of the pendulum respectively. The dynamic of the pendulum is given by

$$\dot{x}_1 = x_2 \tag{37}$$

$$\dot{x}_2 = -\frac{g}{L}\sin(x_1) \tag{38}$$

The measurement is the angular acceleration:

$$y = -\frac{g}{L}\sin(x_1) \tag{39}$$

The nonlinear vector f(x) is partitioned as

$$A_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad A(\hat{x}) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\frac{\sin(\hat{x}_1)}{\hat{x}_1} - 1 & 0 \end{bmatrix} \tag{40}$$

$$H_0 = [0 \quad 1], \quad H(\hat{x}) = \begin{bmatrix} -\frac{g}{L}\frac{\sin(\hat{x}_1)}{\hat{x}_1} & -1 \end{bmatrix} \tag{41}$$

The parameterization chosen for the SDREF is [3]

$$F(\hat{x}) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\frac{\sin(\hat{x}_1)}{\hat{x}_1} & 0 \end{bmatrix} \quad H(\hat{x}) = \begin{bmatrix} -\frac{g}{L}\frac{\sin(\hat{x}_1)}{\hat{x}_1} & -1 \end{bmatrix} \tag{42}$$

Remark:

In the case that $H(x) = \frac{h(x)}{x}$ is not regular at x=0, we can take the same method as that in [9]. For example, consider $h(x) = \cos(x)$, it can be written as $\cos(x) - 1 + 1 = \left[\frac{\cos(x) - 1}{x}\right]x + \frac{1}{z}z$ in which z is an instrumental variable satisfying $\dot{z} = -\lambda z$ with $\lambda > 0$. Now $H(x) = \frac{\cos(x) - 1}{x}$ is well behaved. For details on such manipulations, please refer to [9].

In the simulation, the parameters chosen for the pendulum problem are

g=32.2ft/sec^2, L=1ft.

The measurement noise power spectral intensity:

V=2 rad^2/s^4

The process noise intensity is selected to be

$$\Gamma^T W \Gamma = \begin{bmatrix} G_{11} & 0 \\ 0 & G_{22} \end{bmatrix} = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.05 \end{bmatrix} \tag{43}$$

The simulation was performed using a fourth order Runge-Kutta scheme with a fixed step size of dt=0.005sec. The initial position is 1rad and initial angular rate is 0.

Figure 1 shows the estimations of position, position rate and their errors compared with the actual states. The initial states are assumed to be the same with the actual states, i.e. x(0)=[1,0]. The D matrices are chosen as

$$D_1 = diag\{1.00001e^{-0.2t}[-\frac{T_0A^T(\hat{x})}{\theta} - \frac{A(\hat{x})T_0}{\theta}], 1.00001e^{-t}[-\frac{T_1A^T(\hat{x})}{\theta} - \frac{A(\hat{x})T_1}{\theta}] \tag{44}$$

$$D_2 = diag\{1.00001e^{-0.1t}[-\frac{T_1A^T(\hat{x})}{\theta} - \frac{A(\hat{x})T_1}{\theta}], 1.00001e^{-t}[-\frac{T_1A^T(\hat{x})}{\theta} - \frac{A(\hat{x})T_1}{\theta}] \tag{45}$$

The construction of D_i in (16)-(19) serves two functions. One of them is to provide an appropriate ε_i to guarantee the convergence of power series expansion $\sum_{i=0}^{\infty} T_i(x,\theta)\theta^i$ and stability of the closed loop system. They are shown in another paper to be presented in this conference [10]. The other purpose is to modulate the system transient performance by adjusting the parameters of k_i and l_i in the D_i.

We only pick the first three terms, e.g. up to T_2 terms in the λ expansion. One can use more terms if needed. Usually three terms are enough for a good approximation. In Figure 2, the estimation of SDREF is shown. By comparison, the position error for both SDREF and $\theta - D$ filter are in the same range. However, $\theta - D$ filter achieves less error in the angular rate estimation. Figure 3 and Figure 4 show the SDRE estimate and $\theta - D$ estimate respectively when initial state estimates are quite different from the actual initial states, i.e. $\hat{x}(0) = [0,1]$. We can see that the position estimate of $\theta - D$ filter can track the actual states very quickly and keep the error small while SDRE filter demonstrates bigger oscillations in the position error plot.

As for the implementation, the θ-D algorithm needs a matrix inverse operation only one time *offline* when solving the linear Lyapunov equations (33)-(36) and solution to the first algebraic Riccati equation (33) only one time, *offline*. That is to say, when solving (34)-(36), we only need to rearrange the left hand side of the equations such that they form a linear matrix equation: $\hat{A}_0 T_n = Q_n(\hat{x},t)$ and then $T_n = \hat{A}_0^{-1} Q_n(\hat{x},t)$ where $\hat{A}_0$ is a constant matrix and $Q_n(\hat{x},t)$ is the right hand side of (34)-(36). When implemented online, this method involves only two 2×2 matrix multiplications and three 2×2 matrix additions if we take three terms. However, in comparison, SDRE needs computation of the 2×2 algebraic Riccati equation at *each* sample time. Note that the number of computations will become excessive if we want to solve higher order problems. For example, in the three dimensional homing missile guidance problem a 9×9 algebraic Riccati equation has to be solved on-line when implementing the SDRE. In an aircraft control design, it demands fast computation of control laws based on estimated states. In such application, the SDRE method is computationally intensive in these practical problems. By contrast, the closed-form controller of the θ-D method needs much less computations and is easy to implement in this real-time applications.

5. CONCLUSIONS

In this paper, a new nonlinear filter technique was proposed. This method takes the dual of the $\theta - D$ controller synthesis. The recursive solution of this filter does not need complex on-line computation compared to SDRE technique. In addition, by manipulating the perturbation terms appropriately, we can achieve both stability and good performance. An illustrative pendulum problem was investigated to compare the $\theta - D$ filter with the SDRE filter. The $\theta - D$ filter demonstrates better result. This technique can be applied to a broad class of nonlinear filter problem.

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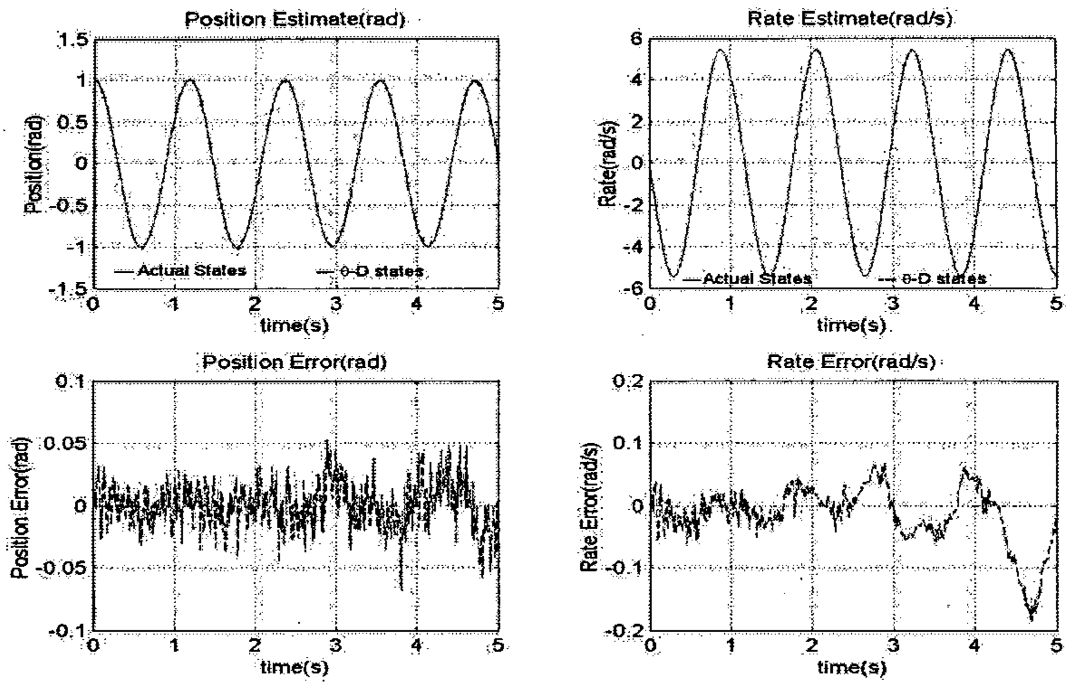

Figure 1: $\theta - D$ filter with the initial estimated states set to $\hat{x}_0 = x_0$

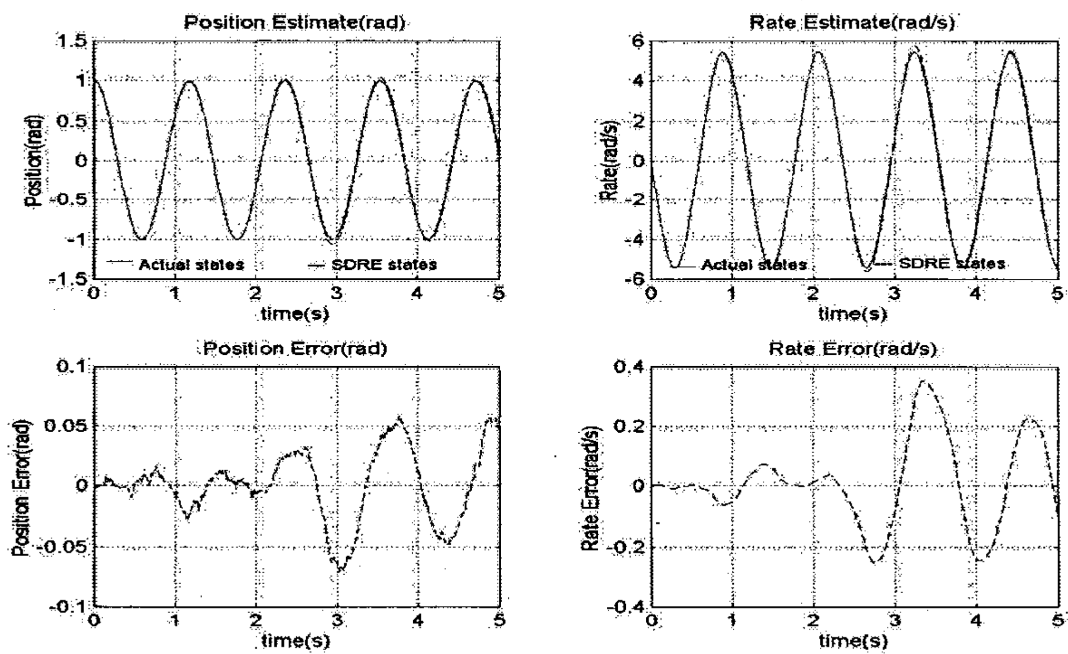

Figure 2: SDRE filter with the initial estimated states set to $\hat{x}_0 = x_0$

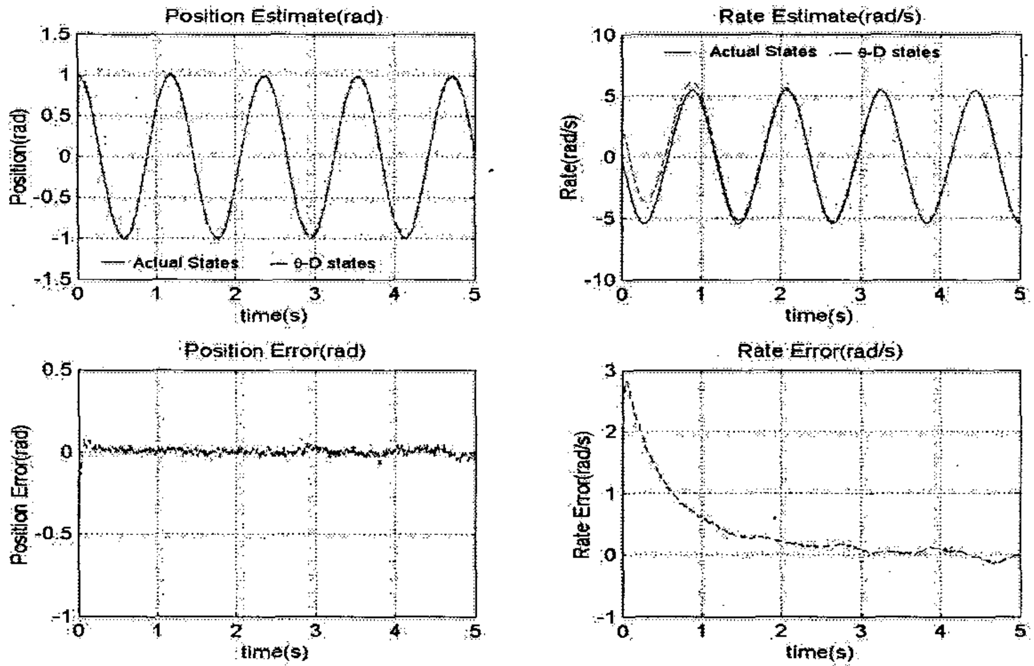

Figure 3: $\theta - D$ filter with initial states: $x(0) = [1, 0]$, $\hat{x}(0) = [0,1]$

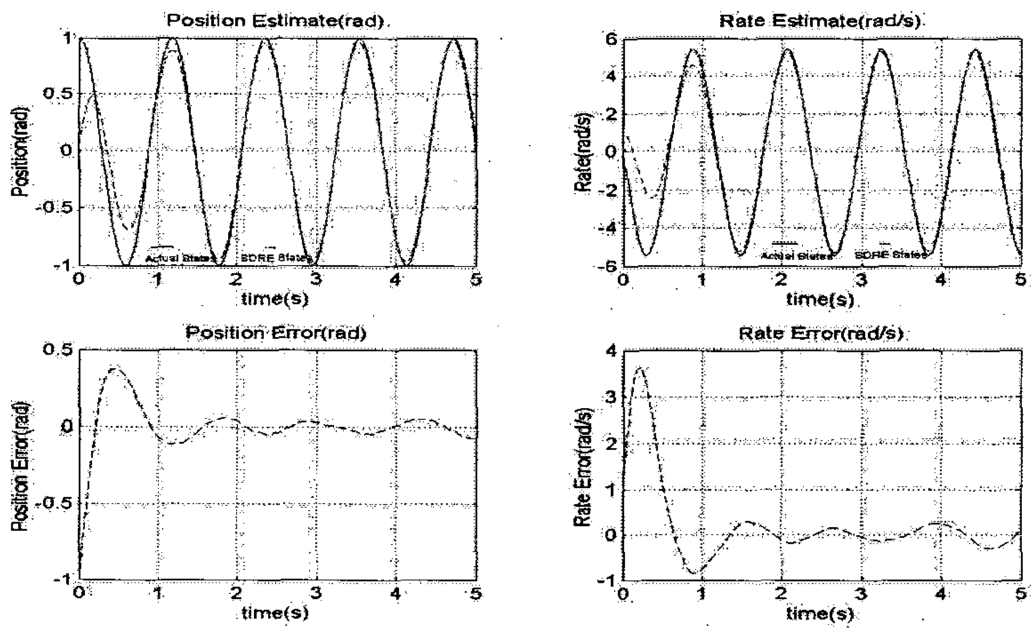

Figure 4: SDRE filter with initial states: $x(0) = [1, 0]$, $\hat{x}(0) = [0,1]$