A study of FMFB demodulator operation in the presence of radio frequency interference and Gaussian noise

Terry D. Wormington

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A STUDY OF FMFB DEMODULATOR OPERATION
IN THE PRESENCE OF RADIO FREQUENCY
INTERFERENCE AND GAUSSIAN NOISE

BY

TERRY D. WORMINGTON, 1949-

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Approved by

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ABSTRACT

A digital computer model of an FMFB demodulator has been used to study the effects of radio frequency interference on the detection of a sinusoidally modulated carrier. The effectiveness of selecting the feedback factor to suppress the interference for three values of modulation index is investigated. Also, the required IF bandwidth for the minimization of the mean-square error of the loop output between a signal corrupted by interference and the same signal without interference was investigated. More work is required before the effects of Gaussian noise plus interference on loop operation can be ascertained.
ACKNOWLEDGEMENTS

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I. INTRODUCTION

The purpose of this thesis is to develop a digital computer model of a frequency modulated feedback (FMFB) demodulator to study the effects of Gaussian noise and continuous wave (CW) interference on demodulator performance. Monte-Carlo simulation is used to obtain a probabilistic approximation of the loop's behavior in noise plus interference. The simulation is accomplished by integrating the loop differential equations using fourth-order Runge-Kutta integration.¹

An FMFB loop is a threshold extension demodulator useful in extracting analog information or a system sub-carrier from a frequency modulated (FM) carrier. The loop locks upon the modulated carrier by using the demodulated output to FM modulate the local oscillator.

Analytical and experimental results for an FMFB loop operating in Gaussian noise are available in the literature.²,³,⁴ In addition, some results are available for the loop operating in CW interference (hereafter referred to as RFI or interference) and multipath backgrounds.⁵ In this thesis, the dependence of loop operation in the presence of RFI and Gaussian noise on the modulation index and loop feedback factor will be ascertained. The performance of the FMFB loop is characterized in terms of the mean-square error (MSE) of the demodulated
signal introduced by the noise and interference.

The following section of this thesis describes the loop operation in the presence of Gaussian noise and includes a discussion of the threshold in the FMFB demodulator. Section III discusses the effects of interference on loop operation. A derivation of the computer model equations is presented in section IV while section V discusses their implementation as a computer simulation. Sections VI and VII present the simulation results and conclusions.

The simulation results show that the MSE decreases as the modulation index of the input and the loop feedback factors are increased. They also verify that the MSE is a function of the intermediate frequency (IF) filter bandwidth.
II. SUMMARY OF FMFB DEMODULATOR PERFORMANCE IN GAUSSIAN BACKGROUNDS

The basic FMFB loop configuration and loop parameters are displayed in Fig. 1. The predetection filter, $H_0$, has a bandwidth of $2(\beta + 1)f_m$ in accordance with Carson's rule,$^5$ where $\beta$ is the modulation index of the input signal and $f_m$ the modulating signal frequency. It is followed by a conventional mixer. The difference frequency term of the mixer output is filtered by $H_{IF}$, the IF filter whose bandwidth is approximately $2(\beta - \hat{\beta} + 1)f_m$ where $\hat{\beta}$ is the modulation index of the reference signal input. The limiter-discriminator demodulates the IF signal and $H_2$, the baseband filter, lowpass filters the resultant. This output signal FM modulates the voltage-controlled oscillator (VCO) which then provides the local reference signal for the mixer.

Let the input to the loop be a combination of signal and Gaussian noise, 

$$y(t) = s(t) + n_g(t)$$

$$= \sqrt{2} \sin (\omega_0 t + \beta \sin \omega_m t) + n_g(t), \quad \text{(II-1)}$$

where $\omega_0$ is the radian carrier frequency, $\beta$ is the modulation index, $\omega_m$ is the radian modulation frequency, and $n_g(t)$ is a white, Gaussian noise component with bandwidth $2(\beta + 1)f_m$ due to the predetection filtering by $H_0$. 
Assume a VCO output signal of the form

\[ v_1(t) = \sqrt{2} \cos(\omega_1 t + \hat{\beta} \sin \omega_m t + \theta_n(t)), \]  

where \( \omega_1 \) is the center VCO radian frequency, 
\( \hat{\beta} \) is the modulation index of the VCO output signal, 
\( \theta_n(t) \) is the phase jitter of the VCO output signal due to the Gaussian noise at the input to the loop. 

If the IF filter passes only the difference-frequency term of the mixer output, its output will be
\[ v_2(t) = \sin \left( (\omega_0 - \omega_1)t + (\beta - \hat{\beta}) \sin \omega_m t - \theta_n(t) + n'_g(t) \right) \]

\[ = \sin \left( \omega_{IF} t + (\beta - \hat{\beta}) \sin \omega_m t - \theta_n(t) \right) + n'_g(t), \]  

(II-3)

where \( \omega_{IF} = (\omega_0 - \omega_1) \) is the IF radian frequency,
(\( \beta - \hat{\beta} \)) is the modulation index of the IF signal,
\( n'_g(t) \) is that portion of the Gaussian noise passed by the IF filter.

Like \( n_g(t) \), \( n'_g(t) \) is also a Gaussian noise process; however, \( n'_g(t) \) can be confined to a bandwidth of 2(\( \beta - \hat{\beta} + 1 \))f_m Hz by the IF filter since, by Carson's rule, this is the effective signal bandwidth of the mixer output. This filter is generally a single-pole filter in order to insure loop stability. However, to avoid distortion, a filter bandwidth slightly greater than that prescribed by Carson's rule is used. Figure 2 illustrates the power spectra for the noise at the predetection filter output and the IF filter output.

The output of the limiter-discriminator is proportional to the derivative of the phase of the signal plus noise at the IF filter output. Thus

\[ v_4(t) = K_d [ (\beta - \hat{\beta}) \omega_m \cos \omega_m t + n_4(t) - \theta_n(t) ], \]  

(II-4)

where \( K_d \) is the discriminator constant in volts/rad/sec,
\( \theta_n(t) \) is the derivative of the phase jitter component, and
\( n_4(t) \) is the output noise component due to \( n'_g(t) \).
which, at high carrier to noise ratios (CNR), is proportional to the component of \( n_g(t) \) in quadrature with the signal.

\[
\text{Figure 2. Spectra of Gaussian noise components at (a) the predetection filter and (b) the IF filter output.}
\]

In order for the output of the VCO to be as specified in Eq. (II-2), its input signal must be of the form

\[
v_4(t) = \frac{1}{K_V} (\hat{\beta}_m \cos \omega_m t + \theta_n(t)), \tag{II-5}
\]

where \( K_V \) is the VCO sensitivity constant in rad/sec/volt. Equating Eq. (II-4) and Eq. (II-5), one obtains

\[
\frac{1}{K_V} [\hat{\beta}_m \cos \omega_m t + \theta'_n(t)] = K_d (\hat{\beta}_m \cos \omega_m t - \theta_n(t)
\]

\[
+ n_4(t)] \tag{II-6}
\]
Setting the noise to zero, the relationship

\[ K \overset{\Delta}{=} K_v K_d = \frac{\beta}{\beta - \beta} \quad (II-7) \]

is obtained, where \( K \) is called the loop feedback factor. As \( K \) is increased, \((\beta - \hat{\beta})\) becomes smaller which indicates that the required bandwidth of the IF filter approaches \( 2f_m \) in the limit of large \( K \). The factor by which the IF filter bandwidth is reduced with respect to the predetection bandwidth is

\[ \frac{\beta}{\beta_T} IF = \frac{(\beta - \hat{\beta} + 1)}{(\beta + 1)} . \quad (II-8) \]

This means that the amount of noise power at the discriminator input is decreased by this factor also, which is the basis for the threshold extension capability in the loop.

Setting the signal to zero and solving Eq. (II-6) for the noise components, one obtains

\[ n_4(t) = \theta_n \frac{K + 1}{K} . \quad (II-9) \]

From Eq. (II-4), \( V_4(t) \) becomes

\[ V_4(t) = K_d \left[ \frac{\beta}{K} \omega_m \cos \omega_m t + n_4(t) - \frac{K}{K + 1} n_4(t) \right] \]

\[ = K_d \left[ \frac{\beta}{K + 1} \omega_m \cos \omega_m t + \frac{K}{K + 1} n_4(t) \right] . \quad (II-10) \]
This output is the same as that of a standard discriminator reduced by the factor \( \frac{1}{K+1} \). Therefore the SNR above threshold may be written as

\[
\frac{S_0}{N_0} = \frac{3}{2} \beta^2 \text{CNR}_z,
\]  

(II-11)

where \( \text{CNR}_z \) is the carrier-to-noise ratio with noise power measured in the bandwidth of \( f_m \) Hz.

Threshold for an FM demodulator is rather arbitrarily defined as the point on its output signal-to-noise ratio (SNR) versus input carrier-to-noise ratio (CNR) characteristic where the output SNR deviates one dB from its linear, high CNR asymptote. Below threshold, the SNR decreases very rapidly with decreasing CNR. Threshold is a point of particular interest because of the rapid degradation of demodulator performance below this point.

One of the properties of an FMFB loop is threshold extension. That is, the threshold is moved, or extended, to a lower CNR value, giving the detector a larger useful range of CNR. Above threshold, the noise component of the discriminator output is approximately Gaussian, as described above. However, as threshold is approached, it has been experimentally determined that a pulse-type, or spike, noise waveform begins to appear. Because the energy associated with the spike noise is much greater at low frequencies
than that of the Gaussian noise component of the output, the appearance of spikes in the output of the discriminator is characteristic of the onset of threshold.

Taub and Schilling\(^5\) have shown that for sinusoidal modulation, the total average number of spikes per second of a conventional demodulator is

\[
N_{\text{disc}} = \frac{B}{2\sqrt{\pi}} \text{erfc} \left( \sqrt{\frac{f_m}{B}} \frac{S_i}{N_m} \right) + 2 \frac{\Delta f}{\pi} \exp \left( - \frac{f_m}{B} \frac{S_i}{N_m} \right), \quad (\text{II-12})
\]

where \(B\) is the bandwidth of the IF filter (\(2(\beta + 1)f_m\) Hz),

- \(f_m\) is the modulation frequency,
- \(S_i\) is the input signal power,
- \(\Delta f\) is the frequency deviation (\(\beta f_m\)),
- \(\text{erfc} (x) = 1 - \text{erf}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} \, du\), and
- \(N_m\) is the noise power in a bandwidth of \(f_m\) Hz.

In deriving this equation it has been assumed that the bandwidth of the IF filter is the same as that of the predetection filter described above. With an FMFB demodulator, the effect of feedback is to decrease the signal deviation which in turn permits a decrease in the IF bandwidth, as previously mentioned. With this in mind, the number of spikes per second for the FMFB case can be obtained from Eq. (II-12) if the IF bandwidth \(B\) without feedback is replaced with \(B'\), the bandwidth with feedback, and the frequency deviation, \(\Delta f\), is replaced by the
frequency deviation of the IF filter output with feedback, \((\beta - \hat{\beta})f_m\). This yields the following equation for the number of spikes per second:

\[
N_{\text{FMFB}} = \frac{B'}{2\sqrt{3}} \text{erfc} \left( \frac{f_m S_i}{B N_m} \right) + \frac{2(\beta - \hat{\beta})}{\pi} \exp \left( - \frac{f_m S_i}{B N_m} \right). \quad (\text{II}-13)
\]

Note that \(\frac{B'}{B} = \frac{\beta + 1}{\beta - \beta + 1} < 1\) and that \(\frac{\Delta f_m}{\Delta f} = \frac{\beta}{\beta - \beta + 1} < 1\).

Therefore, \(N_{\text{FMFB}}\) will be much less than \(N_{\text{Disc}}\), and consequently the threshold is extended.

Schilling and Billig\(^3\) give the following equation for the SNR output of the FMFB demodulator:

\[
S_0 \frac{N_0}{N_0} = \frac{3\beta^2 \frac{S_i^2}{\sigma^2}}{1 + 4\sqrt{3} (\beta - \hat{\beta} + 1)^2 \left( \frac{\beta + 1}{\beta - \beta + 1} \right) \frac{S_i}{\sigma^2} \text{erfc} \left( \frac{\beta + 1}{\beta - \beta + 1} \frac{S_i^2}{\sigma^2} \right)}.
\]

(\text{II}-14)

where \(\sigma^2\) is the variance of the Gaussian noise at the prediction filter output and all other quantities have been defined previously. One will note that this equation giving \(S_0/N_0\) for an ordinary discriminator when \(\hat{\beta}\) is set equal to zero in the above equation. Eq. (\text{II}-14) is plotted in Fig. 3 for three values of \(\beta\) with \(\beta - \hat{\beta} = 1\).

Note that the second term of the denominator of Eq. (\text{II}-14) involving the complementary error function is
negligible when the CNR, $\frac{S_1}{\sigma^2}$, is large, representing the case above threshold. One of the effects of the feedback is to increase the argument of the error function by the factor $\frac{\beta + 1}{\beta - \beta + 1}$ which tends to extend threshold onset to a lower CNR.
Figure 3. Threshold Curves for FMFB Demodulator

$\beta - \hat{\beta} = 1$ for all cases
III. FMFB LOOP OPERATION IN THE PRESENCE OF INTERFERENCE

In this chapter the effects of interference on the demodulation of an FM signal by FMFB will be analyzed.

To begin the analysis, let the input signal to the loop, $y(t)$, be

$$y(t) = S(t) + n_i(t) \tag{III-1}$$

where $S(t) = \sqrt{2} \sin (\omega_0 t + \beta \sin \omega_m t)$ is the signal component, and

$$n_i(t) = \sqrt{2} \alpha \sin ((\omega_0 + \Delta \omega)t + \delta) \tag{III-3}$$

is the interference component.

In Eq. (III-3), $\alpha$ is the RMS value of the interference component relative to the signal, $\Delta \omega$ is the radian frequency offset of the interference from the signal carrier, and $\delta$ is the phase difference between the signal and the interference. Either $\alpha$ or $\delta$ may be time varying: $\delta(t)$ is indicative of an FM interference signal, and $\alpha(t)$ represents an AM interference signal. When neither of these parameters varies with time, the interference is said to be continuous wave (CW), the type of interference with which this thesis is concerned.

As a first-order approximation, the effect of the interference on the VCO output can be considered negligible if $\alpha$ is small. Therefore, the output of the VCO may be written as
\[ v_1(t) \approx \sqrt{2} \cos (\omega_1 t + \hat{\beta} \sin \omega_m t). \] (III-4)

Assuming that the IF filter is tuned to the difference frequency of the mixer output, \( \omega_0 - \omega_1 \triangleq \omega_{IF} \), the above approximation will yield an IF signal output of

\[ v_2(t) = \sin (\omega_{IF} t + (\beta - \hat{\beta}) \sin \omega_m t) \]
\[ + \alpha \sin ((\omega_{IF} + \Delta \omega) t + \delta - \hat{\beta} \sin \omega_m t). \] (III-5)

To more easily see what portion of \( v_2(t) \) is passed by the IF filter, the above signal plus interference can be expanded in a Bessel function series. In general, an FM signal can be expanded as

\[ \sin (\omega t + \gamma \sin \omega_m t + \theta) = \sum_{n=-\infty}^{\infty} J_n(\gamma) \sin ((\omega + n\omega_m) t + \theta) \] (III-6)

where \( J_n(\gamma) \) is the Bessel function of the first kind and order \( n \). When expanded, Eq. (III-5) becomes

\[ v_2(t) = \sum_{n=-\infty}^{\infty} J_n(\beta - \hat{\beta}) \sin (\omega_{IF} + n\omega_m) t \]
\[ + \sum_{n=-\infty}^{\infty} J_n(\hat{\beta}) \sin ((\omega_{IF} + \Delta \omega - n\omega_m) t + \delta) \] (III-7)

If \(|\beta - \hat{\beta}| << 1\), then the signal portion is a narrowband FM signal which, in general, consists primarily of the carrier at frequency \( \omega_{IF} \), and a pair of sidebands at frequency \( \omega_{IF} \pm \omega_m \) rad/sec. To pass most of the signal...
power, the IF filter should be just slightly greater than
$2(\beta - \hat{\beta} + 1)f_m$. For the case of the interference, the
effect of the feedback when $\hat{\beta} >> 1$ is to spread out the
interference power among the sidebands of a wideband FM
signal with modulation index $\hat{\beta}$ and center frequency of
$\omega_{IF} + \Delta \omega$ radians/sec. This means that very little of the
interferring signal power will be permitted to pass
through the IF filter if $\hat{\beta} >> 1$.

Figure 4 illustrated the IF filter passband and the
spectral components of the signal and interference.

In Figure 4, the signal power passed by the IF
filter is approximately .994 while the interference power
passed is .3838. The output signal-to-interference ratio
is then 4.14 dB.

This example is not necessarily a typical case for
the following reasons: 1) the filter was assumed ideal
bandpass; 2) the factor of noise jitter at the VCO output
has been completely ignored which will cause additional
distortion of the signal; and 3) very rarely will the
interference be equal to the signal ($\alpha = 1$), and $\Delta \omega = 0$
is a worse case condition.

In the simulations, the IF bandwidth used was
$2(\beta - \hat{\beta} + 1)$. For the purpose of the following discussion,
this value will be assumed to be fixed with $\beta - \hat{\beta} = 1$. If
this is the case, the interference power passed by the
Figure 4. Spectra of the (a) signal and (b) interference at the IF filter output. $\beta = 6$, $\hat{\beta} = 5$, $\alpha = 1$, $\Delta \omega = 0$, and $\delta = 0$. IF bandwidth $2(\beta - \hat{\beta} + 2)f_m$.

IF filter is approximately

$$P_I = \alpha^2 (1/2 J_0^2 (\hat{\beta}) + J_1^2 (\hat{\beta}) + J_2 (\hat{\beta}) + 1/2 J_3^2 (\hat{\beta}))$$

(III-8)

where $J_3(\hat{\beta})$ has been multiplied by one-half because
this sideband will be at the cutoff point of the filter, and the sideband components beyond the third have been neglected. With these assumptions, $\beta$ has no effect upon the number of sideband pairs that are passed. The actual interference power that will then be passed will be a function only of $\hat{\beta}$. Figure 5 shows the interference power passed by the IF filter as a function of $\hat{\beta}$ under these assumptions. Note that the noise power decreases rather rapidly once $\hat{\beta}$ is above three and that it tends to level off above six or seven. This is intuitively correct because, as $\hat{\beta}$ becomes larger than two or three, the spectral components of the interference begin to spread out, placing more and more power outside the pass-band of the IF filter.

Figure 5. Interference Power Passed by the IF Filter as a Function of $\hat{\beta}$. 

![Figure 5. Interference Power Passed by the IF Filter as a Function of $\hat{\beta}$](image-url)
IV. DERIVATION OF THE COMPUTER MODEL EQUATIONS

In this section, the equations for the computer model for operation in interference and Gaussian noise are obtained by deriving the FMFB loop differential equations.

For the computer model, let the loop input be

\[ y(t) = \sqrt{2} R(t) \cos (\omega_1 t + \theta_1(t)) \]  \hspace{1cm} (IV-1)

where \( R(t) \) is the envelope, and \( \theta_1(t) \) is the phase for the signal plus interference. \( R(t) \) and \( \theta_1(t) \) are defined by the following two equations:

\[ R(t) \cos \theta_1(t) = \cos \phi_m(t) + I_1(t) \overset{\Delta}{=} A_1(t) \]  \hspace{1cm} (IV-2)

and

\[ R(t) \sin \theta_1(t) = \sin \phi_m(t) + I_2(t) \overset{\Delta}{=} A_2(t). \]  \hspace{1cm} (IV-3)

\( \phi_m(t) \) is the modulating signal, which for the case considered here, is sinusoidal:

\[ \phi_m(t) = \beta \sin \omega_m t. \]  \hspace{1cm} (IV-4)

\( I_1(t) \) and \( I_2(t) \) are the in-phase and quadrature interference plus Gaussian noise components and are described as

\[ I_1(t) = \alpha \cos (\Delta \omega + \delta) + n_s(t) \]  \hspace{1cm} (IV-5)

and
\[ I_2(t) = \alpha \sin(\Delta \omega + \delta) + n_c(t) \]  
\hspace{1cm} (IV-6)

where \( n_s(t) \) and \( n_c(t) \) are statistically independent inphase and quadrature components of the input noise:

\[ n(t) = n_c(t) \cos(\omega_1 t) + n_s(t) \sin(\omega_1 t) . \]  
\hspace{1cm} (IV-7)

If the power spectral density of \( n(t) \) is \( N_0/2 \) watts/Hz in the transmission bandwidth about \( \omega_1 \) rad/sec, then the power spectral densities of \( n_c \) and \( n_s \) are \( N_0 \) watts/Hz centered around zero rad/sec\(^5\).

Assuming that the input to the limiter-discriminator is of the form

\[ v_2(t) = V(t) \cos(\omega_{IF} t + \psi(t)), \]  
\hspace{1cm} (IV-8)

the output of the discriminator will then be

\[ V_3(t) = K_d \frac{d\psi(t)}{dt} . \]  
\hspace{1cm} (IV-9)

The output of \( h_2(t) \), which represents any filtering done by the discriminator, will be

\[ V_4(t) = h_2(t) * K_d \frac{d\psi(t)}{dt}, \]  
\hspace{1cm} (IV-10)

where "*" indicates the convolution operator. Since \( v_4(t) \) is the input to the VCO, its output must be

\[ v_1(t) = \sqrt{2} \cos(\omega_1 t + K_v \int K_d h_2(t) * \frac{d\psi(t)}{dt}) \]

\[ = v_2 \cos(\omega_1 t + Kh_2(t) * \psi(t)) \]  
\hspace{1cm} (IV-11)
where \( K = K_v K_d \) is called the feedback factor. Because \( \psi \) is a function of \( \tau \) only inside the convolution, \( \frac{d}{dt} \) can be interchanged with the convolution operation.

This signal is then mixed with the input signal, \( y(t) \). The resultant low-pass part of the mixer output is

\[
v_2'(t)_{LP} = R(t) \cos (\omega_{IF} t + \theta_1(t) - K h_2(t) \ast \psi(t)).
\]

(IV-12)

The output of the IF filter, \( v_2(t) \), is related to its input \( v_2(t) \) by the convolution operation:

\[
v_2(t) = h_1(t) \ast v_2'(t),
\]

(IV-13)

where \( v_2'(t) \) is described by Eq. (IV-12) and \( v_2(t) \) is described by Eq. (IV-8).

It is now desirable to make some approximations for a narrowband, high Q filter impulse response\(^6\). Let \( h_1(t) \) for this narrowband filter be written in terms of a complex envelope, \( \tilde{h}_1(t) \):

\[
h_1(t) = 2 \text{Re} \tilde{h}_1(t) e^{j \omega_{IF} t}.
\]

(IV-14)

Figure 6 illustrates this process. It is now desirable to write \( v_2(t) \) from Eq. (IV-8) in terms of its complex envelope:

\[
v_2(t) = \text{Re} v_2'(t)e^{j \omega_{IF} t} = \text{Re} V(t)e^{j \psi(t)} e^{j \omega_{IF} t}.
\]

(IV-15)

\( v_2'(t) \) can be written similarly as
\[ v'_2(t) = \text{Re} \ v'_2(t) \ e^{j \omega_{IF} t}, \quad (IV-16) \]

where \( v'_2(t) = R(t) \ e^{j(\theta_1(t) - K h_2(t) * \psi(t))}. \quad (IV-17) \)

It can be shown that the complex envelope of the output of a narrowband filter with impulse response \( h_1(t) \) can be written as the complex envelope of the impulse response convolved with the complex envelope of the input, or that

\[ \tilde{v}_2(t) = h_1(t) * \tilde{v}_2(t). \quad (IV-18) \]

The proof is given in Appendix A.

Figure 6. The IF Filter with Complex Envelope Representation.

Substituting the expression for \( \tilde{v}_2(t) \) and \( \tilde{v}_2'(t) \) from Eq. (IV-15) and Eq. (IV-17), respectively, into Eq. (IV-18), the differential equation describing the FMFB loop operation is obtained:

\[ V(t) \ e^{j \psi(t)} = h_1(t) * (R(t) \ e^{j(\theta_1(t) - K h_1(t) * \psi(t))}. \quad (IV-19) \]
In the simulation the IF filter is a single pole filter with complex envelope

$$h_1(t) = \omega_c e^{-\omega_{IF}t} u(t) \quad \text{(IV-20)}$$

or impulse response

$$h_1(t) = \omega_c e^{-\omega_c t} \cos (\omega_{IF}t) u(t), \quad \text{(IV-21)}$$

where $\omega_c$ is the 3dB cut-off frequency of the baseband equivalent of the IF filter in rad/sec and $u(t)$ is the unit step. The filtering done by the discriminator which is represented by $h_2(t)$ is assumed negligible in comparison to the filtering done by $h_1(t)$. Thus, $h_2(t)$ may be written as

$$h_2(t) = \delta(t), \quad \text{(IV-22)}$$

where $\delta(t)$ is the unit impulse. Substituting Eq. (IV-20) and Eq. (IV-22) into Eq. (IV-19) of the preceding section, the loop differential equation is obtained:

$$V(t) e^{j\psi(t)} = (\omega_c e^{-\omega_c t}) \ast (R(t) e^{j(\theta_1(t) - K\delta(t)\psi(t))}). \quad \text{(IV-23)}$$

As shown in Appendix B, Eq. (IV-23) can be reduced to the set of differential equations

$$\dot{V}(t) = -\omega_c V(t) + \omega_c (A_1(t) \cos \phi + A_2(t) \sin \phi) \quad \text{(IV-24)}$$

and
\[ \phi(t) = (K + 1) \omega_c (A_2(t) \cos \phi - A_1 \sin \phi)/V(t) \quad (IV-25) \]

where

\[ \phi(t) = (K + 1) \psi(t) \]

and the dot represents differentiation with respect to time. Equations (IV-24) and (IV-25) are the simulation equations for the FMFB demodulator.
V. IMPLEMENTATION OF LOOP EQUATIONS AS A COMPUTER SIMULATION

This section describes the simulation model by briefly presenting a short flow chart.

Figure 7 shows a summary flow chart for the FMFB simulation. A detailed flow chart is given in Appendix C and a program listing appears in Appendix D.

The simulation consists of a Runge-Kutta integration routine which integrates the loop differential equations presented in the preceding section and an output section which compares the demodulated signal-plus-noise components to the signal demodulated alone by computing the mean-squared error (MSE). Every half cycle of the signal, the SNR of the output, the root-mean-square (RMS) error, and the RMS signal values are computed and printed.

The subroutine \( F(T_{\cdots}) \) computes the right-hand sides of the loop differential equations, Eq. (IV-24) and Eq. (IV-25). This subroutine consists of two sections: the signal generator, and the interference generator.

The Gaussian noise components, \( n_c(t) \) and \( n_s(t) \), are approximated as staircase processes which change only at each integration step. The method of generation of these Gaussian components involves the generation of two uniformly distributed random numbers between zero and one using the multiplicative method. These two random numbers, RN1
Figure 7. A Simplified FMFB Loop Flow Chart
and RN2, are used in the following algorithm to generate two independent Gaussian random numbers of standard deviation sigma:

\[ G = \sqrt{2 \log_{10}(RN2)} \]  
\[ NC = G \cdot \sigma \cdot \cos (2\pi \cdot RN1) \]  
\[ NS = G \cdot \sigma \cdot \sin (2\pi \cdot RN1) \]

These are the samples of the two Gaussian processes, \( n_c(t) \) and \( n_s(t) \).

The probability density function (pdf) of the noise samples is

\[ P(v) = \frac{e^{-\frac{v^2}{2\sigma_v^2}}}{\sqrt{2\pi} \sigma_v} \]  

where \( \sigma_v^2 \) is the variance of the noise process. Because the noise process is sampled at the beginning of each step increment, it appears as a step process, as shown in Figure 8. The power spectral density of this staircase

![Figure 8. A Sampled Noise Process](image-url)
process can be shown to be

\[ S_v(j\omega) = H \sigma_v^2 \left( \frac{\sin \left( \frac{H \cdot \omega}{2} \right)}{\frac{H \cdot \omega}{2}} \right)^2, \]  

(V-5)

which is shown in Figure 9.

Figure 9. The Power Spectral Density of a Sampled Noise Process

From Figure 9, it is obvious that if the noise process is to approximate a white noise process, the bandwidth of any system with such a staircase input must be much less than $1/H$ Hz. For this to be true for the FMFB loop,

\[ \frac{1}{H} \gg B_T, \]

where $H$ is the sampling period and $B_T$ is the bandwidth of the input.

The noise power in the modulating signal bandwidth is

\[ P_N = 2 \cdot S(0) f_m = N_0 f_m = 2 \cdot H \sigma_v^2 f_m, \]  

(V-6)
where $N_0$ is the single-sided noise power spectral density. However, it is more convenient to express $N_0$ in terms of the SNR referred to the modulating signal bandwidth given by

$$\text{SNR} = \frac{1}{2N_0 f_m} \quad (V-7)$$

where the input signal power is assumed unity for convenience. Substituting Eq. (V-7) into Eq. (V-6) and solving for $\sigma^2_v$, one obtains

$$\sigma^2_v = \frac{N_0}{2H} = \frac{1}{2 \cdot \text{SNR} \cdot H \cdot f_m} \quad (V-8)$$

Note that by this equation, the SNR is expressed in terms of the baseband equivalent SNR.

The output of the loop, $\phi(t)$, is filtered before the RMS error is computed. To accomplish this, a digital filter routine is used whereby a Butterworth filter of order one to four may be selected. The filter was obtained by using the bilinear transform:

$$= C \frac{1 - z^{-1}}{1 + z^{-1}} \quad (V-9)$$

where $C$ is the mapping constant and $z^{-1}$ is a single unit delay operator. The constant $C$ is given by

$$C = \frac{\omega}{\omega_r} \cot \left( \frac{\omega_r H}{2} \right) \quad (V-10)$$
where $\overline{\omega_r}$ is the 3 dB cutoff frequency in rad/sec of the standard filter prototype function used, $\omega_r$ is the desired 3 dB frequency in rad/sec, and $H$ is the sampling period. An example for a second-order filter is given in Appendix E.
VI. RESULTS

In this section, the results of the simulations are presented. The basic measure of performance is the mean-square error between the output of the FMFB loop with interference present at the input and the output with no interference at the input, normalized by the mean-square signal. The normalized mean-square error will be referred to as an MSE in the following discussion.

The MSE data for the FMFB loop operating in RFI is presented in Tables I-III. The values of $\hat{\beta}$ are varied over a small range for each of three $\beta$'s; $\alpha$ and $\Delta \omega$ are fixed at 0.4 and 0.11, respectively. To guarantee that all significant signal spectral components are passed by the IF filter while as many of the interference spectral components as possible are rejected, the IF filter bandwidth is varied as $BW = 2(\beta - \hat{\beta} + 2)f_m$. The results from these tables have been plotted in Figure 10. Note that for small values of $\beta$, the MSE is strongly dependent on $\hat{\beta}$, as shown by the slope of the curves. For larger values of $\beta$, the change in the MSE with $\hat{\beta}$ is much less rapid until the value of $\beta - \hat{\beta}$ becomes less than one-half. Also, the MSE decreases rapidly with increasing $\beta$ for moderate values of $\beta$. For example, increasing the value of $\beta$ by only 1.3 from 5 to 6.3 resulted in a 400% change in MSE. For large values of $\beta$, such as $\beta = 8$, the MSE
### TABLE I. MSE as a Function of $\hat{\beta}$ for $\beta = 5$

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\hat{\beta}$</th>
<th>Signal</th>
<th>MSE</th>
<th>SNR out</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.0</td>
<td>3.8</td>
<td>17.9</td>
<td>0.0054237</td>
<td>22.64</td>
</tr>
<tr>
<td>5.0</td>
<td>4.0</td>
<td>17.9</td>
<td>0.005399</td>
<td>22.67</td>
</tr>
<tr>
<td>5.0</td>
<td>4.2</td>
<td>17.9</td>
<td>0.005317</td>
<td>22.73</td>
</tr>
<tr>
<td>5.0</td>
<td>4.4</td>
<td>17.91</td>
<td>0.0052394</td>
<td>22.808</td>
</tr>
<tr>
<td>5.0</td>
<td>4.6</td>
<td>17.9</td>
<td>0.0051637</td>
<td>22.87</td>
</tr>
<tr>
<td>5.0</td>
<td>4.8</td>
<td>17.91</td>
<td>0.0050997</td>
<td>22.926</td>
</tr>
<tr>
<td>5.0</td>
<td>4.9</td>
<td>17.917</td>
<td>0.0058695</td>
<td>22.85</td>
</tr>
</tbody>
</table>

$\alpha = 0.4, \Delta \omega = 0.11, \omega_m = 1.0$

### TABLE II. MSE as a Function of $\hat{\beta}$ for $\beta = 6.3$

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\hat{\beta}$</th>
<th>Signal</th>
<th>MSE</th>
<th>SNR out</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.3</td>
<td>5.1</td>
<td>22.582</td>
<td>0.0012898</td>
<td>28.895</td>
</tr>
<tr>
<td>6.3</td>
<td>5.3</td>
<td>22.58</td>
<td>0.0012695</td>
<td>28.96</td>
</tr>
<tr>
<td>6.3</td>
<td>5.3</td>
<td>22.58</td>
<td>0.0012400</td>
<td>29.066</td>
</tr>
<tr>
<td>6.3</td>
<td>5.7</td>
<td>22.583</td>
<td>0.0012142</td>
<td>29.157</td>
</tr>
<tr>
<td>6.3</td>
<td>5.9</td>
<td>22.58</td>
<td>0.0011547</td>
<td>29.38</td>
</tr>
<tr>
<td>6.3</td>
<td>6.1</td>
<td>22.58</td>
<td>0.001066</td>
<td>29.722</td>
</tr>
<tr>
<td>6.3</td>
<td>6.2</td>
<td>22.581</td>
<td>0.0010555</td>
<td>29.785</td>
</tr>
</tbody>
</table>

$\alpha = 0.4, \Delta \omega = 0.11, \omega_m = 1.0$
### TABLE III. MSE as a Function of $\hat{\beta}$ for $\beta = 8.0$

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\hat{\beta}$</th>
<th>Signal</th>
<th>MSE</th>
<th>SNR out</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.0</td>
<td>6.8</td>
<td>28.711</td>
<td>.001095</td>
<td>29.605</td>
</tr>
<tr>
<td>8.0</td>
<td>7.0</td>
<td>28.718</td>
<td>.001089</td>
<td>29.628</td>
</tr>
<tr>
<td>8.0</td>
<td>7.2</td>
<td>28.716</td>
<td>.00107738</td>
<td>29.676</td>
</tr>
<tr>
<td>8.0</td>
<td>7.4</td>
<td>28.717</td>
<td>.0010678</td>
<td>29.715</td>
</tr>
<tr>
<td>8.0</td>
<td>7.6</td>
<td>27.969</td>
<td>.001077369</td>
<td>29.676</td>
</tr>
<tr>
<td>8.0</td>
<td>7.8</td>
<td>28.713</td>
<td>.00103412</td>
<td>29.854</td>
</tr>
<tr>
<td>8.0</td>
<td>7.9</td>
<td>28.712</td>
<td>.001021</td>
<td>29.909</td>
</tr>
</tbody>
</table>

$\alpha = .4$ $\Delta \omega = .11$ $\omega_m = 1.0$

### TABLE IV. MSE as a Function of IF Bandwidth

<table>
<thead>
<tr>
<th>Cutoff</th>
<th>b</th>
<th>Signal</th>
<th>MSE</th>
<th>SNR out</th>
</tr>
</thead>
<tbody>
<tr>
<td>37.699</td>
<td>2.0</td>
<td>17.918</td>
<td>.00518603</td>
<td>22.852</td>
</tr>
<tr>
<td>28.274</td>
<td>1.5</td>
<td>17.913</td>
<td>.00523724</td>
<td>22.808</td>
</tr>
<tr>
<td>22.619</td>
<td>1.2</td>
<td>17.909</td>
<td>.00533833</td>
<td>22.619</td>
</tr>
<tr>
<td>18.848</td>
<td>1.0</td>
<td>17.904</td>
<td>.00540553</td>
<td>22.672</td>
</tr>
<tr>
<td>15.079</td>
<td>0.8</td>
<td>17.898</td>
<td>.00550189</td>
<td>22.595</td>
</tr>
<tr>
<td>13.194</td>
<td>0.7</td>
<td>17.889</td>
<td>.00559295</td>
<td>22.524</td>
</tr>
<tr>
<td>11.309</td>
<td>0.6</td>
<td>17.880</td>
<td>.0056548</td>
<td>22.473</td>
</tr>
<tr>
<td>9.424</td>
<td>0.5</td>
<td>17.867</td>
<td>.00579453</td>
<td>22.370</td>
</tr>
</tbody>
</table>

$\alpha = .4$ $\Delta \omega = .11$ $\omega_m = 1.0$
Figure 10. MSE as a Function of $\beta - \hat{\beta}$
was less dependent on $\hat{\beta}$. For $\beta - \hat{\beta} = 1$, increasing $\beta$
from 6.3 to 8.0 results in a decrease in MSE of less than
10%. However, for small values of $\beta - \hat{\beta}$, little advantage
is exhibited by the larger values of $\beta$. Also, the
larger $\beta$'s require a considerably wider transmission band-
width.

Some simulation runs which show the effects of the
IF bandwidth on the output MSE are presented in Table IV
and Figure 11. For this data, $\beta = 5$ and $\hat{\beta} = 4$ were not
changed. $\alpha$ was 0.4 and $\Delta \omega$ was 0.11, as before. From
Figure 12, it is noted that the IF bandwidth for the
preceeding interference runs was not necessarily wide
enough ($b = 1$, where $b$ is the bandwidth factor:
$BW = b[2(\beta - \hat{\beta} + 2)f_m]$). However, increasing $b$ to two
gives less than a 10% improvement in MSE. Since the IF
filter also serves the purpose of eliminating Gaussian
noise to give threshold extension, $b = 1$ is probably a
good compromise.

Variations of MSE as a function of frequency offset,
$\Delta \omega$, are presented in Table V and plotted in Figure 12.
Figure 12 also contains the theoretical values for the
interference power passed by the IF filter, normalized
by the interference power at the input, $\alpha^2/2$. In general,
these curves have a similar appearance. As $\hat{\beta}$ increases,
both the interference power and the MSE decrease and the
slopes of these curves decrease. It was not expected
### TABLE V. Mean-Square Error as a Function of $\Delta \omega$

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\hat{\beta}$</th>
<th>$\Delta \omega$</th>
<th>Signal</th>
<th>MSE</th>
<th>SNR out</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.0</td>
<td>4.9</td>
<td>0.11</td>
<td>17.917</td>
<td>.005870</td>
<td>22.95</td>
</tr>
<tr>
<td>5.0</td>
<td>4.9</td>
<td>1.00</td>
<td>17.917</td>
<td>.002955</td>
<td>25.292</td>
</tr>
<tr>
<td>5.0</td>
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<td>2.00</td>
<td>17.917</td>
<td>.003052</td>
<td>24.143</td>
</tr>
<tr>
<td>6.3</td>
<td>6.2</td>
<td>0.11</td>
<td>22.581</td>
<td>.001055</td>
<td>29.785</td>
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<tr>
<td>6.3</td>
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<td>22.581</td>
<td>.001565</td>
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<td>2.00</td>
<td>22.581</td>
<td>.001214</td>
<td>29.150</td>
</tr>
<tr>
<td>8.0</td>
<td>7.9</td>
<td>0.11</td>
<td>28.712</td>
<td>.001021</td>
<td>29.909</td>
</tr>
<tr>
<td>8.0</td>
<td>7.9</td>
<td>1.00</td>
<td>28.712</td>
<td>.000673</td>
<td>31.673</td>
</tr>
<tr>
<td>8.0</td>
<td>7.9</td>
<td>2.00</td>
<td>28.712</td>
<td>.000763</td>
<td>31.176</td>
</tr>
</tbody>
</table>

$a = 0.4$  \hspace{2cm} $\omega_m = 1$

### TABLE VI. SNR Out as a Function of CNR In

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\hat{\beta}$</th>
<th>CNR in</th>
<th>SNR out</th>
<th>Assymptote</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.0</td>
<td>2.0</td>
<td>50.</td>
<td>52.86</td>
<td>61.30</td>
</tr>
<tr>
<td>6.0</td>
<td>5.0</td>
<td>50.</td>
<td>58.89</td>
<td>67.32</td>
</tr>
<tr>
<td>6.3</td>
<td>5.3</td>
<td>48.43</td>
<td>55.63</td>
<td>66.10</td>
</tr>
<tr>
<td>6.3</td>
<td>5.3</td>
<td>38.43</td>
<td>46.242</td>
<td>56.17</td>
</tr>
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<td>5.3</td>
<td>28.43</td>
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<td>6.3</td>
<td>5.3</td>
<td>23.43</td>
<td>27.035</td>
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<td>5.3</td>
<td>13.43</td>
<td>-2.186</td>
<td>31.17</td>
</tr>
</tbody>
</table>
that the two curves would entirely agree, as one curve represents the input interference to the discriminator while the second represents the distortion in the output signal due to this interference. However, both show the same trend.

Figure 11. MSE versus IF bandwidth
\[ b = \text{bandwidth factor}, \]
\[ \beta = 5, \beta = 4, \alpha = .4, \]
\[ \Delta \omega = .11 \]

Data for Gaussian noise are presented in Table IV and is graphed in Figure 14. Also, the theoretical asymptotes for the high input CNR are presented in
Figure 12. MSE as a Function of $\hat{\beta}$
Figure 14 for comparison. It should be noted that the data deviates from the asymptotes by approximately 8.43 dB in all cases. However, the correct slope of unity is maintained above threshold. One possible reason for the low output SNR is that a predetection filter was not used ahead of the loop in the simulation because it was originally thought that the loop equivalent bandwidth would provide sufficient bandwidth limiting of the noise samples. When this was not found to be true, computer time availability did not permit the implementation of a predetection filter. One should note that the curve for $\beta = 6.3$ was closer to 10 dB below the correct asymptotic curve. Also, the value of output SNR for $\beta = 6.3$ at a CNR of 50 dB was less than the value of the output SNR for $\beta = 6.0$ at the same CNR, which is not correct. There does not appear to be any apparent cause for either of these occurrences other than the absence of the pre-detection filter, as discussed above. More work needs to be done with the Gaussian noise portion of this program.
Figure 13. Experimental Threshold Curves for FMFB
VII. CONCLUSION

It has been shown that the computer model developed for the FMFB demodulator is useful in evaluating the effects of RFI on the demodulator performance. Estimates of the required IF bandwidth have been given and the effect of interference frequency offset, $\Delta \omega$, upon the MSE has been shown. However, more work is needed in modeling the Gaussian noise at the input before the effects of noise plus interference on loop operation can be ascertained.
BIBLIOGRAPHY


VITA

The author, Terry D. Wormington, was born on February 8, 1949 in St. Louis County, Missouri. He graduated from the University of Missouri at Rolla on May 23, 1971 with his Bachelor of Science degree in Electrical Engineering.

He enrolled in Graduate School at the University of Missouri at Rolla during the spring semester of 1971 and has been employed as a Graduate Teaching Assistant during the 1971-72 academic year.
APPENDIX A

Proof of the Envelope Representation for the Convolution Integral

The purpose of this appendix is to show Eq. (IV-18), i.e., that the complex envelope of the output of a narrow-band filter is equal to the convolution of the complex envelope of the filter impulse response with the complex envelope of the input.

From the convolution representation of a filter response, the output, \( v_2(t) \), of the IF filter is

\[
v_2(t) = h_1(t) * v_2'(t),
\]

where \( h_1(t) \) is its impulse response and \( v_2'(t) \) its input. \( h_1(t) \) and \( v_2'(t) \) can be written in terms of their complex envelopes as

\[
h_1(t) = 2 \text{Re} \left( \tilde{h}_1(t) e^{j\omega_{IF} t} \right) = \tilde{h}_1(t) e^{j\omega_{IF} t} + \tilde{h}_1^*(t) e^{-j\omega_{IF} t}
\]

(A-2)

and

\[
v_2'(t) = \text{Re} \left( \tilde{v}_2'(t) e^{j\omega_{IF} t} \right) = 1/2 \tilde{v}_2'(t) e^{j\omega_{IF} t}
\]

\[+ 1/2 \tilde{v}_2'(t) e^{-j\omega_{IF} t}, \quad (A-3)\]

respectively, which is true for any real quantity. Using these expressions in the convolution integral (A-1) for
$v_2(t)$, the following is obtained:

$$v_2(t) = \left[ h_1(\tau) e^{j\omega_{IF} \tau} + h_1^*(\tau) e^{-j\omega_{IF} \tau} \right]$$

$$\cdot 1/2 [v_2(t-\tau) e^{j\omega_{IF}(t-\tau)} + v_2^*(t-\tau) e^{-j\omega_{IF}(t-\tau)}] d\tau$$

(A-4)

An expansion of this product yields

$$v_2(t) = \frac{1}{2} \left[ \int_{-\infty}^{\infty} h_1(\tau) v_2^*(t-\tau) e^{j\omega_{IF} t} d\tau + \int_{-\infty}^{\infty} h_1^*(\tau) v_2(t-\tau) e^{-j\omega_{IF} (t-2\tau)} d\tau + \int_{-\infty}^{\infty} h_1(\tau) v_2^*(t-\tau) e^{j\omega_{IF} (t-2\tau)} d\tau + \int_{-\infty}^{\infty} h_1^*(\tau) v_2(t-\tau) e^{-j\omega_{IF} t} d\tau \right].$$

(A-5)

If the complex envelope functions of $h(t)$ and $v_2(t)$ vary slowly with respect to $e^{-j\omega_{IF} t}$, the second and third integrals are approximately zero. To see this, consider the second integral in Eq. (A-5):

$$\int_{-\infty}^{\infty} h_1(\tau) v_2^*(t-\tau) e^{-j\omega_{IF}(t-2\tau)} d\tau = \int_{-\infty}^{\infty} h_1(\tau) v_2^*(t-\tau) \left( \cos \omega_{IF}(t-2\tau) - j \sin \omega_{IF}(t-2\tau) \right) d\tau \approx 0,$$

(A-6)
since \( \hat{h}_1(\tau) \) and \( \tilde{v}_2^*(t-\tau) \) are nearly constant compared with the sine and cosine factors. This permits Eq. (A-5) to be written as

\[
v_2(t) = \frac{1}{2} \left[ \int_{-\infty}^{\infty} \hat{h}_1(\tau) \tilde{v}_2(\tau) e^{j\omega F t} \, d\tau \right. \\
+ \left. \int_{-\infty}^{\infty} \hat{h}_1^*(\tau) \tilde{v}_2^*(\tau) e^{-j\omega F t} \, d\tau \right]
\]

Equating Eq. (A-3) and Eq. (A-7), one obtains

\[
\text{Re} [v_2(t) e^{j\omega F t}] = \text{Re} \int_{-\infty}^{\infty} \hat{h}(\tau) \tilde{v}_2(\tau) e^{j\omega F t} \, d\tau \tag{A-8}
\]

or

\[
\tilde{v}_2(t) = \int_{-\infty}^{\infty} \hat{h}(\tau) \tilde{v}_2(\tau) \, d\tau. \tag{A-9}
\]

This is equivalent to (IV-18).
APPENDIX B

Detailed Derivation of the Differential Equations
for the Computer Model

The purpose of this appendix is to show the steps involved in deriving Eq. (IV-24) and (IV-25), the differential equations for the computer model.

From Eq. (IV-23), \( v_2(t) \) can be written as

\[
V(t) e^{j\psi(t)} = [\omega_c e^{-\omega_c t}] * [R(t) e^{j(\theta_1(t)-K\delta(t))}\psi(t)]
\]

(B-1)

Rewriting this in terms of the convolution integral, one obtains the following equation:

\[
V(t) e^{j\psi(t)} = \int_{-\infty}^{\infty} \omega_c e^{-\omega_c (t-\tau)} R(\tau) e^{j(\theta_1(\tau)-K\psi(\tau))} d\tau
\]

(B-2)

By the Fundamental Theorem of Calculus, Eq. (B-2) can be differentiated to yield:

\[
\frac{d}{dt} (V(t) e^{j\psi(t)}) = \omega_c V(t) e^{j\psi(t)} + \omega_c R(t) e^{j(\theta_1(t)-K\psi(t))}.
\]

(B-3)

The left hand side becomes the following when the derivative is taken:

\[
\frac{d}{dt} (V(t) e^{j\psi(t)}) = \dot{V}(t) e^{j\psi(t)} + j V(t) \dot{\psi}(t) e^{j\psi(t)}.
\]

(B-4)
where the dot means differentiation with respect to time. Equating Eq. (B-4) and Eq. (B-5), and dividing both sides by \( e^{-j\psi(t)} \), one obtains

\[
\dot{V}(t) + j\psi(t) \ V(t) = -\omega_c \ V(t) + \omega_c R(t) \ e^{j(\theta_1(t) - K\psi(t) - \psi(t))}.
\]

(B-6)

Equating real and imaginary portions of Eq. (B-6), one obtains the following pair of differential equations which describe the FMFB loop:

\[
\dot{V}(t) = -\omega_c \ V(t) + \omega_c R(t) \cos (\theta_1(t) - K\psi(t) - \psi(t))
\]

(B-7)

and

\[
\dot{\psi}(t) \ V(t) = \omega_c R(t) \sin (\theta_1(t) - K\psi(t) - \psi(t)).
\]

(B-8)

Using the trigonometric identities, \( \cos (\alpha + \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \) and \( \sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \) in Eq. (B-7) and Eq. (B-8), one obtains

\[
\dot{V}(t) = -\omega_c \ V(t) + \omega_c R(t) \left[ \cos (\theta_1(t)) \cos ((K + 1)\psi) + R(t) \sin ((K + 1)\psi) \right]
\]

(B-9)

and

\[
\dot{\psi}(t) \ V(t) = \omega_c R(t) \left[ \sin (\theta_1(t)) \cos ((K + 1)\psi) - \cos (\theta_1(t)) \sin ((K + 1)\psi) \right].
\]

(B-10)
Let \((K + 1)\psi(t) = \phi\) and recall the definition of

\[\begin{align*}
A_1(t) &\triangleq R(t) \cos (\theta_1(t)) \quad \text{and} \quad A_2(t) \triangleq R(t) \sin (\theta_1(t))
\end{align*}\]

from Eq. (IV-2) and Eq. (IV-3). This yields

\[\dot{V}(t) = -\omega_c V(t) + \omega_c [A_1 \cos \phi + A_2 \sin \phi]\]  \hspace{1cm} (B-11)

and

\[\phi(t) \psi(t) = (K + 1) \omega_c [A_2 \cos \phi - A_1 \sin \phi].\]  \hspace{1cm} (B-12)
APPENDIX C

Flow Chart for the FMFB Simulation
START

READ
input & loop parameters

INITIALIZE
ZERO arrays & counters
CALCULATE loop parameters

LOOP INITIAL CONDITIONS
V=1 \( V_1 = 1 \)
PHI=0 \( \Phi_1 = 0 \)

WRITE loop & input param & other loop constants

RESET integration
save values

NO

YES

GAUSS=0

GENERATE 2 GAUSSIAN NOISE COMPONENTS

CONTINUE

GENERATE signal-plus-noise & signal components (generated by routine F)

FILTER LOOP OUTPUTS
CALL FILTER for S+N
CALL FILTER for S only

WRITE outputs

4th ORDER RUNGE KUTTA
INTEGRATION ROUTINE
FORM NEW LOOP CONDITIONS
FOR ROUTINE F:
\( V, V_1, \Phi, \Phi_1 \)

OUTPUT SECTION
Calculation of RMS error and output SNR
CALCULATE SQUARED ERROR
\[ \text{DISQ} = (\text{PHIDUM} - \text{PH1DUM})^2 \]
\[ \text{SUMSO} = \text{SUMSO} + \text{DISQ} \]

CALCULATE SQUARED SIGNAL
\[ \text{SIGSO} = \text{PH1DUM} \times \text{PH1DUM} \]
\[ \text{SIGSOS} = \text{SIGSOS} + \text{SIGSO} \]

NO

CALCULATION OF RMS QUANTITIES: RMSDUM, RMSERR, RMSIGG

YES

CALCULATE SNR OUT
WRITE \text{snrout snrgsd}

NO

YES
WRITE \text{rmserr=0, snrout}

CONTINUE

\[ T = \text{RTM} \]

\[ <0 \]

RMSDUM

RMSERR

EPS

CONTINUE

\[ <0 \]

\[ \text{RTM-T} \]

\[ >0 \]

\[ T = T + H \]

\[ \text{IND} = \text{IND} + 1 \]
CALCULATE SIN & COS OF PHI
\[ \sin\phi, \cos\phi \]

NO \quad BETA = 0.

MOD = BETA \cdot \sin(\text{PHIMOD})
A1MOD = \cos(\text{MOD})
A2MOD = \sin(\text{MOD})

NO \quad ALPHA = 0.

A1CW = ALPHA \cdot \cos(\text{PHASE})
A2CW = ALPHA \cdot \sin(\text{PHASE})

INTERFERENCE COMPONENTS
\[ A1I = A1CW + A1\text{MP} + \text{NS} \]
\[ A2I = A2CW + A2\text{MP} + \text{NC} \]

SIGNAL + INTERFERENCE
\[ A1 = A1MOD + A1I \]
\[ A2 = A2MOD + A2I \]

CALC. OF DERIVATIVE VALUES:
\[ \text{PHIDOT} = WCO \cdot Kp1 \cdot (A2\cos\phi - A1\sin\phi) / V \]
\[ \text{VDOT} = WCO \cdot (-V + A1\cos\phi + A2\sin\phi) \]
\[ \text{PH1DT} = WCO \cdot Kp1 \cdot (A2\text{MOD} \cdot \cos\phi - A1\text{MOD} \cdot \sin\phi) / V1 \]
\[ \text{V1DOT} = WCO \cdot (-V1 + A1\text{MOD} \cdot \cos\phi + A2\text{MOD} \cdot \sin\phi) \]

(RETURN)
APPENDIX D

Computer Program for the FMPB Simulation
REAL K,KP1,NS,NC
DIMENSION FILT1X(10),FILT1Y(10),FILT2X(10),FILT2Y(10)
COMMON WCO,WMOD,KP1,TWOP1,IX,BETA,ALPHA,RHO,DELTAW,TD,DEL
C THIS IS VERSION 1 PROGRAM
READ (1,200,END=69) BETAH,BETA,FMOD
READ (1,200) GAUSS, SNRGS, ALPHA, DELTAW, DEL, RHO, TD
READ (1,200) BWFAC, WCOFAC, BBFAC, RTM, H, EPS
READ (1,201) NPT, NPLOT, IMAX, NMOD, K1, K2, K3, K4, K5, IX, NPOINT, NSKIP, NOR
10ER
READ (1,201) IPRINT
C INPUTS AND CONSTANTS
RMSERR=10000.
BWBB=BBFAC*FMOO
SNRGS=10.**(SNRGS/10.)
PI=3.1415927
TWOP1=2.*PI
WMOD=TWOP1*FMOD
WCO=WMOD*(2.*ALPHA/(BETA-BETAH))
WCO=WCOFAC*WCO
K=BETAH/(BETA-BETAH)
KP1=K+1.
NS=GAUSS
NC=GAUSS
SIGMA=1./SQRT(2.*SNRGS*BWBB*H)
CUTOFF=BWFAC*FMOD
C INITIAL CONDITIONS
PHI=0.
V=1.
V1=1.
PHI1=0.
SIGSOS=0.
T=0.
A1MOD=0.
A2MOD=0.
RUNGE-KUTTA INTEGRATION ROUTINE

1  PHIS=PHI
  VS=V
  PH1S=PH1
  V1S=V1
  IF(GAUSS.EQ.0.) GO TO 14
IY = IX * 65539
IF (IY) 3, 4, 4
3 IY = IY + 2147483647 + 1
4 RNUM1 = IY / 2147483647.
5 IY = IY * 65539
IF (IY) 5, 6, 6
6 IY = IY + 2147483647 + 1
7 RNUM2 = IY / 2147483647.
8 IX = IY
9 G = SQRT(-2. * ALOG(RNUM2))
10 NC = G * SIGMA * COS(TWOPI * RNUM1)
11 NS = G * SIGMA * SIN(TWOPI * RNUM1)
12
13 CONTINUE
14 ICOUNT = MOD(N, IPRINT)
15 CALL FIT, VDOT, PHIDOT, PHI, V, V1DOT, PHI1DT, PHI1, V1)
16 CALL FILTER(T, H, PHIDOT, PHIDUM, FILT1X, FILT1Y, CUTOFF, NORDER)
17 CALL FILTER(T, H, PHID1, PHIDUM, FILT2X, FILT2Y, CUTOFF, NORDER)
18 IF (ICOUNT .NE. 0) GO TO 77
19 WRITE (3, 100) T, PHIDUM, PHI, V, VDOT, PHIDUM, PHI1, V1, V1DOT, A1I, A2I, DIF
20 ISQ
21
22 CONTINUE
23 CV1 = H * VDOT
24 CP1 = H * PHIDOT
25 C1V1 = H * V1DOT
26 C1P1 = H * PHI1DT
27 CALL FIT + H / 2., VDOT, PHIDOT, PHI + CP1 / 2., VS + CV1 / 2., V1DOT, PHI1DT, PHI1S
28 + C1P1 / 2., V1S + C1V1 / 2.)
29 CV2 = H * VDOT
30 CP2 = H * PHIDOT
31 C1V2 = H * V1DOT
32 C1P2 = H * PHI1DT
33 CALL FIT + H / 2., VDOT, PHIDOT, PHI + CP2 / 2., VS + CV2 / 2., V1DOT, PHI1DT, PHI1S
34 + C1P2 / 2., V1S + C1V2 / 2.)
35 CV3 = H * VDOT
36 CP3 = H * PHIDOT
37 C1V3 = H * V1DOT
94 C1P3=H*PHI1DT
95 CALL F(T+H,VDOT,PHIDOT,PHIS+CP3,VS+CV3,V1DOT,PHI1DT,PHIS+C1P3,V1S
1+C1V3)
96 CV4=H*VDOT
97 CP4=H*PHIDOT
98 C1V4=H*V1DOT
99 C1P4=H*PHI1DT
100 V=VS+(C1V1+2.*(C1V2+C1V3)+C1V4)/6.
101 PHI=PHIS+(CP1+2.*(CP2+CP3)+CP4)/6.
102 V1=V1S+(C1V1+2.*(C1V2+C1V3)+C1V4)/6.
103 PHI1=PHI1S+(C1P1+2.*(C1P2+C1P3)+C1P4)/6.

C EVALUATION AND PRINT OF DATA OUTPUT
104 N=N+1
105 IF(N.LE.NSKIP) GO TO 16
106 NCOUNT=MOD(N,NMOD)
107 DIFSQ=(PHIDUM-PH1DUM)**2
108 SUMSQ=SUMSQ+DIFSQ
109 SIGSQ=PH1DUM*PH1DUM
110 SIGSQS=SIGSQS+SIGSQ
111 IF(NCOUNT.NE.0) GO TO 16
112 RMSDUM=RMSERR
113 RMSERR=SQRT(SUMSQ/FLOAT(N-NSKIP))
114 RMSSIG=SQRT(SIGSQS/FLOAT(N-NSKIP))
115 IF (RMSERR.NE.0.) GO TO 20
116 WRITE(3,303)
117 GO TO 21
118 20 CONTINUE
119 SNROUT=20.*ALOG10(RMSSIG/RMSERR)
120 WRITE (3,302) SNROUT,SNRGS
121 21 CONTINUE
122 WRITE (3,300) RMSERR,RMSSIG
123 IF(ABS(RMSDUM-RMSERR)-EPS) 19,19,16
124 19 T=RTM
125 16 CONTINUE
126 IF (RTM-T) 2,2,7
7 T=T+H
GO TO 1
2 CONTINUE
GO TO 99
69 STOP
100 FORMAT(5X,5F10.5,5X,6F10.5,E12.5)
101 FORMAT(9X, TIME, 3X, PHIDOT, 4X, PHI, 7X, V, 6X, VDOT, 10X, IPHIDOT, 4X, PHI, 7X, V, 6X, VDOT, 7X, NS, 6X, NC)
102 FORMAT (///,14X, '++++++SIGNAL PLUS NOISE PARAMETERS++++++',4X, -1------SIGNAL ONLY PARAMETERS--------NOISE COMPONENTS')
200 FORMAT (7F10.5)
201 FORMAT(9I5,I1O,3I5)
300 FORMAT (' ', '++++++RMS ERROR=',1PE12.5,5X,'RMS SIGNAL=',1PE12.5)
302 FORMAT(' THE OUTPUT SNR IS',E12.5,' WHEN THE INPUT SNR IS ',E12.5)
303 FORMAT (' ', '++++++RMS ERR=0. SNROUT IS INFINITY')
500 FORMAT (7(5X,F12.6))
501 FORMAT (///,T40, '++++++SIGNAL PARAMETERS++++++',/,
1, BETA, T43, ' KP1, T60, BETAH')
502 FORMAT (///,T37, '++++++INTERFERENCE PARAMETERS++++++',/,
2HA, T26, DELTAW, T43, DEL, T60, RHO, T77, TD)
503 FORMAT (///,T41, '++++++LOOP PARAMETERS++++++',/,
3 H, T43, WCO, T60, CUTOFF, T77, BW88)
505 FORMAT (///,T41, '++++++NOISE PARAMETERS++++++',/,
5 SNRGS, T42, SNRGS, T60, SIGMA)
509 FORMAT(///, 'NPT=',I5,5X,' NPLOT=',I3,5X,' IMAX=',I4,5X,' NMOD=',I9I4,5X,' IX=',I10,5X,' NPOINT=',I5,5X,' NSKIP=',I5,4X,' NORDER=',I12)
END
SUBROUTINE FCT(VDOT,PHIDOT,PHI,V1DOT,PHI1D,T,PHI1,V1)
REAL KP1,NS,NC,MOD
COMMON WCO,WMOD,KP1,TWOP,IX,BETA,ALPHA,RHO,DELTAW,TD,DEL
SINPHI=SIN(PHI)
COSPHI=COS(PHI)
SIN1PHI=SIN1(PHI)
COS1PH=COS(PHI1)
11 IF(BETA.EQ.0.) GO TO 12
PHIMOD=WMOD*T
MOD=BETA*SIN(PHIMOD)
A1MOD=COS(MOD)
A2MOD=SIN(MOD)
12 IF (ALPHA.EQ.0.) GO TO 13
PHASE=DELTA*T+DEL
A1CW=ALPHA*COS(PHASE)
A2CW=ALPHA*SIN(PHASE)
13 IF (RHO.EQ.0.) GO TO 14
TDD=T-TD
PHIMOT=WMOD*TDD
PHASMP=-WCO*TDD+BETA*COS(PHIMOT)
A1MP=RHO*COS(PHASMP)
A2MP=RHO*SIN(PHASMP)
14 CONTINUE
A1=A1CW+A1MP+NS
A2=A2CW+A2MP+NC
A1=A1MOD+A1I
A2=A2MOD+A2I
15 PHIDOT=WCO*KP1*(A2*COSPHI-A1*SINPHI)/V
VDOT=WCO*(-V+A1*COSPHI*A2*SINPHI)
16 PHIDOT=WCO*KP1*(A2MOD*COS1PH-A1MOD*SIN1PH)/V1
V1DOT=WCO*(-V1+A1MOD*COS1PH+A2MOD*SIN1PH)
17 RETURN
18 END
SUBROUTINE FILTER (T,H,X,Y,XD,YD,FC,NP)
C
DIMENSION XD(10), YD(10), A(10), B(10)
PI=3.14159265
WC=2.*PI*FC
WA=TAN(WC*H/2.)
11 IF (T.GT.0) GO TO 11
12 DO 20 J=1,10
A(J)=0.
B(J)=0.

20 CONTINUE
GO TO (1,2,3,4),NP

C FIRST-ORDER COEFFICIENTS
1 AK= 1.+WA
AO=WA/AK
A(1)=AO
B(1) = (1. - WA)/AK
GO TO 11

C SECOND-ORDER COEFFICIENTS
2 AK=1.+SQRT(2.)*WA+WA**2
AO=WA**2/AK
A(1)=2.*AO
A(2)=AO
B(1) = 2.*(1. - WA**2)/AK
B(2) = (-1. + SQRT(2.)*WA - WA**2)/AK
GO TO 11

C THIRD-ORDER COEFFICIENTS
3 AK=1.+2.*WA+2.*WA**2+WA**3
AO=WA**3/AK
A(1)=3.*AO
A(2)=A(1)
A(3)=AO
B(1) = (-3.*WA**3 - 2.*WA**2 +2.*WA +3.)/AK
B(2) = (-3.*WA**3 +2.*WA**2 +2.*WA -3.)/AK
B(3)=(-WA**3+2.*WA**2-2.*WA+1.)/AK
GO TO 11

C FOURTH-ORDER COEFFICIENTS
4 AK = WA**4 + 2.613*WA**3 + 3.414*WA**2 + 2.613*WA + 1.
AO = WA**4/AK
A(1) = 4.*AO
A(2) = 6.*AO
A(3) = 4.*AO
A(4) = AO
B(1) = (-4.*WA**4 - 5.226*WA**3 + 5.226*WA + 4.)/AK
B(2) = (-6.*WA**4 + 6.828*WA**2 - 6.)/AK
B(3) = (-4.*WA**4 + 5.226*WA**3 - 5.226*WA + 4.)/AK
B(4) = (-WA**4 + 2.613*WA**3 - 3.414*WA**2 + 2.613*WA - 1.)/AK
GO TO 11
11 CONTINUE
SUM=0.
DO 30 I=1,NP
SUM=SUM+A(I)*XD(I)+B(I)*YD(I)
30 CONTINUE
Y=AO*X+SUM
IF(NP.EQ.1) GO TO 40
50 DO 60 I=2,NP
K=NP+2-I
XD(K)=XD(K-1)
YD(K)=YD(K-1)
60 CONTINUE
40 CONTINUE
XD(1)=X
YD(1)=Y
RETURN
END
/DATA
APPENDIX E

An Example of the Use of the Bilinear Transform to Obtain the Digital Filter Equations

A demonstration of the use of the bilinear transform to implement the digital filters used in the simulation is presented in this appendix.

The transfer function of a second order Butterworth filter with 3 dB cutoff frequency at 1 rad/sec is

\[ H(s) = \frac{1}{s^2 + \sqrt{2} s + 1} \]  \hspace{1cm} (E-1)

To obtain a digital filter with cutoff at \( \omega_r \) rad/sec, one uses Eq. (V-9) and Eq. (V-10) to replace \( s \) in Eq. (E-1) by \( \bar{s} \) as:

\[ H(z) = H(s) \bigg|_{s = \bar{s}} = \frac{(1 + z^{-1})^2}{c^2(1 - z^{-1})^2 + \sqrt{2} c (1 - z^{-1})(1 + z^{-1}) + (1 + z^{-1})^2} \]  \hspace{1cm} (E-2)

For a design 3 dB cutoff frequency, \( \omega_r \), of 314 rad/sec, when the sampling period, \( H \), is .002 sec,

\[ c = \frac{\omega_r}{\omega} \cot \left( \frac{\omega H}{2} \right) = \cot \left( \frac{\pi}{10} \right) = 3.07768. \]  \hspace{1cm} (E-3)

Substituting this value into Eq. (E-2), the final result for a 2nd order Butterworth Filter is obtained.
\[ H(z) = \frac{0.0674553 (1 + 2z^{-1} + z^{-2})}{1 - 1.14298 z^{-1} + 0.412802 z^{-2}} \]  

(E-4)

Designs for filters of order other than second are approached in a similar manner.