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Force analysis of involute spur gears

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FORCE ANALYSIS OF INVOLUTE SPUR GEAR

BY

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ABSTRACT

This study is devoted to the fundamental action which occurs between spur gears with involute tooth profiles when power is transmitted at a constant angular velocity ratio. Various aspects of the interaction between two mating teeth are considered. Analytic expressions for the variation of the force transmitted between two teeth are derived. Two cases are considered which include and exclude the effects of sliding friction between the teeth. The parametric equations of the involute curves for two mating teeth are derived in terms of body-fixed coordinates. Analytical expressions for Hertz compressive stresses in the teeth were also obtained. The same equations were used for a dynamic case to show that they can be used for static as well as dynamic analysis. A comparison is made of the force transmitted between the teeth for different values of coefficient of friction. In a similar fashion a comparison is made for the value of the Hertz contact stress for two different pressure angles of the system. Suggestions are made for further work.
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LIST OF SYMBOLS
(In order of appearance)

X, Y  Axes of the fixed co-ordinate system of gear 1
U, V  Axes of the fixed co-ordinate system of gear 2
x₁, y₁ Axes of the moving co-ordinate system of gear 1
x₂, y₂ Axes of the moving co-ordinate system of gear 2
\hat{i}, \hat{j}  Unit vectors for XY and UV set of axes
\hat{i}_1, \hat{j}_1  Unit vectors for the system of axes x₁y₁
\hat{i}_2, \hat{j}_2  Unit vectors for the system of axes x₂y₂
θ₁  Angular displacement of gear 1 with respect to a set of fixed axes XY
θ₂  Angular displacement of gear 2 with respect to a set of fixed axes UV
\vec{R}_P  Position vector of P
P  Point of contact of two gears
φ  Pressure angle of the gear system
x₃, y₃ A set of fixed axes with origin at initial point of contact
\vec{R}_{P₁}  Radius of the base circle of the gear 1
\vec{R}_{P₂}  Radius of the base circle of the gear 2
\vec{R}_{P₁}  Radius of the pitch circle of gear 1
\vec{R}_{P₂}  Radius of the pitch circle of gear 2
ω₁ = \dot{θ₁}  Magnitude of the angular velocity of gear 1
\( \omega_2 = \dot{\theta}_2 \)  
Magnitude of the angular velocity of gear 2

VR  
Angular velocity ratio = \( \frac{\omega_1}{\omega_2} \)

\( \hat{N}_1 \)  
Unit vector in the direction of an outward normal

\( r_1 \)  
Instantaneous radius of curvature of the gear 1

\( r_2 \)  
Instantaneous radius of curvature of the gear 2

F  
Magnitude of the normal force acting on the tooth

\( \mu \)  
Coefficient of friction between the two gears
I. INTRODUCTION

A. GENERAL DISCUSSION

The purpose of gearing is to transmit motion and power from one shaft to another. Usually power is transmitted at a constant angular velocity ratio. Gears often are the weakest link in the machine-element chain. Noise and vibration in gearing are related and can be real measures of quality. Whereas noisy gears may indicate premature wear and failure, smooth and quiet operation indicates long life. A good lubricant helps provide smooth and quiet operation of gears.

Gears play an important role in our society. In fact, Darle W. Dudley\(^1\) has stated, "The people who design and build gears comprise only a small fraction of the people engaged in manufacturing work. However, the people who use gears include almost everyone. It has been said quite truthfully that you can't tell the time of day without gears, you can't wash your clothes without gears, you can't drive to work without gears and you can't make anything in your shop without gears."

Basically, there are three types of gears. The type depends on the specific surfaces in contact to yield a constant angular velocity ratio. The three types are spur, bevel, and skew gears. For spur gears the pitch surface of each gear is a cylinder. In the case of bevel gears the pitch surfaces are represented by two truncated cones.
rolling together without slippage. In the case of skew gears, hyperboloids are used which have a combination of rolling motion normal to the line of contact and sliding motion parallel to the line of contact. This yields the required angular velocity ratio. Also, positioning of the shafts distinguishes each gear type from the other. The spur gears have parallel shafts, while bevel gears have intersecting shafts, and skew gears have non intersecting, non parallel shafts.

It is interesting to note that while entire volumes have been devoted to the study of gear systems, certain basic fundamentals are absent from the existing literature on gears. In particular this study is devoted to the fundamental action which occurs between spur gears with involute tooth profiles when power is transmitted at a constant angular velocity ratio. The study includes the derivation of parametric equations which define the involute tooth profile of two mating gears. The co-ordinates of the profiles are expressed in terms of co-ordinates fixed to the rotating gears.

The variation of the force transmitted between two mating teeth is considered for cases with and without friction. The value of the Hertz contact stress for the maximum compressive stress in the region of the point of contact is computed. The radius of curvature of each involute tooth at the point of contact must be known for this calculation.
To demonstrate that the equations developed apply to the dynamic analysis of spur gears as well as to the static analysis, an example of a flywheel driven by a pair of spur gears through a flexible shaft is considered.

B. REVIEW OF LITERATURE

Earl Buckingham\(^2\) was the pioneer for the development of the analytical mechanics of gears. According to him, the tooth profiles most commonly used are cycloidal and involute. Ham, Crane, and Rogers\(^3\) note that the involute has almost completely replaced the cycloidal profile.

Buckingham\(^2\) gives the following reasons for the dominance of involute profiles in the field:

1. The involute curve is easy to manufacture and is dependent only on the size of the base circle.

2. The path of contact of two involutes acting against each other is a straight line and it is also a common tangent to both the base circles. It is also the line of action.

3. Due to the fact that the path of contact is a straight line, it follows that the pressure angle is always constant.

4. The pitch diameter of two involutes acting together is proportional to their base circle diameters.

However, according to Lavoie\(^4\), there is a trend away from the involute tooth profile, ranging from partial
modification of involute to an apostasy. Also he discusses the use of circular arc tooth profiles.

In his paper 'The synthesis of gear tooth curves', Dr. Beggs\(^5\) gives a method of arriving analytically at a pair of gear tooth curves which will fulfill certain prescribed conditions.

In his dissertation Dr. C. L. Edwards\(^6\), studied a plane tooth mating with a convex tooth profile. According to Dareing\(^7\), these profiles are better from the standpoint of lower contact stresses and better lubricating film formation.

Regardless of the type of tooth profiles selected, all profiles must satisfy the fundamental law of gearing. Most authors state the fundamental law of gearing as: The common normal of two gear teeth in contact passes through a fixed point on the line of centers called the pitch point. However, C. L. Edwards\(^6\) considers that the above law of gearing is applicable only to spur gears. According to him the general law of gearing should satisfy the following criteria:

1. The distance between origins of the co-ordinate systems for each gear is constant.

2. The angular velocity ratio between two gears is constant.

3. At points of contact of two meshing gear teeth, the components of each velocity vector along the common normal are equal.
These criteria were satisfied in the analysis presented here for spur gears with involute profiles.
II. DERIVATION OF GENERAL EQUATIONS

A. POSITION VECTOR EQUATION

Figure 1 contains two fixed co-ordinate systems XY and UV. It should be born in mind that X and U lie on the same line while Y and V are parallel axes. Also, in the same figure there are two moving co-ordinate systems $x_1 y_1$ and $x_2 y_2$. These systems are fixed to gears 1 and 2 respectively. Gear 1 rotates in a counterclockwise sense about the point $0_1$ while gear 2 rotates in a clockwise sense about point $0_2$.

The tooth surfaces of two gears are in contact at some point $P$ as shown in fig. 1. $\vec{R}_1$ is the position vector of $P$ relative to the $x_1 y_1$ system. $\vec{R}_2$ is the position vector of $P$ relative to the $x_2 y_2$ system. $\vec{R}$ is a constant position vector from $0_1$ to $0_2$.

The closed position vector loop equation for the loop $0_1 P 0_2 0_1$ will be,

$$\vec{R}_1 = \vec{R} + \vec{R}_2 \quad (1)$$

Let

$$\vec{R}_1 = x_1 \hat{i}_1 + y_1 \hat{j}_1$$

$$\vec{R}_2 = x_2 \hat{i}_2 + y_2 \hat{j}_2$$

$$\vec{R} = D \hat{i} .$$
Figure 1. Co-ordinate Systems
Where \( x_1, y_1, x_2, y_2 \) are to be determined and \( D \) is the center distance between gears 1 and 2. \( \hat{i}, \hat{i}_1, \hat{i}_2, \hat{j}, \hat{j}_1, \hat{j}_2 \) are all unit vectors along the \( X, x_1, x_2, Y, y_1, y_2 \) axes respectively.

From fig. 2 and fig. 3 we obtain the following relationships:

\[
\hat{i}_1 = \hat{i} \cos \theta_1 + \hat{j} \sin \theta_1
\]

\[
\hat{j}_1 = \hat{j} \cos \theta_1 - \hat{i} \sin \theta_1
\]

\[
\hat{i} = \hat{i}_2 \cos \theta_2 + \hat{j}_2 \sin \theta_2
\]

\[
\hat{j} = -\hat{i}_2 \sin \theta_2 + \hat{j}_2 \cos \theta_2
\]

Using the above relations,

\[
\hat{i}_1 = \cos(\theta_1 + \theta_2) \hat{i}_2 + \sin(\theta_1 + \theta_2) \hat{j}_2
\]

\[
\hat{j}_1 = -\sin(\theta_1 + \theta_2) \hat{i}_2 + \cos(\theta_1 + \theta_2) \hat{j}_2
\]

Substituting these unit vector relations into eq. (1) yields two scalar equations such that,

\[
x_2 = x_1 \cos(\theta_1 + \theta_2) - y_1 \sin(\theta_1 + \theta_2) - D \cos \theta_2 \tag{2}
\]

\[
y_2 = x_1 \sin(\theta_1 + \theta_2) + y_1 \cos(\theta_1 + \theta_2) - D \sin \theta_2 \tag{3}
\]

Equations (2) and (3) must be satisfied in order for the point of contact to be common to both the gears.
Figure 2. Unit vector transformations, Gear 1
Figure 3. Unit vector transformations, Gear 2
B. INVOLUTE TOOTH PROFILE EQUATIONS

The gears considered in this study have an involute tooth profile. By its definition, an involute is a curve traced out by a fixed point on a string when the string is rolled or unrolled from a base circle. By the property of involutes, the path of the point of contact is a straight line tangent to both the base circles. The assumption was made that the point of contact starts at the base circle of gear 1, at an angle $\alpha$, below the X axis. From the configuration of fig. 4, it should be noted that,

$$\alpha = \phi$$

where $\phi$ = pressure angle of the two gears.

Let $x_3y_3$ be a fixed reference frame with its origin at $O_3$, the initial point of contact. Let $P$ be the point of contact which moves along the $x_3$ axis. The location of any point of contact can be given as $(s,0)$ in the $x_3y_3$ system where $s$ varies along the $x_3$ axis. It should be noted that the path of contact will always be tangent to both base circles,

i.e. $y_3 = 0$.

Also let us assume that initially at $O_3$, $s=0$ and $\theta_1=0$ which implies that $\theta_2=0$. Then for any $s$ thereafter,

$$s = R_1 \theta_1,$$
Figure 4. Path of Contact
where: $R_{b_1}$ - Radius of base circle of gear 1.

$\theta_1$ - Angle of rotation of gear 1.

The location of any point of contact $P$ will be given by

$$x_3 = R_{b_1} \theta_1,$$

$$y_3 \equiv 0.$$

It should also be noted that the maximum value of $\theta_1$ occurs when $P$ is at the other extreme, i.e. at $(L,0)$.

But $L = 0_1 0_2 \sin \phi$

$$R_{b_1} \theta_1 \text{(max)} = D \sin \phi$$

$$\theta_1 \text{(max)} = D \sin \phi / R_{b_1}.$$

Now let us transform these co-ordinates of the point of contact into the body fixed system of axes $x_1 y_1$.

As shown by fig. 5 the position vector of $P$ is,

$$\vec{R}_p = (R_{p_1} \cos \phi \hat{i} - R_{p_1} \sin \phi \hat{j})$$

$$+ R_{b_1} \theta_1 (\sin \phi \hat{i} + \cos \phi \hat{j}).$$

Using the relations between the unit vectors obtained earlier and simplifying the expression,

$$\vec{R}_p = R_{b_1} [\cos \theta_1 (\cos \phi + \theta_1 \sin \phi) - \sin \theta_1 (\sin \phi$$

$$- \theta_1 \cos \phi)] \hat{i}_1.$$
Recalling that

\[ R_{p} = R_1 = x_1 i_1 + y_1 j_1; \]

\[ x_1 = R_{b_1} [\cos(\phi + \theta_1) + \theta_1 \sin(\phi + \theta_1)]; \quad (4) \]

\[ y_1 = R_{b_1} [\theta_1 \cos(\phi + \theta_1) - \sin(\phi + \theta_1)]. \quad (5) \]

Equations (4) and (5) are the parametric equations of the involute curve in the \( x_1 y_1 \) frame. They are valid in the region \( 0 \leq \theta_1 \leq \theta_1(\text{max}). \)

When the values of \( x_1 \) and \( y_1 \) are computed from eq. (4) and (5), the profile of the involute tooth on gear 1 is defined. The profile of the tooth on gear 2 can be determined in a similar manner by substitution of \( x_1 \) and \( y_1 \) into eqs. (2) and (3). It must be remembered that \( \theta_2 \) is related to \( \theta_1 \) by the requirement of a constant angular velocity ratio so that

\[ VR = \frac{\omega_1}{\omega_2} = \frac{\dot{\theta}_1}{\dot{\theta}_2} = \frac{\Delta \theta_1}{\Delta \theta_2} = \frac{\theta_1}{\theta_2}. \quad (6) \]

where \( VR = \text{Angular velocity ratio}. \)

C. VERIFICATION THAT INVOLUTE PROFILES SATISFY THE FUNDAMENTAL LAW OF GEARING

It is not difficult to show that the equations of the
Figure 5. Relative orientation of three co-ordinate systems
tooth profiles derived here do indeed satisfy the fundamental law of gearing. If $P_1$ is the point in gear 1 that is in contact with point $P_2$ in gear 2, then

$$\mathbf{R}_{P_1/XY} = x_1\mathbf{i}_1 + y_1\mathbf{j}_1$$  \hspace{1cm} (7a)$$

$$\mathbf{R}_{P_2/UV} = x_2\mathbf{i}_2 + y_2\mathbf{j}_2$$  \hspace{1cm} (7b)$$

where $x_1, y_1, x_2,$ and $y_2$ are constants because they define a particular point in gear 1 and gear 2 respectively. Then the velocity vectors of points $P_1$ and $P_2$ can be written as

$$\mathbf{V}_{P1/XY} = (-y_1\mathbf{i}_1 + x_1\mathbf{j}_1)\dot{\theta}_1$$  \hspace{1cm} (8a)$$

$$\mathbf{V}_{P2/XY} = (y_2\mathbf{i}_2 - x_2\mathbf{j}_2)\dot{\theta}_2$$  \hspace{1cm} (8b)$$

The component of $\mathbf{V}_{P1/XY}$ along the outward normal to the tooth on gear 1 can be obtained by

$$\mathbf{V}_{P1/XY} \cdot \mathbf{N}_1 = \mathbf{V}_{P1/XY} \cdot [\sin(\phi + \theta_1)\mathbf{i}_1 + \cos(\phi + \theta_1)\mathbf{j}_1]$$ (9)$$

The outward normal $\mathbf{N}_1$ is defined in eq. (20) found in chapter III.

Substituting from eq. (8a) for $\mathbf{V}_{P1/XY}$ yields

$$\mathbf{V}_{P1/XY} \cdot \mathbf{N}_1 = [x_1 \cos(\theta_1 + \phi) - y_1 \sin(\theta_1 + \phi)]\dot{\theta}_1$$  \hspace{1cm} (10)$$

which can be reduced after substituting from eq. (4) and (5) for $x_1$ and $y_1$. 
The component of $\vec{V}_{p1/XY}$ along the same outward normal if $\hat{N}_1$ is written as,

$$\hat{N}_1 = \sin(\phi - \theta_2) \hat{i}_2 + \cos(\phi - \theta_2) \hat{j}_2$$  \hspace{1cm} (12)

so that $\vec{V}_{p2/XY} \cdot \hat{N}_1 = [y_2 \sin(\phi - \theta_2) - x_2 \cos(\phi - \theta_2)] \dot{\theta}_2$ (13)

Eq. (13) can then be simplified by substituting from eqs. (2) and (3) for $x_2$ and $y_2$ to yield the result

$$\vec{V}_{p2/XY} \cdot \hat{N}_1 = R_{b_2} \dot{\theta}_2$$  \hspace{1cm} (14)

The fundamental law of gearing requires that the velocity components given by eqs. (14) and (11) must be the same. Hence we obtain

$$R_{b_1} \dot{\theta}_1 = R_{b_2} \dot{\theta}_2$$  \hspace{1cm} (15)

Eq. (15) can be written as

$$\frac{\dot{\theta}_1}{\dot{\theta}_2} = \frac{R_{b_2}}{R_{b_1}} = \frac{R_{p_2} \cos \phi}{R_{p_1} \cos \phi} = \frac{R_{p_2}}{R_{p_1}}$$  \hspace{1cm} (16)

where $R_{p_1}$ and $R_{p_2}$ are radii of pitch circles of gears 1 and 2 respectively. Eq. (16) is a form of eq. (6). This verifies that involute profiles satisfy this requirement and hence the fundamental law of gearing.
III. FORCE AND STRESS EQUATIONS

A. DERIVATION OF FORCE EQUATION

In the absence of friction, the force acting between two teeth is normal to the face of each tooth and its direction is opposite to that of an outward normal.

From fig. 6(a) we have,

\[ \mathbf{F} = F(-\hat{N}_1) \]

where

- \( F \) - magnitude of force acting
- \( \hat{N}_1 \) - unit vector in the direction of an outward normal which is to be determined.

From fig. 6(b), we note that

\[ \tan \gamma = \frac{dy_1}{dx_1} \]

But from eq. (4),

\[ x_1 = R_{b_1} \left[ \cos(\theta_1 + \phi) + \theta_1 \sin(\theta_1 + \phi) \right] \]

Therefore, \[ \frac{dx_1}{d\theta_1} = R_{b_1} \left[ - \sin(\theta_1 + \phi) + \sin(\theta_1 + \phi) \right. \]

\[ \left. + \theta_1 \cdot \cos(\theta_1 + \phi) \right] \]

\[ \frac{dx_1}{d\theta_1} = R_{b_1} \cdot \theta_1 \cdot \cos(\theta_1 + \phi) \quad (17) \]

Similarly, from eq. (5)

\[ y_1 = R_{b_1} \left[ \theta_1 \cdot \cos(\theta_1 + \phi) - \sin(\theta_1 + \phi) \right] \]
Figure 6. Outward Normal Unit Vector
and

$$\frac{dy_1}{d\theta_1} = R_{b_1} [\cos(\theta_1 + \phi) - \cos(\theta_1 + \phi) - \theta_1 \sin(\theta_1 + \phi)]$$

$$= - R_{b_1} \cdot \theta_1 \cdot \sin(\theta_1 + \phi) \quad (18)$$

\[\therefore \tan \gamma = \frac{dy_1}{dx_1} \]

$$= - \frac{\sin(\theta_1 + \phi)}{\cos(\theta_1 + \phi)}$$

The slope of the outward normal will be

$$- \left[ \frac{dy_1}{dx_1} \right] = - \frac{dx_1}{dy_1}$$

Referring to fig. 6(b),

$$\tan \beta = - \frac{dx_1}{dy_1} = \frac{\cos(\theta_1 + \phi)}{\sin(\theta_1 + \phi)} \quad (19)$$

\[\hat{N}_1 = \text{Unit vector in the direction of outward normal to tooth 1 will be,} \]

$$\hat{N}_1 = \sin(\theta_1 + \phi) \hat{i}_1 + \cos(\theta_1 + \phi) \hat{j}_1 \quad (20)$$

The outward normal to the tooth on gear 2 is $-\hat{N}_1$.

Therefore, the force transmitted between the teeth is,

$$\bar{F} = F (- \sin(\theta_1 + \phi) \hat{i}_1 - \cos(\theta_1 + \phi) \hat{j}_1) \quad (21)$$

where $F$ = magnitude of the force on the tooth.

Now let us analyze the variation of the force $\bar{F}$ as the gears rotate and the point of contact moves along the tooth.
1. Neglecting Friction

Setting up the system as in fig. 7(a), if a constant input torque $\bar{T}_1$ is acting on gear 1, then:

$$\sum \bar{T} = I_1 \ddot{\omega}_1 = \vec{0} \text{ since } \dot{\omega}_1 = \omega_1 = \text{constant.}$$

$$\bar{T}_1 + \bar{R}_1 \times \vec{F} = \vec{0}$$

$$T_1 \hat{k}_1 + (x_1 \hat{i}_1 + y_1 \hat{j}_1) \times (-F \sin(\theta_1 + \phi)) \hat{i}_1$$

$$-F \cos(\theta_1 + \phi) \hat{j}_1 = \vec{0} \quad (22)$$

$$\hat{R} \times \vec{F} = \begin{pmatrix} \hat{i}_1 & \hat{j}_1 & \hat{k}_1 \\ x_1 & y_1 & 0 \\ -F \sin(\theta_1 + \phi) & -F \cos(\theta_1 + \phi) & 0 \end{pmatrix}$$

$$= [F \ y_1 \sin(\theta_1 + \phi) - F \ x_1 \cos(\theta_1 + \phi)] \hat{k}_1 \quad (23)$$

From eq. (22),

$$T_1 \hat{k}_1 + [F \ y_1 \sin(\theta_1 + \phi) - F \ x_1 \cos(\theta_1 + \phi)] \hat{k}_1 = \vec{0}$$

Therefore,

$$T_1 = F [x_1 \cos(\theta_1 + \phi) - y_1 \sin(\theta_1 + \phi)]$$

Substituting for $x_1$ and $y_1$ from eqs. (4) and (5),

$$T_1 = F \cdot R_{b_1}$$

or,
Figure 7. Normal and friction forces
Equation (24) shows that although the position vector $\overline{R}_1$ changes at every instant, the magnitude of the force acting normal to tooth surface is constant for the assumption of constant input torque and neglecting friction.

2. With Friction

Now let us assume that a frictional force is present. Let $\mu$ be the co-efficient of friction between the two gear teeth in contact. The force transmitted between the teeth is then modified as shown in fig. 7(b) to include a component, tangent to the gear tooth.

As before,

$$\sum \overline{T} = \dot{I}_1 \overline{\dot{R}}_1 = \overline{0} \quad \text{for} \quad \dot{\theta}_1 = \omega_1 = \text{constant}.$$  

$$\overline{T}_1 + \overline{R}_1 \times \overline{F}_1 + \overline{R}_1 \times \overline{F}_F = \overline{0} \quad \text{(25)}$$

$$\overline{R}_1 \times \overline{F}_F = (x_1 \hat{i}_1 + y_1 \hat{j}_1) \times (\mu F_1 \{\cos(\theta_1 + \phi) \hat{i}_1 - \sin(\theta_1 + \phi) \hat{j}_1\})$$

$$\begin{vmatrix} \hat{i}_1 & \hat{j}_1 & \hat{k}_1 \\ x_1 & y_1 & 0 \\ + \mu F_1 \cos(\theta_1 + \phi) & -\mu F_1 \sin(\theta_1 + \phi) & 0 \end{vmatrix}$$

$$= \mu F_1 \left[-x_1 \sin(\theta_1 + \phi) - y_1 \cos(\theta_1 + \phi)\right] \hat{k}_1 \quad \text{(26)}$$

Using eqs. (4) and (5) to substitute for $x_1$ and $y_1$, we have
Substituting back in eq. (25)

\[ T_1 = R_{b_1} \cdot F_1 + \mu F_1 R_{b_1} \cdot \theta_1 \]

\[ \therefore T_1 = F_1 \cdot R_{b_1} [1 + \mu \theta_1] \tag{28} \]

For the given gear system the radius of the base circle is constant. Imposing a constant input torque, we have both, torque and radius of base circle as constants. Defining a non-dimensional quantity,

\[ Q = F_1 \cdot \frac{R_{b_1}}{T_1} \]

Eq. (28) becomes

\[ Q = \frac{1}{1 + \mu \theta_1} = \frac{F_1 (\text{with friction})}{F_1 (\text{without friction})} \tag{29} \]

It can be seen that \( F_1 \) is directly proportional to \( Q \).

A sample curve of the variation of the ratio of force with friction to force without friction is shown in fig. (8). A computer program was devised to compute this variation. The maximum variation depended on the maximum value of \( \theta_1 \) which in turn was dependent on the pressure angle \( \phi \) and center distance \( D \) between the two gears.

The curves in fig. (8) are for a center distance of 15", \( VR = 4 \), and a pressure angle of 20°. Three values of coefficient of friction were used with this data. For the above data the minimum ratio of \( F_1 \) was 0.652 as seen from
Figure 8. Curve for Variation of Ratio of Forces
fig. (8) for a $\mu = 0.3$.

B. CONTACT STRESS EQUATION

The gear teeth must be sufficiently strong to carry the wear load $F_w$ arising from the constant stress between the teeth.

As shown in fig. (9) stress is usually calculated by considering the mating teeth as two parallel cylinders in contact. Approximating the teeth as cylinders is fairly satisfactory, for although the compressive stresses are very high, they decrease rapidly at locations removed from the area of contact.

The maximum compressive stress $P_o$, in the contact zone is given by

$$P_o = 0.591 \sqrt{\frac{P_1 E_1 E_2}{E_1 + E_2} \left(\frac{1}{r_1} + \frac{1}{r_2}\right)}$$  (30)

where $r_1$ and $r_2$ are the radii of curvatures of the two teeth respectively.

In our case

$$P_1 = \frac{F}{t}$$

where $t = \text{width of tooth}$

and

$$E_1 = E_2 = E$$

$$P_o = 0.591 \sqrt{\frac{F \cdot E}{2 \cdot t} \left[\frac{1}{r_1} + \frac{1}{r_2}\right]}$$

As shown earlier, neglecting friction,
Figure 9. Contact Stresses
\[ F = \frac{T_1}{R_{b_1}} \]

\[ \therefore P_0 = 0.591 \sqrt{\frac{T_1 \cdot E}{t}} \sqrt{\frac{1}{2 \cdot R_{b_1}} \left( \frac{1}{r_1} + \frac{1}{r_2} \right)} \] (32)

In the above expression, for a given set, \( T_1, E, \) and \( t \) are constants.

Let \( A = P_0 \cdot \sqrt{\frac{t}{T_1 \cdot E}} \)

\[ \therefore A = 0.591 \sqrt{\frac{1}{2 \cdot R_{b_1}} \left( \frac{1}{r_1} + \frac{1}{r_2} \right)} \] (33)

Now let us calculate \( r_1 \) and \( r_2 \), i.e. radii of curvature of gear 1 and gear 2 respectively. From calculus,

\[
\begin{align*}
    r_1 &= \left[ 1 + \left( \frac{dy_1}{dx_1} \right)^2 \right]^{3/2} \\
    &= \frac{\left[ 1 + \left( \frac{dy_1}{dx_1} \right)^2 \right]^{3/2}}{d^2y_1/dx_1^2}
\end{align*}
\]

Similarly

\[
\begin{align*}
    r_2 &= \left[ 1 + \left( \frac{dy_2}{dx_2} \right)^2 \right]^{3/2} \\
    &= \frac{\left[ 1 + \left( \frac{dy_2}{dx_2} \right)^2 \right]^{3/2}}{d^2y_2/dx_2^2}
\end{align*}
\]

As before,

\[
\begin{align*}
    \frac{dy_1}{d\theta_1} &= -R_{b_1} \cdot \theta_1 \cdot \sin(\theta_1 + \phi) \\
    \frac{dx_1}{d\theta_1} &= R_{b_1} \cdot \theta_1 \cdot \cos(\theta_1 + \phi)
\end{align*}
\]
\[ \frac{d^2y_1}{dx_1^2} = \frac{d(y')}{d\theta_1} / \left( \frac{dx_1}{d\theta_1} \right) \]

After simplification,

\[ \frac{d^2x_1}{d\theta_1^2} = -R_b \left[ \theta_1 \cos(\theta_1 + \phi) + \sin(\theta_1 + \phi) \right] \] (34)

\[ \frac{d^2x_1}{d\theta_1^2} = R_b \left[ \cos(\theta_1 + \phi) - \theta_1 \cdot \sin(\theta_1 + \phi) \right] \] (35)

After simplification,

\[ \frac{d^2y_1}{d\theta_1^2} \cdot \frac{dx_1}{d\theta_1} - \frac{dy_1}{d\theta_1} \cdot \frac{d^2x_1}{d\theta_1} = -R_b^2 \cdot \theta_1^2 \]

\[ \frac{dx_1}{d\theta_1} = R_b \cdot \theta_1 \cdot \cos(\theta_1 + \phi) \]
\[ \cdot \left( \frac{dx_1}{d\theta_1} \right)^3 = R_{b_1}^3 \cdot \theta_1^3 \cdot \cos^3(\theta_1 + \phi) \]

\[ \cdot \frac{d^2 y_1}{dx_1^2} = - \frac{1}{R_{b_1} \cdot \theta_1 \cdot \cos^3(\theta_1 + \phi)} \]

To get \( r_2 \), using equations (2) and (3),

\[ x_2 = x_1 \cos(\theta_1 + \theta_2) - y_1 \sin(\theta_1 + \theta_2) - D \cos \theta_2 \]

\[ y_2 = x_1 \sin(\theta_1 + \theta_2) + y_1 \cos(\theta_1 + \theta_2) - D \sin \theta_2 \]

Substituting for \( x_1 \) and \( y_1 \) from equations (4) and (5)

\[ x_2 = R_{b_1} \cdot \cos[\theta_1(1 + 1/VR)][\cos(\theta_1 + \phi) + \theta_1 \cdot \sin(\theta_1 + \phi)] \]

\[ - R_{b_1} \cdot \sin[\theta_1(1 + 1/VR)][\theta_1 \cdot \cos(\theta_1 + \phi) - \sin(\theta_1 + \phi)] \]

\[ - D \cos \theta_2 \]

Let for convenience say \((1 + 1/VR) = g\)

\[ \cdot \frac{dx_2}{d\theta_1} = R_{b_1} \cdot \theta_1 \cdot \cos(\phi + \theta_1 - g\theta_1) \]

\[ + g \cdot R_{b_1} \cdot \sin(\phi + \theta_1 - g\theta_1) \]

\[ - g \cdot R_{b_1} \cdot \theta_1 \cdot [\cos(\phi + \theta_1 - g\theta_1)] \]

\[ + \frac{D}{VR} \cdot \sin(\theta_1/VR) \]
\[
\frac{d^2 x_2}{d\theta_1^2} = (1 - g) R_{b_1} \cdot \theta_1 \cdot \cos(\phi + \theta_1 - g\theta_1)
\]
\[
+ g \cdot R_{b_1} \cdot \sin(\phi + \theta_1 - g\theta_1)
\]
\[
+ \frac{D}{VR} \cdot \sin(\theta_1/VR)
\]  

(36)

\[
\frac{d^2 x_2}{d\theta_1^2} = (1 - g) R_{b_1} \cdot \cos(\phi + \theta_1 - g\theta_1)
\]
\[
- (1 - g)^2 R_{b_1} \cdot \theta_1 \cdot \sin(\phi + \theta_1 - g\theta_1)
\]
\[
+ g(1 - g) R_{b_1} \cdot \cos(\phi + \theta_1 - g\theta_1)
\]
\[
+ \frac{D}{(VR)^2} \cdot \cos(\theta_1/VR)
\]  

(37)

In a similar fashion, we have

\[
y_2 = R_{b_1} \cdot \sin g\theta_1 [\cos(\theta_1 + \phi) + \theta_1 \sin(\theta_1 + \phi)]
\]
\[
+ R_{b_1} \cdot \cos g\theta_1 [\theta_1 \cdot \cos(\theta_1 + \phi) - \sin(\theta_1 + \phi)]
\]
\[
- D \sin(\theta_1/VR)
\]

\[
\frac{dy_2}{d\theta_1} = -(1 - g) R_{b_1} \cdot \theta_1 \cdot \sin(\phi + \theta_1 - g\theta_1)
\]
\[
+ g \cdot R_{b_1} \cdot \cos(\phi + \theta_1 - g\theta_1)
\]
\[ \frac{d^2 y_2}{d \theta_1^2} = -(1 - g) \cdot R_{b_1} \cdot \sin(\phi + \theta_1 - g\theta_1) \]
\[ - (1 - g)^2 R_{b_1} \cdot \theta_1 \cdot \cos(\phi + \theta_1 - g\theta_1) \]
\[ - g(1 - g)R_{b_1} \cdot \sin(\phi + \theta_1 - g\theta_1) \]
\[ + \frac{D}{(VR)^2} \sin(\frac{\theta_1}{VR}) \]  

Since all the terms on the right hand side of the above equation are known, it is easy to compute \( \frac{d^2 y_2}{dx_2^2} \).

Then,

\[ r_1 = \left| \frac{[1 + \left( \frac{dy_1}{dx_1} \right)^2]^{3/2}}{\frac{d^2 y_2}{dx_2^2}} \right| \]

and,

\[ \text{...} \]
Computing $r_1$ and $r_2$ from eqs. (20) and (41) and using them in eq. (33) we can obtain the variation of contact stress along the tooth face.

As seen from fig. 10, the curve is an asymptote to the vertical at both end points. Also the curve shows the stress for a $14^{10}_2$ pressure angle tooth and $20^0$ pressure angle tooth for the same system of gears. The center distance was 15" and VR was 4.

It should also be noted that, the shape of both the curves is the same, but the values of stress changes considerably for larger values of $\theta_1$. The value of contact stress at the pitch point in both cases is shown in fig. 10.
\[ A = \text{(Hertz Stress)} \cdot \text{constant} \]

\[ \phi = 14.5^\circ \]

\[ \phi = 20^\circ \]

Hertz stress at pitch points

Figure 10. Comparison of Stress for Various Pressure Angles
IV. DYNAMIC RESPONSE

To demonstrate that the equations obtained can also be used for dynamic analysis, let us attach a flywheel and elastic shaft to gear 2 as shown in fig. (11). Gear 1 is driven at a constant angular rate by an applied torque $T_1$. The shaft of gear 1 and the gear teeth themselves are assumed to be rigid.

The flywheel has a mass moment of inertia $I$ - in lb·sec$^2$-in. Its value is given by $\frac{1}{2} \frac{W}{g} r^2$, where $r$ is the radius of the flywheel in inches and $W$ is the weight of flywheel in lbs. Let $K$ be the torsional stiffness of the shaft. Its value is $\frac{G J}{l}$ where $G$ is the modulus of rigidity of the shaft material in psi. $J$ is the polar moment of inertia of the cross-sectional areas. It is given by $\frac{\pi}{32} d^4$ where $d$ is the diameter of the shaft in inches and $l$ is the length of shaft in inches.

Setting up the equations of motion from fig. (11), we have,

\[-K(\theta_3 - \theta_2) = I \ddot{\theta}_3\]  \hspace{1cm} (42)

\[\ddot{\theta}_3 + K \theta_3 = K \theta_2\]  \hspace{1cm} (43)

It should be noted that $\theta_2 = \omega_2 \cdot t$, where $\omega_2$ is constant. The general solution of eq. (43) will consist of two parts; a complimentary, and a particular solution.
Figure 11. Dynamic System
\[ \theta_3 = c_1 \cos \sqrt{\frac{K}{I}} \cdot t + c_2 \sin \sqrt{\frac{K}{I}} \cdot t \quad (44) \]

\[ \dot{\theta}_3 = \omega_2 \cdot t \quad (45) \]

c_1 and c_2 are arbitrary constants to be determined.

Let the initial conditions be \( \theta_3(0) = 0.1 \) radian and \( \dot{\theta}_3(0) = \omega_2 \).

\[ \therefore \quad \theta_3 = c_1 \cos \sqrt{\frac{K}{I}} \cdot t + c_2 \sin \sqrt{\frac{K}{I}} \cdot t + \omega_2 t \]

\[ c_1 + 0 + 0 = 0.1 \]

\[ c_1 = 0.1 \]

\[ \dot{\theta}_3 = -c_1 \sqrt{\frac{K}{I}} \sin \sqrt{\frac{K}{I}} \cdot t + c_2 \sqrt{\frac{K}{I}} \cos \sqrt{\frac{K}{I}} \cdot t + \omega_2 \]

\[ \dot{\theta}_3(0) = c_2 \sqrt{\frac{K}{I}} + \omega_2 \]

\[ c_2 \sqrt{\frac{K}{I}} = 0 \quad (\therefore \quad \dot{\theta}_3(0) = \dot{\theta}_2(0) = \omega_2) \]

\[ \sqrt{\frac{K}{I}} \neq 0 \]

\[ \therefore \quad c_2 = 0 \]

\[ \theta_3 = 0.1 \cos \sqrt{\frac{K}{I}} \cdot t + \omega_2 \cdot t \]

\[ \theta_3 = 0.1 \cos \sqrt{\frac{K}{I}} \cdot t + \dot{\theta}_2 \quad (46) \]

The natural frequency of vibration of this system is
$\omega_n = \sqrt{\frac{K}{I}}$ where $K = GJ/1$

Now,

$-K (\theta_3 - \theta_2) = T_{\text{shaft}}$

Therefore,

$-K[\theta_2 + 0.1 \cos \sqrt{\frac{K}{I}} \cdot t - \theta_2] = T_{\text{shaft}}$

\[\therefore T_{\text{shaft}} = -0.1 K \cos \sqrt{\frac{K}{I}} \cdot t \quad (47)\]

Ignoring the mass of the shaft itself, the torque $T_{\text{shaft}}$ is exerted on gear 2 and consequently the torque of gear 1 can be determined from eq. (24) as

$T_1 = \frac{R_{b1}}{R_{b2}} T_2$

\[\therefore F = -0.1 \frac{K}{R_{b2}} \cos \sqrt{\frac{K}{I}} \cdot t \quad (48)\]

For a particular gear set (the data appears in Appendix A) the variation of force $F$ was studied. The curve in fig. (12) shows this variation for different values of $\theta_1$. This demonstrates that the equations developed earlier can be applied to the dynamic analysis of a gear system as well as to the static analysis.
Figure 12. Response of the Dynamic System
V. CONCLUSION

The following conclusions for involute tooth profile spur gears can be drawn from the preceding analysis:

1. The involute tooth profiles defined by the parametric equations derived here satisfy the fundamental law of gearing.

2. For a constant input torque and neglecting friction, the magnitude of the force transmitted between two teeth does not vary with the position of the point of contact on the tooth face.

3. Assuming a tangential friction force proportional to the normal force magnitude, the force transmitted between the teeth is reduced for a constant torque applied to gear 1. The extent of reduction depends upon the value of the coefficient of friction between the two mating teeth.

4. The Hertz contact stress is infinite when the point of contact lies at the base circle of gear 1 or gear 2. However, the value of the stress drops rapidly as the point of contact moves away from the base circles. The stress is almost constant during part of the cycle.

5. In general a 20° pressure angle gives lower contact stress values than a 14.5° system transmitting the same power at the same speed.
6. The equations derived for the static force analysis can also be applied to problems involving dynamic force analysis.

This investigation could be expanded to include the case where more than one pair of teeth are in contact at a given instant. Similar considerations could be given to tooth profile forms other than the involute. The work could also be expanded to cover systems other than spur gears.
BIBLIOGRAPHY


VITA

The author was born on March 25, 1949, in Ahmedabad, India. He graduated from the New Era High School, Baroda, India, in June, 1965. He received the Bachelor of Engineering degree in Mechanical Engineering in June, 1970. He came to the United States for further studies and since January 1971 has been working towards Master of Science degree in Mechanical Engineering at the University of Missouri-Rolla.
APPENDIX A

GEAR DATA

The gear set and flywheel in the foregoing computation following data was used.

Pinion Diameter Gear 1 = 6.0 in.
Gear Diameter Gear 2 = 24.0 in.
Angular velocity of Gear 1 = 2800 rpm
Modulus of rigidity of shaft G = 11.5 x 10^6 psi
Diameter of the shaft d = 8.0 in.
Length of the shaft L = 40.0 in.
Weight of the flywheel w = 1500 lbs.
Radius of the flywheel r = 30.0 in.
Pressure angle between two gears = 20°
Initial rotation of the flywheel = 0.1 radian