Comparison of plastic stress-strain material properties as determined from conventional experiments

Wen Mo Chen
COMPARISON OF PLASTIC STRESS-STRAIN MATERIAL PROPERTIES
AS DETERMINED FROM CONVENTIONAL EXPERIMENTS

BY
WEN MO CHEN - 1939-

A
THESIS
submitted to the faculty of
THE UNIVERSITY OF MISSOURI AT ROLLA
in partial fulfillment of the requirements for the
Degree of
MASTER OF SCIENCE IN ENGINEERING MECHANICS
Rolla, Missouri
1966

Approved by
Robert L. Davis (advisor)  R. A. Schaefer

Wm. A. Andrews  John T. Best
I. PURPOSE AND OBJECTIVES

In practice, the stress-strain curves of materials, as obtained from different loading conditions, are generally not the same. However, in theory, it can be shown that by appropriate selection of stress and strain related parameters, the stress-strain curves obtained from a single material, subjected to tension, compression, torsion and bending loads, will coincide. Like materials have been tested under the aforementioned loading conditions, and these observations have been used to construct the appropriate stress-strain diagrams. It is the primary purpose of this study to present and discuss the relative agreement between the theoretical and experimental results.
ABSTRACT

In problems of engineering design, it is usually assumed that the stress-strain diagrams of a given material, as obtained from different loading conditions, are the same. In fact, by using the classical theory of plasticity, it can be shown that, with an appropriate selection of stress and strain related parameters, the diagrams should coincide.

Three different materials - magnesium, aluminum and steel, have been tested under tension, compression, torsion and bending loads. Stress-strain diagrams were constructed from these observations. This thesis presents an analysis of the relative agreement between the theoretical and experimental results of these materials.
ACKNOWLEDGEMENT

The author wishes to express his sincere appreciation to his advisor, Dr. Robert L. Davis, Assistant Professor of Engineering Mechanics, for his assistance in the completion of this study.

The author would also like to take this opportunity to thank all the other professors and his friends who have sincerely instructed and helped him to overcome the difficulties that he has encountered.
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II. INTRODUCTION

When a material is subjected to external forces its behavior depends not only upon the magnitudes of the forces and the strength and the shape of the material itself, but also upon the manner in which the forces are applied. Metals, for example, generally exhibit relatively high tenacity and are therefore better suited to resisting tensile loads. Brittle materials, such as mortar, concrete, brick and cast iron, have tensile strengths that are low compared with their compressive strengths, and are principally employed to resist compressive forces. Again, the tensile strength of wood is relatively high, but it cannot always be used in structural members because of its low shear resistance, which causes failure before the full tensile resistance of a member can be developed.

Before an engineer designs a structure, a stress analysis must be made. It is usually assumed that the material follows Hooke's Law, and the modulus of elasticity of the material is known. But for selecting the safe dimension of a structure, this knowledge is not sufficient. The designer must know the behavior of the material beyond the proportional limit under various stress conditions. Information of this type can be obtained only by experimental investigations. Material-testing laboratories are equipped with test machines for performing tensile, compressive, torsional and bending tests of a specimen that
will provide added information on material behavior.

In plastic design, it is generally assumed that the stress-strain curves for tension, compression and bending are identical. Thus, when an engineer designs a structure, the property of the material to be used is obtained from a simple tension test, and these results represent the basis of design regardless of the nature of loading or state of stress in the structure. In practice, the behavior of a given material, when subjected to various loading conditions, is not the same, and these deviations must be accounted for. This fact is extremely important in plastic analysis, and is discussed further in Section V.
III. THEORY OF STRESS-STRAIN CURVES OBTAINED
FROM VARIOUS LOADING CONDITIONS

1. Stress-Strain Curves for Tension and Compression.

In securing stress-strain diagrams for a given material, the three usual types of tests taken are the tension of a rod, the compression of a short cylindrical block, and the twisting of a thin-walled tube. The results of such tests are represented by plotting the mean stress acting in the current cross-sectional area against some measurement of the total strain.

For an ideal tensile test, the volume of the tensile specimen is practically constant

$$\pi r_0^2 l = \pi r^2 l$$

that is,

$$r^2 = \frac{l_0}{l} r_0^2 \quad (a)$$

where

$r_0$ and $r$ are the initial and current radii, respectively.

$l_0$ and $l$ are the initial and current length, respectively.

The total work ($W$) done by the tensile force is

$$W = \int \pi r_0^2 \sigma_e dl + \int \pi r^2 Y_t dl$$

where $\sigma_e$ is the elastic stress, and $Y_t$ is the tensile yield stress. Comparing the elastic work done with the plastic work, the elastic work is so small that it
can be neglected. Thus, the total work done is considered to be

\[ W = \int \pi r^2 Y_t \, dl. \]  

(b)

Combining equations (a) and (b), the work done per unit volume \((w)\) is

\[ w = \int \frac{Y_t}{E_t} \, \frac{dl}{l}. \]

The natural strain \((\varepsilon)\) is defined as

\[ d\varepsilon = \frac{dl}{l}. \]

Hence, the work done per unit volume is just the area under the true stress-natural strain curve.

Now, the amount of recoverable elastic work done is approximately equal to the shaded area \((A_s)\) shown in Figure III-1, which can be expressed mathematically as

\[ A_s = Y_t \frac{dY_t}{E_t}. \]

where \(E_t\) is the modulus of elasticity for tension.

Thus, the total plastic work done per unit volume is

\[ w_p = \int dw_p = \int \frac{Y_t}{E_t} \, \frac{dl}{l} - \frac{Y_t^2}{2E_t}. \]

According to theory of the "equivalent of plastic work", the yield stress \(Y_t\) is related to the plastic work, thus
Fig. III-1

EFFECTIVE STRESS-EFFECTIVE STRAIN DIAGRAM
\[ Y_t = F(w_p) = F\left( \int_{l_i}^{l} Y_t \, \frac{dl}{l} - \frac{Y_t^2}{2E_t} \right) \]

where the symbol \( F \) represents some arbitrary function.

If \( Y_t \) is plotted against \( \ln \frac{l}{l_i} - \frac{Y_t}{E_t} \), the argument of \( F \) is the area under the curve up to the stress level \( Y_t \).

Similarly, for simple compression, the compressive yield stress is

\[ Y_c = F\left( \int_{h}^{h_0} Y_c \, \frac{dh}{h} - \frac{Y_c^2}{2E_c} \right) \]

where

\( Y_c \) is the compressive yield stress

\( h_0 \) and \( h \) are the initial and current heights of the specimen, respectively

\( E_c \) is the modulus of elasticity for compression.

It can be shown that the true stress \( Y_t \) is the same function of \( \ln \frac{l}{l_i} - \frac{Y_t}{E_t} \) as \( Y_c \) is of \( \ln \frac{h_0}{h} - \frac{Y_c}{E_c} \). Thus, theoretically, the stress-strain curves for simple tension and simple compression will coincide if \( Y_t \) is plotted against \( \ln \frac{l}{l_i} - \frac{Y_t}{E_t} \), and \( Y_c \) against \( \ln \frac{h_0}{h} - \frac{Y_c}{E_c} \). That is, the two curves coincide only when true stress is plotted against the natural strain, but not when plotted against the engineering strains \( \frac{l - l_i}{l_i} \) and \( \frac{h_0 - h}{h} \).

* Superscript numbers refer to references listed on page 60.
2. Stress-Strain Curves for Torsion

It is very interesting to find that, after selection of the appropriate stress and strain related parameters, the stress-strain curves for torsion coincide with that for tension. This fact is demonstrated in the following paragraphs.

Hooke's law for three dimensional stresses in the plastic range is written as

\[ \varepsilon_i = \beta \left[ \sigma_i - \mu \left( \sigma_j + \sigma_k \right) \right] \]
\[ \varepsilon_2 = \beta \left[ \sigma_2 - \mu \left( \sigma_i + \sigma_3 \right) \right] \]
\[ \varepsilon_3 = \beta \left[ \sigma_3 - \mu \left( \sigma_i + \sigma_2 \right) \right] \]  \hspace{1cm} (III-1)²

where \( \mu \) is Poisson's ratio, and \( \beta \) is, in general, a function of the strains, and can be determined experimentally. The quantities \( \sigma_i, \sigma_2, \) and \( \sigma_3 \) are the principal stresses, and \( \varepsilon_i, \varepsilon_2, \) and \( \varepsilon_3 \) are the principal strains.

The equation of volume constancy is written in terms of the total strains as

\[ \varepsilon_i + \varepsilon_2 + \varepsilon_3 = 0 \]  \hspace{1cm} (III-2)

If equations (III-1) and (III-2) are combined, it can be shown that Poisson's ratio \( \mu \) remains constant and equal to one half in the plastic range.
Now, it is desired to find some relationships between shear stress and normal stress, and shear strain and normal strain. According to the Von Mises yield criterion

$$(q_1 - q_2)^2 + (q_2 - q_3)^2 + (q_3 - q_1)^2 = 2Y_t^2 = 6\psi^2 \quad (III-3)$$

where

$Y_t$ is the tensile yield stress

$k$ is the yield shear stress

thus

$$Y_t = \sqrt{3}k. \quad (III-4)$$

Making use of the known relations of the material constants gives

$$G = \frac{E_t}{2(1+\mu)} = \frac{E_t}{3} = \frac{Y_t}{3\epsilon} = \frac{k}{\psi}$$

where

$G$ is the modulus of rigidity

$E_t$ is the modulus of elasticity for tension

$\psi$ is the shearing strain at any radius $r$

Combining the above equations gives

$$\epsilon_r = \frac{\psi}{3} \frac{Y_t}{k} = \frac{\psi}{3\sqrt{3}}.$$
Now, consider and isotropic cylinder of unit gage length and outside radius $r_0$, subjected to a plastic torsional strain. (See Figure III-2) Torsion theory assumes that the diameter of the bar remains constant during the loading process, thus, for incompressible materials, the gage length also remains fixed. At any radius $r$, the shear strain is $\gamma = \tan\phi_1$. And, for a given angle of twist per unit length $\theta$

$$r\theta = \tan\phi_1 = \gamma$$

or

$$\theta = \frac{\gamma}{r} = \frac{\gamma_0}{r_0} \quad (c)$$

where $\gamma_0$ is the shear strain at the outside radius, and $\phi_1$ is angle by which a generatrix of the cylinder is tilted after twist.

Since the total strain ($\varepsilon_p$) is the sum of the strains in the elastic ($\varepsilon_e$) and plastic ($\varepsilon_p$) ranges, then the plastic strain can be expressed as

$$\varepsilon_p = \varepsilon_r - \varepsilon_e = \frac{\gamma}{\sqrt{3}} - \frac{Y_t}{E_t} = \frac{r\theta - G}{\sqrt{3}}. \quad (III-5)$$

From equations (III-4) and (III-5), it is seen that the stress-strain curves for the torsion of a circular shaft, constructed by plotting $\sqrt{3}k$ versus $r\theta - \frac{k}{\sqrt{3}}$, will coincide with the usual curves for tension.
Fig. III-2 TORSION OF SOLID CIRCULAR SHAFT
Referring again to Figure III-2, the resisting

torque \( T \), acting on a given cross section, is written as

\[
T = \int_0^{r_0} k(2 \pi r \, dr) \, r. \tag{III-6}
\]

Substituting equation (c) into equation (III-6) gives

\[
\frac{T \theta^3}{2 \pi} = \int_0^{r_0} k \, r^2 \, dr. \tag{III-7}
\]

Thus, the twisting moment \( T \) is proportional to the

second moment of the shaded area under the stress-strain

curve for shear, \( k = f(r) \), shown in Figure III-3. If the

moment-twist curve, \( T = f(\theta) \), is known from observation

of a torsion test, the unknown material stress-strain

curve for shear, \( k = f(r) \), may be determined from equation

(III-7). Differentiating equation (III-7) with respect

to \( \theta \)

\[
\frac{d}{d\theta} \left( \frac{T \theta^3}{2 \pi} \right) = \int_0^{r_0} \frac{d}{d\theta} [f(r) \, r^2] \, dr + f(r_0 \theta) (r_0 \theta)^2 \, r_0. \]

Since \( r \) is independent of \( \theta \), the first term on the right

side is zero, thus

\[
k_0 = f(r_0 \theta) = \frac{1}{2 \pi r_0^2} \left( \theta \frac{dT}{d\theta} + 3T \right).
\]

* This result is obtained by employing the Leibnitz

formula given on page 20.
Fig. III-3 SHEAR STRESS-SHEAR STRAIN CURVE
If the $T-\theta$ curve is known, the $k-r$ curve can be derived. Note that $k_0 = \frac{3T}{\pi r^2}$ is the usual simple expression for torque due to a maximum yield shear stress of $k_0$ in the fully plastic condition of a non-work-hardening material.

From Figure III-4, the slope at point B on the $T-\theta$ curve is given by

$$\frac{dT}{d\theta} = \frac{BC}{DC}$$

or

$$BC = \theta \frac{dT}{d\theta}.$$

Hence, the shear stress existing at the outside cylindrical surface, corresponding to an angle of twist $\theta$, is

$$k_0 = \frac{1}{2\pi r^2} (3BN + 3C). \quad (III-8)$$

Therefore, by drawing tangents to points on the $T-\theta$ diagram, the $k-r$ diagram can be derived. A more complete discussion of this procedure is given in Appendix I, where a sample problem has been solved.
$\theta$ - Angle of Twist per Unit Length

Fig. III-4 TORQUE-ANGLE OF TWIST CURVE FOR CIRCULAR BAR
3. Stress-Strain Curves for Bending

In an ideal bending test, it is generally assumed that the extreme fibers on the tensile and compressive sides of the beam are under the sole influence of tensile and compressive stresses, respectively. In other words, theoretically, the stress-strain curves for the extreme fibers on both sides of the beam are the same as those obtained from direct tensile and compressive tests. Although yielding of the extreme fiber is marked by the supporting effect of the less highly stressed fibers near the neutral axis, it will be shown later that, for certain materials, this effect is small.

Consider a long prismatic beam having a rectangular cross section of width b and depth d subjected to two equal and symmetrically spaced concentrated loads as shown in Figure III-5. It is assumed that a uniaxial state of stress exists at all points of the beam with the longitudinal stress being the only nonvanishing principal stress along the pure bending portion. If two adjacent vertical lines m-m and n-n are drawn along the side of the beam, experiment shows that these lines remain essentially straight during bending, and rotate so as to remain perpendicular to the longitudinal fibers of the beam.
Fig. III-5 BEAM IN PURE BENDING
The theory of bending is based on the assumption that not only such lines as m-m remain straight, but that the entire transverse section of the beam, originally plane, remains plane and normal to the longitudinal fibers of the beam after bending.

From Figure III-6 it is seen that

\[ \varepsilon = \frac{y}{\rho} \]  \hspace{1cm} (d)

and

\[ \frac{1}{\rho} = \frac{\varepsilon_c + \varepsilon_t}{d} \]

where \( \rho \) is the radius of curvature, \( d \) is the beam depth and \( \varepsilon_c \) and \( \varepsilon_t \) are the unit strains on the top and bottom surfaces, respectively. Further, from Figure III-7

\[ 2\phi = \frac{l_o}{\rho} \]  \hspace{1cm} (e)

where \( \phi \) is the slope of the tangent to the elastic line of the bent bar \( l_o \) is the gage length. Thus

\[ \phi = \frac{l_o}{2d} (\varepsilon_c + \varepsilon_t) \]  \hspace{1cm} (III-9)

The equilibrium equations for any section of the beam are

\[ \int_A \sigma \, dA = 0 \]  \hspace{1cm} (f)

and

\[ \int_A \sigma y \, dA = M \]  \hspace{1cm} (g)
Fig. III-6 ELEMENT TAKEN FROM DEFORMED BEAM

Fig. III-7 DEFORMED BEAM IN PURE BENDING
where the nomenclature is

\[ A = \text{cross sectional area of beam} \]
\[ M = \text{resisting moment at a given cross section} \]

If it is assumed that the relation between stress and strain is

\[ \sigma_c = f(\epsilon) \quad (h) \]

then

\[ \sigma_c = -f(-\epsilon_c) \quad (i) \]

\[ \sigma_t = f(\epsilon_t) . \]

Combining equations, \( (d) \), \( (h) \), and \( (i) \), and substituting these values into equation \( (f) \) gives

\[ \int_{-\epsilon_c}^{+\epsilon_t} f(\epsilon) \, d\epsilon = 0 \quad (j) \]

Differentiating equation \( (j) \) with respect to \( \phi \), and following the generalized Leibnitz formula*, gives

\[ \frac{\sigma_t}{\sigma_c} = \frac{d \epsilon_c}{d \epsilon_t} \quad (III-10) \]

An appropriate combination of equations \( (d) \), \( (e) \), \( (g) \), and \( (j) \) results in

\[ \frac{hM\phi^2}{b} \left[ \int_{-\epsilon_c}^{+\epsilon_t} f(\epsilon) \epsilon \, d\epsilon \right] = (k) \]

If equation \( (k) \) is differentiated with respect to \( \phi \), again proceeding according to the generalized Leibnitz formula, then

* The generalized Leibnitz formulas is written as

\[ \int_a^b f(x, \alpha) \, dx = \int_a^b \frac{\partial f(x, \alpha)}{\partial \alpha} \, dx + f(b, \alpha) \frac{db}{d\alpha} - f(a, \alpha) \frac{da}{d\alpha} . \]
\[
\frac{b^2}{l^2} \frac{d}{d\phi} (M \phi^2) = f(\varepsilon_t) \varepsilon_t \frac{d\varepsilon_t}{d\phi} - f(-\varepsilon_c) \varepsilon_c \frac{d\varepsilon_c}{d\phi} = \varepsilon_t \varepsilon_t \frac{d\varepsilon_t}{d\phi} + \varepsilon_c \varepsilon_c \frac{d\varepsilon_c}{d\phi}.
\]

Using the equations (III-9) and (III-10), the strain increments can be written as

\[
d\varepsilon_c = \frac{2d}{\lambda_0} \frac{\varepsilon_t}{\varepsilon_c + \varepsilon_t} d\phi
\]

\[
d\varepsilon_t = \frac{2d}{\lambda_0} \frac{\varepsilon_c}{\varepsilon_c + \varepsilon_t} d\phi.
\]

The desired moment equation is now in the form

\[
\frac{b^2}{l^2} \frac{\varepsilon_c \varepsilon_t}{\varepsilon_c + \varepsilon_t} = 2M + \phi \frac{dM}{d\phi}. \quad (III-11)
\]

The right side of this equation is known if, from observed test data, the bending moment \( M \) is a known function of the slope \( \phi \). From the observed strains \( \varepsilon_t \) and \( \varepsilon_c \), the ratio \( \frac{d\varepsilon_c}{d\varepsilon_t} \), and, from equation (III-10), the quotient \( \frac{\varepsilon_t}{\varepsilon_c} \) may be determined for various values of the observed angle \( \phi \). These expressions permit the calculation of the stresses \( \sigma_c \) and \( \sigma_t \) farthest from the neutral axis as functions of the strains or, in other words, the construction of the stress-strain diagrams for tension and compression from bending tests.

NOTE:

In the case of beams with circular cross sections,
equation (f) becomes

\[ \int_{-\epsilon_c}^{\epsilon_t} f(\epsilon) \sqrt{r_o^2 - (\epsilon \rho)^2} \, d\epsilon = 0 \]  \hspace{1cm} (III-12)

where \( r_o \) is the radius of the circular cross section. According to equation (e), \( \rho \) depends on \( \phi \). If equation (III-12) is differentiated with respect to \( \phi \), it can be seen that the first term of the right side of the generalized Leibnitz formula does not vanish. That is, it is not as convenient to find the relationship between \( \sigma_c \) and \( \sigma_t \) for beams of circular cross section. Considerable effort has been put forth by this author in an attempt to solve the beam problem for circular cross section; however, since the completion of this thesis does not depend upon the knowledge of circular bar behavior, this problem has been left as a future thesis project. The reason for pursuing this problem was to make possible the selection of tension, compression, torsion, and bending specimens from a single piece of round bar stock.

4. Summary

From the preceding discussion of the theoretical stress-strain curves for tension, compression, torsion and bending, it is concluded that by appropriate
selection of stress and strain related parameters, these four curves will coincide. Table III-1 shows the necessary quantities for stress and strain that must be used in constructing these graphs for various loading conditions.
<table>
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<tr>
<th>Loading Conditions</th>
<th>Stress</th>
<th>Strain</th>
</tr>
</thead>
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<td>Simple Tension</td>
<td>$Y_t$</td>
<td>$\ln \frac{l}{l_0} - \frac{Y_t}{E_t}$</td>
</tr>
<tr>
<td>Simple Compression</td>
<td>$Y_c$</td>
<td>$\ln \frac{h_0}{h} - \frac{Y_c}{E_c}$</td>
</tr>
<tr>
<td>Simple Torsion</td>
<td>$\sqrt[3]{k}$</td>
<td>$\tau_0 - \frac{k}{\sqrt[3]{J_k}}$</td>
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<tr>
<td>Pure Bending</td>
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<td></td>
<td>$\sigma_c$</td>
<td>$\ln \frac{h_0}{h} - \frac{\sigma_c}{E_b}$</td>
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</tbody>
</table>

**Table III-1**

STRESS-STRAIN PARAMETERS FOR

CONSTRUCTING THEORETICALLY IDENTICAL CURVES
5. Stress-Strain Curves for Biaxial Tension

In addition to the four types of loads discussed previously, it is interesting to note that the stress-strain curves of a metal membrane subjected to biaxial tension can also be reduced to an identical diagram.

![Diagram of a membrane under pressure](image)

**Fig. III-8 PRESSURE-LOADED MEMBRANE**

Figure III-8 shows a circular membrane clamped around its periphery with fluid pressure acting on one side. It is assumed that the thickness-to-diameter ratio of the membrane is so small that bending and shearing stresses can be neglected. From the equilibrium condition of the membrane element shown in Figure III-9
Fig. III-9 ELEMENT TAKEN FROM DEFORMED MEMBRANE

\[ \sigma_h = \frac{p_0r}{2t \sin \lambda} = \frac{p_0 \rho}{2t} \]

where the nomenclature is

- \( \sigma_h \) = hoop stress, psi
- \( p_0 \) = internal pressure, psi
- \( t \) = current thickness, in
- \( \rho \) = radius of curvature, in

From symmetry, the hoop strain \( \varepsilon_h \) at the pole is equal to the radial strain \( \varepsilon_r \). The thickness strain \( \varepsilon_t \) is obtained from the equation of incompressibility, equation (III-2), as

\[ \varepsilon_t = -2 \varepsilon_h = -\ln \frac{t_0}{t} \]
where $t_0$ is the initial thickness, inches.

Considering the stress system at the pole for a thin membrane, the stress normal to the sheet is negligible, and the material at the pole is therefore subjected to a balanced bi-axial tensile stress. That is,

$$\sigma_t << \sigma_h \text{ and } \sigma_r$$

where the symbology is

- $\sigma_t = \text{thickness stress, psi}$
- $\sigma_h = \text{hoop stress, psi}$
- $\sigma_r = \text{radial stress, psi}$

Thus $\sigma_t$ can be neglected. Since at the pole, the hoop stress and radial stress are equal, the following diagram can be used to express the state of stress.

![Diagram showing equivalent states of stress](image)

Fig. III-10 EQUIVALENT STATES OF STRESS

Since it is assumed that a hydrostatic pressure has no effect on yielding, the system is therefore equivalent to a simple compressive stress $\sigma_h$ acting normal to the sheet.
The equations for effective stress $\bar{\sigma}$ and effective strain $\bar{\varepsilon}$ are

$$\bar{\sigma} = \sqrt[3]{\frac{\left(\varepsilon_r - \varepsilon_h\right)^2 + \left(\varepsilon_h - \varepsilon_t\right)^2 + \left(\sigma_r - \sigma_h\right)^2}{3}}$$

$$\bar{\varepsilon} = \frac{2}{3} \left[ (\varepsilon_h - \varepsilon_t)^2 + (\varepsilon_r - \varepsilon_t)^2 + (\varepsilon_t - \varepsilon_h)^2 \right].$$  \hspace{1cm} (III-13)^1

It can be shown from equations (III-13), that the hoop stress $\sigma_h$ and thickness strain $\varepsilon_t$ are just the effective stress and effective strain, respectively. If the quantities $\sigma_h$ and $\ln \frac{t}{t_0} - \frac{\sigma_h}{E_{bi}}$ ($E_{bi}$ is modulus of elasticity for biaxial tension) are chosen as the ordinate and abscissa, respectively, the resulting stress-strain curve will coincide with the curves described in Table III-1.
IV. EXPERIMENTAL FACILITIES AND PROCEDURES

Three different materials - magnesium (AZ 61A), aluminum (24ST-4), and (AISI C1018) steel have been tested under tension, compression, torsion and bending loads. All of the test specimens, for a given material, are cut from the same bar. The subject testing materials were purchased in the form of a 12-foot section of bar, having a 1½-inch by 3/4-inch rectangular cross section. The bending test specimens were made by cutting a section of appropriate length from the original bar. The tension, compression, and torsion specimens were also made from the original bar, but they required that a machining operation be made.

A. Objective:

To obtain experimental stress-strain curves for tension, compression, torsion and bending.

B. Equipment Necessary:

1. Tension Test (shown in Figure IV-1)

   Testing machine with grips to fit specimen

   Two inch dial gage (calibrated with increments of 0.0001")

   Pair of dividers

   One inch micrometer

   Gage length punch

2. Compression Test (shown in Figure IV-2)

   Testing Machine
Fig. IV-1  TENSILE TESTING APPARATUS
Fig. IV-2 COMPRESSION TESTING APPARATUS
Baldwin-Lima-Hamilton portable Model 120 strain indicator and the Model 225 Switching and Balancing Unit (this instrument has been designed and calibrated especially to work with resistance strain gages having an initial resistance of 120 ohms)

Micrometers

Lubricant (graphite)

Bearing plates (these plates are essentially rigid in comparison to the compression specimens)

Budd Metalfilm strain gages (post yield, high elongation, type HE-141 gages having a gage factor of 2.02 ± 0.5 % and an initial resistance of 120 ± 0.2 ohms)

3. Torsion Test (shown in Figure IV-3)

Testing machine

Angular-measurement device

Micrometers

Pair of dividers

Gage length punch

4. Bending Test (shown in Figure IV-4)

Testing machine

Baldwin-Lima-Hamilton portable Model 120 strain indicator and the Model 225 Switching and Balancing Unit

Budd Metalfilm strain gages (these gages are identical to the ones described in paragraph 2 of this Section)

Ruler

Two hard-material rollers

C. Specimen Requirements:

1. Tension
Fig. IV-3  TORSION TESTING APPARATUS
Fig. IV-4  BENDING TESTING APPARATUS
a. The central portion of the length is usually of smaller cross section than the end portions in order to cause failure at a section where the stresses are not affected by the gripping device.

b. The specimen must be symmetrical with respect to the longitudinal axis throughout its length in order to avoid bending during application of the load.

2. Compression:

a. Compressive specimens are limited to such a length that column action is not a factor.

b. In order to get uniformly distributed stresses over the cross section, the ends of the specimen must be flat and normal to the longitudinal axis of the specimen. It is necessary to put a good lubricant between the compression plates and the ends of the specimen to prevent the development of shear stresses which will disrupt the assumed uniform normal stress distribution.

3. Torsion:

a. In order to prevent the occurrence of complicated stress distributions, specimens of circular cross section were used.
b. The specimen must be symmetrical with respect to the longitudinal axis throughout its length.

4. Bending:
   a. The beam under test must be so proportioned that it will not fail in shear or by lateral deflection before it reaches its ultimate flexural strength. To prevent a shear failure, the specimen must not be too short with respect to the beam depth. The test specimen were made in accordance with ASTM designation: E 1b-39.
   b. For results in which the measured performance is independent of the thickness of the specimen, rectangular specimens with widths at least one and one-half times the thickness must be used. The edges must be smoothed such that localized fracture will not take place.

D. Procedure:

1. Tension Test.
   Using a 2-inch punch, two points, symmetrical with respect to the specimen center, were made, and the dial gage mounted in position. The specimen, and dial gage, were then
mounted in the testing machine, and a check made to see that the gage and gripping devices functioned properly. Making sure that the testing machine initially registered zero load, the tension test was performed by measuring and documenting the diameter of the mid-point of the specimen, the specimen elongation as recorded on the dial gage, and the corresponding machine load. This procedure was repeated at predetermined load increments until fracture of the specimen.

2. Compression Test.
Before testing, the ends of the specimen and the face of the bearing plates were cleaned with acetone and then coated with graphite. Next, the strain indicator was balanced and its initial reading recorded. The diameter of the specimen and corresponding strain at increasing loads were recorded. The alignment and centering of the heads of the testing machine, the bearing plates, and the axis of the specimen were preserved throughout the test.

3. Torsion Test.
Before testing, the gripping device of the torsion testing machine was cleaned and then the specimen and angular measurement device
mounted. Both the torque and angular measurement device were zeroed and checked to see that they were functioning properly. The specimen diameter, angle of twist $\theta$, and magnitude of torque for predetermined increments of torque were recorded.

4. Bending Test.

In order to produce pure bending, two hard-material rollers were set at equal distances from the end supports. A second beam was mounted on these two rollers, and a concentrated load, supplied by the compression machine, applied at the center of the top, as shown in Figure IV-4. The strain gages located on the top and bottom sides of the beam are connected to individual channels of the switching and balancing unit. Before testing, both channels were balanced and the initial readings recorded. A load was applied to the beam and held constant. The strain indicator was balanced and a new strain reading obtained from both the tension and compression gages. This process was continued at predetermined load increments.
V. RESULTS

Three materials - magnesium, aluminum, and steel - have been tested under tension, compression, torsion, and bending. Figures V-1 through V-4 are constructed from these experimental observations. It is evident that initially the relationship between load and deformation is essentially linear. However, on further straining beyond the elastic limit, the relation between load and deformation is no longer linear, and plastic flow has started. In order to predict the modulus of elasticity and modulus of rigidity, Figures V-5, V-6 and V-7 have been constructed for tension and compression. Figures V-3 and V-4 can be used to determine the modulus of elasticity for bending and the modulus of rigidity for torsion, respectively.

Within the elastic limit, for bending,

\[ M = E_b I \frac{dy}{dx^2} = E_b I \frac{1}{\rho} = E_b I \frac{2\phi}{l_o} \]

where

- \( M \) is the applied moment
- \( I \) is the centroidal moment of inertia of the cross section
- \( l_o \) is the gage length \( \left(\frac{1}{4}a\right)\)
- \( E_b \) is the modulus of elasticity for bending
- \( x \) and \( y \) are the horizontal and vertical coordinates defining a point in the beam

Thus,

\[ E_b = \frac{l_o}{2I} \frac{M}{\phi} = 0.593 \frac{M}{\rho} \]
Fig. V-1 EFFECTIVE STRESS-EFFECTIVE STRAIN RELATIONS FOR TENSION
Fig. V-2 EFFECTIVE STRESS-EFFECTIVE STRAIN RELATIONS FOR COMPRESSION
Fig. V-3  EXPERIMENTAL TORQUE-ANGLE OF TWIST DATA FOR CIRCULAR BARS
Fig. V-4 MOMENT-CURVATURE DATA FOR BENDING
Fig. V-5 ELASTIC PORTION OF STRESS-STRAIN CURVES FOR ALUMINUM
Fig. V-6  ELASTIC PORTION OF STRESS-STRAIN CURVES FOR MAGNESIUM
Engineering stress (ksi)

Engineering strain (in/in) $10^5$

○ Tension ($E_t = 30.6 \times 10^6$ psi)
△ Compression ($E_c = 29 \times 10^6$ psi)
--- Bending ($E_b = 28.5 \times 10^6$ psi)

(See Fig. V-4 for data)

Fig. V-7 ELASTIC PORTION OF STRESS-STRAIN CURVES FOR STEEL
Hence the slope of the straight line portion of the $M$ versus $\phi$ curve, multiplied by the constant 0.593, gives the desired modulus, $E_b$.

Similarly, for torsion,

$$G = \frac{1}{I_p} \cdot \frac{T}{\theta}$$  \hspace{1cm} (V-3)

where $I_p$ is the polar moment of inertia of the circular cross section. Thus, the slope of the straight line section of the $T$ versus $\theta$ curve, multiplied by the factor $1/I_p$, gives a value for the modulus of rigidity, $G$.

It is evident that the magnesium tensile specimens (right specimens in Figures V-8 and V-9) fractured along a $45^\circ$ plane, which is indicative of a shear fracture. Figure V-10 shows that a tensile steel specimen will neck down locally before fracture. The stress distribution over the smallest section of the neck is no longer uniaxial, but triaxial. So once necking down occurs, the stress-strain diagram has no meaning for simple tension.

The general types of observations and records of test in torsion are similar to those of tension and compression. However, the resulting shear fracture is quite distinct from either the tension or compression fracture. A solid rod of magnesium that fails during a torsion test will break along a plane normal to the axis of the rod, as shown in the center specimen of Figure V-11. For the steel bar (specimen on the left in Figure V-11), the fracture is silky in texture, and
left side - steel
center - aluminum
right side - magnesium

Fig. V-8 TENSILE FRACTURED SPECIMENS
left side - steel

center - aluminum

right side - magnesium

Fig. V-9 COMPRESSIVE FRACTURED SPECIMENS
Fig. V-10 NECKING IN TENSILE STEEL SPECIMEN
left side - steel
center - magnesium
right side - aluminum

Fig. V-11 TORSION FRACTURED SPECIMENS
the axis about which the final twisting takes place can be seen. Since the surface of the break may not be quite smooth, the outer portions, acting like cams, push the piece apart in the direction of its longitudinal axis. The center portion of the specimen that is not yet broken by shear is possibly broken in tension by this cam action.

Elastic stress-strain diagrams for the three different materials are obtained from various loading conditions and are shown in Figures V-5, V-6 and V-7. For aluminum, the modulus of elasticity in bending is reasonably close to that in tension. However, for magnesium and steel, the modulus of elasticity in flexure tends to be slightly below those for tension and compression. Since some error is involved in setting up the bending test, slight amounts of shear can be present which will induce shearing strains, thereby causing an increase in the observed strain over that due to fiber strains alone.

Substituting the values of the moduli of elasticity and rigidity into the equation

$$G = \frac{E}{2(1+\nu)}$$

Poisson's ratio for all three materials can be found. The results are shown in Table V-1.
Table V-1 ELASTIC VALUES OF POISSON’S RATIO

From this table, it can be seen that in the elastic range, Poisson’s ratio is always below one half.

The stress diagrams shown in Figures V-12, V-13 and V-14 are constructed according to the stress and strain parameters shown in Table III-1. A sample problem, showing how these stress-strain curves are constructed, is presented, with discussion, in Appendix I. From these stress-strain diagrams, it is seen that the torsion tests produce a distinctively different curve (especially for the soft materials—aluminum and magnesium). The reason for this is that in simple torsion, the principal axes of stress and strain do rotate relative to the element during the loading process. A more complete discussion of this is given in the next section. Another explanation is that the center portion of a cylinder of originally soft ductile metal, after it has been subjected to a plastic torque, is permanently strained
Fig. V-12 EXPERIMENTAL COMPARISON OF PLASTIC STRESS-STRAIN PROPERTIES OF ALUMINUM
Fig. V-13 EXPERIMENTAL COMPARISON OF PLASTIC STRESS-STRAIN PROPERTIES OF STEEL
Experiment 14: Experimental Comparison of Plastic Stress-Strain Properties of Magnesium
in the axial direction. Thus the assumed distribution of shearing stresses does not represent the complete system of stresses since a system of secondary normal stresses occurring in the axial, and possibly also in the radial and tangential directions, must be present. For this reason, the effective strain equation (III-13) will not be satisfied with the actual large deformation of the torsional specimen.

The stress-strain diagrams in compression always appear below those determined from tension tests. The reason for this is that lateral expansion at the end of the specimen is prevented by the friction force that exists on the surfaces of contact between the specimen and the bearing plates, thereby causing a complicated non-uniform stress distribution. When this occurs, the total plastic work done by the triaxial stresses are greater than that due to the axial stress alone. In tension, if the specimen satisfies the requirements mentioned in section IV-C, the stress distribution is always considered uniaxial until necking down occurs.
VI SUMMARY AND CONCLUSIONS

This study shows that with the appropriate selection of stress and strain related parameters the stress-strain curves obtained from a single material subjected to tension, compression, torsion and bending loads coincide. Further, the stress-strain diagrams obtained from experimental observation are presented, and these diagrams show the relative agreement for magnesium, aluminum and steel.

With the exception of the torsional loads, the stress-strain diagrams constructed from the various loading tests show relatively little deviation for steel and aluminum materials. The discrepancy in the torsion test results is to be expected since the effective strain equation used in this analysis is only valid if the following two conditions are satisfied: (1) the principal axes of stress and strain for a particle do not rotate with respect to the particle during the process of straining; and, (2) the strain increments are proportional to the total strains. Since the torsion of a circular bar does not satisfy these conditions, the experimental results will show some deviation from the expected curve. A more appropriate torsional analysis can be made by utilizing the incremental effective strain equation. However, the results of this analysis are valuable in that the magnitude of the error involved in assuming the validity of equation (III-13) for torsion problems is not fully discussed in the literature.
The stress-strain diagrams constructed from the various load tests for magnesium have very little resemblance. This was suspected at the outset of this program since the handbook data for magnesium indicates that its tensile and compressive properties are different. Magnesium was purposely selected to illustrate the care that must be exercised in applying the simplified equations of plasticity to all materials.

In making a more critical review of the results achieved for the steel and aluminum, it is seen that the aluminum stress-strain data gives the closest correlation. Using the extent of strain as measured in a tensile test as an index of ductility, it appears, from the results of this study, that the materials that exhibit the greatest ductility are most likely to satisfy the simplified laws of plasticity previously presented.

Tensile tests are generally used as the basis of design regardless of the nature of loading or state of stress in the structure. Fortunately, the tensile stress-strain curves are always in the highest position (see Figures V-12, V-13, and V-14). Thus, if an engineer uses tensile stress-strain data for all design work, there is a degree of assurance that this will be a conservative design.
REFERENCES


SELECTED BIBLIOGRAPHY

VITA

The author of this thesis, Wen-mo Chen, son of Mr. and Mrs. Mou-lin Chen, was born on June 20, 1939, at Yin-ko, Taipei shen, Taiwan, Free China.

The author graduated from the Civil Engineering Department of Provincial Cheng Kung University in 1963. After he received the B. S. degree, he served in the Chinese Army for one year.

After leaving the Chinese Army, he studied Applied Mechanics at the Graduate School of Engineering, College of Chinese Culture, for one year, and then transferred to the Department of Engineering Mechanics, University of Missouri at Rolla, for continued graduate study, in September, 1965.
APPENDIX I
SAMPLE RESULTS

As indicated earlier, in order to analyze the relative agreement of the stress-strain properties obtained from basically different experimental tests of a given material, some appropriate stress and strain related parameters should be selected and the corresponding diagrams constructed from experimental observations. These diagrams have been mentioned and discussed in previous sections. Here, the aluminum specimen is taken as an example to show how these stress-strain curves were constructed for various loading conditions—tension, compression, torsion and bending.

A. Construction of stress-strain diagrams for tension.
   1. Loads, current diameters and elongations (or strains) are the quantities of observations.
   2. In order to determine the modulus of elasticity, \(E_t\), the stress-strain diagrams in the elastic range are constructed and the slope is calculated. This construction is shown in Figure V-5.
   3. From the measured diameter, the current area of the specimen can be determined. Next, the corresponding load acting on the specimen is divided by the current area, giving a value of the true material stress (\(Y_t\)).
4. Knowing the initial gage length $l_0$, plus the elongation at any load, the current length $l$ of the specimen can be obtained. The natural strain is defined as being the natural logarithm of the quotient, current length divided by initial length. Subtracting the ratio $\frac{X}{E_t}$ from the natural strain gives the desired strain parameter.

5. The calculation of the above parameters for the plastic strain of an aluminum tensile specimen is presented in Table A-1.

B. Construction of stress-strain diagrams for compression. The procedure of constructing stress-strain diagrams for compression is the same as that for tension and needs no further description here.

C. Construction of stress-strain diagrams for torsion.

1. Twist moment and angle are the quantities of observation.

2. In order to determine the modulus of rigidity, a curve of torque $T$ versus angle of twist $\theta$, in the elastic range, is constructed first. Then, the method described in Section V, equation V-3, is employed.

3. The equation used for determining the shear
Material: Aluminum

Gage length: \( l_0 = 1.80 \) inches

Modulus of elasticity = 10,000,000 psi

Formula:

\[
\text{Yield stress: } Y_t = \frac{P}{A}
\]

\[
\text{Strain: } \ln \frac{l}{l_0} - \frac{Y_t}{E_t}
\]

<table>
<thead>
<tr>
<th>( P ) (lb.) Loads</th>
<th>( D ) (in.) Diameter</th>
<th>( Y_t ) (psi)</th>
<th>( \frac{Y_t}{E_t} )</th>
<th>( e \times 10^4 ) Elongation</th>
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**Table A-1** PLASTIC TENSILE DATA FOR ALUMINUM
stress-shear strain curve has been derived; equation (III-3), and is rewritten here as

\[ \sqrt{3} k_0 = \frac{\sqrt{3}}{2 \pi r_0} (3 T + \theta \frac{dT}{d\theta}). \]

4. A curve of torque \( T \) versus angle of twist per unit length \( \theta \) can be constructed from the experimental data as shown in Figure A-1. By drawing tangents at points along the curve, the slope \( \frac{dT}{d\theta} \) can be found. Substituting these values of \( T \), \( \theta \) and the corresponding slope, for a number of different points into equation (III-3), the yield shear stress is determined. Further, for a given value of \( \theta \), the corresponding value of the shear strain \( \gamma \) can be calculated by the equation

\[ \gamma = r\theta = 0.3443\theta. \]

Using this, the values of the strain parameter

\[ \frac{r\theta - \frac{k_0}{3}}{\sqrt{3}} \]

can be calculated.

5. The calculation of the above stress-strain parameter is presented in Table A-2.

D. Construction of stress-strain diagrams for bending

1. Loads, and the strains occurring in the tensile
Material: Aluminum

Radius: \( r_0 = 0.3443 \) inch

Modulus of rigidity: \( G = 392,000 \) psi

Formulas:

\[
\sqrt{3} k = \frac{\sqrt{3}}{2\pi r_0^2} (2T + \theta \frac{dT}{d\theta})
\]

\[
\text{Strain: } \frac{\theta - \frac{k}{G}}{\sqrt{3}}
\]

<table>
<thead>
<tr>
<th>Points</th>
<th>( T ) (in-lb)</th>
<th>( \theta ) (rad/in)</th>
<th>( \frac{dT}{d\theta} )</th>
<th>( \sqrt{3} k )</th>
<th>( \frac{\theta - \frac{k}{G}}{\sqrt{3}} )</th>
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Table A-2 PLASTIC TORSIONAL DATA FOR ALUMINUM
and compressive side of the beam, are the qualities of observation.

2. In order to determine the modulus of elasticity $E_p$ for bending, an elastic curve of moment $M$ versus curvature $\phi$ is constructed first, and then the method described in Section V equation (V-2) is applied.

3. The applicable equations for determining the stress-strain curve have been derived previously, and are written again as

$$\frac{\sigma_t}{\sigma_c} = \frac{d\varepsilon_t}{d\varepsilon_c} \quad (A-1)$$

$$bd^2 \frac{\varepsilon_c \sigma_t}{\sigma_c + \sigma_t} = (2M + \phi \frac{dM}{d\phi}). \quad (A-2)$$

In order to find the relation between $\sigma_t$ and $\sigma_c$, a curve of $\varepsilon_c$ versus $\varepsilon_t$ must be constructed, as shown in Figure A-2. The results of this figure indicate that

$$\sigma_t = 1.012 \sigma_c. \quad (A-3)$$

In order to establish the value of the right side of equation (A-2), a curve of bending moment versus curvature must be constructed, which is shown in Figure A-3. By drawing tangents at individual points along this curve, and substituting the values of $M$, $\phi$
Slope = $\frac{d\varepsilon_\sigma}{d\varepsilon_t} = 1.012$

**Fig. A-2** EXTERNAL FIBER STRAINS FOR ALUMINUM BEAM
Compressive strain, $\epsilon_c = 0.504 \phi$

Tensile strain, $\epsilon_t = 0.497 \phi$

Fig. A-4 BEAM CURVATURE-STRAIN DATA FOR ALUMINUM
and the corresponding slopes into equation (A-2), the results can be combined with equation (A-3) to find the values of $C_t$ and $C_c$. Further, by using the diagram of curvature $\phi$ versus $\epsilon_c$ and $\epsilon_t$, shown in Figure A-4, the corresponding strains can be found. The required strain parameters are then found by subtracting the quantities $C_t/E_b$ and $C_c/E_b$. Thus, the stress-strain curves can now be constructed.

4. The calculation of the above stress-strain parameter is presented in Table A-3.
Material: Aluminum

Modulus of elasticity = 109,000,000 psi

Formulas:

For stress:
\[ \sigma_c = 1175 \times (2H + \phi \frac{dM}{d\phi}) \]
\[ \tau_c = 1.012 \sigma_c \]

For strain:
\[ \varepsilon_c = \varepsilon_c - \frac{\sigma_c}{E_b} = 0.504 \times 10^{-6} \]
\[ \varepsilon_t = \varepsilon_t - \frac{\sigma_t}{E_b} = 0.497 \times 10^{-6} \]

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<th>Point</th>
<th>( M ) (kip-in)</th>
<th>( \phi ) 12000</th>
<th>( \frac{dM}{d\phi} ) 1/12000</th>
<th>( \sigma_c ) (ksi)</th>
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Table A-3: PLASTIC BENDING DATA FOR ALUMINUM