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An Effective Local Absorbing Boundary for 3D FEM Time Domain Analyses

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ABSTRACT

The aim of this paper is to investigate and develop alternative methods for analyzing transient problems in dynamic soil-structure-interaction, (SSI) within the FEM context. Development of a simple and efficient FEM procedure for the solution directly in the time domain of SSI problems is the main issue. Considering Direct Method of analysis, a new formulation of local transmitting boundaries for transient three-dimensional (3D) FEM analysis is presented based on the radiation criterion and strength-of-materials theory. These numerical devices can be considered as doubly asymptotic, (DA) approximations and are given in terms of first order differential operators. Different formulations for volume and surface waves are considered. A computer code employing an implicit FEM for solving 3D elastic wave propagation problems is developed to investigate the effectiveness of the proposed boundary conditions, (BC's). Numerical examples for homogeneous hafspace in full 3D are presented in comparison to extended mesh and fundamental solutions, a classical approach. As the effort for implementing them is the same as for the impedance BC, standard assembly procedure can be used. Due to the local nature they also preserve the overall structure of the global equations of motions.

INTRODUCTION

The well-known numerical problem in dynamic SSI analysis is how to simulate computationally the far field medium at infinity. The radiation condition in this case leads to a boundary-value problem with a unique solution. To analyze the semi-infinite domain of the soil numerically, a surface or zone is chosen called 'interaction horizon', (Sandler, 1981). In the Direct Method of analysis the interaction horizon is identical to an artificial boundary up to which the soil is modeled with e.g. FE's. The constitutive at the nodes on this horizon represents the significant features of the far field. The rigorous BC is global in space and time and is described through integro-differential operators. As this is computationally expensive and in the Direct Method artificial boundaries are placed far from the energy source, approximate or local BC's can be formulated using only differential operators with respect to space and time. Numerical procedures for the dynamic SSI analysis may be classified as either time harmonic or transient. However a direct time integration approach is necessary whenever nonlinearities occur and may be advantageous for some classes of linear problems.

A detailed analysis of these techniques including different areas of application is given by (Wolf, 1988) (Givoli, 1991) etc.

The simplest local absorbing BC is the classical normal impedance (Lysmer and Kuhlemeyer, 1969). Its performance is known to deteriorate when approaching the source of perturbation and it fails for static loads. In (White et al, 1977) the dashpot coefficients are given as functions of Poisson's ratio. (Akiyoshi, 1978) proposed a viscous boundary for shear waves involving convolution integral in its formulation losing the local character. (Smith 1974) and (Kumar and Marti, 1981) proposed superposition boundary based on the principle of the virtual image. (Engquist and Majda, 1977) and (Clayton and Engquist, 1980) derived a sequence of BC's of increasing order using rational approximations to pseudo-differential operators. (Reynolds 1978) derived the paraxial boundary equations making another factorization of the scalar wave differential operator. These schemes are usually applied by finite differences and cannot readily be implemented in the FE calculations. (Cohen and Jennings, 1983) tried to extend the paraxial boundary applicable for FE calculations, however their formulation was not very clear. (Bayliss and Turkel, 1980) have introduced another sequence of boundary operators based on the asymptotic expansion of the solution for the acoustic problem. (Liao and Wong, 1984) and (Cheng and Cheng, 1995) formulated an explicit-time-integration scheme in connection with extrapolation algorithm. (Underwood and Geers, 1981) developed DA approximation using boundary element method to define the static stiffness coefficients. (Keys, 1985) derived BC's for the acoustic wave equation based on a vectorial representation of incoming and outgoing waves. In another way (Higdon, 1990, 1991) constructed multi-directional (MD) boundary by concatenating several of these operators applicable for acoustic and elastic media. When the number of operators goes to infinity a global formulation results. In practice only the product of two or three operators is taken. (Wolf and Song, 1996) formulated doubly asymptotic multi-directional (DAMD) boundary that combines the advantages of DA approximation and MD formulation. It is investigated that the accuracy is governed by the static behavior and to increase it the boundary should be situated rather far from the energy source. Lastly (Kellezi, 2000) formulated simple BC's for 2D plane strain and axisymetric analysis based on the strength of materials theory. This formulation is further extended here for full 3D dynamic SSI analysis.

3D FEM TIME DOMAIN ANALYSIS

From the investigated literature, boundary operators of order two or more give nonsymmetrical matrices when implemented in the FEM On the other hand it is noted that the accuracy of the boundary is governed by the static behavior. So the need for local transmitting boundaries accurate for low and high frequencies using first order differential operators, arises.

When σ_{ij} and u_i denote the stress and displacement components respectively, ρ soil density and p(t) the applied time function stress, the equations of motion for elasto-dynamic can be given as

$$\frac{\partial \sigma_{ij}}{\partial x_j} - \rho \frac{\partial^2 u_i}{\partial t^2} + p(t) = 0 \tag{1}$$

FEM formulations are based on the weak form of the field Eq. 1. The equations of motion are multiplied by a weight function in the form of a virtual displacement field $\bar{u}_i$ followed by integration over the volume and reformulation using the divergence theorem.

$$\int_\Gamma \bar{u}_i (\sigma_{ij} n_j) d\Gamma - \int_\Omega \left(\bar{\varepsilon}_{ij} \sigma_{ij} + \rho \bar{u} \ddot{u} - \bar{u} p \right) d\Omega = 0 \tag{2}$$

A dot denotes differentiation with respect to time. The integral identity, Eq. 2, reduces to a set of linear equations when the spatial variation of the actual and virtual displacement fields is represented by shape functions of the form

$$\begin{aligned} u_i(x_j, t) &= \bar{N}_i(x_j) u(t) \\ \bar{u}_i(x_j, t) &= \bar{N}_i(x_j) \bar{u}(t) \end{aligned} \tag{3}$$

The stress σ_{ij} at the boundary integral in Eq. 2 should represent the stiffness of the far field and geometrical damping. In vector form it can be written

$$\{\sigma\} = [D_K]\{u\} + [D_C]\{u_{,t}\} \tag{4}$$

$[D_K]$ and $[D_C]$ represent the constitutive relation between near and the far field. Taking the same shape functions for actual and virtual displacement fields, substitution of Eq. 3 and Eq. 4 into Eq. 2 gives the equations of motions in matrix form

$$\begin{aligned} &[M]\{u_{,tt}\} + ([C] + [C]_\infty)\{u_{,t}\} + \\ &([K] + [K]_\infty)\{u\} = \{P(t)\} \end{aligned} \tag{5}$$

$\{u\}$, $\{u_{,t}\}$ and $\{u_{,tt}\}$ are the system vectors for displacement, velocity and acceleration respectively. [M] is the well known mass matrix given by a part of volume integral in Eq. 2. The system stiffness consists of the volume contribution [K] and the stiffness arising from the integral over artificial or transmitting boundary $[K]_\infty$. The FE consistent stiffness at the boundary surface derives as

$$[k]_\infty = \int_\Gamma \left[\bar{N}\right]^T [D_K] \left[\bar{N}\right] d\Gamma \tag{6}$$

The damping of the system consists of [C] which models material damping for the near and the far field and the boundary integral, which model radiation damping $[C]_\infty$. Material damping matrix modeled as Raleigh damping is given in the form

$$[C] = \alpha[M] + \beta([K] + [K]_\infty) \tag{7}$$

α and β are the so-called Rayleigh damping coefficients related to the modal damping ratio γ of the i-th mode by the relation

$$\gamma_i = \frac{\alpha + 4\pi^2 \beta f_i^2}{4\pi f_i}. \tag{8}$$

The FE consistent damping at the boundary surface derives as

$$[c]_\infty = \int_\Gamma \left[\bar{N}\right]^T [D_C] \left[\bar{N}\right] d\Gamma \tag{9}$$

The load vector {P(t)} is the usual weighted integral of the surface traction. The matrices in Eq. 6 and Eq. 9 are formulated in terms of the element shape function matrix, constitutive matrix for the far field stiffness $[D_K]$ and constitutive matrix for the far field geometrical or radiation damping $[D_C]$.

Local Transmitting Boundary Conditions

The outline of the new transmitting boundaries stands on how the constitutive matrices $[D_K]$ and $[D_C]$ are formulated. The attention is focused on the 1D or physical modeling of the unbounded soil domain and construction of a numerical device at the boundary, which can approximate far field behavior in a 3D FEM analysis.

When an impulse is acting on an elastic halfspace medium, radiation of energy occurs when the displacement amplitude decays in inverse proportion to the square root of the surface area at infinity. This is radiation criterion. In 3D analysis, the surface at infinity for body waves is a large hemisphere with radius $r \rightarrow \infty$. For R-waves the surface at infinity is a flat cylinder with radius r and height approximately one R-wave length λ_R.

So the amplitude of the body waves can be approximated to decreases in proportion to the ratio 1/r where r is the distance from the input source and the amplitude of the R-waves in proportion to $1/\sqrt{r}$ where r is the radius of the cylindrical surface

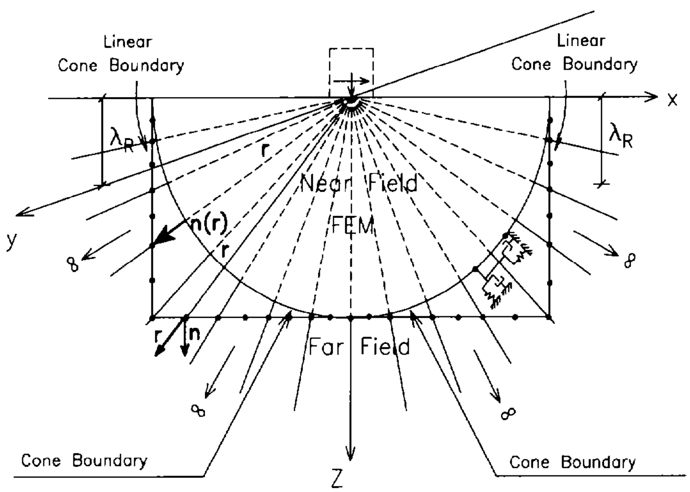

Fig. 1. Transmitting boundaries for 3D FEM analysis. Plane view.

<u>Body Waves.</u> From the soil mechanics point of view a load applied at the free surface of a halfspace leads to stresses acting on an area that increases with depth. The semi-infinite rod with variable cross section, specifically with its area changing with depth as a cone and truncated could be an approximate model for the soil. From this theory hemispherical P- or S wave fronts travelling in the positive z-direction could be closely approximated by

$$u_i(z,t) = \frac{1}{z} f\left(z - c_{P(S)} t\right) \tag{10}$$

For the conical horn, taking the equilibrium of the infinitesimal element, (Graff, 1975), the differential equation of motion for a P or S wave reduces to

$$\frac{1}{c_{P(S)}} u_{i,tt} - \frac{2}{z} u_{i,z} - u_{i,zz} = 0 \tag{11}$$

The equation of motion Eq. 11 could be given also as a product of two complementary operators as

$$\left[\frac{1}{c_{P(S)}}\frac{\partial}{\partial t} + \left(\frac{1}{z} + \frac{\partial}{\partial z}\right)\right]\left[\frac{1}{c_{P(S)}}\frac{\partial}{\partial t} - \left(\frac{1}{z} + \frac{\partial}{\partial z}\right)\right] u_i = 0 \tag{12}$$

The residual term $1/z^2$, which grows smaller as $z_i \rightarrow \infty$ is neglected. Considering only outgoing waves given as in Eq. 10 the boundary differential equation reduces to

$$\left[\frac{\partial}{\partial t} + \frac{c_{P(S)}}{z} + c_{P(S)}\frac{\partial}{\partial z}\right] u_i = 0 \tag{13}$$

The boundary stress at location z derives as

$$\sigma_i(x,t) = -\left[\frac{\rho c^2_{P(S)}}{z} u_i(z,t) + \rho c_{P(S)} u_{i,t}(z,t)\right] \tag{14}$$

So the missing part of the cones from the boundary location z to infinity is modeled by a mechanical system which contains a spring and a damper with frequency independent coefficients. The stiffness terms are in inverse proportion to the apex heights of the cones. It is known, (Meek and Wolf, 1993), that these models called translational cone models can sufficiently represent body waves in a dynamic analysis.

In the 3D case the power of P and S-waves is maximum under the source in the dilatational and shear windows resp. The rest of the halfspace transmits a small part of the radiated power in the far field except for R-waves, which propagate horizontally along the surface. From this investigation cone models could be used as transmitting boundary for body waves in a 3D FEM dynamic analysis of the halfspace medium, Fig. 1. They should be employed for all degrees of freedoms (DOF's) at the boundary defining the distance r of each Gauss integration point from the energy source, and knowing the coordinates of wave direction vectors **r** and vectors **n** normal to the boundary. So $[D_K]$ in Eq. 4 and Eq. 6 derives as

$$[D_K] = \frac{\rho}{r}(n * r)\left\{c_P^2[N] + c_S^2([I] - [N])\right\} \tag{15}$$

and $[D_C]$ in Eq. 4 and Eq. 9 derives as

$$[D_C] = \rho(n * r)\left\{c_P[N] + c_S([I] - [N])\right\} \tag{16}$$

considering a general boundary surface. In Fig. 1 two cases are shown, a spherical boundary for which (Wolf, 1994) gave this idea too, and a flat (box model) boundary. The last one is used in the numerical computations carried out later using Cartesian coordinates. [I] is the identity matrix and [N] is a 3x3 matrix. Its elements are products of coordinates of vectors **n** ($\mathbf{n_x},\mathbf{n_y},\mathbf{n_z}$).

So transmitting boundary for body waves in a 3D analysis can be modeled by a bunch of cones simulated from springs and dashpots attached to the boundary nodes and connected to a rigid base. Apexes of the cones derive from the geometry of the model and source location. The cones will have the same dimensions for all nodal DOF's different from vertical and horizontal cone models used in foundation vibration analysis.

<u>Surface Waves</u>. As it was mentioned before, in a 3D dynamic analysis these waves propagate with a cylindrical wave front and decay more slowly with distance than the body waves

In the context of 1D wave theory a cylindrical wave travelling in the positive x-direction can be closely approximated by

$$u_i(x,t) = \frac{1}{\sqrt{x}} f(x - ct) \tag{17}$$

where c is the wave velocity. In case of soil halspace model $c=c_R$ for example for the horizontal component of the in-plane motion. From the strength of materials theory the differential equation satisfied from a cylindrical wave front could be

$$\frac{1}{c}u_{i,tt} - \frac{1}{x}u_{i,x} - u_{i,xx} = 0 \quad (18)$$

which is the equation of motion of a cone with linear area variation, (Graff, 1975). Similarly to the cone models this equation can be written as product of two complementary operators as

$$\left[\frac{1}{c}\frac{\partial}{\partial t} + \left(\frac{1}{2x} + \frac{\partial}{\partial x}\right)\right]\left[\frac{1}{c} - \left(\frac{1}{2x} + \frac{\partial}{\partial x}\right)\right]u_i = 0 \quad (19)$$

disregarding the term $1/4x^2$ which grows smaller as x increases. Considering only outgoing waves the boundary differential equations equals

$$\left[\frac{\partial}{\partial t} + \frac{c}{2x} + c\,\frac{\partial}{\partial x}\right]u_i = 0 \quad (20)$$

from where the boundary stress derives as

$$\sigma_i(x,t) = -\left[\frac{\rho c^2}{2x}u_i(x,t) + \rho c\; u_{i,t}(x,t)\right] \quad (21)$$

So the missing part of the linear cones from the boundary location x to infinity is modeled also by a mechanical system The stiffness terms are in inverse proportion to double apex axis of the models. These models can be used as transmitting boundary for surface waves in similar way as for the body waves. For the box geometry for example the constitutive stiffness matrix $[D_K]$ is given as

$$[D_K] = \frac{\rho}{r}(n*r)\left\{\frac{1}{2}\left(sc_R^2 n_x^2 + c_R^2\left(1-n_z^2\right)\right) + c_S^2 n_y^2\right\} \quad (22)$$

and constitutive damping matrix $[D_C]$ as

$$[D_C] = \rho(n*r)\left\{sc_R n_x^2 + c_R\left(1-n_z^2\right) + c_S n_y^2\right\} \quad (23)$$

The matrices in Eq. 22 and Eq. 23 should be used in Eq. 6 and Eq. 9 resp. when building lateral boundary matrices till a depth equal to one λ_R based on the radiation criterion. λ_R is constant and known for a harmonic pulse but varies for a transient one. In this case the predominant frequency of the pulse should be considered to determine the length of the lateral boundary where surface waves will be absorbed. It looks as if the boundary becomes frequency dependent, however this dependence exists already in the FE discretization where a certain number of elements should be used to cover the predominant wavelength. In Eq. 22 and Eq. 23 s is the ratio of P- to S wave velocities.

COMPUTATIONAL RESULTS

To see the effectiveness of the proposed BC's, numerical experiments are performed first in time domain within the FE models, comparing the results of the small mesh with those of the extended mesh which is taken big enough to prevent reflections from the boundary to the interesting location. As the error due to FE discretization is also present in the extended mesh, the differences are caused by the inherent error from the transmitting boundary. A quarter of the problem is modeled using symmetry conditions when vertical loads is considered. A Hammer pulse with predominant period T_p=0.1s is chosen as transient source of vibration. The linear homogeneous, isotropic soil halfspace model has c_s=180m/s, ρ=1800kg/m^3 and Poisson's ratio ν=0.4 to test the stability of the boundary. A small 3D FE box model in Cartesian coordinates is built, which contains 1000 8-node cubic FE's. BC's are implemented at the bottom and the lateral sides of the model. The point load hammer pulse is applied at (0,0,0). Based on the time history of the pulse and shear wave velocity of the soil, element discretization is carried out taking dx=dy=dz=λ_S/9=2m. So the boundaries of the model are placed at a distance 1.1λ_S=20m from the source. The behavior at a lateral boundary node, at the free surface of the model is given in Fig. 2. Different BC's like stress free or Dirichlet, normal impedance, tensor impedance BC developed by (Krenk et al, 1999), and the new transmitting boundary based on cone and linear cone models are considered. Comparison is made with the extended mesh solution

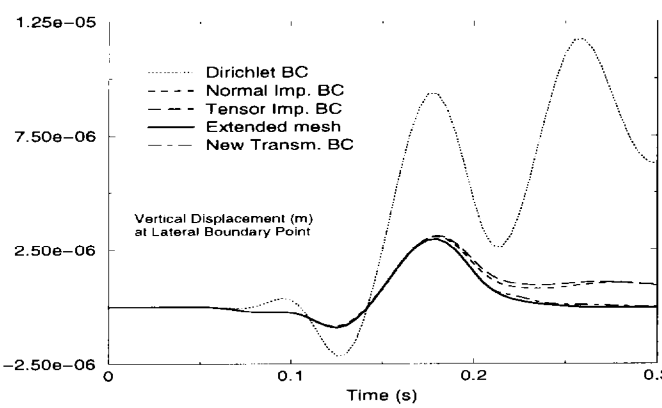

Fig. 2 Time history at x=20m, y=0m, z=0m for different BC's

It is obvious that the new BC's approach the extended mesh solution better than previous numerical absorption devices. Very little material damping with γ=2% was included in those computations to see clear the effect of the new boundary.
A 3D representation of the free surface of the model is given in Fig. 3 and Fig. 4 for Dirichlet and the new transmitting BC at different instants of time following the wave propagation in the soil model. Vertical displacement is chosen, as the point source is a vertical load. Implicit time integration is performed with Δt=0.005s. From these results, the necessity of absorbing BC's is obvious and the great power of the new boundary in absorbing the energy is clear. From Fig. 4 at t=0.3s, which is the case when the waves have left the model and are propagating in the far field we see that the response for stress free BC is unacceptable and

the behavior for new BC is very realistic.

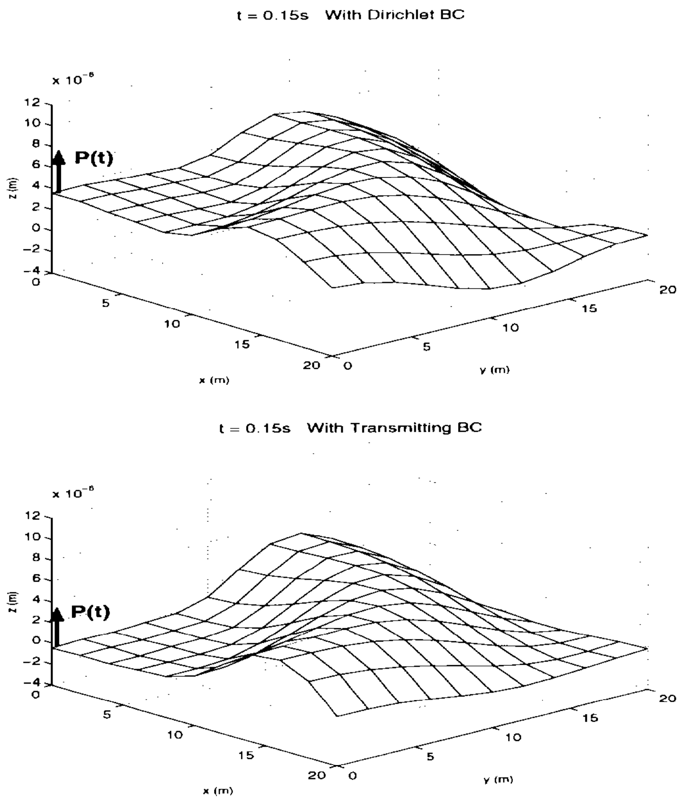

Fig. 3 Vertical displacement at z=0 when the waves are inside the model

Another application is carried out in cylindrical coordinates for horizontal loading. A Heaviside impulse load at t=0s is considered acting on a circular surface of r=1m in such a way that the total load P=1kN. The FE model is built with dx=dz≤4m and the boundary is placed at $x_b=z_b=116$m from the axis of symmetry to include nodes placed at a circle of radius R=100m from the origin where the closed form solutions are calculated. This type of load with constant time history is a test for the new boundary, which models approximately also the stiffness of the unbounded soil domain. The material data for the model are taken c_S=224m/s, ρ=2000kg/m^3, ν=0.25, γ=2%. The displacement response near the boundary for vertical w, horizontal u and tangential v components are given in nondimensionalized form in Fig. 5 in comparison to closed form solutions, (Takemiya and Steinfeld, 1993). These solutions are derived by use of the integral transform technique introducing the displacement potentials, which are expanded into the Fourier series along the azimuth direction. The Laplace transform is taken with respect to time. The Hankel transform for the radial direction decouples the wave field into the in-and the out-of-plane motion. In Fig. 5 Psinφ should be understood at the ordinate for displacement v. Values of φ=0 and φ=π/2 are excluded. The results seem to be quite acceptable considering FE approximation from the mesh discretization. U is used to symbolize all components of displacements.

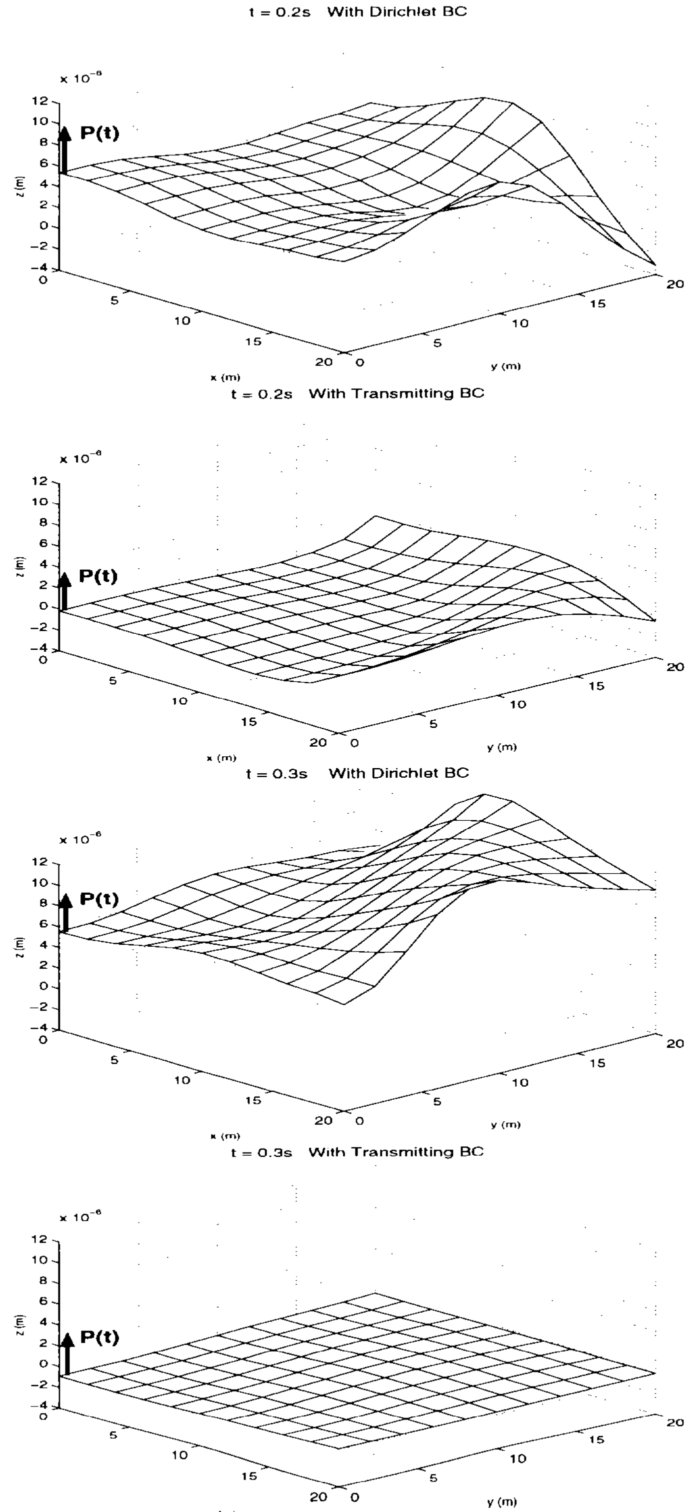

Fig. 4 Vertical displacement at z=0 for waves at the boundary and outside the model

CONCLUSIONS

BC's formulated here can be considered as DA approximations. MD boundary or viscous boundary satisfies Kreiss criterion, which determines well possedness of initial

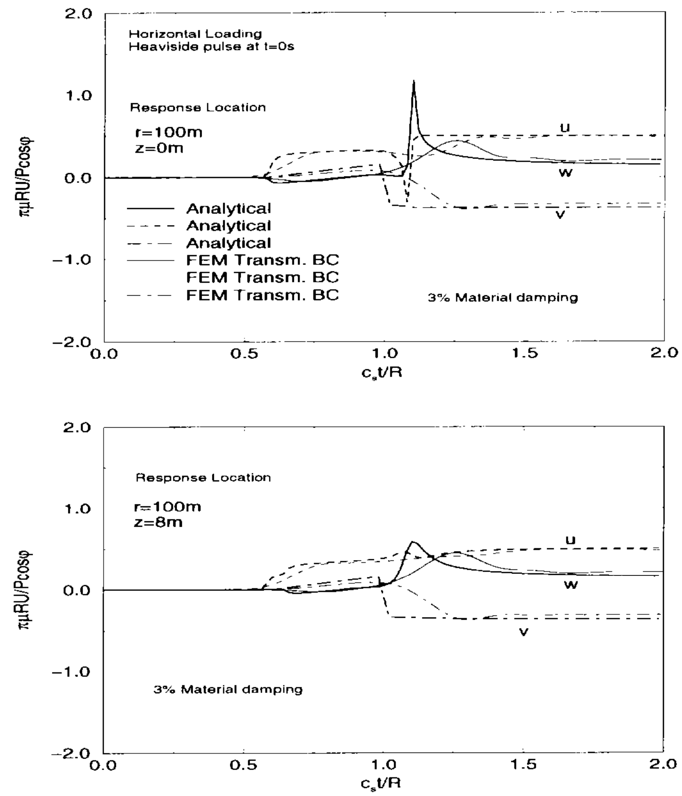

Fig. 5 Response at near boundary location

boundary value problem for first order linear hyperbolic systems. However this criterion is not satisfied for zero frequency that means static case. The reason for this exception is the fact that any differential operator is zeroing at frequency zero if this operator has not a constant term. When boundary differential equations are formulated based on linear cone or cone models then Kreiss criterion is satisfied also for static case, which makes the operator DA. So the formulated boundaries for 3D analysis represent an attempt to construct a local stiffness matrix for the unbounded soil domain. They are accurate for homogeneous halfspace conditions and localized source of vibration. The accuracy is sufficient when the boundary is placed at least $(1\div1.5)\lambda_S$ from the source of vibration.

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