Apr 2nd - Apr 7th

Longitudinal Waves in a Rod in an Elastic Medium

A. Bodare
Roy Institute of Technology, Stockholm, Sweden

I. K. Eriksson
Roy Institute of Technology, Stockholm, Sweden

K. Samuelsson
Roy Institute of Technology, Stockholm, Sweden

Proceedings: Third International Conference on Recent Advances in Geotechnical Earthquake Engineering and Soil Dynamics, April 2–7, 1995, Volume II, St. Louis, Missouri

Longitudinal Waves in a Rod in an Elastic Medium

Paper No. 10.11

A. Bodare and I.K. Eriksson
Assistant Professor and Research Eng., Div. of Soil and Rock Mechanics, Roy Institute of Technology, Stockholm, Sweden

K. Samuelsson
Research Eng., Division of Num. Analysis, Roy Institute of Technology, Stockholm, Sweden

SYNOPSIS The general equations for the propagation of longitudinal waves in an infinite cylindrical rod including the effect of a surrounding elastic medium have been derived in order to obtain the relationship between the circular frequency and the wave number. A comparison has then been made with the Smith model in order to investigate the limits of the model and to refine the expressions for the spring and damming constants.

INTRODUCTION

The scope of this study has been to derive the general equations for the propagation of longitudinal waves in an infinite cylindrical rod including the effect of a surrounding elastic medium. The relationship between the circular frequency and the wave number has been obtained. This relationship can be used to give a rheologic model of the soil to be used in pile analysis. It can also be used to understand the limits of the Smith model.

WAVES IN A ROD WITH A SURROUNDING MEDIUM

A rod with radius, a, in an infinite elastic medium is subjected to a propagating wave. See figure 1. In order to solve the wave equations for the longitudinal elastic stress waves, expressed in the potential functions ϕ and $\overline{H}$ in cylindrical coordinates, the following expressions are introduced;

$$\phi(r,z,t) = f(r)\, e^{i(kz-\omega t)}\ ,\quad H_\theta(r,z,t) = h(r) e^{i(kz-\omega t)} \qquad (1)\text{-}(2)$$

where k is the wave number in the axial direction in the rod and ω the angular frequency.

After substitution the wave equations then appear like;

$$\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} + \left(\frac{\omega^2}{c_p^2} - k^2\right) f = 0 \qquad (3)$$

$$\frac{d^2h}{dr^2} + \frac{1}{r}\frac{dh}{dr} + \left(\frac{\omega^2}{c_s^2} - k^2 - \frac{1}{r^2}\right) h = 0 \qquad (4)$$

where c_p and c_s are the velocity of propagation of the P-wave and S-wave, respectively.

Equation (3) and (4) are the Bessel differential equations with the following solutions,

$$f(r) = A\, J_0(l\, r) + B\, Y_0(l\, r)\ ;\ l^2 = \frac{\omega^2}{c_p^2} - k^2 \qquad (5)$$

$$h(r) = C\, J_1(m\, r) + D\, Y_1(m\, r)\ ;\ m^2 = \frac{\omega^2}{c_s^2} - k^2 \qquad (6)$$

where, A, B, C and D are constants. J_0 and J_1 are Bessels functions of the first kind and of the zero and first order, respectively, and Y_0 and Y_1 Bessel functions of the second kind (also called Weber's functions) of zero and first order. l and m are the wave number in the radial direction, in the rod and the surrounding medium, respectively.

Substituting equation (5) and (6) into (1) and (2) gives the following expression for the displacements;

$$u_r = \begin{bmatrix} -A\, l\, J_1(l\, r) - B\, l\, Y_1(l\, r) - \\ -C\, i\, k\, J_1(m\, r) - D\, i\, k\, Y_1(m\, r) \end{bmatrix} \qquad (7)$$

$$u_z = \begin{bmatrix} A\, i\, k\, J_0(l\, r) + B\, i\, k\, Y_0(l\, r) + \\ + C\, m\, J_0(m\, r) + D\, m\, Y_0(m\, r) \end{bmatrix} \qquad (8)$$

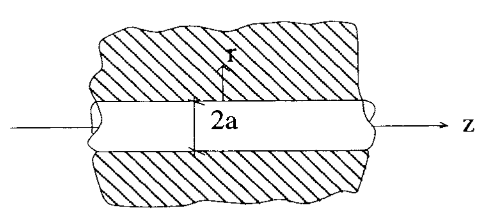


Figure 1. *A rod with a surrounding medium.*

With the use of Hooke's law in three dimensions, the normal and shear stresses can be expressed as;

$$\sigma_{rr} = 2G \begin{bmatrix} \left[-\frac{1}{2}(m^2-k^2)\,J_0(l\,r)+\frac{l}{r}\,J_1(l\,r)\right]A + \\ +\left[-\frac{1}{2}(m^2-k^2)\,Y_0(l\,r)+\frac{l}{r}\,Y_1(l\,r)\right]B + \\ +i\left[-k\,m\,J_0(m\,r)+\frac{k}{r}\,J_1(m\,r)\right]C + \\ +i\left[-k\,m\,Y_0(m\,r)+\frac{k}{r}\,Y_1(m\,r)\right]D \end{bmatrix} \quad (9)$$

$$\tau_{rz} = 2G \begin{bmatrix} -2\,i\,k\,l\,J_1(l\,r)\,A - 2\,i\,k\,l\,Y_1(l\,r)\,B + \\ +(k^2-m^2)\,J_1(m\,r)\,C + (k^2-m^2)\,Y_1(m\,r)\,D \end{bmatrix} \quad (10)$$

Equations (9) and (10) hold for both the rod and the surrounding medium, i.e. totally four expressions, σ_{rr1}, σ_{rr2}, τ_{rz1} and τ_{rz2}. (The entities with subscripts 1 and 2 refer to the rod and the surrounding medium, respectively.)

The constants A and C are in combinations of Bessel functions J_0 and J_1 and the constants B and D combinations of Weber functions, Y_0 and Y_1. As the origo is set within the rod, B_1 and D_1 must be zero as Y_0 and Y_1 go to infinity. The expressions for the surrounding medium contain the whole set of constants, A_2, B_2, C_2, D_2, i.e. there are totally six unknown constants, which means that six boundary conditions will be needed for the sytem of equations to be solved.

The boundary conditions, valid for this case, are the following,

$$\sigma_{rr1}(r=a) = \sigma_{rr2}(r=a) \;,\; \tau_{rz1}(r=a) = \tau_{rz2}(r=a) \quad (11)\text{-}(12)$$

$$u_{r1}(r=a) = u_{r2}(r=a) \;,\; u_{z1}(r=a) = u_{z2}(r=a) \quad (13)\text{-}(14)$$

Finally, the radiation condition, (only outgoing waves in infinity)

$$\begin{matrix} u_r(r\to\infty,t) \approx e^{i\left(\frac{1}{m}r-\omega t\right)} \\ u_z(r\to\infty,t) \approx e^{i\left(\frac{1}{m}r-\omega t\right)} \end{matrix} \text{ where } \frac{1}{m} > 0 \quad (15)$$

For the solution the Hankel function needs to be introduced;

$$H_n(r) = J_n(r) + i\,Y_n(r)\,,\; n = 0,1 \quad (16)$$

which corresponds to a Bessel function of the third order.

The expression for stresses and displacements can be written as;

$$\sigma_{rr} = G \begin{bmatrix} \left[-\frac{1}{2}(m^2-k^2)\,H_0^{(1)}(l\,r)+\frac{l}{r}\,H_1^{(1)}(r)\right]\mathcal{A} + \\ +\left[-\frac{1}{2}(m^2-k^2)\,H_0^{(2)}(l\,r)+\frac{l}{r}\,H_1^{(2)}(r)\right]\mathcal{B} + \\ +i\left[-k\,m\,H_0^{(1)}(m\,r)+\frac{k}{r}\,H_1^{(1)}(m\,r)\right]\mathcal{C} + \\ +i\left[-k\,m\,H_0^{(2)}(m\,r)+\frac{k}{r}\,H_1^{(2)}(m\,r)\right]\mathcal{D} \end{bmatrix} \quad (17)$$

$$\tau_{rz} = G \begin{bmatrix} -2\,i\,k\,l\left[H_1^{(1)}(l\,r)\,\mathcal{A} + H_1^{(2)}(l\,r)\,\mathcal{B}\right] + \\ +(k^2-m^2)\left[H_1^{(1)}(m\,r)\,\mathcal{C} + H_1^{(2)}(m\,r)\,\mathcal{D}\right] \end{bmatrix} \quad (18)$$

$$u_r = \frac{1}{2} \begin{bmatrix} -l\,H_1^{(1)}(l\,r)\,\mathcal{A} - l\,H_1^{(2)}(l\,r)\,\mathcal{B} - \\ -i\,k\,H_1^{(1)}(m\,r)\,\mathcal{C} - i\,k\,H_1^{(2)}(m\,r)\,\mathcal{D} \end{bmatrix} \quad (19)$$

$$u_z = \frac{1}{2} \begin{bmatrix} i\,k\,H_0^{(1)}(l\,r)\,\mathcal{A} + i\,k\,H_0^{(2)}(l\,r)\,\mathcal{B} + \\ +m\,H_0^{(1)}(m\,r)\,\mathcal{C} + m\,H_0^{(2)}(m\,r)\,\mathcal{D} \end{bmatrix} \quad (20)$$

where

$$\mathcal{A} = A + \frac{B}{i} = A - iB \;,\; \mathcal{B} = A - \frac{B}{i} = A + iB \quad (21)\text{-}(22)$$

$$\mathcal{C} = C + \frac{D}{i} = C - iD \;,\; \mathcal{D} = C - \frac{D}{i} = C + iD \quad (23)\text{-}(24)$$

In the rod, $B_1 = D_1 = 0$, as explained above, i.e. $A_1 = \mathcal{A}_1 = \mathcal{B}_1$ and $C_1 = \mathcal{C}_1 = \mathcal{D}_1$. The radiation condition (15) means that $\mathcal{B}_2 = \mathcal{D}_2 = 0$ as

$$H_n^{(2)}\left(\frac{1}{m}r\right) \to \sqrt{\frac{2}{\pi r}}\,e^{i\left(\frac{1}{m}r+\frac{\pi}{4}+\frac{n\pi}{2}\right)} \text{ where } r\to\infty$$

Consequently $B_2 = i\,A_2$ and $D_2 = i\,C_2$.

The boundary conditions (11)-(15) can be written as a homogeneous sytem of linear equations. For the system to be solvable the determinant of the matrix must be zero.

By introducing the following notations,

$$\gamma = \frac{G_1}{G_2}\ , \quad \kappa = k\,a \qquad (25)\text{-}(26)$$

$$\lambda_i = l_i\,a\ , \quad \mu_i = m_i\,a \quad i = 1, 2 \qquad (27)\text{-}(28)$$

$$\left(\kappa^2 + \lambda_i^2\right) = \left(\frac{\omega\,a}{c_{pi}}\right)^2 \quad i = 1, 2 \qquad (29)$$

$$\left(\kappa^2 + \mu_i^2\right) = \left(\frac{\omega\,a}{c_{si}}\right)^2 \quad i = 1, 2 \qquad (30)$$

$$H_i \equiv H_i^{(1)} = J_i + i\,J_i \quad i = 1, 2 \qquad (31)$$

where γ is the stiffness ratio, G_1 and G_2 the shear modulus of the rod and the surrounding medium, respectively, κ the dimensionless wave number in the axiell direction of the rod, λ and μ dimensionless radial wave numbers the following equation is obtained;

$$\det\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix} = 0 \qquad (32)$$

where $a_{1,1} = \gamma\left[\frac{1}{2}\left(\kappa^2 - \mu_1^2\right)J_0(\lambda_1) + \lambda_1 J_1(\lambda_1)\right]$ $\quad a_{3,1} = -\lambda_1 J_1(\lambda_1)$

$a_{1,2} = i\gamma\left[-\kappa\mu_1 J_0(\mu_1) + \kappa J_1(\mu_1)\right]$ $\quad a_{3,2} = -i\kappa J_1(\mu_1)$

$a_{1,3} = \frac{1}{2}\left(\kappa^2 - \mu_2^2\right)H_0(\lambda_2) + \lambda_2 H_1(\lambda_2)$ $\quad a_{3,3} = -\lambda_2 H_1(\lambda_2)$

$a_{1,4} = i\left[-\kappa\mu_2 H_0(\mu_2) + \kappa H_1(\mu_2)\right]$ $\quad a_{3,4} = -i\kappa H_1(\mu_2)$

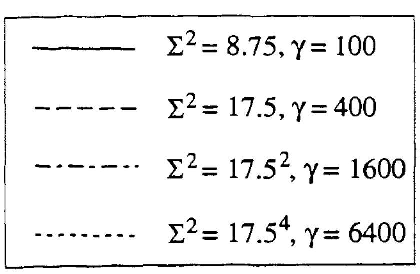
$a_{2,1} = -2i\gamma\kappa\lambda_1 J_1(\lambda_1)$ $\quad a_{4,1} = i\kappa J_0(\lambda_1)$

$a_{2,2} = \gamma\left(\kappa^2 - \mu_1^2\right)J_1(\mu_1)$ $\quad a_{4,2} = \mu_1 J_0(\mu_1)$

$a_{2,3} = -2i\kappa\lambda_2 H_1(\lambda_2)$ $\quad a_{4,3} = i\kappa H_0(\lambda_2)$

$a_{2,4} = \left(\kappa^2 - \mu_2^2\right)H_1(\mu_2)$ $\quad a_{4,4} = \mu_2 H_0(\mu_2)$

The solution of this equation gives the relationship between the wave number, k, and the angular frequency, ω, or rewritten, κ (which is a comlex function) as a function of the dimensionless angular frequency, Ω. These relationships plotted, see figure 2-5, give the values of the parameters for a homogeneous elastic soil medium as a function of the angular frequency.



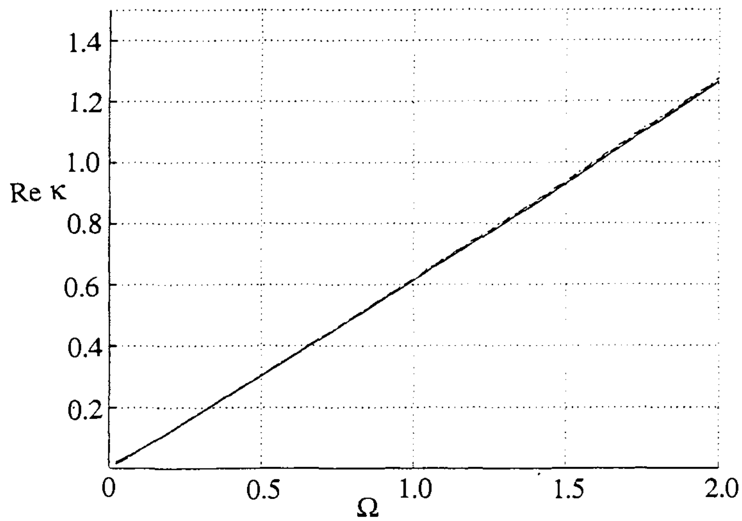


Figure 2. *Relationship between Ω and Re κ.*

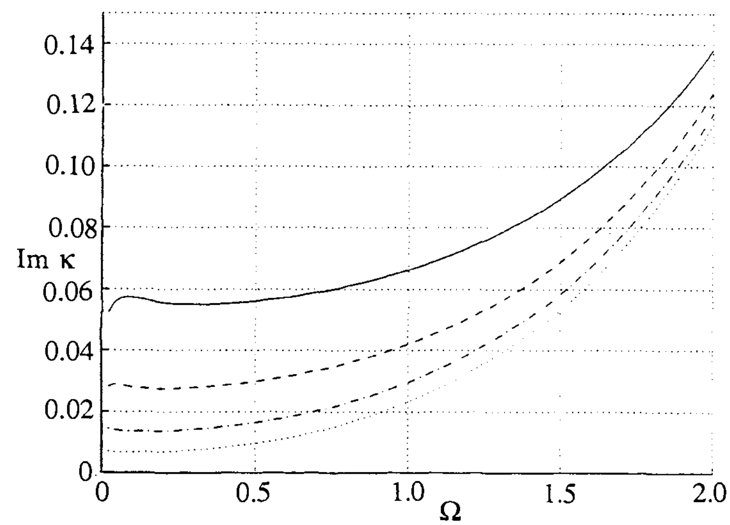


Figure 3. *Relationship between Ω and Im κ.*

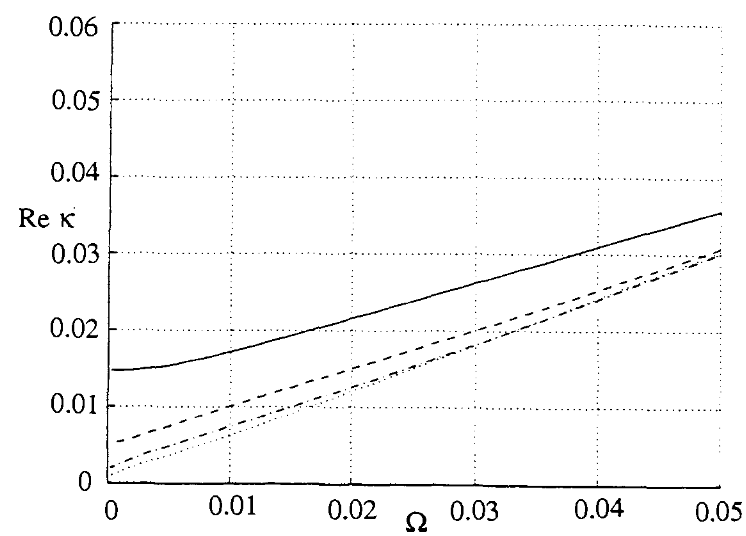


Figure 4. *Relationship between Ω and Re κ.*

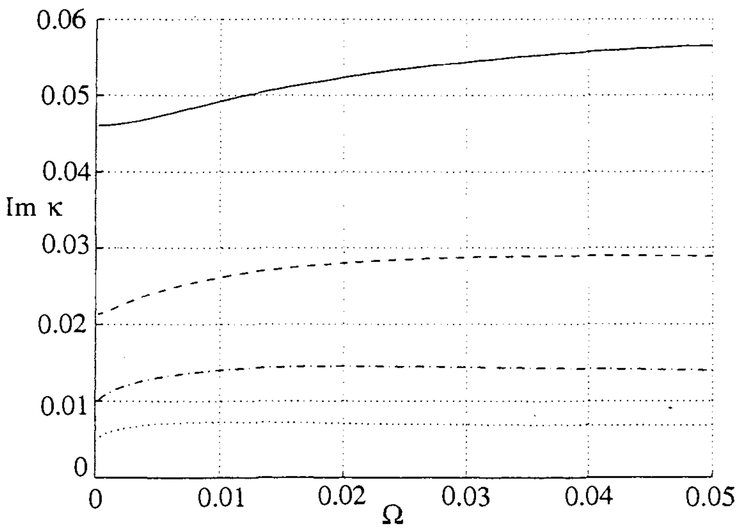


Figure 5. *Relationship between Ω and Im κ.*

SMITH'S SOIL MODEL

Having derived the general equation for the propagation of longitudinal waves in a cylindrical rod including the effect of a surrounding elastic medium, we now want to compare this relationship between the excited circular frequency and the wave number with the one derived from the Smith soil model.

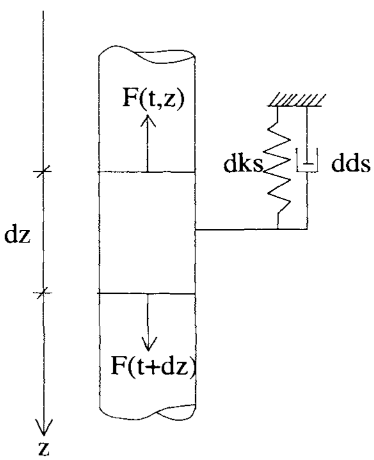


Figure 6. *Smith's model. Spring stiffness and damping coefficient per unit of length.*

$$dk_S = k_S \, dz \ , \ dd_S = d_S \, dz \qquad (33)\text{-}(34)$$

where k_S is the spring constant per unit of length and d_S the damping coefficient per unit of length.

The equation of motion of the rod when it has been disturbed through a distance w, in the z-direction, can be written from Newton's second law of motion;

$$\ddot{w} + \beta \, \dot{w} + \alpha^2 w = \frac{d^2 w}{dz^2} c_b^2 \qquad (35)$$

Where α is the root of the spring constant, β the damping coefficient and c_b the velocity of the wave propagation in a cylindrical bar.

$$\alpha = \sqrt{\frac{k_S}{A\rho}} \ , \ \beta = \frac{d_S}{A\rho} \ , \ c_b = \sqrt{\frac{E}{\rho}} \qquad (36)\text{-}(38)$$

Where A is the cross-sectional area of the cylindrical rod, ρ the density of the rod and E, Young's modulus.

Suppose that,

$$w = \hat{w} \, e^{i\left(k_S z - \omega t\right)} \qquad (39)$$

Derivations of equation (39) and substitution into equation (35) gives us the following general relation,

$$\omega^2 - \alpha^2 + i \, \omega \, \beta = k_S^2 c_b^2 \qquad (40)$$

As ω is real, excited by a vibrator, k_S must be imaginary.

A dimensionless expression of the wave number as a function of the frequency is

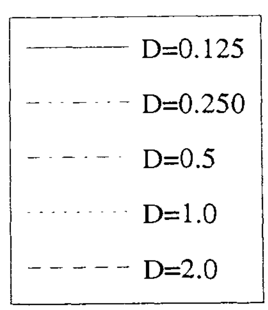
$$\kappa_S = \pm \left(\frac{\Omega_S^2 - 1^2 + \left(\left(\Omega_S^2 - 1^2\right)^2 + \left(2\Omega_S D\right)^2 \right)^{0.5}}{2} \right)^{0.5} + i \left(\frac{-\Omega_S^2 + 1^2 + \left(\left(\Omega_S^2 - 1^2\right)^2 + \left(2\Omega_S D\right)^2 \right)^{0.5}}{2} \right)^{0.5} \qquad (41)$$

where

$$\kappa = k_S \, c_b \ , \ \kappa_S = \frac{\kappa}{\alpha} \ , \ \Omega_S = \frac{\omega}{\alpha} \ , \ D = \frac{\beta}{2\alpha} \qquad (42)\text{-}(45)$$

Where κ_S is the normalized wave number, Ω_S the dimensionless angular frequency in the Smith model, and D the damping ratio.

Expression (41) plotted, for different values of the damping ratio, D, gives us the curves in figure 7 and 8.



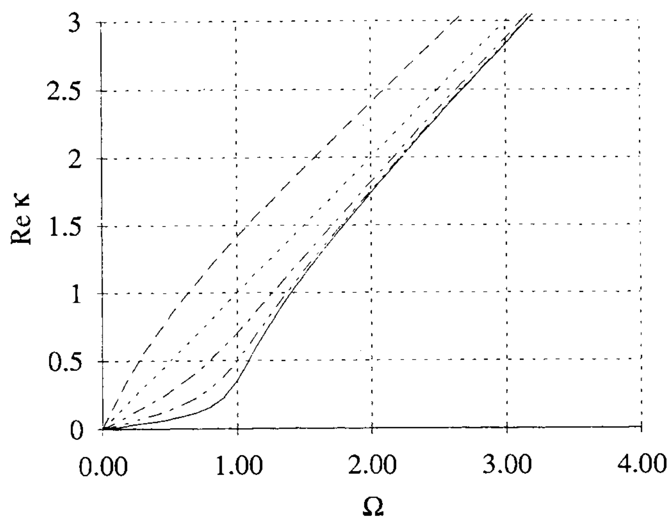


Figure 7. *Relationship between Ω and Re κ according to Smith's model.*

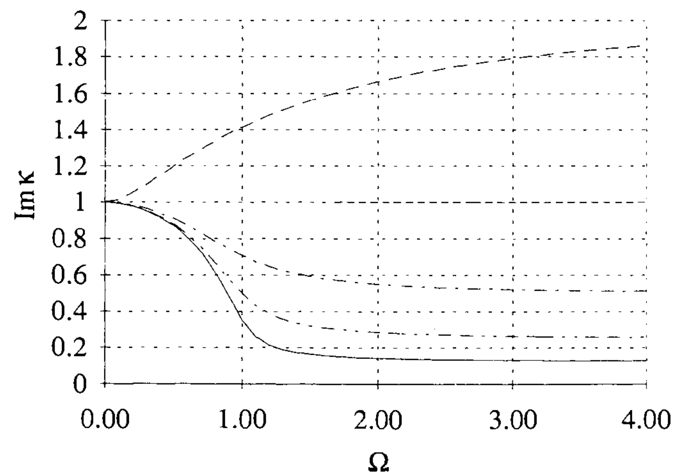


Figure 8. *Relationship between Ω and Im κ according to Smith's model.*

SUMMARY

When comparing the Smith model with the derived general case, regarding the relationship between Ω and Re κ, see figure 2 and 7, we find that the Smith model well corresponds to the general case, when a damping ratio, D= 1, is chosen. However, when the area around origo is enlarged, see figure 4, (which, concerning piling activities, is the area of interest), something unexpected is revealed, namely that the curves in the general case do not pass origo.

The correspondence between the Smith model and the derived general case, regarding the relationship between Ω and Im κ, see figure 3 and 8, is on the other hand very rough, but for small values of Ω, see figure 5, the correspondence is good.

To improve the expression for the damping effect in the Smith model, let us compare figure 5 with the curve, D=1 in figure 8, to find a relationship between the imaginary part of κ and $\gamma = G_1/G_2$. This leads us to the conclusion that when the Smith model is used, the following values of the spring and damping constants should be chosen;

$$k_S = 1.184\left(1+\nu_1\right)^{0.252} E_1^{0.748} G_2^{0.252} \tag{46}$$

$$d_S = 2\sqrt{\rho_1 A_1}\sqrt{1.184\left(1+\nu_1\right)^{0.252} E_1^{0.748} G_2^{0.252}} \tag{47}$$

where ν_1 is Poisson's ratio of the rod.

Table 1. A numerical example.

Pile material	ρ_1 [t/m^3]	E_1 [GPa]	ν_1 [-]	G_2* [MPa]	k_S [kg/s^{2}m]	d_S [kg/sm]
Steel	7.0-7.8	210	0.3	3.5-54	(16.6-33.1)E9	(8.78-13.09)E6
Concrete	2.4	30	0.2	3.5-54	(3.80-7.56)E9	(2.46-3.47)E6

* Range from saturated clay to saturated till.

CONCLUSION

When having compared the general equations for the propagation of longitudinal waves in a cylindrical rod including the effect of a surrounding elastic medium with the equations in the Smith model, we have found that the expression for the damping coefficient is very rough. In order to get a better correspondance to the general equations we have proposed a new approximate expression for the spring and damping constants in the Smith model, as functions of the shear modulus of the soil and the modulus of elasticity of the pile, see equation (46) and (47).

SYMBOLS

Roman letters

A= cross-sectional area
A, B, C, D, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ = constants
c_b = velocity of the wave propagation in a cylindrical rod
c_p = velocity of propagation of the P-wave
c_s = velocity of propagation of the S-wave
D = damping ratio
d_S = damping coefficient per unit of length, Smith model
E = Young's modulus
f(r) = solution of the Bessel function of the zero order
G = shear modulus
$\overline{H}$ = potential for the S-wave
H_n = Hankel function of the n order
h(r) = solution of the Bessel function of the first order
J_n = Bessel function of the first kind and n order
k = scale factor
k = wave number in the axiell direction of the rod
k_S = spring constant per unit of length, Smith model
l = wave number in the radiell direction of the rod
m = wave number in the radiell direction of the rod

u_r = displacement in the radial direction
u_z = displacement in the axial direction
r = distance along radial axis
t = time
w = displacement
Y_n = Bessel function of the second kind and order n
z = distance along Z-axis

Greek letters

α = the root of the spring constant in the Smith model
β = damping coefficient in the Smith model
γ = stiffness ratio
θ = spherical coordinate
κ = dimensionless wave number in the axiell direction of the rod
κ_S = normalized wave number, Smith model
λ = dimensionless radial wave number
μ = dimensionless radial wave number
ν = Poisson´s ratio
ρ = density
Σ = constant
σ_{rr} = normal stress in the r-direction
τ_{rz} = shear stress in the z-r plane
ϕ = potential for the P-wave
Ω = dimensionless angular frequency
Ω_S = dimensionless angular frequency, Smith model
ω = angular frequency

REFERENCES

Graff, K. F. (1975), *Wave motion in elastic solids.* Clarendon Press Oxford.

Love, A. E. H. (1944), *A treatise on the mathematical theory of elasticity.* Dover Publications NewYork.

Pochhammer, L. (1876), Über die Fortpflanzungsgeschwindigkeiten kleiner Schwingungen in einem unbegrenzten isotropen Kreiszylinder. *J. reine angew. Math. 81*, p. 324-336.

Råde, L. and Westergren, B. (1988), *Beta β Mathematics Handbook.* Studentlitteratur, Sweden.