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MASS TRANSFER FROM SPHERICAL GAS BUBBLES AND LIQUID
DROPLETS MOVING THROUGH POWER-LAW FLUIDS
IN THE LAMINAR FLOW REGIME

by

Cheng-chun Huang, 1942-

A

Dissertation

presented to the faculty of

UNIVERSITY OF MISSOURI - ROLLA

in partial fulfillment of the requirements for the

Degree of


DOCTOR OF PHILOSOPHY IN CHEMICAL ENGINEERING

Rolla, Missouri

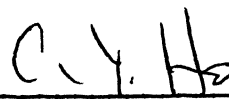
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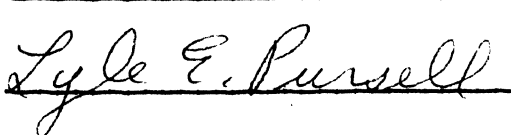
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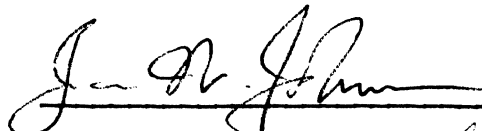
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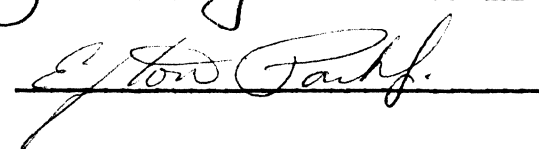
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Abstract

Continuous phase mass transfer coefficients are presented for internally circulating spheres of a Newtonian fluid traveling through a power-law type continuous phase in the so-called creeping-flow region. The Nakano and Tien stream functions allow the Sherwood number to be determined as a function of the Peclet number, power-law index, and a viscosity ratio parameter. The Hirose and Moo-Young relation is shown to be a limiting case of this solution. Mass transfer rates increase as the fluids become more pseudoplastic and/or the continuous phase consistency index increases, all other factors held constant.

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MASS TRANSFER FROM SPHERICAL GAS BUBBLES AND LIQUID
DROPLETS MOVING THROUGH POWER-LAW FLUIDS
IN THE LAMINAR FLOW REGIME

Because of its importance in many chemical engineering processes, the problem of mass transfer from bubbles and droplets has been investigated by numerous workers. In almost all previous theoretical and experimental studies, the fluids under consideration have been Newtonian in character. In one recent study, a theoretical relation was developed which described the continuous phase mass transfer mechanism for a fully circulating gas bubble moving through a non-Newtonian liquid continuous phase which was described by the power-law rheological model (Hirose and Moo-Young, 1969).

It is the purpose of this paper to present a solution for the continuous phase mass transfer coefficient for a wide range of dispersed and continuous phase properties. The primary investigation will be based upon the use of the Nakano and Tien (1968) continuous phase stream functions which were developed for a power-law type, non-Newtonian fluid in the continuous phase and a Newtonian fluid in the dispersed phase. The solution will be applicable for both a gas and liquid Newtonian dispersed phase and for a range of non-Newtonian continuous phase characteristics.

The results will also be compared with the continuous phase mass transfer coefficients obtained using the more restrictive non-Newtonian stream function of Hirose and Moo-Young (1969) for fully circulating gas bubbles rising in a power-law fluid, and also, with the approximate non-Newtonian stream functions developed by Tomita (1959) for the case of no internal circulation within the dispersed phase and a power-law type, continuous phase fluid. In all cases the models will be compared with existing models based upon a Newtonian continuous phase.

Mass Transfer Model

The mathematical model for mass transfer from spheres in the creeping flow regime is described by the following dimensionless partial differential equation and boundary conditions using spherical coordinates with origin at the center of the sphere:

$$V_y \frac{\partial C}{\partial y} + \frac{V_\theta}{y} \frac{\partial C}{\partial \theta} = \frac{2}{Pe} \left[\frac{\partial^2 C}{\partial y^2} + \frac{2}{y} \frac{\partial C}{\partial y} \right] \quad (1)$$

with boundary conditions

$$C = 0 \quad \text{at} \quad y = 1 \quad (2a)$$

$$C = 1 \quad \text{at} \quad y \rightarrow \infty \quad (2b)$$

$$\frac{\partial C}{\partial \theta} = 0 \quad \text{at} \quad \theta = 0, \pi \quad (2c)$$

The following assumptions have been made:

- (i) The flow is axially symmetric, isothermal, and steady state has been reached.
- (ii) The dispersed phase is a Newtonian fluid, while the continuous phase obeys a power-law model, i.e.,

$$\tau_j^i = -\mu \Delta_j^i \quad \text{for dispersed phase}$$

$$\tau_j^i = -K \left(\frac{1}{2} \Delta_k^m \Delta_k^m \right)^{\frac{n-1}{2}} \Delta_j^i \quad \text{for continuous phase}$$

- (iii) Natural convection, molecular diffusion in the angular direction and resistance to mass transfer inside the sphere are negligible.
- (iv) The solute is dilute. The solute diffusivity and density of both phases are constant.

In order to complete the description of the forced-convective mass transfer model, the velocity components V_θ and V_r have to be specified. The following continuous phase velocity components were derived in this work from the Nakano and Tien (1968) stream functions:

$$V_y = (-1 + 2A_1 y^{\sigma-2} + 2 \frac{A_2}{y^3}) \cos \theta + (B_1/y^3 + B_2/y^4)(2 \cos^2 \theta - \sin^2 \theta) \quad (3a)$$

$$V_\theta = (1 - A_1 \sigma y^{\sigma-2} + A_2/y^3) \sin \theta + (B_1/y^3 - 2 B_2/y^4) \cos \theta \sin \theta \quad (3b)$$

The coefficients A_1 , A_2 , B_1 , B_2 , and the exponent σ were tabulated for different values of flow behavior index n and a viscosity ratio parameter X . The parameter X is defined below in Equation 4.

$$X = \frac{\mu_d}{K} \left(\frac{a}{V_\infty} \right)^{n-1} \quad (4)$$

These coefficients were so chosen that for the Newtonian case ($n = 1$), the

Nakano and Tien stream functions would become identical with the Hadamard (1911) stream functions for Newtonian continuous and dispersed phase fluids.

Nakano and Tien (1968) in their study were concerned with the prediction of drag coefficients for a fluid sphere. Drag coefficients determined by Nakano and Tien agreed very well with the experimental results of Fararoui and Kintner (1961) for liquid droplets of a nitrobenzene and tetrachlorethane mixture falling through an aqueous solution of carboxymethylcellulose.

The velocity components based on the Hadamard stream functions were also used in this work as a check upon the mass transfer rates determined using the Nakano and Tien stream function when the continuous phase was Newtonian. They are as follows:

$$V_y = \left[-1 + \left(\frac{3R+2}{2R+2} \right) \frac{1}{y} - \frac{1}{2} \left(\frac{R}{R+1} \right) \frac{1}{y^3} \right] \cos\theta \quad (5a)$$

$$V_\theta = \left[1 - \frac{1}{2} \left(\frac{3R+2}{2R+2} \right) \frac{1}{y} - \frac{1}{4} \left(\frac{R}{R+1} \right) \frac{1}{y^3} \right] \sin\theta \quad (5b)$$

In this case ($n = 1$), the viscosity ratio parameter of Nakano and Tien, X , becomes

$$X = \frac{\mu_d}{\mu_c} \equiv R \quad (6)$$

where μ_d and μ_c are the viscosity of the dispersed phase and the continuous phase, respectively.

By use of Equations 3a and 3b, the boundary value problem of Equations 1 and 2 can be solved numerically for the solute concentration profiles in the continuous phase around the bubble or droplet. The average

Sherwood number is then calculated as follows:

$$Sh = \frac{k_c 2a}{D} = \int_0^\pi \left(\frac{\partial C}{\partial y} \right)_{y=1} \sin\theta d\theta \quad (7)$$

Continuous phase mass transfer coefficients were also calculated based upon velocity components derived from the works of Hirose and Moo-Young (1969) and Tomita (1959). These velocity components are given later in this work [See Equations 14 and 16] along with a discussion of their conditions of applicability.

Numerical Solution

Because of its strong stability, the Crank-Nicolson implicit numerical method (1947) was used to solve the mass transfer models described above. To initiate the solution, boundary condition Equation 2c was first applied to obtain the solution for $\theta = 0$. In this case, Equation 1 becomes

$$V_y \frac{dC}{dy} = \frac{2}{Pe} \left(\frac{d^2 C}{dy^2} + \frac{2}{y} \frac{dC}{dy} \right) \quad (8)$$

or written in the finite difference form using central difference approximations:

$$a_{i,1} C_{i-1,1} + b_{i,1} C_{i,1} + d_{i,1} C_{i+1,1} = 0 \quad (9)$$

where the second subscript, 1, represents the solution for $\theta = 0$, and

$$a_{i,1} = \frac{2}{Pe\Delta y} + \frac{V_y}{2} - \frac{2}{Pey} \quad (10a)$$

$$b_{i,1} = -\frac{4}{Pe\Delta y} \quad (10b)$$

$$d_{i,1} = \frac{2}{Pe \Delta y} - \frac{V_y}{2} - \frac{2}{Pe y} \quad (10c)$$

Applying boundary condition Equations 2a and 2b, Equation 9 can be reduced to a system of linear, homogeneous, simultaneous equations having a triangular coefficient matrix. Thus the solution is readily obtained for $\theta = 0$ by use of any well established numerical technique (e.g., the Gauss elimination method or the Gauss-Jordan method with pivotal condensation.)

For $\theta > 0$, Equation 2 is written in the following finite difference form using central difference approximations:

$$f_{i,j} C_{i-1,j+1} + \ell_{i,j} C_{i,j+1} + p_{i,j} C_{i+1,j+1} = q_{i,j} \quad (11)$$

where

$$f_{i,j} = \frac{1}{Pe(\Delta y)^2} + \frac{V_y}{4(\Delta y)} - \frac{1}{Pe y(\Delta y)} \quad (12a)$$

$$\ell_{i,j} = -\frac{2}{Pe(\Delta y)^2} - \frac{V_\theta}{y(\Delta \theta)} \quad (12b)$$

$$p_{i,j} = \frac{1}{Pe(\Delta y)^2} - \frac{V_y}{4(\Delta y)} + \frac{1}{Pe y(\Delta y)} \quad (12c)$$

$$q_{i,j} = -f_{i,j} C_{i-1,j} + t_{i,j} C_{i,j} - p_{i,j} C_{i+1,j} \quad (12d)$$

$$t_{i,j} = \frac{2}{Pe(\Delta y)^2} - \frac{V_\theta}{y(\Delta \theta)} \quad (12e)$$

Equations 11 and 12 represent an implicit numerical scheme obtained by use of the Crank-Nicolson method. To solve the equations for any step in the angular direction, the solution for the previous step is needed. Therefore, with the known solution of Equation 8 or 9 for $\theta = 0$,

the solution for any $\theta > 0$ can be obtained by solving Equation 11 and 12 repeatedly with boundary condition Equations 2a and 2b.

Results and Discussion

The boundary value problem of Equation 1 and 2 for the forced-convective mass transfer problem was solved numerically for various values of Peclet numbers ranging between 10^{-3} and 10^5 . The results are presented in terms of Sherwood numbers as a function of the Peclet number, the continuous phase flow behavior index n , and the viscosity ratio parameter X . The upper Pe limit, 10^5 , was the largest value of Pe possible for ordinarily encountered systems which still satisfy the creeping flow criterion $Re < 1$.

This wide range of Peclet numbers creates a stability problem in the numerical solution of the partial differential equation if an attempt is made to use one specific value of the radial increment, Δy , for all values of the Peclet number. Therefore, it was found essential to use different values of Δy for different values of the Peclet number; it was found by trial that a smaller radial increment has to be used for a larger Peclet number in order to obtain the most stable concentration profiles. The Stokes stream function was used as a standard for this preliminary search for optimum values of Δy at each Peclet number level. The values of Δy used in this work are presented in Table III.*

*For tabular results (Table III), order NAPS Document XXXXX from ASIS National Publication Service, c/o CCM Information Sciences, Inc., 22 West 34th Street, New York, New York 10001, remitting \$1.00 for microfiche or \$3.00 for photocopies.

As expected for larger Peclet numbers, the concentration boundary layer is thinner. The outer limits of the boundary layer used in the present work are 0.5440 dimensionless radii for $Pe = 10^4$ and 0.1176 dimensionless radii for $Pe = 10^5$ from the sphere surface. For such high Peclet numbers, these are believed to be sufficiently far away from the sphere surface where the concentration is unity.

For the angular direction, an increment of 3 degrees was used. A reduction of the angular increment by a factor of two resulted in a change of only about 0.5% in the calculated value of Sh when Pe was equal to 10^4 .

By choosing n equal to unity in the Nakano and Tien stream function (or directly using the Stokes and Hadamard stream functions), the Sherwood number relations obtained numerically by the present model will first be compared with previous studies of circulating droplets and bubbles moving through Newtonian fluids. This preliminary study will then be followed by the presentation of results for a non-Newtonian continuous phase.

Newtonian continuous phase. For the non-circulating spheres, the above numerical technique when combined with the Stokes velocity components

$$V_y = \left[-1 + \frac{3}{2} \left(\frac{1}{y} \right) - \frac{1}{2} \left(\frac{1}{y} \right)^3 \right] \cos \theta \quad (13a)$$

$$V_\theta = \left[1 - \frac{3}{4} \left(\frac{1}{y} \right) - \frac{1}{4} \left(\frac{1}{y} \right)^3 \right] \sin \theta \quad (13b)$$

gave Sherwood numbers almost identical with those obtained by Friedlander (1957, 1961) using boundary layer theory and the Stokes (1880) stream functions. Bowman et al. (1961) obtained nearly identical values for the Sherwood number using the same technique as Friedlander's, but with a

four-term polynomial used for the concentration profile in contrast to the two-term profile used by Friedlander. In comparison with Friedlander's early work (1957), Friedlander's later work (1961) and that of Levich (1952) agree more closely with the results of the present work for the thin boundary layer case, $Pe > 10^2$. [Yuge (1956) obtained Sherwood numbers for the case of $Pe = 0.3, 1.0, 3.0, 10.0$ which are nearly identical with the results in this paper, except at $Pe = 0.30$.] A comparison of the Sherwood number - Peclet number relations discussed above is indicated as case (3) on Figure 1; and formulae representing these relations are summarized in Table I.

For internally circulating spheres, the present numerical scheme when used with the Hadamard stream functions for Newtonian fluids gave results in very good agreement with the Sherwood numbers calculated by Bowman et al. (1961) using a four-term concentration profile except when Pe is greater than about $5(10)^3$. Comparisons between these two approaches are presented in Figure 1 for a circulating gas bubble of carbon dioxide rising in water [case (1)] and a circulating liquid droplet of ethyl acetate rising in water [case(2)].

As indicated in Figure 1, the ratio of the dispersed to the continuous phase viscosity has a definite effect on the intensity of internal circulation and, therefore, the mass transfer rates. The Sherwood number relation is inversely proportional to the value of the viscosity ratio. As expected, the value of the Sherwood number approaches that of non-circulating spheres when the viscosity ratio approaches infinity or when the Peclet number becomes very small. The former case implies that the spheres become

essentially rigid, the latter case, Sh equal to two, implies a very thick concentration boundary layer or diffusion into an essentially stagnant medium. In either case, the effect of internal circulation is negligible.

Analytical solutions for mass transfer from circulating spheres have been obtained by a number of previous workers assuming a very thin concentration boundary layer, which is usually encountered in the high Peclet number region. Formulae for the Sherwood number are listed in Table II, and a comparison of the present model with the relations of previous workers is illustrated in Figure 2 for this Peclet number region.

It should be understood that each of the analytical expressions was derived through several stages of approximation. For example, Griffith (1960) and Ward (1961) using different approximate forms for the concentration distribution arrived at different relations for predicting the Sherwood number. Also because of different assumptions made in one stage of the approximation procedure, their relations have different regions of validity.

The Peclet number criteria for the range of applicability of these relations will be considered first. To simplify the calculations, the case of a zero viscosity ratio ($R = 0$) will be considered as an illustration. For this case, Griffith's (1958, 1960) development suggests $1 < Pe < 38.4$ and $Pe > 38.4$ for Equation 21 and 23, respectively; whereas the ranges for the development of Ward (1961) are $1 < Pe < 1010.8$ and $Pe > 1010.8$ for Equations 22 and 24, respectively. The close similarity of their Sherwood number relations and yet the wide difference in their region of validity combine to suggest the uncertain, approximate nature of the analytical solutions.

The inconsistency of the results of the previous workers can be seen from another point of view. For the case of $Pe = 10^4$, Figure 2 indicates that Griffith (1958, 1960) predicts a switch-over of the power to which the Peclet number is raised (one-half to one-third) at $R \cong 6.55$, whereas Ward (1961) predicts that this occurs at $R \cong 1.55$. This accounts, in part, for the wide discrepancies between their results in the region, $1 < R < 7$.

It should be noted that both Griffith and Ward used Friedlander's (1957) approach of polynomial approximation for the concentration distribution. However, Ward (1961) using a four-term polynomial claimed more accurate results than Griffith (1960) who used a two-term polynomial. Ward's relation predicts lower mass transfer rates than Griffith in the low viscosity ratio region and higher transfer rates in the high viscosity ratio region. Thus, Ward's results are closer to the present model than Griffith's (1960) model, except for the region, $1 < R < 7$. In fact, for the case of $Pe = 10^4$, Figure 2 indicates that the results of the present model are asymptotic to those of Ward for $R > 10$. However, the abrupt change from the one-half power Peclet number relation to the one-third power relation in Ward's work does not seem to be realistic. Thus, for the region, $1 < R < 100$, the present model should be used to predict the mass transfer rate from circulating spheres.

Ward's Equation 22 can be used for $R > 100$ as it reduces essentially to Equation 19 when R approaches infinity, which Friedlander (1961) claimed to be more realistic than Equation 18a. For the same reason, Griffith's Equation 21 cannot be used as it reduces identically to Equation 18a when $R \rightarrow \infty$.

For the region $R < 1$, the present model at high Peclet numbers predicts lower transfer rates than Griffith (1960) and Ward (1961). In an article coauthored with Ward, Bowman et al. (1961) suggested the use of a numerical method for circulating spheres, i.e., for spheres in the low viscosity ratio region. This numerical method predicts lower mass transfer rates than Ward's (1961) analytical solutions and, thus, compares more favorably with the present model. Six points from the Bowman et al. (1961) results taken from Figure 1 are replotted in Figure 2. It is interesting to note that for this region ($R < 1$) Griffith's earlier work (1958) results in the closest agreement with the present model for $Pe = 10^4$, but it is Griffith's later work (1960) that predicts almost the same transfer rate as the present model for $Pe \leq 10^3$.

On the other hand, Levich (1962) used a different approach from Friedlander (1957) and arrived at an analytical solution which predicts lower mass transfer rates than the present model for all values of Peclet numbers. However, in his derivation Levich used an approximate form of the stream function and also neglected the second term on the right-hand side of Equation 1. It is interesting to note in Figure 2 that for $Pe = 10^4$, the results of the present model are between those of Griffith (1960) and Levich (1962). Levich's criterion for a thin concentration boundary layer is $Pe > 100$. Therefore, the exact region of validity for the thin boundary layer assumption is still open to further investigation.

Due to the various aspects of uncertainty in the analytical solutions in Table II, it is suggested that the present model, free from any limitations (except the numerical, finite-difference approximation) be used for

predicting mass transfer from circulating spheres. For approximate purposes, Ward's (1961) relationship can be used for $R > 100$ and $Pe > 1000$, and Griffith's (1960) results can be used for $R < 1.0$ and $Pe < 1000$. For $R < 1.0$ and Peclet numbers higher than about $5(10)^3$, both Ward's (1961) and Griffith's (1960) relations tend to predict high mass transfer rates which are too large when compared with the present model; and thus, these relations should not be used in this region [i.e., $R < 1.0$ and $Pe \geq 5(10)^3$].

The above discussion and comparison of continuous phase mass transfer models for Newtonian fluids in both phases indicates that the numerical technique used in this work leads to a Sherwood number-Peclet number relation which agrees well with the conflicting solutions of previous workers, particularly for Peclet numbers less than about 10^3 . Thus, this investigation was extended with confidence to situations involving a non-Newtonian continuous phase.

Non-Newtonian Continuous Phase. The results obtained using the velocity components derived from the Nakano and Tien (1968) stream functions will be discussed first, followed by comparisons and studies of Sherwood number relations based upon more restrictive stream functions [Hirose and Moo-Young, 1969; Tomita, 1959].

As described earlier, the Nakano and Tien (1968) stream functions are applicable to either liquid droplets or gas bubbles [assumed to be Newtonian fluids] traveling under laminar flow conditions through a non-Newtonian continuous phase described by a power-law constitutive equation. The continuous phase Sherwood number $\left(\frac{k_c^{2a}}{D}\right)$ was calculated as a function

of the Peclet number, Pe , flow behavior index, n , and a viscosity ratio parameter, X .* The Sherwood number relation obtained by considering the flow behavior index n equal to unity was discussed in the previous section for specific examples [CO_2 bubbles in water; ethyl acetate droplets in water; and rigid spheres in any Newtonian fluid]. Mass transfer results using the Nakano and Tien stream functions are presented as a function of Pe in Figures 3a, 3b, and 3c for values of n equal to 1.0, 0.8, and 0.6, respectively. On each figure, three values of the viscosity ratio parameter $X = (\mu_d/k)(a/V_\infty)^{n-1}$ are employed: 0.01, 1.0, and 10.0. The Sherwood number relations are also presented in Figure 4 to more clearly describe the effect of the viscosity ratio parameter X and the power-law index n .

As may be seen in Figures 3a, 3b, and 3c, the Sherwood number increases with increases in Pe for both non-Newtonian and Newtonian continuous phases. When the Peclet number is less than about 10^{-2} , there is virtually no effect of changes of Pe on droplet mass transfer rates.

The effect of the power-law index n on mass transfer from the dispersed phase becomes extremely important when the Peclet number is greater than about 500. As n decreases from unity, the Sherwood number increased, i.e., increased pseudoplasticity results in an enhancement of mass transfer - all other variables held constant. A decrease in the value of n from 1.0 to 0.6 results in about a 25% increase in the Sherwood number. Nearly one-half of the increase in the value of the Sherwood number occurs when n decreases from 1.0 to 0.9. The results indicate, however, that the index n has a negligible effect on the continuous phase mass transfer

rate for physical situations when the Peclet number is less than about five.

The dependency of the Sherwood number on the viscosity ratio parameter X is noticeable at Peclet number greater than about 50, and the effect is very pronounced when the Peclet number is greater than about 500. As may be seen in Figure 4, the Sherwood number is most sensitive to changes of X for values of the viscosity parameter between 0.1 and 10. This figure and the tabulated results* indicate that the effect of the viscosity ratio parameter is negligible when $X < 0.01$, almost regardless of the value of the Peclet number. One could define the mass transfer results for $X < 0.01$ as being indicative of "fully" circulating gas bubbles or liquid droplets. The effect of the viscosity ratio parameter asymptotically diminishes as X increases beyond a value of about ten; this reflects the decrease of circulation velocity within the dispersed phase, partly as a result of the relative increase in dispersed phase viscosity which in a practical sense would most likely be encountered with certain liquid droplet systems.

Hirose and Moo-Young (1969), in a recent study of drag and mass transfer characteristics of a spherical gas bubble moving in creeping flow through a power-law non-Newtonian fluid, derived the following continuous phase velocity components:

$$V_y = - \left\{ \left[1 - \left(\frac{1}{y} \right) \right] - \frac{6n(n-1)}{(2n+1)} \left[\frac{1}{y} \ell^{ny} + \frac{1}{6} \left(\frac{1}{y} \right)^3 - \frac{1}{6} \left(\frac{1}{y} \right) \right] \right\} \cos \theta \quad (14a)$$

$$V_\theta = + \left\{ \left[1 - \frac{1}{2} \left(\frac{1}{y} \right) \right] - \frac{3n(n-1)}{(2n+1)} \left[\frac{1}{y} \ell^{ny} - \frac{1}{5} \left(\frac{1}{y} \right)^3 + \frac{5}{6} \left(\frac{1}{y} \right) \right] \right\} \sin \theta \quad (14b)$$

In the derivation of the above relations, it was assumed that the gas phase may be regarded as inviscid relative to the external liquid phase. At one

point in their derivation, it was also assumed that the deviation from Newtonian flow is small, i.e., $|n-1| \ll 1$. Upon application of Baird and Hamielec (1962) relation for short-range diffusion from fully circulating spherical bubbles, Hirose and Moo-Young obtained the following relation between the Sherwood number, Peclet number, and power-law index, n :

$$Sh = 0.65 \left[1 - \frac{4n(n-1)}{(2n+1)} \right]^{1/2} Pe^{1/2} \quad (15)$$

The viscosity ratio parameter X is not considered in the above derivation. Hence, situations in which the dispersed phase viscosity is not negligible compared with the continuous phase non-Newtonian fluid (e.g., for some dispersed phase liquids) will not be properly described by Equation 15.

Equation 15 is compared with the results obtained in this investigation (for $X = 0.01$) in Figure 5. The two approaches agree very well for this special case of a dispersed phase which is essentially inviscid compared with the continuous phase, except for Peclet numbers larger than about $5(10)^3$. The exact reason for this deviation at very high Peclet numbers is not completely understood at this time. However, for fully circulating spheres ($R = 0$) moving through Newtonian fluids ($n = 1$), Equation 15 reduces to a form which predicts mass transfer rates between those predicted by Equations 23b and 24 [developed by Griffith (1960) and Ward (1961), respectively]. From the discussion presented for the Newtonian case with circulation in an earlier section of this paper, it can also be reasonably deduced that Equation 15 will predict erroneously high Sherwood numbers in the high Peclet number region. Attempts to correct the original Hirose and Moo-Young expression (Equation 15) by replacing

the constant 0.67 by the constant 0.58 (see Equation 23a) will lead to an expression which agrees fairly closely with the results obtained in this work based upon the Nakano and Tien velocity function for the case when n is equal to unity and in the very high Peclet number region $[Pe > 5(10)^3]$. But, this modified expression (or any other simple modification) will not accurately estimate the effect on the Sherwood number of either the power law index n (when n is not equal to one) or the Peclet number when the Peclet number is less than about 10^3 . It is concluded that the Hirose and Moo-Young relation (Equation 15) may only be used when the Peclet number is less than about $5(10)^3$ and when X is less than about 0.01. Almost all gas bubble systems will satisfy these criteria for X , however, not all liquid droplet systems will satisfy these criteria.

In an attempt to further study and possibly explain this discrepancy of the Hirose - Moo-Young model, the author of this paper directly used the velocity components developed by Hirose and Moo-Young (1969) [i.e., Equations 14a and 14b] with the differential mass balance and boundary conditions used in this investigation [Equations 1 and 2] to numerically solve for the Sherwood number at various values of the Peclet number and power-law index, n . The results are indicated in Figure 5. In contrast to Hirose and Moo-Young's analytic solution which was based upon some approximations in the integration procedure, the Sherwood number relation obtained numerically using their velocity components agrees very closely with the numerical results obtained in this work using the Nakano and Tien (1968) velocity components (for $X \leq 0.01$) at all levels of the Peclet number.

It is tentatively concluded that the Hirose and Moo-Young velocity components are correct, but there probably is an inaccurate approximation in their integration procedure which becomes important when $Pe > 5(10)^3$.

Tomita (1959) studied the drag force on a very slowly moving sphere (with no internal circulation) moving through a power-law type fluid by assuming an expression for the continuous phase stream function which satisfied certain boundary conditions. The corresponding velocity components based upon these stream functions are given as follows:

$$V_y = - \left[1 - (1/y)^n \right]^2 \cos\theta \quad (16a)$$

$$V_\theta = \left[1 - (1/y)^n \right] \left[1 - (1-n)(1/y)^n \right] \sin\theta \quad (16b)$$

The approximate nature of the stream function is demonstrated by the fact that when n is equal to unity, the above velocity components do not reduce identically to those derived from the Stokes stream function [let $R \rightarrow \infty$ in Equation 5] which were based on Newtonian fluid behavior. Despite this difficulty, Sherwood numbers were calculated numerically using Equations 1, 2, and 16. The results are presented in Figure 6 as the ratio of the Sherwood numbers obtained using the present model based on the Tomita stream functions to those Sherwood numbers based on the Stokes stream function plotted as a function of the power-law index n , with the Peclet number as a parameter. At n equal to unity, all curves should coincide at the point $[1.0, 1.0]$; however, as seen in Figure 6 the curves do not coincide at this point indicating the error in the Tomita analysis. In fact, the discrepancy increases as Pe increases. It is also interesting to

observe that in the pseudoplastic region the Sherwood number based on Tomita's stream functions decreases as n decreases, contrary to the results obtained in this work for all values of X or by the development of Hirose and Moo-Young (1969).

Conclusions and Summary

The Nakano and Tien (1968) stream functions which were developed for spheres of a Newtonian fluid moving in power-law type, non-Newtonian fluid were used in this investigation to solve for the continuous phase rates of mass transfer in the creeping-flow region. The Sherwood number relation was developed for a wide range of the Peclet number, power-law index, and viscosity ratio parameter. The solution applies to both liquid droplets and gas bubbles. In the pseudoplastic region, the Sherwood number decreases as the power-law index increases; but this effect significantly decreases at low Peclet numbers.

The Hirose and Moo-Young analytic mass transfer solution for fully circulating bubbles and the numerical results of this investigation (for $X < 0.01$) are in close agreement except at very high Peclet numbers [$Pe > 5(10)^3$]. However, for viscosity ratio parameters greater than about 0.05, the results of this investigation rather than the Hirose and Moo-Young relation must be used to consider the relative viscous nature of the two phases.

The Tomita stream functions for non-circulating spheres cannot be used to accurately predict continuous phase mass transfer coefficients.

Nomenclature

a	=	spherical radius of the dispersed phase
$a_{i,1}, b_{i,1}, d_{i,1}$	=	coefficients in Equation 9, defined in Equations 10a - 10c
A_i, B_i	=	coefficients in Equations 3a and 3b
C	=	concentration of the solute transferred from sphere into continuous phase, dimensionless
$C_{i,j}$	=	concentration of solute in finite difference Equations 9 and 11
D	=	diffusivity of solute in continuous phase
$f_{i,j}, l_{i,j}, p_{i,j}, q_{i,j}$	=	coefficients in Equation 11, defined in Equations 12a - 12d
k_c	=	continuous phase mass transfer coefficient
K	=	consistency index of the power-law fluid
n	=	flow behavior index
Pe	=	Peclet number, $2aV_\infty/D$, dimensionless
R	=	μ_d/μ_c , dimensionless
Re	=	Reynolds number, $2aV_\infty\rho/\mu$, dimensionless
Sh	=	average Sherwood number, $k_c 2a/D$, dimensionless
$t_{i,j}$	=	coefficients in Equation 12d, defined in Equation 12e
V_y	=	radial velocity, dimensionless
V_θ	=	tangential velocity, dimensionless
V_∞	=	relative velocity between the sphere and continuous phase fluid
X	=	$\frac{\mu_d}{K} \left(\frac{a}{V_\infty} \right)^{n-1}$

y = radial displacement from center of sphere

Greek Letters

θ = angular displacement from the front stagnation point

μ = viscosity

ρ = density

σ = exponent in Equations 3a and 3b

τ_j^i = stress tensor

Δ_j^i = rate of deformation tensor

Δy = radial increment in finite difference equations

$\Delta\theta$ = angular increment in finite difference equations

Subscripts

c = continuous phase

d = dispersed phase

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Table I. Analytical Solutions for a Non-circulating Sphere:

Newtonian Continuous Phase

Equation Number	Equation for Sherwood number	Region of Validity	Physical Description	Reference
17a	$2\left(1 + \frac{Pe}{4} + \frac{Pe^2}{12} + \dots\right)$	$Pe < 1$	thick B.L.*	Friedlander (1957)
17b	$2 + \frac{9}{16}Pe + \frac{9}{64}Pe^2 + \dots$	$Pe < 1$	thick B.L.	Bowman, Ward, Johnson, Trass (1961)
18a	$0.89 Pe^{1/3}$	$Pe > 10^3$	thin B.L.	Friedlander (1957)
18b	$0.978 Pe^{1/3}$	$Pe > 10^3$	thin B.L.	Bowman, Ward Johnson, Trass (1961)
19	$0.991 Pe^{1/3}$	$Pe > 10^2$	thin B.L.	Friedlander (1961)
20a	$0.997 Pe^{1/3}$	$Pe > 10^2$	thin B.L.	Levich (1952)
20b	$0.64 Pe^{1/3}$	$Pe \gg 1$	thin B.L.	Levich (1962)

*B.L. = boundary layer

Table II. Analytical Solutions for a Circulating Sphere:
Newtonian Continuous Phase; Thin Concentration Boundary Layer

Equation Number	Equation for Sherwood Number	Region of Validity, $Re < 1$	Reference
21	$0.89 Pe^{1/3} \left(\frac{R + 1.33}{R + 1} \right)^{1/3}$	$1 < Pe < 2.4(3R + 4)^2 (R + 1)$	Griffith (1960)
22	$0.98 Pe^{1/3} \left(\frac{R + 1.58}{R + 1} \right)^{1/3}$	$10^3 < Pe < 2.8(12R + 19)^2 (R + 1)$	Ward (1961)
23a	$0.58 Pe^{1/2} \left(\frac{1}{R + 1} \right)^{1/2}$	$Pe > 2.4(3R + 4)^2 (R + 1)$	Griffith (1958)
23b	$0.67 Pe^{1/2} \left(\frac{1}{R + 1} \right)^{1/2}$	$Pe > 2.4(3R + 4)^2 (R + 1) > 1$	Griffith (1960)
24	$0.61 Pe^{1/2} \left(\frac{1}{R + 1} \right)^{1/2}$	$Pe > 2.8(12R + 19)^2 (R + 1)$	Ward (1961)
25	$\frac{2}{\sqrt{6\pi}} Pe^{1/2} \left(\frac{1}{R + 1} \right)^{1/2}$	$Pe \gg 1$	Levich (1962)

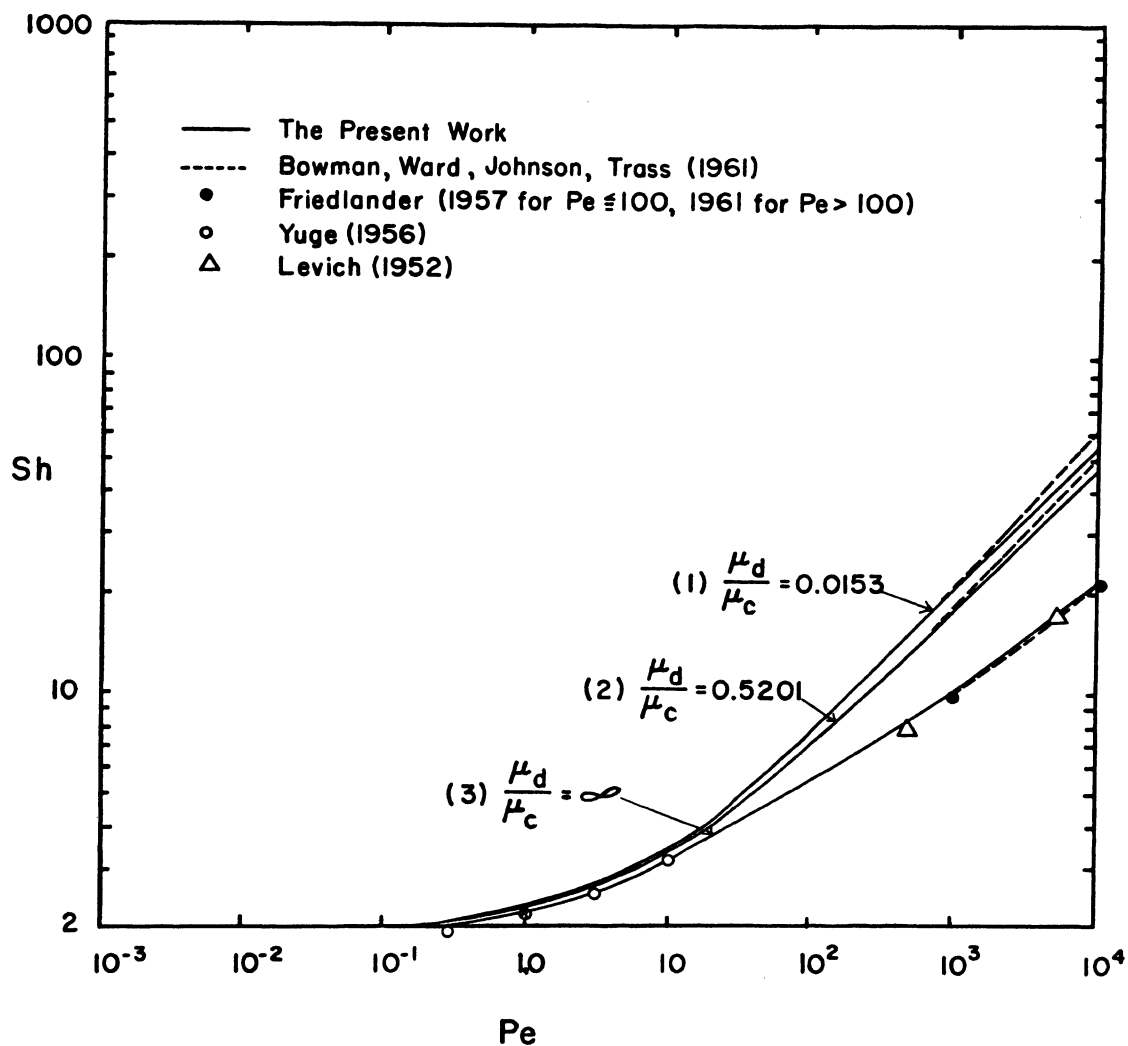


Figure 1. Sherwood Number as a Function of Peclet Number for Selected Newtonian Systems: (1) carbon dioxide bubble in water; (2) ethyl acetate droplet in water; (3) any non-circulating system.

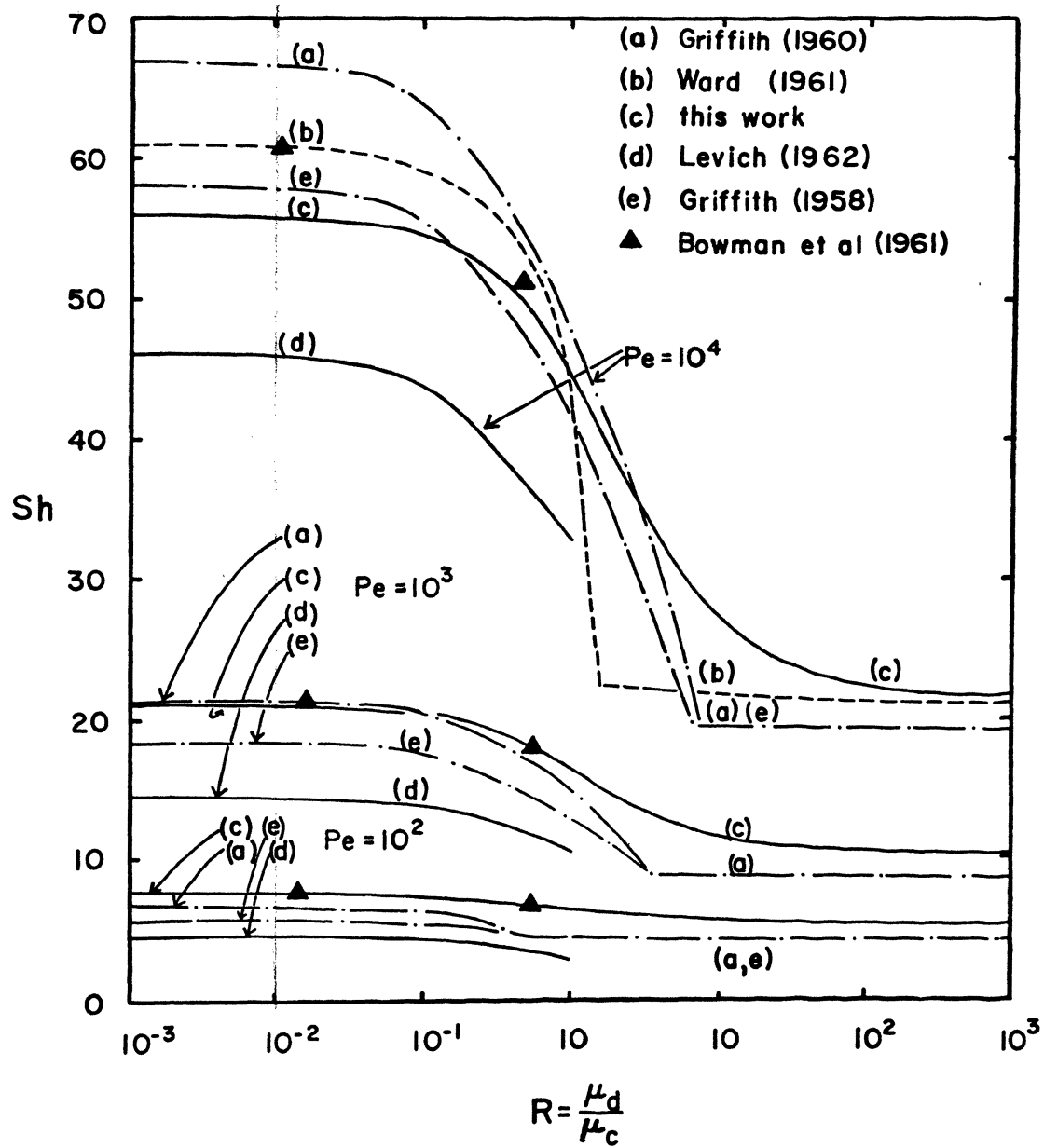


Figure 2. Comparison of Current Model with Previous Studies for a Newtonian Continuous Phase: Effect of Viscosity Ratio in High Peclet Number Region.

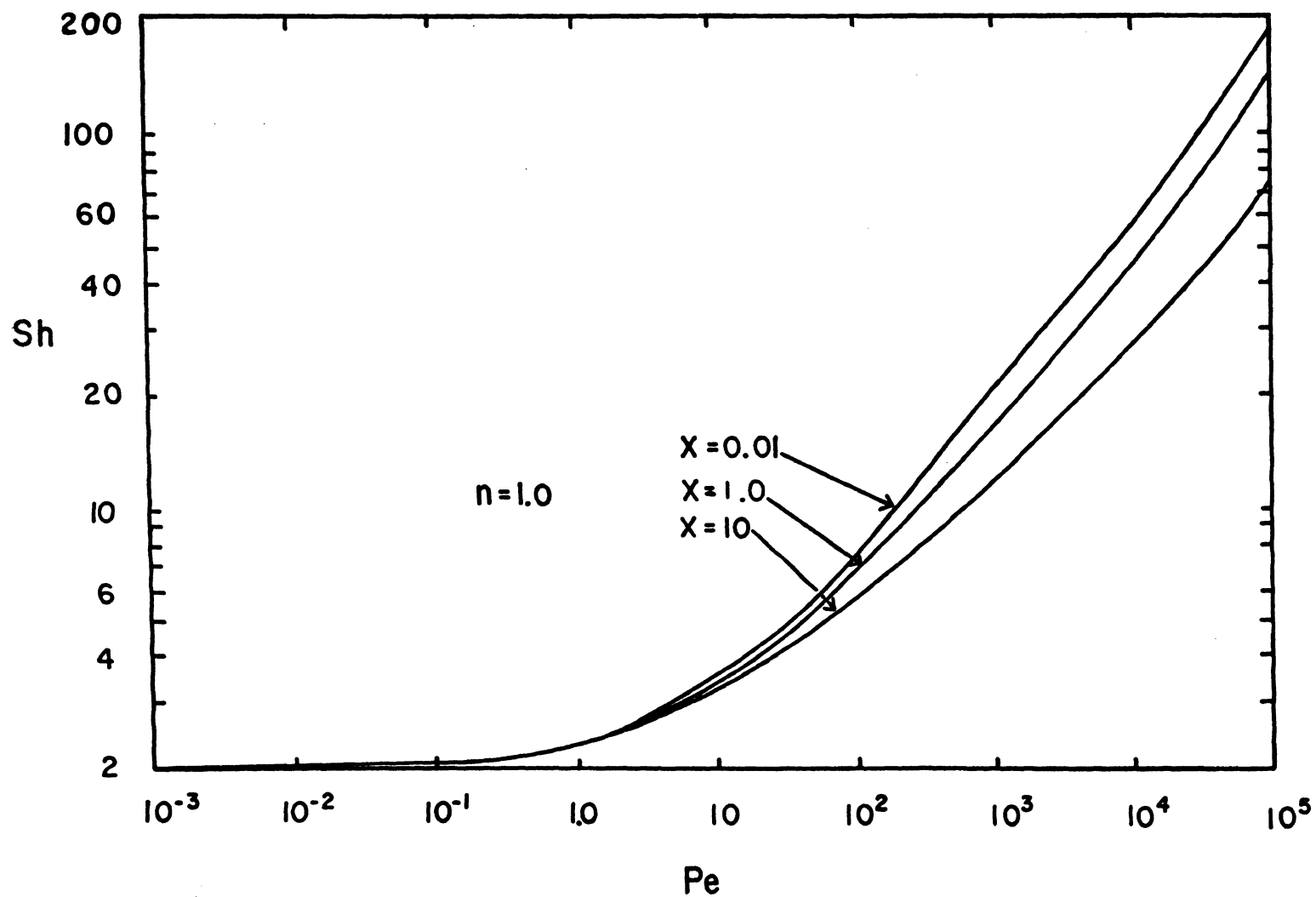


Figure 3(a). Sherwood Number as a Function of Peclet Number and the Viscosity Ratio
Parameter: based on the Nakano and Tien Stream Functions ($n=1.0$).

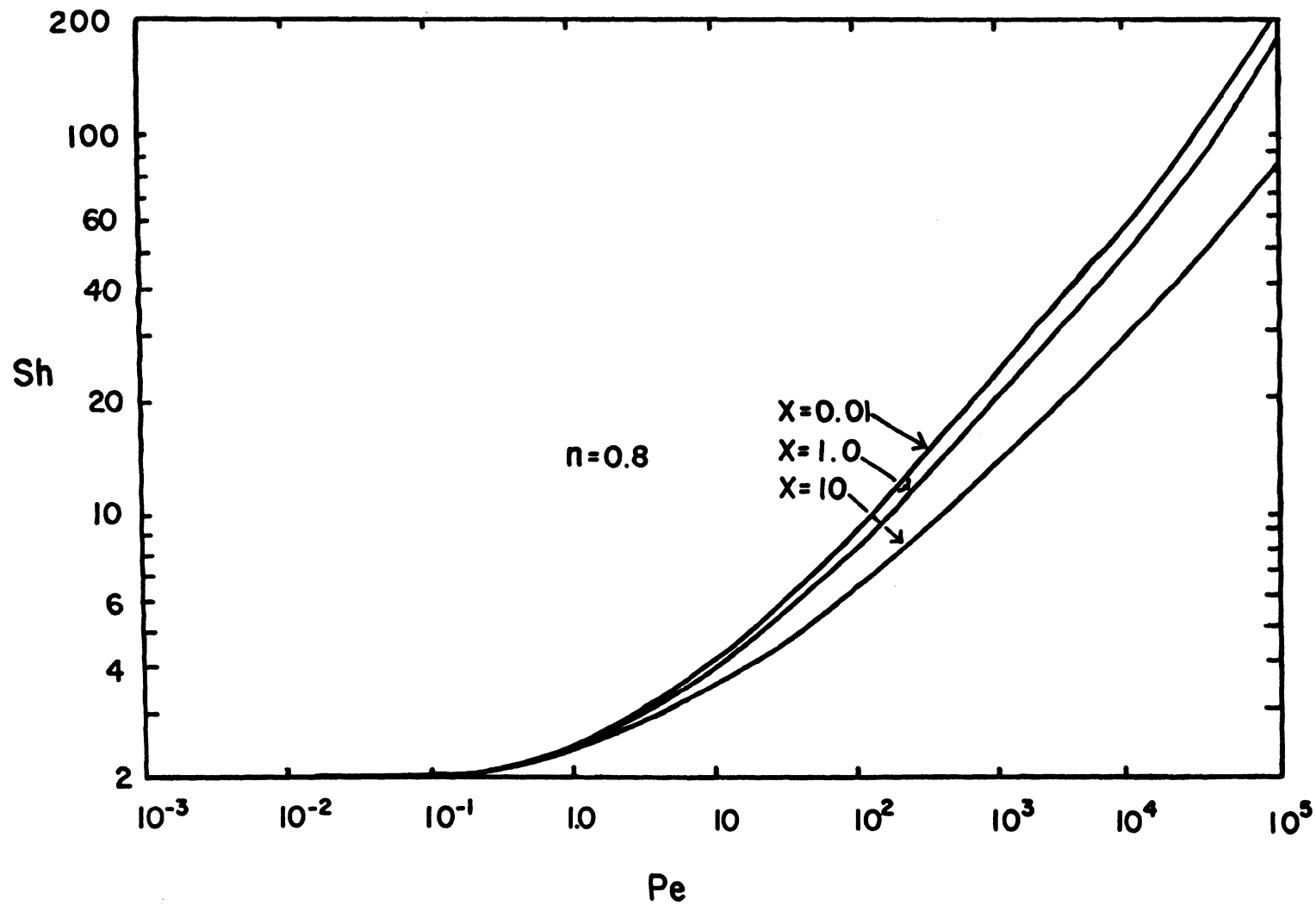


Figure 3(b). Sherwood Number as a Function of Peclet Number and the Viscosity Ratio
Parameter: based on Nakano and Tien Stream Functions ($n=0.8$).

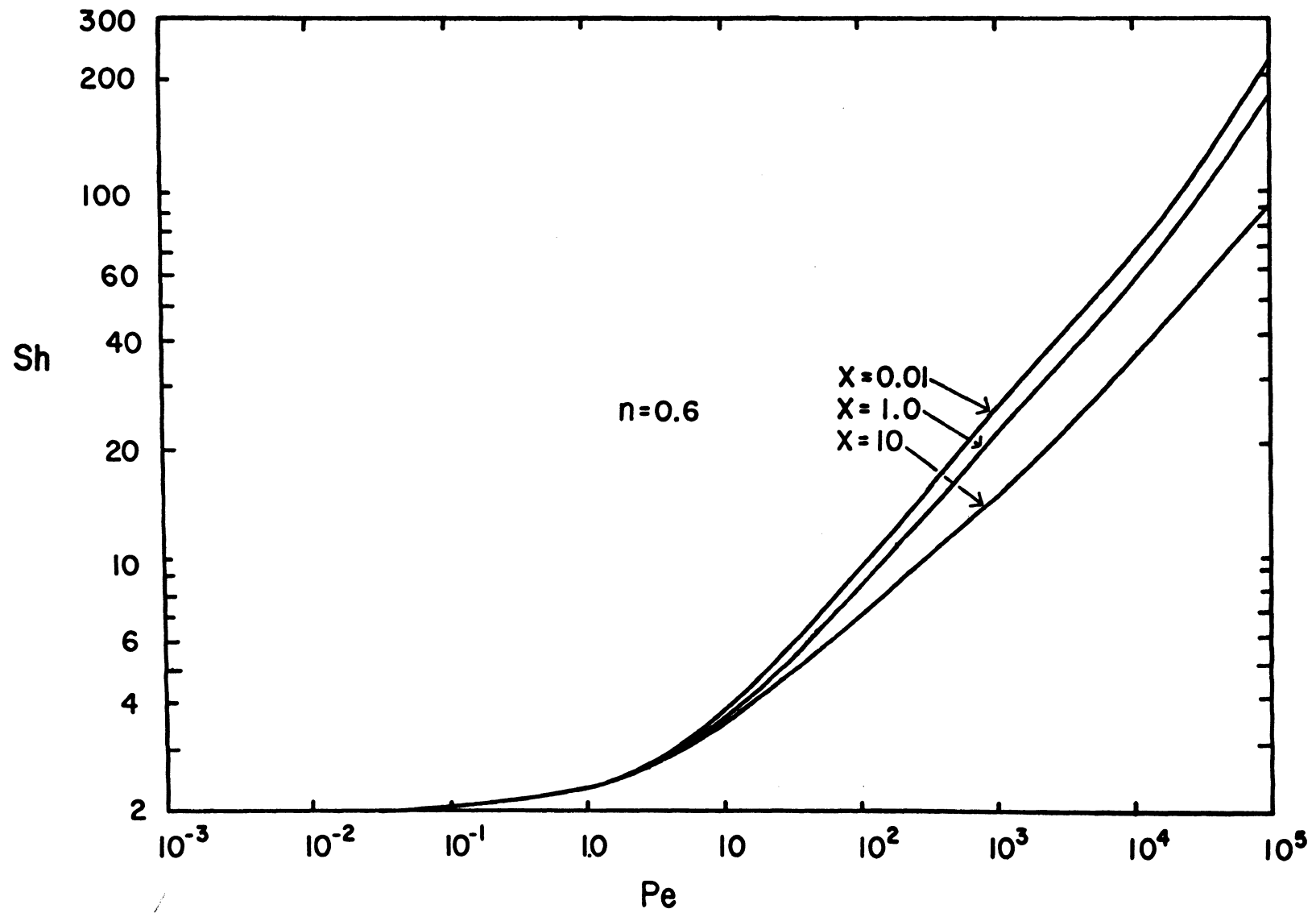


Figure 3(c). Sherwood Number as a Function of Peclet Number and the Viscosity Ratio
Parameter: based on the Nakano and Tien Stream Functions ($n=0.6$).

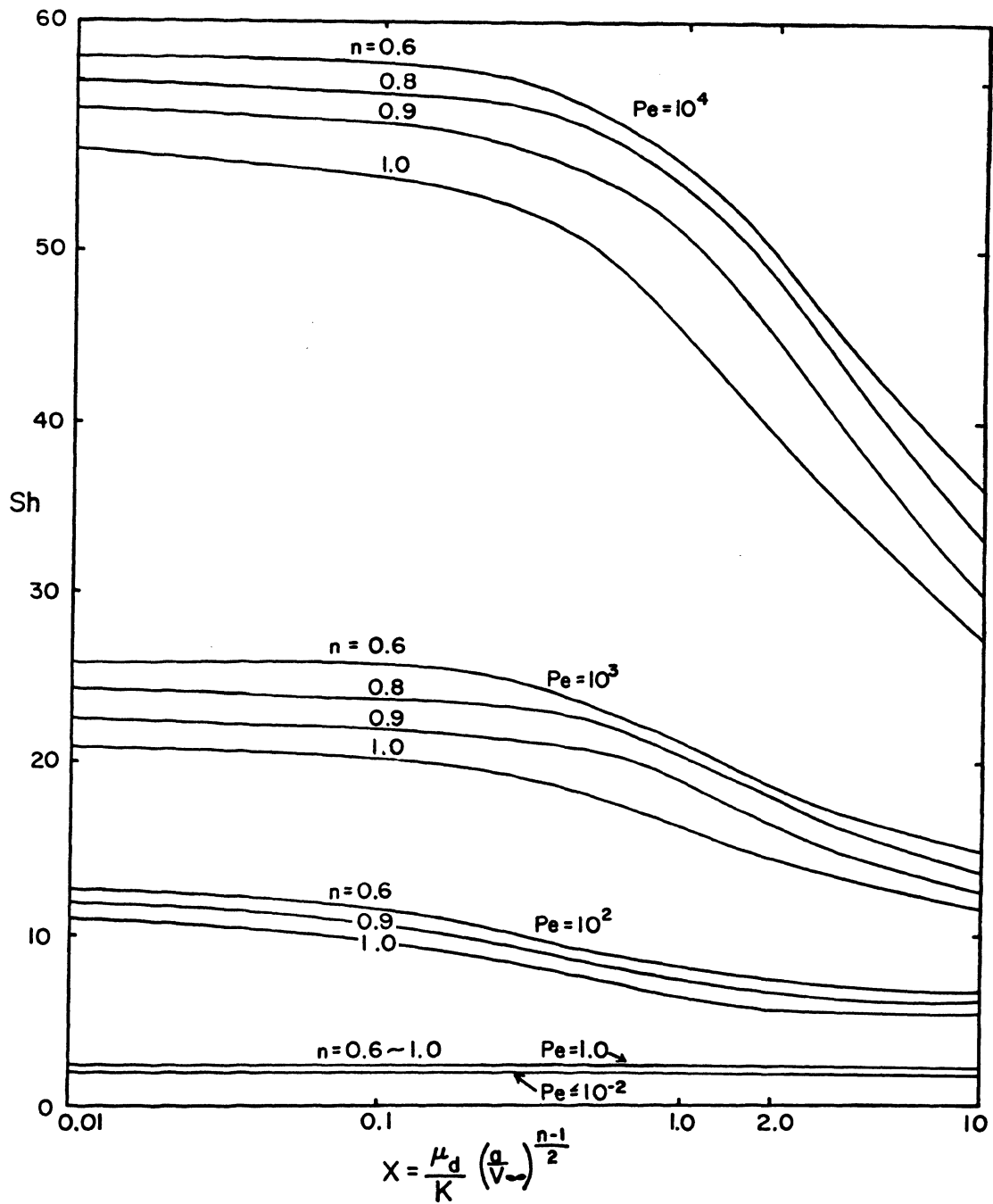


Figure 4. The Effect of the Viscosity Ratio Parameter X on the Sherwood Number for Representative Values of the Power-law index n and Peclet Number: based on the Nakano and Tien Stream Functions.

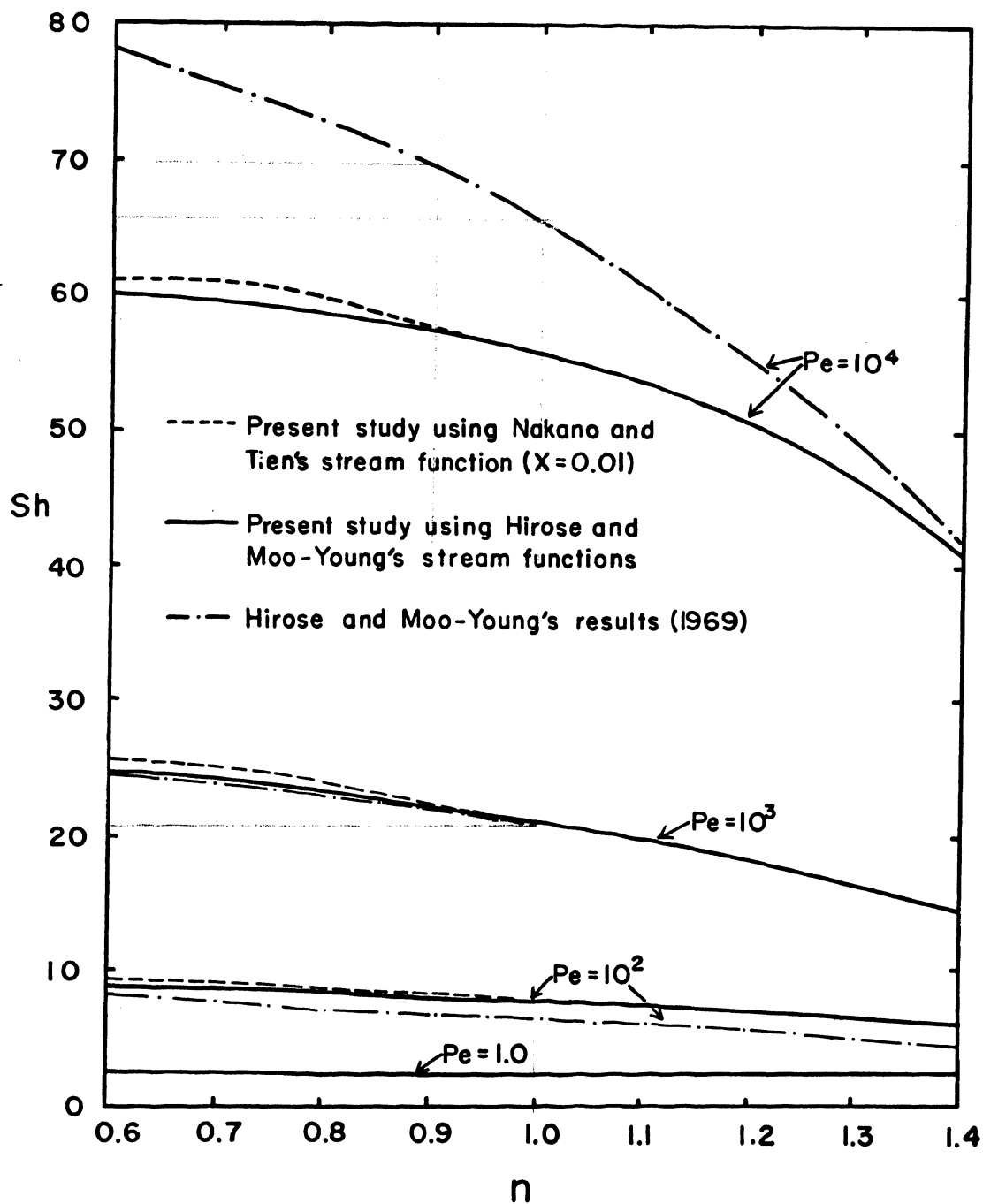


Figure 5. Effect of Non-Newtonian Behavior on Mass Transfer from Circulating Gas Bubbles: Comparison of Various Models.

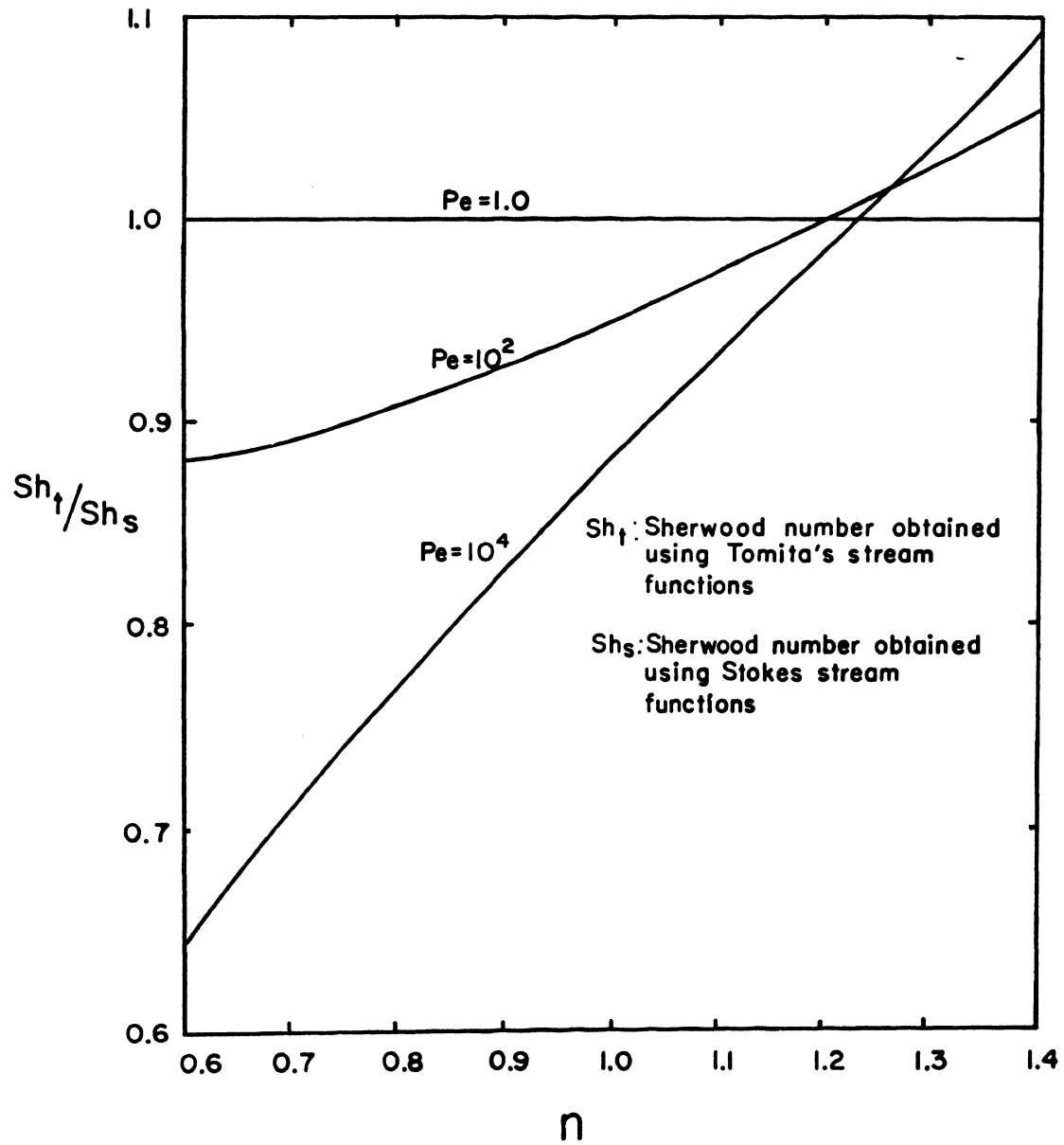


Figure 6. Effect of Non-Newtonian Behavior for Non-Circulating Spheres using the Tomita Stream Functions.

Appendix A. Derivation of Mathematical Model

The Governing Equation. In a binary diffusion system where

- (i) no chemical reaction occurs
- (ii) no thermal, pressure and forced diffusion
- (iii) density and diffusivity are constant
- (iv) the solution is dilute

the equation of continuity (Bird, Stewart, Lightfoot, 1963) can be written as

$$\frac{\partial C'_A}{\partial t} + (\nu' \cdot \nabla C'_A) = D_{AB} \nabla^2 C'_A \quad (\text{A.1})$$

Rewriting Equation A.1 in spherical coordinates with the origin at the center of the sphere, one obtains

$$\begin{aligned} & \frac{\partial C'_A}{\partial t} + \left(\nu'_r \frac{\partial C'_A}{\partial r} + \frac{\nu'_\theta}{r} \frac{\partial C'_A}{\partial \theta} + \nu'_\phi \frac{1}{r \sin \theta} \frac{\partial C'_A}{\partial \phi} \right) \\ &= D_{AB} \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial C'_A}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial C'_A}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 C'_A}{\partial \phi^2} \right] \quad (\text{A.2}) \end{aligned}$$

Assuming that

- (i) steady state conditions have been reached, i.e. $\frac{\partial C'_A}{\partial t} = 0$
- (ii) the flow around the sphere is axially symmetric, i.e.

$$\frac{\partial C'_A}{\partial \phi} = 0, \quad \frac{\partial^2 C'_A}{\partial \phi^2} = 0$$

- (iii) diffusion in tangential direction is negligible, i.e.

$$D_{AB} \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial C'_A}{\partial \theta} \right) = 0$$

then from Equation A.2 we obtain

$$V'_r \frac{\partial C'_A}{\partial r} + \frac{\nu'_\theta}{r} \frac{\partial C'_A}{\partial \theta} = D_{AB} \left(\frac{2}{r} \frac{\partial C'_A}{\partial r} + \frac{\partial^2 C'_A}{\partial r^2} \right) \quad (\text{A.3})$$

Introducing dimensionless variables

$$y = \frac{r}{a}, \text{ dimensionless radial distance}$$

$$\text{Pe} = 2aV_\infty/D, \text{ the Peclet number}$$

$$C = (C'_A - C^*_A)/(C'_\infty - C^*_A), \text{ dimensionless concentration}$$

$$V_y = V'_r/V_\infty, \quad V_\theta = V'_\theta/V_\infty, \text{ dimensionless velocities}$$

we can rewrite the governing equation A.3 in dimensionless form as follows:

$$V_y \frac{\partial C}{\partial y} + \frac{V_\theta}{y} \frac{\partial C}{\partial \theta} = \frac{2}{\text{Pe}} \left(\frac{\partial^2 C}{\partial y^2} + \frac{2}{y} \frac{\partial C}{\partial y} \right) \quad (\text{A.4})$$

The Boundary Conditions.

(i) The solute concentration has reached its equilibrium value at the surface of the sphere, i.e.

$$C'_A = C^*_A \quad \text{at} \quad r = a \quad (\text{A.5})$$

or in dimensionless form

$$C = 0 \quad \text{at} \quad y = 1 \quad (\text{A.6})$$

(ii) The solute concentration at a far distance from the sphere surface is essentially equal to the bulk concentration, i.e.

$$C'_A = C'_\infty \quad \text{at} \quad r \rightarrow \infty \quad (\text{A.7})$$

or in dimensionless form

$$C = 1 \quad \text{at} \quad y \rightarrow \infty \quad (\text{A.8})$$

(iii) The concentration profile is symmetric with respect to the θ

axis at $\theta = 0, \pi$, i.e.

$$\frac{\partial C'_A}{\partial \theta} = 0 \quad \text{at} \quad \theta = 0, \pi \quad (\text{A.9})$$

or in dimensionless form

$$\frac{\partial C}{\partial \theta} = 0 \quad \text{at} \quad \theta = 0, \pi \quad (\text{A.10})$$

Velocity Components. The velocity components used in this work are derived from the Nakano and Tien external stream function as follows:

$$\psi_c = \left(-\frac{1}{2} r^2 + A_1 r^\sigma + \frac{A_2}{r}\right)(1 - \cos^2 \theta) + \left(\frac{B_1}{r} + \frac{B_2}{2}\right)\cos\theta(1 - \cos^2 \theta) \quad (\text{A.11})$$

where the exponent σ and the coefficients, A_1 , A_2 , B_1 , and B_2 are determined by the hydrodynamic boundary conditions of the system (see Appendix D).

The basic assumptions for the system are listed in Section 2. The velocity components are derived by use of the following relationships:

$$V_y = \frac{1}{y^2 \sin\theta} \frac{\partial \psi_c}{\partial \theta} \quad (\text{A.12})$$

$$V_\theta = \frac{-1}{y \sin\theta} \frac{\partial \psi_c}{\partial y} \quad (\text{A.13})$$

Similarly, the velocity components describing a circulating sphere in creeping flow where both continuous and dispersed phases are Newtonian fluids are derived from the Hadamard external stream function as follows:

$$\psi_c = -\frac{1}{2} \left[1 - \left(\frac{3X+2}{2X+2}\right) \frac{1}{y} + \frac{1}{2} \left(\frac{X}{X+1}\right) \frac{1}{y^3} \right] y^2 \sin^2 \theta \quad (\text{A.14})$$

The Average Sherwood Number. From Fick's first law of diffusion in a binary system, the molar diffusion flux of the component A is given by

$$N_A = x_a (N_A + N_B) - C'D_{AB} \nabla x_A \quad (\text{A.15})$$

where C' is molar density of solution and x_A is molar fraction of the solute.

If the molar flux of the solute resulting from the bulk motion of the fluid is negligible in the creeping flow region, then

$$N_A = -C'D_{AB} \nabla x_A \quad (\text{A.16})$$

For spherical coordinates if we neglect diffusion in tangential direction and assume axisymmetry of the flow, then Equation A.16 becomes, i.e.

$$N_A = -C'D_{AB} \frac{\partial x_A}{\partial r} \quad (\text{A.17})$$

Again assuming the density constant, we have

$$N_A = -D_{AB} \frac{\partial C'_A}{\partial r} \quad (\text{A.18})$$

If we focus our attention on the sphere surface, $r = a$, through which the solute enters the stream at the local rate N_A moles/(unit area) (unit time), then Equation A.18 becomes

$$N_{AO} = -D_{AB} \left. \frac{\partial C'_A}{\partial r} \right|_{r=a} \quad (\text{A.19})$$

Now, the molar flux across an interface can be obtained through the definition of a mass transfer coefficient as follows:

$$N_{AO} = -k_c (C'_\infty - C'_A) \quad (\text{A.20})$$

If we consider the sphere surface as the interface, then at $r = a$, $C'_A = C_A^*$ and Equation A.20 becomes

$$N_{AO} = -k_c (C'_\infty - C_A^*) \quad \text{at} \quad r = a \quad (\text{A.21})$$

Equating Equations A.19 and A.21 we obtain

$$k_c (C'_\infty - C_A^*) = D_{AB} \left. \frac{\partial C'_A}{\partial r} \right|_{r=a} \quad (\text{A.22})$$

Introducing dimensionless variables

$$y = \frac{r}{a}, \quad c = \frac{C'_A - C_A^*}{C'_\infty - C_A^*} \quad (\text{A.23})$$

one obtains after rearrangement the local Sherwood number as follows:

$$\text{Sh}_{\text{loc}} = \frac{k_c 2a}{D_{AB}} = 2 \left. \frac{\partial C}{\partial y} \right|_{y=1} \quad (\text{A.24})$$

The average Sherwood number for a sphere can be obtained by integration over the surface of the sphere, i.e.

$$\begin{aligned} \text{Sh} &= \frac{\int_0^{2\pi} \int_0^\pi \text{Sh}_{\text{loc}} a^2 \sin\theta d\theta d\phi}{\int_0^{2\pi} \int_0^\pi a^2 \sin\theta d\theta d\phi} = \frac{2\pi a^2 \cdot \int_0^\pi 2 \left(\frac{\partial C}{\partial y} \right)_{y=1} \sin\theta d\theta}{4\pi a^2} \\ &= \int_0^\pi \left(\frac{\partial C}{\partial y} \right)_{y=1} \sin\theta d\theta \end{aligned} \quad (\text{A.25})$$

Appendix B. Discussion of the Numerical Solution

Derivation of Finite Difference Equations. The boundary value problem of Equations 1 and 2a - 2c was solved in this work by use of Crank-Nicolson implicit method. Thus Equation 1 is rewritten as

$$\left(V_y - \frac{4}{Pe y}\right) \frac{\partial C}{\partial y} + \frac{V_\theta}{y} \frac{\partial C}{\partial \theta} = \frac{2}{Pe} \frac{\partial^2 C}{\partial y^2} \quad (B.1)$$

or in finite difference form

$$\begin{aligned} & \left(V_y - \frac{4}{Pe y}\right) \left[\frac{C_{i+1, j+1} - C_{i-1, j+1}}{2(\Delta y)} + \frac{C_{i+1, j} - C_{i-1, j}}{2(\Delta y)} \right] \left(\frac{1}{2}\right) \\ & + \left(\frac{V_\theta}{y}\right) \left[\frac{C_{i, j+1} - C_{i, j}}{(\Delta \theta)} \right] \\ & = \frac{2}{Pe} \left(\frac{1}{2}\right) \left[\frac{C_{i+1, j+1} - 2C_{i, j+1} + C_{i-1, j+1}}{(\Delta y)^2} \right. \\ & \quad \left. + \frac{C_{i+1, j} - 2C_{i, j} + C_{i-1, j}}{(\Delta y)^2} \right] \quad (B.2) \end{aligned}$$

where the derivatives $\frac{\partial C}{\partial y}$ and $\frac{\partial^2 C}{\partial y^2}$ are approximated by finite difference

form according to Crank-Nicolson technique. After rearrangement

Equation B.2 becomes

$$\begin{aligned} & \left[\frac{1}{Pe(\Delta y)^2} + \frac{V_y}{4(\Delta y)} - \frac{1}{Pe y(\Delta y)} \right] C_{i-1, j+1} + \left[\frac{-2}{Pe(\Delta y)^2} - \frac{V_\theta}{y(\Delta \theta)} \right] C_{i, j+1} \\ & + \left[\frac{1}{Pe(\Delta y)^2} - \frac{V_y}{4(\Delta y)} + \frac{1}{Pe y(\Delta y)} \right] C_{i+1, j+1} \\ & = - \left[\frac{1}{Pe(\Delta y)^2} + \frac{V_y}{4(\Delta y)} - \frac{1}{Pe y(\Delta y)} \right] C_{i-1, j} - \left[\frac{-2}{Pe(\Delta y)^2} + \frac{V_\theta}{y(\Delta \theta)} \right] C_{i, j} \end{aligned}$$

$$- \left[\frac{1}{\text{Pe}(\Delta y)^2} - \frac{V_y}{4(\Delta y)} + \frac{1}{\text{Pe}_y(\Delta y)} \right] C_{i+1, j} \quad (\text{B.3})$$

It should be noted that all the coefficients are to be evaluated at the point $(i\Delta y, j\Delta\theta)$. Thus let

$$f_{i, j} = \frac{1}{\text{Pe}(\Delta y)^2} + \frac{V_y}{4(\Delta y)} - \frac{1}{\text{Pe}_y(\Delta y)} \quad (\text{B.4})$$

$$\ell_{i, j} = \frac{-2}{\text{Pe}(\Delta y)^2} - \frac{V_\theta}{y(\Delta\theta)} \quad (\text{B.5})$$

$$p_{i, j} = \frac{1}{\text{Pe}(\Delta y)^2} - \frac{V_y}{4(\Delta y)} + \frac{1}{\text{Pe}_y(\Delta y)} \quad (\text{B.6})$$

$$q_{i, j} = -f_{i, j} C_{i-1, j} + \ell_{i, j} C_{i, j} - p_{i, j} C_{i+1, j} \quad (\text{B.7})$$

$$t_{i, j} = \frac{2}{\text{Pe}(\Delta y)^2} - \frac{V_\theta}{y(\Delta\theta)} \quad (\text{B.8})$$

then Equation B.3 becomes

$$f_{i, j} C_{i-1, j+1} + \ell_{i, j} C_{i, j+1} + p_{i, j} C_{i+1, j+1} = q_{i, j} \quad (\text{B.9})$$

Since this is an implicit scheme, we need the solution for $\theta = 0$ to initiate the solutions for $\theta > 0$. From the boundary condition of Equation 2c

$$\frac{\partial C}{\partial \theta} = 0 \quad \text{at} \quad \theta = 0, \pi$$

Equation B.1 becomes

$$\left(V_y - \frac{4}{\text{Pe}_y} \right) \frac{dC}{dy} = \frac{2}{\text{Pe}} \frac{d^2 C}{dy^2} \quad (\text{B.10})$$

This is an one-dimensional problem. We may use the explicit method with central differences for approximation.

$$(V_y - \frac{4}{Pe_y}) \frac{C_{i+1,1} - C_{i-1,1}}{2(\Delta y)} = \frac{2}{Pe} \left[\frac{C_{i-1,1} - 2C_{i,1} + C_{i+1,1}}{(\Delta y)^2} \right] \quad (B.11)$$

After rearrangement Equation B.11 becomes

$$\left[\frac{2}{Pe(\Delta y)} + \frac{V_y}{2} - \frac{2}{Pe(\Delta y)} \right] C_{i-1,1} + \left[\frac{-4}{Pe(\Delta y)} \right] C_{i,1} + \left[\frac{2}{Pe(\Delta y)} - \frac{V_y}{2} + \frac{2}{Pe(\Delta y)} \right] C_{i+1,1} = 0 \quad (B.12)$$

Notice that the above equation is written for the point $(i\Delta y, \Delta\theta)$. Thus let

$$a_{i,1} = \frac{2}{Pe(\Delta y)} + \frac{V_y}{2} - \frac{2}{Pe(\Delta y)} \quad (B.13)$$

$$b_{i,1} = \frac{-4}{Pe(\Delta y)} \quad (B.14)$$

$$d_{i,1} = \frac{2}{Pe(\Delta y)} - \frac{V_y}{2} + \frac{2}{Pe(\Delta y)} \quad (B.15)$$

Then Equation B.12 becomes

$$a_{i,1} C_{i-1,1} + b_{i,1} C_{i,1} + d_{i,1} C_{i+1,1} = 0 \quad (B.16)$$

Solution of the Finite Difference Equations. Equation B.16 represents a set of algebraic equations which can be solved simultaneously with the boundary conditions of Equations 2a and 2b as follows:

$$C = 0 \quad \text{at} \quad y = 1 \quad (B.17)$$

$$C = 1 \quad \text{at} \quad y \rightarrow \infty \quad (B.18)$$

or in finite difference form

$$C_{i,1} = 0 \quad \text{at} \quad i = 1 \quad (B.19)$$

$$C_{i,1} = 1 \quad \text{at} \quad i \rightarrow \infty \quad (B.20)$$

To satisfy the boundary condition of Equation B.20 sixteen steps from the sphere surface are considered sufficient to approximate the infinite distance where the concentration approaches the bulk concentration provided that the step size is correctly chosen for different Peclet numbers. The choice of Δy and $\Delta \theta$ will be discussed later (see Appendix C). Once the solution for $\theta = 0$ is obtained, the solution for $\theta > 0$ can be obtained by solving Equation B.9, B.19 and B.20 simultaneously. The system of algebraic equations resulting from Equations B.16, B.19 and B.20 are as follows:

$$\begin{aligned}
 b_{1,1} C_{1,1} + d_{1,1} C_{2,1} &= 0 \\
 a_{2,1} C_{1,1} + b_{2,1} C_{2,1} + d_{2,1} C_{3,1} &= 0 \\
 a_{3,1} C_{2,1} + b_{3,1} C_{3,1} + d_{3,1} C_{4,1} &= 0 \\
 &\vdots \\
 &\vdots \\
 &\vdots \\
 a_{n-1,1} C_{n-2,1} + b_{n-1,1} C_{n-1,1} + d_{n-1,1} C_{n,1} &= 0 \\
 a_{n,1} C_{n-1,1} + b_{n,1} C_{n,1} &= -d_{n,1}
 \end{aligned} \tag{B.21}$$

Similarly, the system of algebraic equations resulting from Equations B.9, B.19 and B.20 are as follows:

$$\begin{aligned}
 \ell_{1,j} C_{1,j+1} + p_{1,j} C_{1,j+1} &= q_{1,j} \\
 f_{2,j} C_{1,j+1} + \ell_{2,j} C_{2,j+1} + p_{2,j} C_{3,j+1} &= q_{2,j} \\
 f_{3,j} C_{2,j+1} + \ell_{3,j} C_{3,j+1} + p_{3,j} C_{4,j+1} &= q_{3,j}
 \end{aligned}$$

$$\begin{aligned}
& \cdot \\
& \cdot \\
& \cdot \\
& f_{n-1,j} C_{n-2,j} + \ell_{n-1,j} C_{n-1,j} + p_{n-1,j} C_{n,j} = q_{n-1,j} \\
& f_{n,j} C_{n-1,j} + \ell_{n,j} C_{n,j} = -p_{n,j} + q_{n,j}
\end{aligned} \tag{B.22}$$

Both Equations B.21 and B.22 can be solved by Gauss elimination method.

Thus the solution for equation B.21 can be calculated from

$$C_{n,1} = \frac{S_{n,1}}{\alpha_{n,1}} \tag{B.23}$$

$$C_{i,1} = \frac{1}{\alpha_{i,1}} (S_{i,1} - d_{i,1} C_{i+1,1}) \quad i = n-1, n-2, \dots, 1 \tag{B.24}$$

where the α 's and S's are given recursively by

$$\alpha_{1,1} = b_{1,1}; \alpha_{i,1} = b_{i,1} - \frac{a_{i,1}}{\alpha_{i-1,1}} d_{i-1,1} \tag{B.25}$$

$$S_{1,1} = d_{1,1}; S_{i,1} = \frac{-a_{i,1}}{\alpha_{i-1,1}} S_{i-1,1} \quad i = 2, 3, \dots, n-1 \tag{B.26}$$

$$S_{n,1} = -d_{n,1} - \frac{a_{n,1}}{\alpha_{n-1,1}} S_{n-1,1} \tag{B.27}$$

Similarly, the solution for Equation B.22 can be calculated from

$$C_{n,j} = \frac{S_{n,j}}{\alpha_{n,j}} \tag{B.28}$$

$$C_{i,j} = \frac{1}{\alpha_{i,j}} (S_{i,j} - p_{i,j} C_{i+1,j}) \quad i = n-1, n-2, \dots, 1 \tag{B.29}$$

where the α 's and S's are given recursively by

$$\alpha_{1,j} = \ell_{1,j}; \alpha_{i,j} = \ell_{i,j} - \frac{f_{i,j}}{\alpha_{i-1,j}} p_{i-1,j} \tag{B.30}$$

$$S_{1,j} = q_{1,j}; \quad S_{i,j} = q_{i,j} - \frac{f_{i,j}}{\alpha_{i-1,j}} S_{i-1,j} \quad i = 2, 3, \dots, n-1 \quad (\text{B.31})$$

$$S_{n,j} = -p_{n,j} + q_{n,j} - \frac{f_{n,j}}{\alpha_{n-1,j}} S_{n-1,j} \quad (\text{B.32})$$

The above systems of equations, B.21 and B.22 can also be solved by Gauss-Jordan method with pivotal condensation. However, the computer time for each data point so obtained is more than three times that obtained by use of Gauss elimination method. Furthermore, the difference between results of the two methods is very insignificant. Thus, the use of Gauss-Jordan instead of Gauss elimination method can not be justified.

Calculation of Average Sherwood Number. Once the local Sherwood numbers for each θ step is obtained for a set of Peclet number and viscosity ratio index, the average Sherwood number can be calculated by integration over the sphere surface according to Equation A.15. This can be done numerically by use of Simpson's rule which is outlined as follows:

Let S_{2N} be the numerical approximation for the average Sherwood number, and

$$f(\theta) = \left(\frac{\partial C}{\partial y} \right)_{y=1} \sin \theta \quad (\text{B.33})$$

If we divide the interval $[0, \pi]$ into $2N$ ($=60$) subintervals of length, $\Delta\theta = \pi/60$, then according to Simpson's rule

$$S_{2N} = \frac{\Delta\theta}{3} (f_0 + 4f_1 + 2f_2 + 4f_3 + \dots + 4f_{59} + f_{60}) \quad (\text{B.34})$$

where f_i is the value of the function $f(\theta)$ evaluated at $\theta = i(\Delta\theta)$.

Appendix C. The Computer Program

Explanation of Computer Program. The computer program for the numerical solution described in Appendix B will be explained statement by statement by reference to statement numbers on the left hand side of the program listing. This program is written for the IBM, 360 model 65 in the language of FORTRAN IV.

In statements 1 through 3 are some of the variable names to be used in double precision. They are

- A(I, J) = coefficient matrix for Equations B.21 and B.22
- Y(I) = radial distance from sphere surface; Y(1) represents
 one step or a distance of Δy away from sphere
 surface.
- U(I) = concentration obtained from solution of Equation B.21
 or B.22, the index I is the number representing
 radial distance of the grid points.

It should be noted that the angular distance for each grid point is controlled by the DO LOOP as will be mentioned later.

- AB(I) = local Sherwood numbers for each angular step
- S(I) = $S_{i,j}$ in Equations B.23 through B.32
- R(I) = $\alpha_{i,j}$ in Equations B.23 through B.32
- PE = Peclet number
- X = viscosity ratio parameter or simply viscosity ratio
 in the case of Newtonian continuous phase
- BN = flow behavior index of power-law fluid

DY	=	step size in radial direction, Δy
FI, A1, A2, B1, B2	=	coefficients or exponents in Nakano and Tien stream function, σ , A_1 , A_2 , B_1 , B_2
DT	=	step size in angular direction, $\Delta \theta$
VY	=	velocity component in radial direction, V_y
VZ	=	velocity component in angular direction, V_θ
AA	=	$\frac{1}{Pe(\Delta y)^2}$
BB	=	$\frac{1}{4(\Delta y)} \left[V_y - \frac{4}{Pe y} \right]$
DD	=	$\frac{V_\theta}{y \Delta \theta}$
SSN	=	average Sherwood number
DABS, DSIN, DCOS	=	names of subroutines absolute, sine and cosine in double precision

In statements 4 through 9 some physical or mathematical control variables are defined. Statement 4 reads in two constants, namely N the number of steps in the radial direction and NO the number of steps in the angular direction. In this particular program N is set to equal 16 and NO equal 60. The definition of three other variables MB, M and DT are presented in statements 5 through 7. Statement 8 starts a DO LOOP for various sets of PE and DY which are read in by statement 9. Once DY is specified for each PE, the radial distance of each grid point can be calculated as shown in statements 10 through 12.

Statement 13 starts another DO LOOP which specifies various X , BN , FI , A_1 , A_2 , B_1 , B_2 in statement 14 for each set of PE and DY . Now, we are in a position to find a solution for the concentration distribution for a set of PE , X and BN . Notice here that the coefficients or exponents α , A_1 , A_2 , B_1 , B_2 can be determined once X and n are known.

The solution of the concentration distribution for $\theta = 0$ is programmed in statements 15 through 42. Statements 15 through 32 calculate the coefficients of each term in Equation B.21. The Gauss elimination method as outlined in Equations B.23 through B.27 is programmed in statements 33 through 42.

It should be mentioned that statements 21 and 26 are the velocity components in the radial direction which are derived from the Nakano and Tien stream function. For other rheological models, these two statements must be changed accordingly.

The solution of the concentration distribution for $\theta = 0$ is stored in $U(I)$ as indicated in statements 39 through 42. It is needed for the solution of the concentration distribution in the range of $0 < \theta < \pi$. Each time the DO LOOP started by statement 43 is executed, the angular distance is increased by a distance DT as indicated by statement 44. The coefficients for each term of Equation B.22 are calculated in statements 45 through 71. Again for different rheological models, the velocity components in statements 47, 48, 59, 60 should be changed. The Gauss elimination scheme of Equations B.28 through B.32 are indicated by statements 72 through 81.

In statement 82 the integrands for the average Sherwood numbers are stored in AB(I) for each value of θ according to Equation B.33. For each set of Pe , X and n the average Sherwood number is then obtained by use of Simpson's rule in statements 84 through 89 according to Equation B.34. The results are printed out by statement 90.

```

CC      MASS TRANSFER FROM SPHERICAL CIRCULATING GAS BUBBLES AND LIQUID
CC      DROPLETS IN LAMINAR FLOW REGION
C       NAKANO AND TIEN MODEL
1       DOUBLE PRECISION A(20,20), Y(20), U(20), AB(70), S(20), R(20)
2       DOUBLE PRECISION PE, X, BN, DY, FI, A1, A2, B1, B2, TT, VY, VZ, DT
3       DOUBLE PRECISION AA, BB, DD, SSN, DABS, DSIN, DCOS
C       PHYSICAL AND MATHEMATICAL CONTROL VARIABLES
4       READ(1,120)N, NO
5       MB=N-1
6       M=N+1
7       DT=3.1415926535898/NO
8       DO 1968 ILOVEF=1, 10
9       READ(1,114)PE, DY
C       DETERMINATION OF RADIAL GRID POINTS
10      Y(1)=1.+DY
11      DO 10 I=1, N
12      10 Y(I+1)=Y(I)+DY
C       SOLUTION FOR A PARTICULAR SET OF (PE, X, N)
13      DO 1942 LB=1, 30
14      READ(1,114)X, BN, FI, A1, A2, B1, B2
C       SOLUTION FOR ANGLE EQUAL TO ZERO
15      TT=0.0
16      DO 22 I=1, N
C       CALCULATION OF COEFFICIENT FOR EACH TERM
17      DO 22 J=1, M
18      22 A(I, J)=0.0
19      J=0
20      DO 47 I=2, MB
21      VY=(-.5+1*(Y(I))**(FI-2.)+(A2+B1)/(Y(I))**3+B2/(Y(I))**4)*2.
22      A(I, J+1)=2./(PE*DY)+(VY-4./(Y(I)*PE))/2.
23      A(I, J+2)=-4./(PE*DY)

```

> A1

↑

```

24      A(I, J+3)=2./(PE*DY)-(VY-4./(Y(I)*PE))/2.
25      47 J=J+1
26      VY=(-.5+A1*(Y(I))**(FI-2.)+(A2+B1)/(Y(I))**3+B2/(Y(I))**4)*2.
27      A(I, 1)=-4./(PE*DY)
28      A(I, 2)=2./(PE*DY)-(VY-4./(Y(I)*PE))/2.
29      VY=(-.5+A1*(Y(N))**(FI-2.)+(A2+B1)/(Y(N))**3+B2/(Y(N))**4)*2.
30      A(N, MB)=2./(PE*DY)+(VY-4./(Y(N)*PE))/2.
31      A(N, N)=-4./(PE*DY)
32      A(N, M)=-2./(PE*DY)-(VY-4./(Y(N)*PE))/2.)
C      SOLUTION OF SIMULTANEOUS EQUATIONS
33      R(1)=A(1, 1)
34      DO 1 I=2, N
35      1 R(I)=A(I, I)-A(I, I-1)/R(I-1)*A(I-1, I)
36      S(1)=A(1, M)
37      DO 2 I=2, N
38      2 S(I)=A(I, M)-A(I, I-1)/R(I-1)*S(I-1)
39      U(N)=S(N)/R(N)
40      N1=N-1
41      DO 3 I=1, N1
42      3 U(N-I)=(S(N-I)-A(N-I, N-I+1)*U(N-I+1))/R(N-I)
C      SOLUTION FOR ANGLE GREATER THAN ZERO
43      DO 25 J=1, NO
44      TT=TT+DT
C      CALCULATION OF COEFFICIENT FOR EACH TERM
45      J1=0
46      DO 471 I=2, MB
47      VY=(-1.+2*A1*(Y(I))**(FI-2)+2*A2/(Y(I))**3)*DCOS(TT)+(B1/(Y(I))**3
2+B2/(Y(I))**4)*(2*(DCOS(TT))**2-(DSIN(TT))**2)
48      VZ=(1-A1*FI*(Y(I))**(FI-2)+A2/(Y(I))**3)*DSIN(TT)+(B1/(Y(I))**3-2*
2B2/(Y(I))**4)*DCOS(TT)*DSIN(TT)
49      AA=1./(PE*DY*DY)

```



```

50      BB=(VY-4./(PE*Y(I)))/(4.*DY)
51      DD=VZ/(Y(I)*DT)
52      A(I,J1+1)=AA+BB
53      A(I,J1+2)=-2.*AA-DD
54      A(I,J1+3)=AA-BB
55      A(I,M)=- (AA+BB)*U(I-1)+(2.*AA-DD)*U(I)-(AA-BB)*U(I+1)
56  471 J1=J1+1
57      K=1
58      DO 472 I=1,2
59      VY=(-1.+2*A1*(Y(K))*(FI-2)+2*A2/(Y(K))**3)*DCOS(TT)+(B1/(Y(K))**3
60      +2*B2/(Y(K))**4)*(2*(DCOS(TT))**2-(DSIN(TT))**2)
61      VZ=(1-A1*FI*(Y(K))*(FI-2)+A2/(Y(K))**3)*DSIN(TT)+(B1/(Y(K))**3-2*
62      2*B2/(Y(K))**4)*DCOS(TT)*DSIN(TT)
63      BB=(VY-4./(PE*Y(K)))/(4.*DY)
64      DD=VZ/(Y(K)*DT)
65      IF(K-1)473,473,474
66  473 A(1,1)=-2*AA-DD
67      A(1,2)=AA-BB
68      A(1,M)=(2*AA-DD)*U(1)-(AA-BB)*U(2)
69      GO TO 472
70  474 A(N,MB)=AA+BB
71      A(N,N)=-2*AA-DD
72      A(N,M)=- (AA+BB)*U(MB)+(2*AA-DD)*U(N)-(AA-BB)*2.
73  472 K=K+MB
74  C      SOLUTION OF SIMULTANEOUS EQUATIONS
75      R(1)=A(1,1)
76      DO 171 I=2,N
77  171 R(I)=A(I,I)-A(I,I-1)/R(I-1)*A(I-1,I)
78      S(1)=A(1,M)
79      DO 272 I=2,N
80  272 S(I)=A(I,M)-A(I,I-1)/R(I-1)*S(I-1)

```

```

78      U(N)=S(N)/R(N)
79      N1=N-1
80      DO 373 I=1,N1
81      373 U(N-I)=(S(N-I)-A(N-I,N-I+1)*U(N-I+1))/R(N-I)
      C      CALCULATION FOR LOCAL SHERWOOD NUMBER
82      AB(J+1)=U(1)/DY*DSIN(TT)
83      25 CONTINUE
      C      CALCULATION FOR AVERAGE SHERWOOD NUMBER
84      AB(1)=0
85      SSN=AB(1)+4*AB(2)
86      NI1=NO/2-1
87      DO 97 L=1,NI1
88      97 SSN=2*AB(1+2*L)+4*AB(2+2*L)+SSN
89      SSN=(SSN+AB(NO+1))*(3.1415963/(NO*3))
90      1942 WRITE(3,117)PE,X,BN,SSN
91      1968 CONTINUE
92      999 STOP
93      114 FORMAT(7D10.1)
94      117 FORMAT(///6X,'PE=',D16.8,4X,'X=',D16.8,4X,'N=',D16.8,4X,'SHERWOOD
      2=',D16.8)
95      120 FORMAT(2I10)
96      END

```

```

PE= 0.10000000D 02    X= 0.10000000D 02    N= 0.90000000D 00    SHEERWOOD= 0.32910266D 01

PE= 0.10000000D 02    X= 0.20000000D 01    N= 0.90000000D 00    SHEERWOOD= 0.34114606D 01

PE= 0.10000000D 02    X= 0.10000000D 01    N= 0.90000000D 00    SHEERWOOD= 0.34936874D 01

```

Determination of Step Sizes for Numerical Solution. Since an attempt to solve Equation B.1 by the explicit method results in a failure to maintain both convergence and stability, the Crank-Nicolson technique has to be used. For a simple parabolic equation such as

$$\frac{\partial C}{\partial \theta} = \frac{\partial^2 C}{\partial y^2}$$

The Crank-Nicolson implicit scheme proves to be convergent and stable for all finite values of the ratio, $\Delta\theta/(\Delta y)^2$. However, this does not apply to a more complicated parabolic equation such as Equation B.1.

As can be seen in Appendix B the coefficients of the partial derivatives in Equation B.1 are not constants but are functions of θ , y and Pe . It is found that convergence and stability can not be ensured for all finite values of $\Delta\theta/(\Delta y)^2$ for this type of equation. In fact, indiscriminate use of $\Delta\theta$ and Δy will result in an average Sherwood number different from the correct value.

In this program an angular increment of three degrees, $\Delta\theta = \pi/60$, is used. Changing the angular increment to one and half degree, $\Delta\theta = \pi/120$, results in only about 0.5% change in the calculated value of average Sherwood number when the Peclet number is equal to 10^4 , the radial increment being kept constant. However, for a fixed value of $\Delta\theta$, the results show a strong dependence on Δy . The Stokes model is used for the above investigation.

The strong dependence of the average Sherwood number on Δy may be explained by the existence of concentration boundary layer outside the sphere surface. The boundary layer thickness decreases with the increase

of the Peclet number which is defined as $Pe = V_{\infty} 2a/D$. It is conceivable that for a certain value of Peclet number the boundary layer thickness is fixed assuming that other factors remain unchanged. Therefore, if the angular increment and the number of steps in the radial direction are held constant, then the step size, Δy , should be properly chosen for each Peclet number so that the maximum radial distance will fall within a realistic range of actual boundary layer thickness. This is necessary because the boundary condition of unity concentration outside the boundary layer must be satisfied according to Equation A.8. Otherwise, the concentration profile would be stretched or contracted should the Δy be too large or too small. (It should be noted that this effect of Δy is not so significant for the low Peclet number region. In this region mass transfer is slow and the concentration boundary layer is so thick that any deviation from the actual thickness will not have a pronounced effect on the value of Sh). Consequently, this would cause inaccuracy in the results of average Sherwood numbers.

Figures C.I and C.II show the effect of Δy on the average Sherwood number for various Peclet numbers. For this program sixteen steps are chosen for the radial direction and sixty steps for the angular direction. As can be seen in the figures, the right hand portion of each curve shows a strong dependence of average Sherwood number on Δy . In these situations Δy is too large. This may have two effects on the results. First, the concentration profile is stretched; second, Δy is not small enough to ensure sufficient convergence. On the other hand, the left hand portion of each curve shows the fact of concentration profile being contracted and a sign of

instability as Δy becomes too small.

The flat portion of each curve indicates numerical stability and shows a result essentially identical to the results of previous workers using Stokes stream function. This gives the correct range of Δy values which may be used. The exact values of Δy , used as shown by dots in the figures, are determined by comparison of the calculated values of average Sherwood numbers with those obtained by latest workers using Stokes stream function. These values are assumed to apply for different stream functions. A list of Δy values for various Peclet numbers is presented in Table C.I.

Tables of Numerical Results. The following tables of the results from numerical solution to different physical systems are listed with the names of previous workers to identify the different rheological models assumed for the continuous phase. In all cases, the dispersed phase is always Newtonian, and the basic assumptions listed in Section 2 or Appendix A apply.

The numerical results show that for X (or R) larger than about 10^{-2} the numerical scheme is convergent and stable. However, for X (or R) less than about 10^{-2} the computer program becomes unstable for some points although convergence is still observed. These can be seen by comparison of average Sherwood numbers between the cases of X (or R) equal 10^{-3} and 10^{-2} .

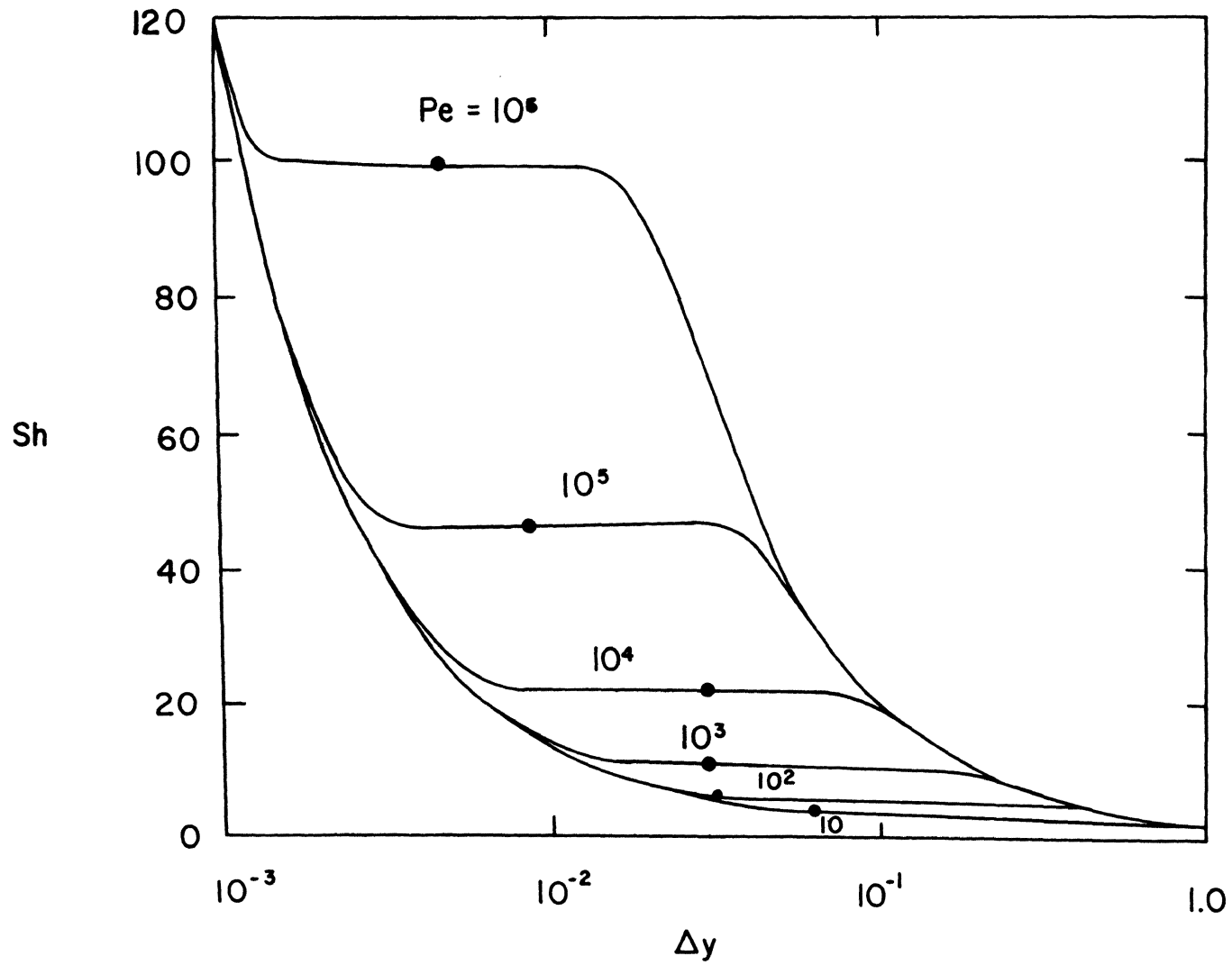


Figure C.1. Effect of Radial Increment on Sherwood Number for Various Peclet Numbers ($Pe = 10 \sim 10^6$).

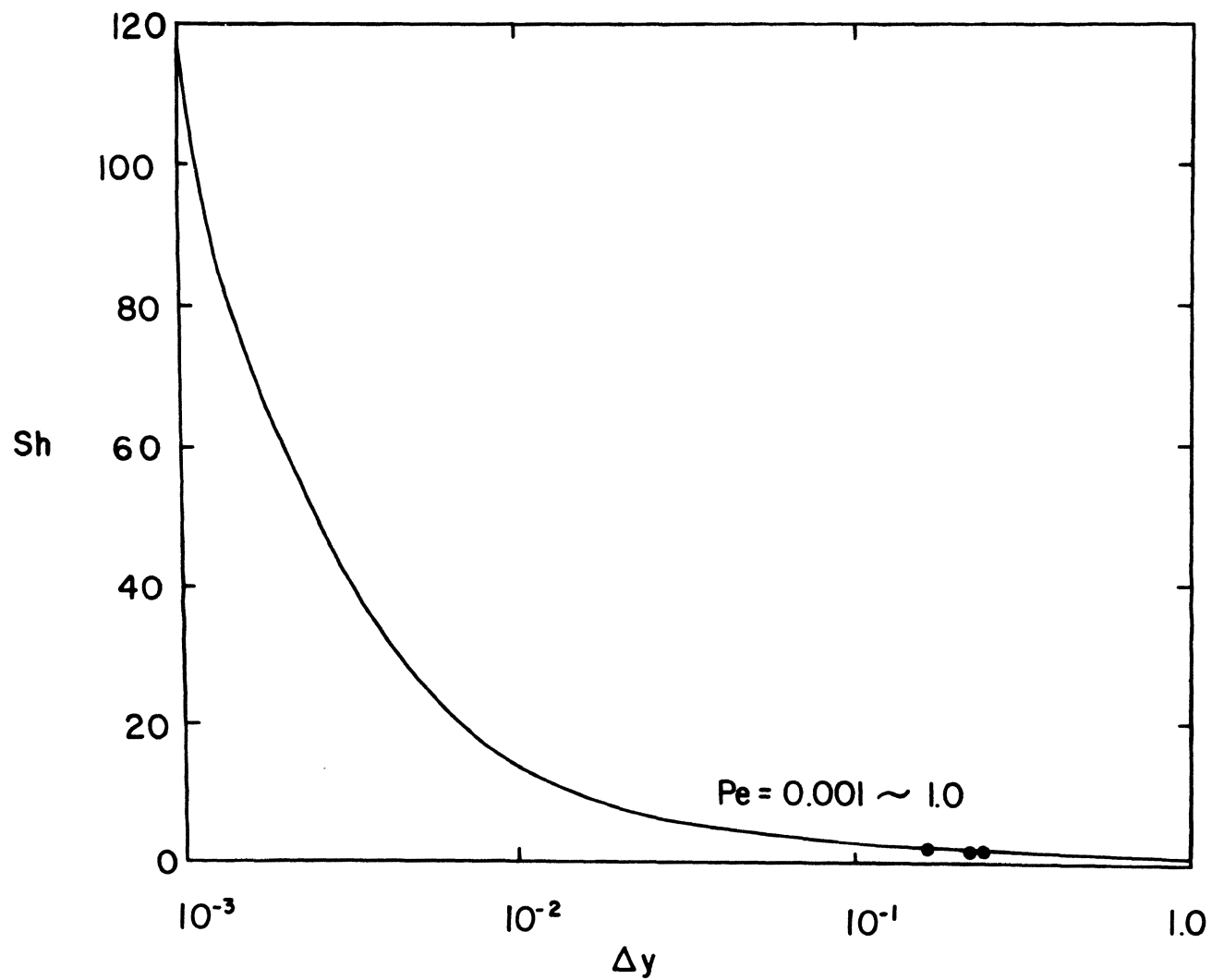


Figure C.2. Effect of Radial Increment on Sherwood Number for Various Peclet Numbers ($Pe = 0.001 \sim 1.0$).

Table C.I
Radial Increment for Various Peclet Numbers

Pe	Δy
0.001	0.2424
0.01	0.2410
0.1	0.2260
1.0	0.1620
10	0.0880
100	0.0400
1000	0.0370
10,000	0.0340
100,000	0.0074
1000,000	0.0030

Table C.II
Stokes Model
($X = \infty$, $n = 1$)

Pe	Sh
10^{-3}	2.00056
10^{-2}	2.00491
10^{-1}	2.05566
1	2.35320
10	3.20851
10^2	5.48417
10^3	10.42634
10^4	21.67877
10^5	46.22968
10^6	98.86883

Table C. III
Hadamard Model
(X = R, n = 1)

Pe	R=10 ⁻³ Sh	R=10 ⁻² Sh	R=10 ⁻¹ Sh	R=1 Sh	R=2 Sh	R=10 Sh
10 ⁻³	2.0006	2.0006	2.0006	2.0006	2.0006	2.0006
10 ⁻²	2.0049	2.0049	2.0049	2.0049	2.0049	2.0049
10 ⁻¹	2.0556	2.0556	2.0556	2.0556	2.0556	2.0556
1	2.3634	2.3633	2.3623	2.3577	2.3560	2.3539
10	3.4889	3.4860	3.4596	3.3351	3.2892	3.2290
10 ²	7.7387	7.7191	7.5392	6.6029	6.2166	5.6735
10 ³	20.9514	20.8764	20.1817	16.2845	14.4715	11.5727
10 ⁴	55.9544	55.8223	54.5277	44.7564	38.7393	27.0859
10 ⁵	193.8298	198.0832	186.0347	142.1475	118.5553	71.4611
10 ⁶	579.3642	577.6736	561.1356	438.6675	364.4255	204.2248

Table C. III cont'd

Hadamard Model

(X = R, n = 1)

Pe	R=10 ² Sh	R=10 ³ Sh	R=10 ⁴ Sh	R=10 ⁵ Sh
10 ⁻³	2.0006	2.0006	2.0006	2.0006
10 ⁻²	2.0049	2.0049	2.0049	2.0049
10 ⁻¹	2.0557	2.0557	2.0557	2.0557
1	2.3533	2.3532	2.3532	2.3532
10	3.2106	3.2087	3.2085	3.2085
10 ²	5.5042	5.4862	5.4844	5.4842
10 ³	10.5516	10.4390	10.4276	10.4264
10 ⁴	22.2904	21.7405	21.6848	21.6793
10 ⁵	49.2172	46.5322	46.2597	46.2325
10 ⁶	112.8270	100.2956	99.0118	98.8831

Table C. IV
Nakano and Tien Model

$n = 1.0$

	$X=10^{-3}$	$X=10^{-2}$	$X=10^{-1}$	$X=1$	$X=2$	$X=10$
Pe	Sh	Sh	Sh	Sh	Sh	Sh
10^{-3}	2.0006	2.0006	2.0006	2.0006	2.0006	2.0006
10^{-2}	2.0049	2.0049	2.0049	2.0049	2.0049	2.0049
10^{-1}	2.0556	2.0556	2.0556	2.0556	2.0556	2.0556
1	2.3634	2.3633	2.3623	2.3577	2.3560	2.3539
10	3.4888	3.4859	3.4597	3.3351	3.2892	3.2289
10^2	7.7374	7.7185	7.5400	6.6030	6.2163	5.6733
10^3	20.9411	20.8713	20.1880	16.2845	14.4700	11.5713
10^4	55.9306	55.8103	54.5443	44.7566	38.7341	27.0799
10^5	193.3889	192.9380	186.5428	142.1920	118.5598	71.4386
10^6	576.0532	576.0152	563.3444	438.6675	364.3582	204.1242

Table C. IV cont'd
Nakano and Tien Model

$n = 0.9$

	$X=10^{-3}$	$X=10^{-2}$	$X=10^{-1}$	$X=1$	$X=2$	$X=10$
Pe	Sh	Sh	Sh	Sh	Sh	Sh
10^{-3}	2.0006	2.0006	2.0006	2.0006	2.0006	2.0006
10^{-2}	2.0049	2.0049	2.0049	2.0049	2.0049	2.0049
10^{-1}	2.0556	2.0556	2.0556	2.0556	2.0556	2.0556
1	2.3686	2.3694	2.3685	2.3662	2.3627	2.3575
10	3.5866	3.5999	3.5772	3.4937	3.4115	3.2910
10^2	8.2082	8.2563	8.1022	7.3876	6.8224	5.9694
10^3	22.5027	22.5857	21.9789	18.8090	16.3912	12.4649
10^4	58.1718	58.2165	57.3219	50.8245	44.0544	29.4968
10^5	211.9687	208.4450	202.0507	166.5473	136.9050	78.3173
10^6	625.6226	609.9635	595.3580	508.7149	420.2894	224.7773

Table C. IV cont'd
Nakano and Tien Model

$n = 0.8$

	$X=10^{-3}$	$X=10^{-2}$	$X=10^{-1}$	$X=1$	$X=2$	$X=10$
Pe	Sh	Sh	Sh	Sh	Sh	Sh
10^{-3}	2.0006	2.0006	2.0006	2.0006	2.0006	2.0006
10^{-2}	2.0049	2.0049	2.0049	2.0049	2.0049	2.0049
10^{-1}	2.0556	2.0556	2.0556	2.0556	2.0556	2.0556
1	2.3745	2.3752	2.3748	2.3715	2.3686	2.3628
10	3.6943	3.7067	3.6923	3.5892	3.5161	3.3819
10^2	8.6867	8.7373	8.6271	7.8242	7.3006	6.3728
10^3	23.9546	24.1041	23.6529	20.2250	17.9007	13.6088
10^4	59.6847	59.8114	59.2759	53.6371	47.9281	32.6458
10^5	219.3981	220.5068	216.3580	184.9600	156.2002	88.5689
10^6	628.7744	630.5960	623.4022	573.8366	500.7400	268.2706

Table C. IV cont'd
Nakano and Tien Model
 $n = 0.7$

Pe	X=10 ⁻³ Sh	X=10 ⁻² Sh	X=10 ⁻¹ Sh	X=1 Sh	X=2 Sh	X=10 Sh
10 ⁻³	2.0006	2.0006	2.0006	2.0006	2.0006	2.0006
10 ⁻²	2.0049	2.0049	2.0049	2.0049	2.0049	2.0049
10 ⁻¹	2.0556	2.0556	2.0556	2.0556	2.0556	2.0557
1	2.3801	2.3812	2.3792	2.3762	2.3711	2.3668
10	3.7956	3.8149	3.7739	3.6741	3.5574	3.4480
10 ²	9.1373	9.2191	8.9929	8.1941	7.4670	6.6442
10 ³	25.3792	25.6286	24.8144	21.3201	18.3530	14.3023
10 ⁴	60.8397	60.9998	60.3091	55.3216	48.8634	34.2780
10 ⁵	229.5931	231.2762	224.9418	189.2216	160.5452	90.6876
10 ⁶	645.5344	647.7576	638.1394	566.2556	515.9271	259.6597

Table C. IV cont'd
Nakano and Tien Model
n = 0.6

Pe	X=10 ⁻³ Sh	X=10 ⁻² Sh	X=10 ⁻¹ Sh	X=1 Sh	X=2 Sh	X=10 Sh
10 ⁻³	2.0006	2.0006	2.0006	2.0006	2.0006	2.0006
10 ⁻²	2.0049	2.0049	2.0049	2.0049	2.0049	2.0049
10 ⁻¹	2.0556	2.0556	2.0556	2.0556	2.0557	2.0557
1	2.3831	2.3821	2.3830	2.3760	2.3732	2.3701
10	3.8505	3.8310	3.8427	3.6676	3.5912	3.5045
10 ²	9.3757	9.2914	9.3016	8.1400	7.5779	6.8677
10 ³	26.0854	25.8581	25.7871	21.0697	18.5078	14.8505
10 ⁴	61.2904	61.1490	61.0061	54.8397	48.8382	35.5256
10 ⁵	231.9253	233.3408	231.7319	186.3816	153.9696	92.9264
10 ⁶	639.6737	649.9659	646.2622	559.0062	469.4186	263.8608

Table C. V
Tomita Model
($X = \infty$)

Pe	n=1.4 Sh	n=1.3 Sh	n=1.2 Sh	n=1.1 Sh
10^{-3}	2.0006	2.0006	2.0006	2.0006
10^{-2}	2.0049	2.0049	2.0049	2.0049
10^{-1}	2.0557	2.0557	2.0557	2.0557
1	2.3574	2.3555	2.3537	2.3520
10	3.2767	3.2437	3.2123	3.1830
10^2	5.7838	5.6291	5.4786	5.3351
10^3	11.2772	10.8257	10.3615	9.8833
10^4	23.6494	22.5666	23.6494	20.3257
10^5	50.4521	48.0983	45.6848	43.2047
10^6	108.0248	102.8829	97.6109	92.1935

Table C. V cont'd

Tomita Model

 $(X = \infty)$

	n=1.0	n=0.9	n=0.8	n=0.7	n=0.6
Pe	Sh	Sh	Sh	Sh	Sh
10^{-3}	2.0006	2.0006	2.0006	2.0006	2.0006
10^{-2}	2.0049	2.0049	2.0049	2.0049	2.0049
10^{-1}	2.0557	2.0557	2.0557	2.0557	2.0558
1	2.3505	2.3492	2.3481	2.3472	2.3465
10	3.1562	3.1325	3.1123	3.0959	3.0834
10^2	5.2017	5.0821	4.9794	4.8962	4.8337
10^3	9.3893	8.8773	8.3445	7.7873	7.2018
10^4	19.1607	17.9602	16.7183	15.4273	14.0765
10^5	40.6493	38.0083	35.2682	32.4119	29.4166
10^6	86.6125	80.8450	74.8623	68.6289	62.1154

Table C. VI
Hirose and Moo-Young Model
(X = 0)

Pe	n=1.4 Sh	n=1.3 Sh	n=1.2 Sh	n=1.1 Sh
10^{-3}	2.0006	2.0006	2.0006	2.0006
10^{-2}	2.0049	2.0049	2.0049	2.0049
10^{-1}	2.0556	2.0556	2.0556	2.0556
1	2.3496	2.3523	2.3556	2.3593
10	2.7208	2.8484	2.9700	3.0834
10^2	5.8036	6.3484	6.8603	7.3251
10^3	14.2714	16.3513	18.1108	19.6321
10^4	40.5598	46.2729	50.5473	53.6910
10^5	130.1800	150.6529	167.5913	181.8681
10^6	405.3200	466.6788	514.5432	551.4694

Table C. VI cont'd
Hirose and Moo-Young Model
(X = 0)

Pe	n=1.0 Sh	n=0.9 Sh	n=0.8 Sh	n=0.7 Sh	n=0.6 Sh
10^{-3}	2.0006	2.0006	2.0006	2.0006	2.0006
10^{-2}	2.0049	2.0049	2.0049	2.0049	2.0049
10^{-1}	2.0556	2.0556	2.0556	2.0556	2.0556
1	2.3634	2.3677	2.3720	2.3761	2.3798
10	3.1873	3.2813	3.3644	3.4358	3.4938
10^2	7.7410	8.1089	8.4297	8.7016	8.9205
10^3	20.9598	22.1184	23.1188	23.9617	24.6367
10^4	55.9691	57.6022	58.7619	59.5735	60.1247
10^5	193.9133	203.9831	212.2608	218.8820	223.9273
10^6	579.5524	600.6550	616.3284	627.7730	635.8445

Appendix D. Review of Literature

The fluid dynamics and solute mass transfer from laminar circulating spheres have been topics of considerable interest in the literature for many years. Some of the previous investigations of continuous mass transfer have been reviewed in Section 4 of the present study. For this particular case, the dispersed phase resistance is negligible. A more extensive review on the analyses of continuous phase transfer can be found in the paper of Sideman and Shabtai (1964). All these analyses assume that both continuous and dispersed phases are Newtonian fluids. Until recently no attempts have been made to extend these studies to non-Newtonian media.

Hirose and Moo-Young (1969) considered the motion of a power-law fluid around a spherical gas bubble and obtained an approximate solution for the drag coefficient and mass transfer rate for this system. The approximate nature of the solution is due to an assumption that the deviation from Newtonian flow is small enough to warrant some quantities to be evaluated according to Newtonian behavior.

The following boundary conditions are involved in the solution of equation of motion for axisymmetric creeping flow:

$$V_r = 0, \quad \frac{\partial}{\partial r} \left(\frac{V_\theta}{r} \right) = 0 \quad @ \quad r = a \quad (D.1)$$

$$V_r = V_\infty \cos \theta, \quad V_\theta = -V_\infty \sin \theta \quad \text{as } r \rightarrow \infty \quad (D.2)$$

The boundary condition $[\partial(V_\theta/r)/\partial r]_{r=a} = 0$ results from the absence of tangential stress at the surface of clean bubbles (Calderbank and Moo-Young, 1961) since the gas phase may be regarded as inviscid relative to

the non-Newtonian continuous phase fluid.

The continuous phase stream function resulting from the solution of the boundary value problem is as follows:

$$\psi' = \frac{1}{2} V_{\infty} a^2 \left[\left\{ \left(\frac{r}{a} \right) - \left(\frac{r}{a} \right)^2 \right\} + \frac{6n(n-1)}{2n+1} \left\{ \frac{r}{a} \ln \left(\frac{r}{a} \right) + \frac{1}{6} \left(\frac{a}{r} \right) - \frac{1}{6} \left(\frac{r}{a} \right) \right\} \right] \sin^2 \theta \quad (D.3)$$

or in dimensionless form

$$\psi = \frac{1}{2} \left[\{ y - y^2 \} + \frac{6n(n-1)}{2n+1} \left\{ y \ln y + \frac{1}{6} \left(\frac{1}{y} \right) - \frac{1}{6} (y) \right\} \right] \sin^2 \theta \quad (D.4)$$

The velocity components can be derived by use of Equations D.4, A.12 and A.13, and are presented in Equations 14a and 14b. For clean bubbles, i.e. in the absence of frictional drag, the drag coefficient is obtained as

$$C_D = \frac{16}{Re'} (2)^{n-1} (3)^{\frac{n-1}{2}} \frac{(13 + 4n - 8n^2)}{(2n+1)(n+2)} \quad (D.5)$$

where

$$Re' = \frac{(2a)^n (V_{\infty})^{2-n} \rho}{K} \quad (D.6)$$

A comparison of Equation D.5 with the drag coefficient for Newtonian fluids results in a correction factor for non-Newtonian behavior as shown below:

$$Y_D = \frac{(C_D)_{\text{non-Newtonian}}}{(C_D)_{\text{Newtonian}}} = (2)^{n-1} (3)^{\frac{n-1}{2}} \frac{(13 + 4n - 8n^2)}{(2n+1)(n+2)} \quad (D.7)$$

A plot of Y_D against n (see Figure D.1) shows that Y_D is greater than unity for pseudoplastic fluid and less than unity for dilatant fluids.

By use of Equation A.13 and relation of Baird and Hamielec (1962) for

mass transfer from a spherical bubble

$$\text{Sh} = \frac{2}{\pi} \left[\int_0^\pi \left(-\frac{V_\theta}{V_\infty} \right)_{r=a} \sin^2 \theta d\theta \right]^{\frac{1}{2}} (\text{Pe})^{\frac{1}{2}} \quad (\text{D.8})$$

a theoretical expression for the average Sherwood number is obtained as shown in Equation 15. Similarly, a comparison of the average Sherwood number of Equation 15 with that for bubbles in Newtonian fluid results in a correction factor for mass transfer in non-Newtonian fluid, Y_M .

$$Y_M = \left\{ 1 - \frac{4n(n-1)}{2n+1} \right\}^{\frac{1}{2}} \quad (\text{D.9})$$

Again a plot of Y_M against n (see Figure D.1) indicates that the mass transfer coefficient is enhanced for pseudoplastics and depressed for dilatants compared to the situation for Newtonian fluids. Hirose and Moo-Young also presented some experimental results to support the relations of Equations D.7 and D.9.

Nakano and Tien (1968) also presented an analysis of the creeping flow of a power-law fluid over a Newtonian circulating sphere. Unlike the work of Hirose and Moo-Young (1969) this analysis applies not only for gas bubbles, but also for liquid droplets. Two equations of motions (for the continuous and dispersed phase, respectively) are solved with the following boundary conditions:

$$(V_y)_d = (V_y)_c = 0 \quad @ \quad y = 1 \quad (\text{D.10})$$

$$(V_\theta)_d = (V_\theta)_c \quad @ \quad y = 1 \quad (\text{D.11})$$

$$(\tau_{y\theta})_d = (\tau_{y\theta})_c \quad @ \quad y = 1 \quad (\text{D.12})$$

$$\psi_c \rightarrow -\frac{1}{2} y^2 \sin^2 \theta \quad \text{as } y \rightarrow \infty \quad (\text{D.13})$$

$$(V_{\theta_d}) \text{ and } (V_{y_d}) \text{ remain finite @ } y = 0 \quad (\text{D.14})$$

The resulting continuous and dispersed phase stream functions are given as follows:

$$\psi_c = \left(-\frac{1}{2}y^2 + A_1 y^\sigma + \frac{A_2}{y}\right)(1 - Z^2) + \left(\frac{B_1}{y} + \frac{B_2}{y^2}\right)Z(1 - Z^2) \quad (\text{D.15})$$

$$\begin{aligned} \psi_d &= (C_1 y^2 + C_2 y^3 + C_3 y^4)(1 - Z^2) \\ &+ (D_1 y^2 + D_2 y^3 + D_3 y^4)Z(1 - Z^2) \end{aligned} \quad (\text{D.16})$$

$$Z = \cos \theta$$

where the coefficients, A_1 , A_2 , B_1 , B_2 ..., and the exponent σ are numerically determined for specified values of n and X through the use of a combination of Galerkin's method and variational integrals and are given in Table D.I. The range of parameters is given as

$$\begin{aligned} 0.001 &< X < 10.0 \\ 0.5 &< n < 1.0 \end{aligned} \quad (\text{D.17})$$

It was found that almost identical results of the coefficients A_1 , A_2 , B_1 , B_2 , σ were obtained for $X = 0.001$ and $X \leq 0.0001$. Consequently, the values for $X = 0.001$ are taken as a lower limit in this work.

The drag coefficient is given as

$$C_D = \frac{24 Y_D}{2^n N_{\text{Re}}} \quad (\text{D.18})$$

where

$$N_{Re} = \frac{\rho V_{\infty}^{2-n} a^n}{K} \quad (D.19)$$

and Y_D , the correction factor is defined as

$$Y_D = \frac{2^{n+1}}{3} \int_{-1}^1 \int_0^1 (X x^2 \psi_d^{\frac{2}{n-1}} + \psi_c^{\frac{n+1}{n-1}} x^{-4}) dx dZ \quad (D.20)$$

Astarita (1965) also considered the motion of a gas bubble through a power-law fluid. A semiquantitative relationship was obtained which may be expressed in terms of the correction factor Y_D as follows:

$$F = \left(\frac{Y_{D \text{ rigid sphere}}}{Y_{D \text{ fluid sphere}}} \right)^{1/n} > 1.5 \quad (D.21)$$

$$X = 0$$

Nakano and Tien (1968) calculated values of F for various values of n by use of Equation D.20 and verified the validity of Equation D.21.

Agreement was also found to be good between the theoretical results of Nakano and Tien and the experimental results of Fararoui and Kinter (1961) for the CMC (carboxymethylcellulose) solution.

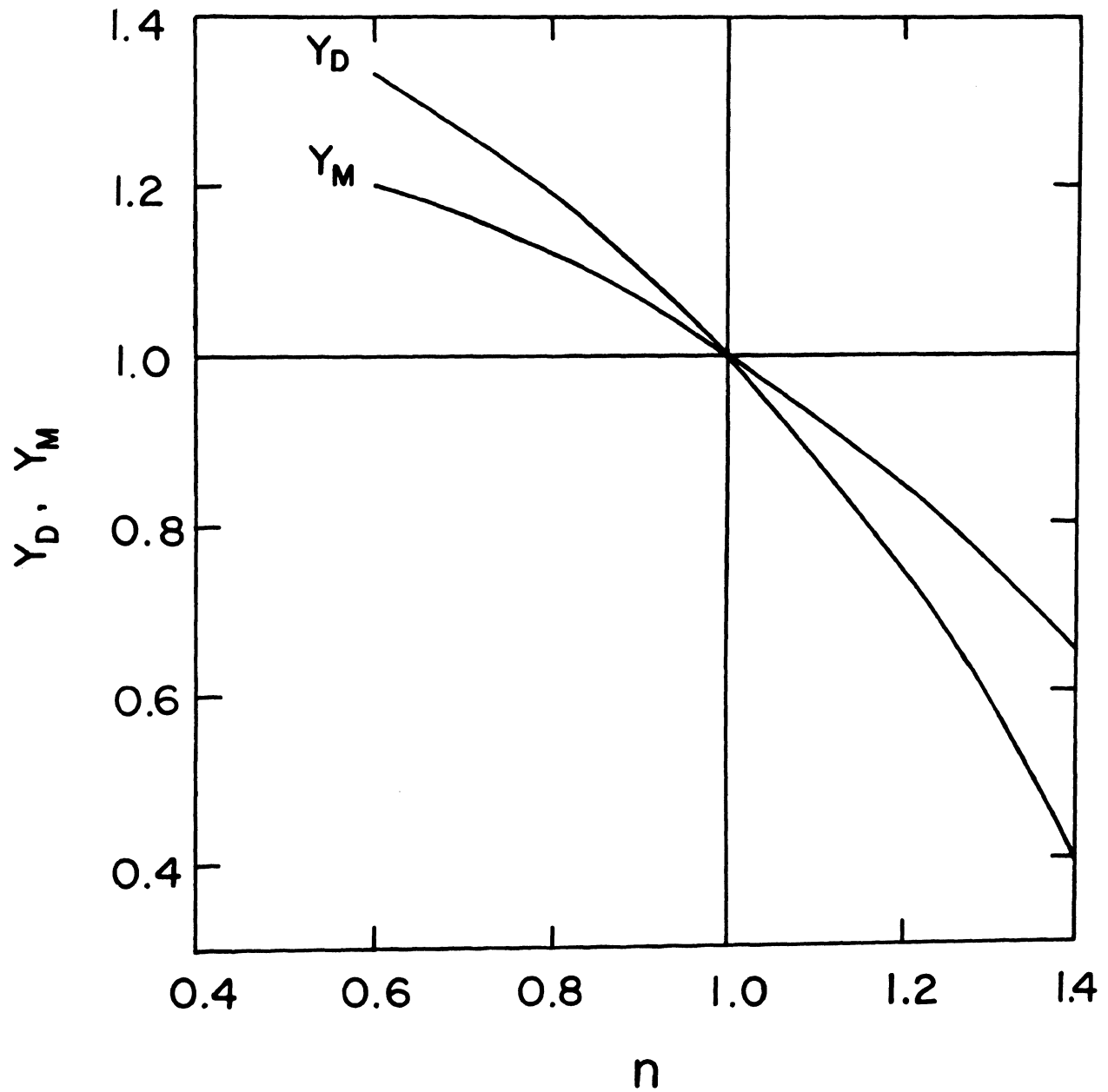


Figure D.1. Results on Bubble Drag and Mass Transfer in
Non-Newtonian Fluids (Hirose and Moo-Young, 1969)

Table D.I
Numerical Values of Coefficients of Stream Functions

X = 10					
n	1.00	0.90	0.80	0.70	0.60
σ	1.000	7.269(-1)	2.942(-1)	-6.937(-2)	-4.284(-1)
A_1	7.273(-1)	8.364(-1)	1.106(0)	1.532(0)	2.492(0)
A_2	-2.273(-1)	-3.364(-1)	-6.061(-1)	-1.032(0)	-1.992(0)
B_1	0.000	-9.439(-8)	-1.378(-7)	-1.816(-7)	-2.376(-7)
B_2	0.000	9.439(-8)	1.378(-7)	1.816(-7)	2.376(-7)
C_1	2.273(-2)	2.781(-2)	3.428(-2)	3.723(-2)	3.794(-2)
C_3	-2.273(-2)	-2.781(-2)	-3.428(-2)	-3.723(-2)	-3.794(-2)
D_1	0.000	-4.719(-8)	-6.892(-8)	-9.079(-8)	-1.188(-7)
D_2	0.000	1.888(-7)	2.757(-7)	3.632(-7)	4.751(-7)
D_3	0.000	-1.416(-7)	-2.068(-7)	-2.724(-7)	-3.564(-7)

X = 2.0					
n	1.00	0.90	0.80	0.70	0.60
σ	1.000	5.562(-1)	1.015(-1)	-1.334(-1)	-3.767(-1)
A_1	6.667(-1)	8.186(-1)	1.112(0)	1.400(0)	1.945(0)
A_2	-1.667(-1)	-3.186(-1)	-6.124(-1)	-9.005(-1)	-1.445(0)
B_1	0.000	-2.472(-7)	-3.179(-7)	-3.735(-7)	-4.539(-7)
B_2	0.000	2.472(-7)	3.179(-7)	3.735(-7)	4.539(-7)
C_1	8.333(-2)	1.130(-1)	1.347(-1)	1.432(-1)	1.437(-1)
C_3	-8.333(-2)	-1.130(-1)	-1.374(-1)	-1.432(-1)	-1.437(-1)
D_1	0.000	-1.236(-7)	-1.590(-7)	-1.868(-7)	-2.269(-7)
D_2	0.000	4.943(-7)	6.359(-7)	7.470(-7)	9.078(-7)
D_3	0.000	-3.707(-7)	-4.769(-7)	-5.603(-7)	-6.808(-7)

Table D.I cont'd
Numerical Values of Coefficients of Stream Functions

X = 1.0					
n	1.00	0.90	0.80	0.70	0.60
σ	1.000	4.767(-1)	7.258(-2)	-3.712(-1)	-3.851(-1)
A_1	6.250(-1)	7.782(-1)	1.017(0)	1.646(0)	1.711(0)
A_2	-1.250(-1)	-2.782(-1)	-5.175(-1)	-1.146(0)	-1.211(0)
B_1	0.000	-2.608(-7)	-3.363(-7)	-4.039(-7)	-4.022(-7)
B_2	0.000	2.608(-7)	3.363(-7)	4.039(-7)	4.022(-7)
C_1	1.250(-1)	1.754(-1)	2.043(-1)	2.326(-1)	2.240(-1)
C_3	-1.250(-1)	-1.754(-1)	-2.043(-1)	-2.326(-1)	-2.240(-1)
D_1	0.000	-1.304(-7)	-1.682(-7)	-2.020(-7)	-2.011(-7)
D_2	0.000	5.216(-7)	6.727(-7)	8.079(-7)	8.044(-7)
D_3	0.000	-3.912(-7)	-5.045(-7)	-6.059(-7)	-6.033(-7)

X = 0.1					
n	1.00	0.90	0.80	0.70	0.60
σ	1.000	6.795(-1)	2.795(-1)	-8.856(-2)	-4.815(-1)
A_1	5.227(-1)	5.656(-1)	6.665(-1)	8.560(-1)	1.379(0)
A_2	-2.273(-2)	-6.557(-2)	-1.665(-1)	-3.560(-1)	-8.789(-1)
B_1	0.000	-6.270(-8)	-6.589(-8)	-7.222(-8)	-7.173(-8)
B_2	0.000	6.270(-8)	6.589(-8)	7.222(-8)	7.173(-8)
C_1	2.273(-1)	2.750(-1)	3.326(-1)	3.599(-1)	3.925(-1)
C_3	-2.273(-1)	-2.750(-1)	-3.326(-1)	-3.599(-1)	-3.925(-1)
D_1	0.000	-3.135(-8)	-3.294(-8)	-3.611(-8)	-3.587(-8)
D_2	0.000	1.254(-7)	1.318(-7)	1.444(-7)	1.435(-7)
D_3	0.000	-9.405(-8)	-9.883(-8)	-1.083(-7)	-1.076(-7)

Table D.I cont'd
Numerical Values of Coefficients of Stream Functions

X = 0.01					
n	1.00	0.90	0.80	0.70	0.60
σ	1.000	6.898(-1)	3.191(-1)	-1.867(-1)	-2.774(-1)
A_1	5.025(-1)	5.392(-1)	6.215(-1)	8.867(-1)	9.765(-1)
A_2	-2.475(-3)	-3.923(-2)	-1.215(-1)	-3.867(-1)	-4.765(-1)
B_1	0.000	-7.052(-9)	-6.690(-9)	-7.722(-9)	-6.673(-9)
B_2	0.000	7.052(-9)	6.690(-9)	7.722(-9)	6.673(-9)
C_1	2.475(-1)	2.944(-1)	3.401(-1)	3.894(-1)	3.972(-1)
C_3	-2.475(-1)	-2.944(-1)	-3.401(-1)	-3.894(-1)	-3.972(-1)
D_1	0.000	-3.526(-9)	-3.345(-9)	-3.861(-9)	-3.336(-9)
D_2	0.000	1.410(-8)	1.338(-8)	1.544(-8)	1.335(-8)
D_3	0.000	-1.058(-8)	-1.003(-8)	-1.158(-8)	-1.001(-8)

X = 0.001					
n	1.00	0.90	0.80	0.70	0.60
σ	1.000	7.415(-1)	3.773(-1)	-7.167(-2)	-3.885(-1)
A_1	5.003(-1)	5.284(-1)	6.016(-1)	7.942(-1)	1.119(0)
A_2	-2.500(-4)	-2.894(-2)	-1.016(-1)	-2.942(-1)	-6.185(-1)
B_1	0.000	-7.020(-10)	-6.980(-10)	-6.705(-10)	-6.440(-10)
B_2	0.000	7.026(-10)	6.980(-10)	6.705(-10)	6.440(-10)
C_1	2.500(-1)	2.894(-1)	3.357(-1)	3.814(-1)	4.080(-1)
C_3	-2.500(-1)	-2.894(-1)	-3.357(-1)	-3.814(-1)	-4.080(-1)
D_1	0.000	-3.513(-10)	-3.490(-10)	-3.352(-10)	-3.220(-10)
D_2	0.000	1.405(-9)	1.396(-9)	1.341(-9)	1.288(-9)
D_3	0.000	-1.054(-9)	-1.047(-9)	-1.006(-9)	-9.661(-10)

Supplementary Nomenclature

C'	=	molar density of solution
C'_A	=	concentration of solute A
C_A^*	=	equilibrium concentration of solute A at interface
C_D	=	drag coefficient, defined in Equation D.5
C_i, D_i	=	coefficients in Equation D.16
C'_∞	=	bulk concentration of solute A
$f(\theta)$	=	$\left(\frac{\partial C}{\partial y}\right)_{y=1} \sin \theta$
N_A	=	molar diffusion flux of solute A
N_{AO}	=	interfacial molar diffusion flux of solute A
N_{Re}	=	Reynolds number, defined in Equation D.19, dimensionless
Re'	=	Reynolds number, defined in Equation D.6, dimensionless
r	=	radial displacement from center of sphere
t	=	contact time
V'	=	velocity
V'_r	=	radial velocity component
V'_ϕ	=	axial velocity component
V'_θ	=	tangential velocity component
$S_{i,j}$	=	defined in Equations B.23 through B.32
S_{2N}	=	defined in Equation B.34
Sh_{loc}	=	local Sherwood number
Y_D	=	correction factor, defined in Equation D.20
Y_M	=	correction factor, defined in Equation D.9
Z	=	$\cos \theta$

Supplementary Greek Letters

$\alpha_{i,j}$ = defined in Equations B.23 through B.32

ψ = stream function

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Vita

Cheng-chun Huang was born on February 27, 1942, in Yilan, Taiwan, Republic of China.

He entered Tunghai University in Taichung, Taiwan in September 1959 and graduated in June 1963 with a Bachelor of Science Degree in Chemical Engineering.

After graduation he joined the Chinese Petroleum Corporation, working for one year as an operating engineer and then two years as a process engineer in Kaohsiung Oil Refinery, Kaohsiung, Taiwan, a subsidiary of Chinese Petroleum Corporation.

He enrolled at the University of Missouri - Rolla in September 1966 and received his Master Degree in Chemical Engineering in February 1968. He is a member of Phi Kappa Phi and the American Institute of Chemical Engineers.

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