An introduction to linear time-variant digital filtering of seismic data

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AN INTRODUCTION TO LINEAR TIME-VARIANT DIGITAL FILTERING OF SEISMIC DATA

A dissertation
Presented to
the Faculty of the Graduate School
University of Missouri at Rolla

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

by
Richard Harold Lassley
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ABSTRACT

The exploration geophysicist is constantly searching for new and better methods for analyzing seismic data. The primary purpose of this dissertation is to introduce linear time-variant digital filters as a technique for the filtering of seismic data.

Methods of characterizing linear time-variant digital filters are discussed in general terms. The advantages of the different impulse and frequency responses are cited. In addition, the meanings and possible interpretations of the time and frequency variables introduced are considered.

The concept of linear time-variant digital delay filters is presented along with their relationship to impulse responses. A pictorial diagram of discrete time-variant impulse responses is included for clarity. Frequency domain concepts are discussed and elucidated with a simple example. In general, the amplitude characteristic and phase-lag characteristic, illustrated in the example, are functions of both frequency and time (i.e., the instant of observation).

The optimization of linear time-variant digital filters is carried out for a nonstationary random input. The ensemble mean-square error criterion is used, assuming that the autocorrelation of the input and the crosscorrelation of the input with the desired output are known.
It is concluded that linear time-variant digital filters will extract additional information from the seismic data but with quite a large increase in computation.
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CHAPTER I

INTRODUCTION

In the search for oil the exploration geophysicist is constantly searching for new and better methods for processing seismic data. To locate smaller and deeper oil traps, greater refinement is necessary to gain the utmost information from the seismic record. The purpose of this dissertation is to introduce linear time-variant digital filters as a technique for the analysis of seismic data.

The application of linear time-variant digital filters to the processing of geophysical data has not, to the writer's knowledge, appeared in the exploration seismic literature. It was not until recently that linear time-invariant digital filters have been utilized on seismic reflection records. Wadsworth, et al(45)\(^1\) introduced the use of linear operators as a means of detecting seismic reflections. More recently, Treitel and Robinson(42),(44) have established a firm foundation for the use of linear time-invariant digital filters on seismic data.

The ground work for the analysis of linear time-invariant continuous systems was laid down in a classical work by Wiener(45) in 1942. In Wiener's theory the assumption is made that the input to a linear system consists of a

\(^1\) A number in parenthesis refers to a number in the Bibliography.
random signal plus a random noise, each of which is stationary in the sense that its statistical properties do not vary with time. Levinson\(^{(30)}\)(\(^{(32)}\)) gave a heuristic discussion of Wiener's work and presented the theory in terms that were easier to comprehend. Bode and Shannon\(^{(4)}\) also used simplified methods to further the understanding of Wiener's smoothing and prediction theory. More recently, Lee\(^{(30)}\) has written an excellent book which gives a clear and concise treatment of statistical communication theory. It deals with the optimization of continuous linear constant-coefficient systems.

In the early 1950's sampled-data theory began to be employed in the analysis and synthesis of linear time-invariant systems. With the development of sampled-data systems, statistical design techniques have been adapted to the case where the input is observed only at discrete instants of time. Franklin\(^{(12)}\) considered the optimum synthesis of a sampled-data feedback control system with the sum of a stationary random message and a stationary random noise as its input. This problem is the sampled-data counterpart of the Wiener problem. The control system problem has also been investigated by Chang\(^{(8)}\). However, as with most sampled-data problems, his synthesis procedure is carried out mostly in the z-domain as contrasted to Franklin's time-domain approach. Jury\(^{(20)}\)(\(^{(21)}\)) has organized the z-domain analysis into a mathematical theory and, in so doing, has increased its effectiveness as a tool in the analysis of
sampled-data systems. Ragazzini and Franklin(36) treat sampled-data control systems in some detail utilizing the z-transform. Recently, Treitel and Robinson(44) have applied the z-transform in the discussion of the stability of digital filters with seismic wavelets as inputs.

In the application of statistical filter theory to seismic data a somewhat different approach must be used. This is primarily due to the fact that in seismic reflection work the signal is not well defined. The earth is excited by a dynamite explosion (or other means) which is somewhat impulsive but not well known analytically. In addition, the mechanics of its propagation through the earth is not completely understood. Thus, the geophysicist must try to interpret a seismic record which is the combined response of the earth and recording equipment to an input which is not well defined. The problem is difficult.

Several approaches have been taken in attempting to decompose the seismic trace into its constituent components. Ricker(40) attempted the inverse filtering or "deconvolution" problem by electrical means. Robinson(41) used digital computer methods to establish a workable technique which he called "predictive decomposition." Since then Foster, Hicks, and Nipper(11); George, Smith, and Bostick(14); and Rice(38) have extended the concept to various types of geophysical problems.
Linear time-variant systems are a generalization of linear time-invariant systems. Since it has been a natural tendency to generalize existing theories, the study of linear time-variant systems was the logical area of study.

Zadeh(47) seems to have been the first to investigate linear time-variant networks. His approach was to analyze the system in the frequency domain. Zadeh generalized these concepts in 1952(48). Booten(6) generalized Wiener's optimization theory to apply to nonstationary input functions and continuous time-varying desired operations. Aseltine (1) presented a method for transforming linear time-varying systems. Several writers (Boxer and Thaler(7); Koschmann (24); Rosenbloom, Heilbron, and Trautman(45); Zadeh(47), et al) have investigated time-varying networks which can be expressed by a differential equation. Friedland(13) investigated time-varying sampled-data systems. These investigations were carried out for linear systems to be used in such fields as electronic control problems, radar, communication channels, etc.

It is the primary purpose of this thesis to introduce linear time-variant digital filters as a method for processing seismic data.

The development begins with a short note on the sampling process followed by the characterization of linear time-variant digital systems in general. The impulse response and frequency response are discussed with several
new means of representing them as well as an interpretation of the variables introduced. An investigation of linear time-variant digital "delay" filters is included as well as their relationship to the impulse responses. In addition, the frequency domain concepts of amplitude characteristic and phase characteristic for the time-variant case are examined in detail. Consideration is given to the problem of optimizing the linear time-variant digital filter when the input is corrupted by unwanted signals (or noise). The ensemble mean-square-error criterion is used in the optimization procedure. A brief summary and several conclusions conclude the dissertation.
CHAPTER II

CHARACTERIZATION OF LINEAR TIME-VARIANT DIGITAL SYSTEMS

A time-variant (or time-varying, or time-variable) digital system is one whose input-output relationship is not invariant under translations in time. By digital (or discrete) we mean that a continuous function has been sampled at discrete points in time and, thus, the input to the system appears as a sequence of impulses or numbers. If, in addition, the superposition principle holds for the system, we have a linear digital system.

The most commonly used methods of characterizing a linear time-variant system, that is, of specifying its input-output relationship, are those that give (a) the differential equation of the system, (b) the impulse response of the system, (c) the frequency response of the system, and (d) special techniques that are valid only for particular classes of systems. This paper, however, will concern itself with the impulse and frequency response only.

In this chapter we include a very brief note on the sampling process. Following this we shall discuss some impulse responses and frequency responses of linear time-variant digital systems. The meanings and possible interpretations of the time and frequency variables that are introduced are considered.
The Sampling Process

Since the seismic trace is usually recorded in the field as a continuous time function, say \( x(t) \), and then converted into a sequence of numbers at equally spaced points in time, a discussion of the sampling process will serve as a background for the material which follows. On this digitized record each number represents the value of the function \( x(t) \) at its particular sampling instant.

Because an extremely narrow pulse represents the physical situation accurately at the sampling instant, and also because of the resultant mathematical simplifications, a common form of sampling-function element is an impulse, or Dirac, delta function, \( \delta(t) \). Mathematically, then, the process may be represented by the expression

\[
2.1 \quad x^*(t) = x(t) \delta_T(t)
\]

where the superscript \( * \) is used to indicate that the function \( x^*(t) \) is a discrete, sampled function, and \( \delta_T(t) \) represents a periodic train of unit impulses spaced \( T \) seconds apart, that is,

\[
2.2 \quad \delta_T(t) = \sum_{n} \delta(t-nT)
\]

If \( x(t) = 0 \) for \( t < 0 \), Equation 2.1 becomes

\[
2.3 \quad x^*(t) = \sum_{n=0}^{\infty} x(nT) \delta(t-nT)
\]
Since, throughout the paper, it is assumed that the input has already been digitized the superscript * is omitted and the input sequence denoted by \( x(nT) \), \( n = 0, 1, 2, \ldots \).

**The Impulse Response and the Frequency Response**

The impulse response of a linear time-variant system is defined as \( h'(nT,jT) \), i.e., the response to a unit impulse input at time \( jT \) measured at time \( nT \). Throughout this paper we shall consider the sampling interval as being constant and equal to one unit of time, that is, \( T = 1 \) and, hence, the impulse response becomes \( h'(n,j) \).

Let us develop the input-output relationship, in the time domain, for a linear time-variant digital system, employing the impulse response concept.

Figure 1(a) illustrates the response \( h'(n,0) \) of the system \( N \) to a unit impulse input at time zero. The system \( N \) is assumed to be a physically realizable one. This means that the system is at rest prior to the time of the unit impulse input. In the present case this means that \( h'(n,0) = 0 \) for \( n \) less than zero.

In Figure 1(b) the unit impulse is delayed by \( s \) seconds, therefore, the impulse response \( h'(n,s) \) will be delayed \( s \) seconds. In this case the realizability condition dictates \( h'(n,s) = 0 \) for \( n \) less than \( s \). Note that the impulse response \( h'(n,s) \) is generally not of the same form.
as the response \( h'(n,0) \). They would be identical in form if \( N \) were a time-invariant system.

Now consider a general input sequence \( x(n), n = -L, -L+1, -L+2, \ldots \). Each element \( x(j) \) in the sequence \( x(n) \) can be considered as an impulse of amplitude \( x(j) \). Thus, the response to this impulse will be \( x(j)h'(n,j) \) (Figure 1(c)). By adding up the responses due to all the elements of \( x(n) \) we obtain an expression for \( y(n) \), the output at time \( n \):

\[
2.4 \quad y(n) = \sum_{j=-L}^{n} h'(n,j)x(j)
\]

Equation 2.4 is the discrete time-variant superposition summation or convolution summation.

The limits on the summation of Equation 2.4 can be extended from minus infinity to plus infinity since \( h'(n,j) = 0 \) for \( n \) less than \( j \). Thus,

\[
2.5 \quad y(n) = \sum_{j=-\infty}^{\infty} h'(n,j)x(j)
\]

In a time-invariant system \( h'(n,j) \) would be a function of \( (n-j) \) only, and not of \( n \) and \( j \) separately.

The frequency response, or transfer function is defined by Zadeh(47) in the continuous case as

\[
2.6 \quad H(\omega,t) = \int_{-\infty}^{\infty} h(t,u)e^{-i\omega(t-u)}du
\]

For the sampled-data case Equation 2.6 reduces to

\[
2.7 \quad H'(\omega,n) = \sum_{j} h'(n,j)e^{-i\omega(n-j)}
\]
Figure 1. Block Diagrams for the Derivation of the Input-Output Relationship in the Time Domain.
where the notation $\sum_j$ indicated a sum over all $j$ from minus infinity to plus infinity$^1$. $H'(\omega,n)$ is called the pulsed frequency response, or pulsed transfer function.

Using Equation 2.5 with $x(n) = e^{i\omega n}$, we have

$$2.9 \quad H'(\omega,n) = \sum_j h'(n,j)e^{-i\omega(n-j)}$$

or

$$2.10 \quad H'(\omega,n) = \frac{\text{response of the filter } N \text{ to } e^{i\omega n}}{e^{i\omega n}}$$

which explains why it is called the frequency response function. For later comparisons, it is convenient to introduce

$$2.11 \quad H'(\omega,\mu) = \sum_n \hat{H}'(\omega,n)e^{-i\mu n}$$

Other Forms for the Impulse Response

In the function $h'(n,j)$ the realizability condition is that the response be identically zero for $n$ less than $j$. This constraint involves both the variables $n$ and $j$ and, therefore, is often inconvenient to use. In alternate forms of the impulse response which will presently be described, the realizability condition involves only one variable. Another feature is that these forms will exhibit

---

$^1$When limits of summation are not explicitly indicated this notation will be used throughout the paper.
Figure 2. Block Diagram of $h''(q,j)$. 

Figure 3. Block Diagram of $h'''(k,n)$. 

Figure 4. A Plot of the Impulse Response $h''(q,j)$. 
direct Fourier transform relationships between the frequency and impulse response functions.

We define

\[ h''(q,j) = \text{response to a unit impulse input at time } j, \text{ measured at time } n = j + q. \]

\[ h'''(k,n) = \text{response measured at time } n \text{ to a unit impulse input at time } n - k. \]

Thus, \( q \) measures elapsed time, and \( k \) measures the age of the input. The realizability conditions are zero response for \( q \) less than zero and \( k \) less than zero, respectively. These are illustrated in Figures 2 and 3.

Of course, \( h'(n,j) \), \( h''(q,j) \), and \( h'''(k,n) \) must all be related. The rules governing transformation from one form to another are given later. They are derived from the relations \( q = n - j = k \) between the time domain variables \( q, n, j, \) and \( k \).

In terms of \( h''(q,j) \) the operation of the linear time-variant system can be conveniently pictured as in Figure 4, which displays on a \( qj \) plane several system responses to impulse inputs at different times \( j \). Notice, again, that the variable \( q \) in \( h''(q,j) \) refers to duration of the input time sequence. If we had a fixed system, the response to a unit impulse at time \( j = 1 \) would be the same as the response to a unit impulse at time \( j = 2 \). In terms of Figure 4, then, it would appear that the variation of \( h''(g,j) \) with \( j \), for fixed \( q \), would be a measure of the
rate of variation of the system. We could find the discrete Fourier transform of \( h''(q,j) \) with respect to \( j \) for fixed \( q \),

\[
2.12 \quad H''(q,\mu) = \sum_j h''(q,j)e^{-j\mu j}
\]

and the variable \( \mu \) would be a frequency domain measure of the variation of the system. If \( \mu \) were confined to low values, the system would be varying slowly; if there were high frequencies in the \( \mu \) domain, the system would be varying rapidly.

We may also define discrete Fourier transforms with respect to \( q \), keeping \( j \) fixed:

\[
2.13 \quad H''(\nu,j) = \sum_q h''(q,j)e^{-j\nu q}
\]

and

\[
2.14 \quad H''(\nu,\mu) = \sum_j H''(\nu,j)e^{-j\mu j}
\]

\[
2.15 \quad = \sum_q H''(q,\mu)e^{-j\nu q}
\]

\[
2.16 \quad = \sum_q \sum_j h''(q,j)e^{-j\nu q}e^{-j\mu j}
\]

Similarly for \( h'''(k,n) \) we obtain by direct Fourier transformation the set of functions

\[
2.17 \quad H'''(\omega,n) = \sum_k h'''(k,n)e^{-j\omega k}
\]

\[
2.18 \quad H'''(k,\mu) = \sum_n h'''(k,n)e^{-j\mu n}
\]
Equations 2.17 and 2.19 are exactly equivalent to \( H'(\omega, n) \) and \( H'(\omega, \mu) \). This is because, by using Equation 2.5, we can write

\[
2.22 \quad H''''(\omega, n) = \frac{\text{response of } N \text{ to } e^{i\omega n}}{e^{i\omega n}} = H'(\omega, n)
\]

and therefore, by virtue of Equations 2.11 and 2.20

\[
2.23 \quad H''''(\omega, \mu) = H'(\omega, \mu)
\]

Thus, the frequency variables for \( h''''(k, n) \) have the same significance as those for \( h'(n, j) \), and therefore, we have used the same symbols, \( \omega \) and \( \mu \), in both cases. However, note that \( H''''(\omega, n) \) and \( h''''(k, n) \) are related directly by Fourier transform, which is not true of \( H'(\omega, n) \) and \( h'(n, j) \).

Finally, the form of the input-output relation (Appendix A)

\[
2.24 \quad y(n) = \sum_{k=0}^{\infty} h''''(k, n)x(n-k)
\]

suggests that \( h''''(k, n) \) can be interpreted as a weighing function by which the signal inputs in the past must be
multiplied to determine their contributions to the present output. The realizability condition, \( h'''(k,n) = 0 \) for \( k \) less than zero, thus reflects the fact that the system cannot weigh portions of the input that have yet to occur.

**Summary**

For convenience, we now list the relationships between the various forms of impulse and frequency response that we have introduced and also give the convolution summations connecting the input and output relations. Proofs of all but the most immediately evident relations listed here are given in Appendix A.

**a. Transformations between** \( h'(n,j) \), \( h''(q,j) \), \( h'''(k,n) \)

2.25 \( h'(n,j) = h''(n-j,j) \)
2.26 \( h''(q,j) = h'(q+j,j) \)
2.27 \( h'''(q,j) = h'''(q,q+j) \)
2.28 \( h'''(k,n) = h''(k,n-k) \)
2.29 \( h'''(k,n) = h'(n,n-k) \)
2.30 \( h'(n,j) = h'''(n-j,n) \)

**b. Input-output relations — time domain**

2.31 \( y(n) = \sum_{j=-\infty}^{n} h'(n,j)x(j) = \sum_{j=0}^{\infty} h'(n,n-j)x(n-j) \)
2.32 \( y(n) = \sum_{q=-\infty}^{n} h''(n-q,q)x(q) = \sum_{q=0}^{\infty} h''(q,n-q)x(n-q) \)
2.33 \( y(n) = \sum_{k=-\infty}^{n} h'''(n-k,n)x(k) = \sum_{k=0}^{\infty} h'''(k,n)x(n-k) \)
c. Transformations between $H'(\omega, \mu)$, $H''(\nu, \mu)$, $H'''(\omega, \mu)$

2.34 $H'(\omega, \mu) = H''(\omega+\mu, \mu) = H'''(\omega, \mu)$

2.35 $H''(\nu, \mu) = H'''(\nu-\mu, \mu) = H(\nu-\mu, \mu)$

2.36 $H'''(\omega, \mu) = H'(\omega, \mu) = H''(\omega+\mu, \mu)$

d. Interpretations of the variables

Time domain

- $n$: time index corresponding to instant of observation
- $j$: time index corresponding to instant of impulse input
- $q$: time index corresponding to elapsed time
- $k$: time index corresponding to age of input

Frequency domain

- $\nu$: variable corresponding to system variation
- $\omega$: variable corresponding to input frequencies
- $\nu$: variable corresponding to output frequencies
CHAPTER III

LINEAR TIME-VARIANT DIGITAL DELAY FILTERS

In recent years there has been an increasing use of the digital computer as a means of filtering geophysical data. The seismic trace, for example, is usually recorded in the field as a continuous function of time and then converted into a sequence of numbers at equally spaced points in time. The filtering techniques are then applied to this "digitized" record, utilizing a digital computer.

In the previous chapter we dealt with the linear time-variant digital filters in a more or less general sense. Now let us investigate them in a somewhat different light and in more detail.

Digital Delay Filters

Using the format that Robinson and Trietel(42) employed for linear time-invariant digital filters we shall develop expressions for linear time-variant digital delay filters.

In their development $z$ is defined as a unit delay filter (or operator), $z^2$ a two-unit delay filter, and $z^N$ a N-unit delay filter. Reference to Figure 5 shows that the output from a unit delay operator is the input delayed one unit of time. Similarly, the output from a two-unit delay operator is the input delayed two units in time.
Finally, a N-unit delay operator delays the input N units of time.

Now what happens when we connect in series with these z operators a filter with time-varying coefficients? Before answering this question the input sequence shall be clearly defined.

\[ y(n) = x(n-1) \]

\[ y(n) = x(n-2) \]

\[ y(n) = x(n-N) \]

Figure 5. Block Diagrams of Delay Operators.

We shall consider the input to be a stable, positive, finite length sequence. This means that \( x(n) \) satisfies the conditions

\[ \sum_{n} |x(n)| < c \]

where \( c \) is a finite constant, and

\[ x(n) = 0, \quad \text{for } n < 0 \text{ and } n > M \]
Equation 3.1 expresses the stability condition and Equation 3.2 the positive, finite length condition.

We now return to the question of delay operators in series with time-variant filters.

First, however, we consider the action of a time-variant filter $h(0,n)$ which is illustrated in the block diagram of Figure 6. In effect $h(0,n)$ acts as a scaling filter with each point of the input being scaled by a different magnitude.

$$x(n) \quad h(0,n) \quad y(n) = h(0,n)x(n)$$

Figure 6. Block Diagram of $h(0,n)$.

Unlike the constant-coefficient filter, the output from a series combination of a time-variant filter and a delay operator will depend on which one the input encounters first. For example, consider the unit delay operator in series with the time-variant filter $h(1,n)$ as illustrated in Figure 7. In the first case, Figure 7(a), the time-variant filter acts on the input and then the result is delayed one unit of time. In the second case, Figure 7(b), only the input is delayed and the time-variant filter acts on this delayed input.

Let us distinguish between the two time-variant delay filters as follows:
Type 2 delay filter: the time-variant filter precedes the delay operator

Type 3 delay filter: the time-variant filter follows the delay operator

Thus, Figure 7(a) depicts a Type 2 first-order time-variant delay filter. Figure 7(b) exhibits a Type 3 first-order time-variant delay filter. The choice of symbols will become clear later.

Consider now the case of the time-variant filter $h(0,n)$ connected in parallel with a Type 2 or Type 3 delay filter. For example, consider the parallel arrangements shown in Figure 8. Figure 8(a) has $h(0,n)$ connected in parallel with a Type 2 first-order delay filter and Figure 8(b) shows a parallel arrangement of $h(0,n)$ with a Type 3 first-order delay filter.

Proceeding in a similar manner, a general time-variant delay filter can be built up for each of the two types of delay filters, $H_2^*$ and $H_3^*$. For filter $H_2^*$, the Type 2 delay filter, the most general Nth-order time-variant delay filter would be

$$
H_2^*(z,n) = h(0,n) + h(1,n-1)z + h(2,n-2)z^2 + \cdots + h(N,n-N)z^N
$$

and for the filter $H_3^*$

$$
H_3^*(z,n) = h(0,n) + h(1,n)z + h(2,n)z^2 + \cdots + h(N,n)z^N
$$
\[ x(n) \xrightarrow{h(1,n)} h(1,n)x(n) \xrightarrow{z} y(n) = h(1,n-1)x(n-1) \]

(a)

\[ x(n) \xrightarrow{z} x(n-1) \xrightarrow{h(1,n)} y(n) = h(1,n)x(n-1) \]

(b)

**Figure 7.** Block Diagrams of the Unit Delay Operator and \( h(1,n) \) in Series.

\[ x(n) \xrightarrow{h(0,n)} h(0,n)x(n) \xrightarrow{z} y(n) = h(0,n)x(n) + h(1,n-1)x(n-1) \]

(a)

\[ x(n) \xrightarrow{h(0,n)} h(0,n)x(n) \xrightarrow{z} h(1,n)x(n) \xrightarrow{z} y(n) = h(0,n)x(n) + h(1,n)x(n-1) \]

(b)

**Figure 8.** Block Diagrams of First-Order Time-Variant Delay Filters.
\[ H^*_3(z,n) = \sum_{k=0}^{N} h(k,n) z^k \]

To introduce some terminology, we will call Equation 3.3 the z-transform of \( h(q,n-q) = h(q,j) \) with respect to \( q \). Similarly, 3.4 represents the z-transform of \( h(k,n) \) with respect to \( k \). A cursory introduction to z-transform theory is given in Appendix C.

The two \( N \)th-order time-variant delay filters, \( H^*_2 \) and \( H^*_3 \), may be illustrated in block diagrams as shown in Figure 9. Filters \( H^*_2 \) and \( H^*_3 \) are displayed in Figure 9(a) and 9(b) respectively. Figure 9 also serves to illustrate the action of the two filters on an input given by the equally spaced sampled values \( x(0), x(1), \cdots, x(M) \).

Writing the output in summation form we have

\[ 3.5 \quad y(n) = \sum_{q=0}^{N} h(q,n-q)x(n-q), \]

for \( n = 0, 1, \cdots, M+N \)

\[ = 0 \quad \text{, otherwise} \]

\[ 3.6 \quad y(n) = \sum_{k=0}^{N} h(k,n)x(n-k), \]

for \( n = 0, 1, \cdots, M+N \)

\[ = 0 \quad \text{, otherwise} \]

for filters \( H^*_2 \) and \( H^*_3 \) respectively. Notice that \( M \) corresponds to the length of the input \( x(n) \) and \( N \) refers to the memory length of the filter.

Now, comparing Equations 3.5 and 3.6 with Equations 2.32 and 2.33 of Chapter II, i.e.
we find that Equations 3.5 and 3.6 are just the finite version of Equations 3.7 and 3.8 respectively. Thus, the impulse response function \( h''(k,n) \) is the same as the coefficients of the Nth-order time-variant delay filter \( H_3^* \). Similarly, the impulse response function \( h''(q,j) \) is identical to the coefficients of the Type 2 Nth-order time-variant delay filter \( H_2^* \).

For the remainder of the paper we shall designate \( h''(q,j) \) as \( h_2(q,j) \) and \( h''(k,n) \) as \( h_3(k,n) \).

The time-variant delay filter, or impulse response function, concept may be illustrated in a form which will look more familiar to a geophysicist.

In Figure 10(a) is a plot of a spike record. The spikes are equal distant apart and all of the same amplitude. Figure 10(b) illustrates the results of convolving the spike record with a symmetric Ricker wavelet(39) whose wavelet breadth is changing as we proceed "down" the record from left to right.

For this particular plot, at record time zero, corresponding to the first spike, the Ricker wavelet has a wavelet breadth \( b \) of 17.75 milliseconds. The breadth is increased by equal increments until the final wavelet
(a) Type 2 Nth-Order Time-Variant Delay Filter.

\[ y(n) = h(0, n)x(n) + h(1, n-1)x(n-1) + h(2, n-2)x(n-2) + \cdots + h(N, n-N)x(n-N) \]

(b) Type 3 Nth-Order Time-Variant Delay Filter.

\[ y(n) = h(0, n)x(n) + h(1, n)x(n-1) + h(2, n)x(n-2) + \cdots + h(N, n)x(n-N) \]

Figure 9. Block Diagrams of Nth-Order Time-Variant Delay Filters.
breadth is \( b = 30.00 \) milliseconds. Thus, the wavelets all have the same shape but are getting progressively broader.

In effect the first wavelet displayed in Figure 10(b) can be visualized as the response to a spike (or impulse) input at time \( t = 0 \).

Subsequent wavelets then may be considered as the response to spike inputs at the corresponding delay times. If we consider the impulse response \( h_2(q,j) \), the response to an impulse at time \( j \) measured at time \( q + j \), the first wavelet would correspond to \( h_2(q,0) \), \( q = 0, 1, 2, \cdots, 25 \), where 25 milliseconds is the length of the wavelet. The second wavelet would correspond to \( h_2(q,26) \), \( q = 0, 1, 2, \cdots, 25 \), etc. Obviously, then, we have one form of a time-variant impulse response or delay filter.

Figure 11 shows a similar plot. The same spike record was used but the wavelet was made to vary in a different manner (see Appendix B). The breadth was held constant at \( b = 25 \) milliseconds but the shape was made to vary by equal increments from an initial form, the first wavelet, to the final form, a symmetric Ricker wavelet. Here, then, is another form of a time-variant impulse response function or delay filter.

**Frequency Domain Concepts**

Many people in seismic work are more accustomed to considering filtering in the frequency domain. Therefore,
(a) Spike Record.

(b) Convolution of Spike Record with Time-Varying Wavelet.

Figure 10. Illustrating the Impulse Response.
(a) Spike Record.

(b) Convolution of Spike Record with Time-Varying Wavelet.

Figure 11. Another Display of the Impulse Response.
the action of time-variant digital filters in the frequency domain will now be considered. Some of the analogies between time-variant filters will be cited.

In this section the Type 3 time-variant delay filter, $H_3^*$, will be utilized to illustrate the discussion, i.e. the filter with impulse response function $h_3(k,n)$.

Recall from Chapter II, Equation 2.22,

3.9 \( \tilde{H}''''(\omega,n) = H_3(\omega,n) = \frac{\text{response of } N \text{ to } e^{i\omega n}}{e^{i\omega n}} \)

as the definition of the frequency response or transfer function. Writing Equation 3.9 in polar form we have

3.10 \( H_3(\omega,n) = A(\omega,n)e^{i\phi(\omega,n)} \)

where $A(\omega,n)$ is the magnitude of the transfer function at each instant of time $n$, that is, $A(\omega,n) = |H_3(\omega,n)|$, and is called the amplitude characteristic of the time-variant delay filter. $\phi(\omega,n)$, the phase angle of the time-variant filter, is called the phase characteristic of the filter. In general, as will be shown, both the amplitude characteristic and phase characteristic are functions of the frequency and time. This differs from the time-invariant filter where, in general, the amplitude and phase characteristics depend only on the frequency.

We first consider the action of the time-variant delay filter $h_3(0,n)$, whose $z$-transform with respect to $k$ is $H_3^*(z,n) = h_3(0,n)$, on the sinusoidal input $e^{i\omega n}$. 
Reference to Figure 9 shows that the output will be

3.11 \[ y(n) = h_3(0,n)e^{i\omega n} \]

and hence

3.12 \[ H_3(\omega,n) = \frac{y(n)}{e^{i\omega n}} = h_3(0,n) \]

Thus, comparing Equation 3.12 with Equation 3.10, the amplitude characteristic is \( A(\omega,n) = h_3(0,n) \) and the phase characteristic is \( \phi(\omega,n) = 0 \) for all \( \omega \) and \( n \). In other words, the input and output are in phase, but the output has a time-varying amplitude.

Next, consider the action of the time-variant delay filter whose \( z \)-transform with respect to \( k \) is \( H^*_3(z,n) = h_3(1,n)z \) on the input \( e^{i\omega n} \).

Again referring to Figure 9 we have

3.13 \[ y(n) = h_3(1,n)e^{i\omega(n-1)} \]

and

3.14 \[ H_3(\omega,n) = \frac{y(n)}{e^{i\omega n}} = h_3(1,n)e^{-i\omega} \]

In this case the amplitude characteristic is \( A(\omega,n) = h_3(1,n) \) and the phase characteristic is \( \phi(\omega,n) = -\omega \).

Thus, the input and output are out of phase by \( -\omega \). The output again has a time-varying amplitude but independent of the frequency.

Finally, consider the time-variant delay filter with \( z \)-transform \( H^*_3(z,n) = h_3(0,n) + h_3(1,n)z \). Its action on
the input $e^{i\omega n}$ will be given by

$$y(n) = h_3(0,n)e^{i\omega n} + h_3(1,n)e^{i\omega(n-1)}$$

and, hence, the transfer function is

$$H_3(\omega,n) = h_3(0,n) + h_3(1,n)e^{-i\omega}$$

By writing Equation 3.16 in the form

$$H_3(\omega,n) = h_3(0,n) + h_3(1,n)\cos \omega - ih_3(1,n)\sin \omega$$

one can readily see that the amplitude characteristic is given by

$$A(\omega,n) = \sqrt{(h_3(0,n) + h_3(1,n)\cos \omega)^2 + (h_3(1,n)\sin \omega)^2}$$

$$= \sqrt{h_3^2(0,n) + h_3^2(1,n) + 2h_3(0,n)h_3(1,n)\cos \omega}$$

and the phase characteristic by

$$\phi(\omega,n) = \arctan \frac{-h_3(1,n)\sin \omega}{h_3(0,n) + h_3(1,n)\cos \omega}$$

Both the amplitude characteristic and the phase characteristic depend on the frequency $\omega$ and the instant of observation $n$ in this case. We can further conclude, therefore, that the amplitude characteristic and phase characteristic depend on frequency and time in general.

We might notice that if we consider a general time-variant delay filter with z-transform

$$H_3^*(z,n) = \sum_{k=0}^{N} h_3(k,n)z^k$$

and make the substitution $e^{-i\omega} = z$ we formally obtain the filter's transfer function, namely (see Appendix C)
3.21 \quad H_3(\omega,n) = \sum_{k=0}^{N} h_3(k,n)e^{-i\omega k}

A simple example will help to illustrate the foregoing discussion.

Let us consider the time-variant delay filter with z-transform

3.22 \quad H^*_3(z,n) = h_3(0,n) + h_3(1,n)z

and let \( h_3(k,n) = 1/n+k+1 \), i.e., \( h_3(0,n) = 1/n+1 \) and \( h_3(1,n) = 1/n+2 \). Utilizing Equation 3.20 and 3.21 the transfer function is obtained from Equation 3.22, that is,

3.23 \quad H_3(\omega,n) = \frac{1}{n+1} + \frac{1}{n+2}e^{-i\omega}

Equation 3.18 gives for the amplitude characteristic

3.24 \quad A(\omega,n) = \sqrt{(1/n+1)^2 + (1/n+2)^2 + (2\cos \omega/(n+1)(n+2))}

and the phase characteristic obtained from Equation 3.19 is

3.25 \quad \phi(\omega,n) = \arctan \frac{-1/n+2 \sin \omega}{1/n+1 + 1/n+2 \cos \omega}

or

3.26 \quad \phi(\omega,n) = \arctan \frac{-\sin \omega}{n+2/n+1 + \cos \omega}

A graph of the amplitude characteristic \( A(\omega,n) \) versus frequency is shown in Figure 12. A family of curves results, one for each value of \( n \). Thus, the filter has a particular amplitude characteristic for each time of observation \( n \). Figure 12 displays five of these curves corresponding to \( n = 0, 1, 2, 3, 4 \) in Equation 3.24.
Figure 12. The Amplitude Characteristic Versus Frequency
Figure 13 is a plot of $-\phi(\omega, n)$ versus the frequency $\omega$. $-\phi(\omega, n)$ has been referred to as the "phase lag" characteristic when the filter is time-invariant or independent of the time $n$. Thus, we have a "time-variant phase lag" characteristic. Designating $Q(\omega, n)$ as the time-variant phase lag characteristic of the filter $H^*_3$, we have from Equation 3.26

$$3.27 \quad Q(\omega, n) = -\phi(\omega, n) = \arctan \frac{\sin \omega}{n+2/n+1 + \cos \omega}$$

Again, a family of curves result, one for each value of $n$. Thus, the filter also has a phase lag characteristic corresponding to each instant of observation $n$. Figure 13 exhibits five of these curves corresponding to $n = 0, 1, 2, 3, 4$.

The primary difference between a time-invariant filter and a time-variant system, then, is that the constant-coefficient filter has only one amplitude characteristic and one phase characteristic which holds for all $n$, while the time-variant filter has an amplitude characteristic and a phase characteristic for each instant of observation $n$. It would appear that much greater refinement can be obtained with a time-variant filter but with a sacrifice of simplicity and much shorter computation time enjoyed by the time-invariant filter.
Figure 13. The Phase-Lag Characteristic Versus Frequency.
OPTIMIZATION OF LINEAR TIME-VARIANT DIGITAL FILTERS

In the processing of seismic data the presence of noise results in a complication of the problem. Hence an optimal filter is generally used in order to get as much information from the seismic data as possible. In addition, the process of operating digitally on a seismic trace to convert it to a continuous velocity log requires an optimization procedure. This process, known as "deconvolution," has received considerable attention in the recent geophysical literature (Backus(2); George, Smith and Bostick(14); Foster, Hicks; and Nipper(11); Robinson(41)). To date, however, only linear time-invariant discrete filters have been considered in the exploration seismic literature. Thus, the optimization of linear time-variant digital filters is derived in the present chapter.

 Definitions and Assumptions

In the synthesis procedure to be developed the following criteria will be employed.

(1) The input sequence, \( x(n) \), is a nonstationary random process.

(2) It is assumed that the autocorrelation of the input and the crosscorrelation of the input and the desired output is known.
(3) The system error, which is defined as the difference between the desired output and the actual output, has zero ensemble mean, that is

\[ \overline{e(n)} = 0 \]

where the bar denotes the ensemble average. The ensemble mean of the desired signal as well as the noise is also zero.

(4) The ensemble mean-square error of the system, \( \overline{e^2(n)} \), is a minimum with respect to the impulse response function \( h_3(k,n) \).

(5) The filter is physically realizable, linear time-variant, and with a finite memory. The last statement implies that the impulse response function should vanish for all time greater than a certain specified number. Thus,

\[ h(k,n) = 0, \quad \text{for } k < 0 \text{ and } k > (N+1) \]

This condition was also assumed in the previous chapter. The subscript on the impulse response function has been dropped for convenience and it will be understood that \( h(k,n) = h_3(k,n) \) in the remaining portion of this chapter.

(6) The actual output is given by Equation 3.6 (Chapter III)

\[ y(n) = \sum_{k=0}^{N} h(k,n)x(n-k) \]
With these specified conditions we proceed to the definition of the correlation functions.

The autocorrelation function $R_{xx}(x,n)$ of the sequence $x(n)$, $n = 0, 1, 2, \cdots, M$, is defined as (see Appendix D)

$$4.4 \quad R_{xx}(x,n) = \overline{x(n)x(n-s)}$$

The foregoing average may be either an ensemble average or a time average over $n$. Since the input is assumed to be nonstationary, the ensemble average must be considered.

Similarly, the crosscorrelation function between two discrete random functions $v(n)$, $n = 0, 1, 2, \cdots, N$, and $x(n)$, $n = 0, 1, 2, \cdots, M$, is defined as (see Appendix D)

$$4.5 \quad R_{vx}(s,n) = \overline{v(n)x(n-s)}$$

If the input were a stationary, ergodic process the ensemble average would be equivalent to the time average and the autocorrelation and the crosscorrelation functions would be given by

$$4.6 \quad R_{xx}(s) = \sum_{n=0}^{M+s} x(n)x(n-s), \quad s = -M, \cdots, 0, \cdots, M$$

and

$$4.7 \quad R_{vx}(s) = \sum_{n=0}^{M+s} v(n)x(n-s), \quad s = -M, \cdots, 0, \cdots, N$$

respectively. We see that they are functions of the displacement variable, $s$, only.
Minimization of the Mean Square Error

The system error has been defined as

\[ e(n) = v(n) - y(n) \]

where \( v(n) \) is the desired output and \( y(n) \) is the actual output at time \( n \). Substituting \( y(n) \) from Equation 4.3 into Equation 4.8 yields

\[ e(n) = v(n) - \sum_{k=0}^{N} h(k,n)x(n-k) \]

The square of the error, then, is given by

\[ e^2(n) = \left[ v(n) - \sum_{k=0}^{N} h(k,n)x(n-k) \right]^2 \]

Expanding Equation 4.10, we have

\[ e^2(n) = v(n)v(n) - 2 \sum_{k=0}^{N} h(k,n)v(n)x(n-k) \]

\[ + \sum_{k=0}^{N} \sum_{r=0}^{N} h(k,n)h(r,n)x(n-k)x(n-r) \]

The ensemble-averaged mean-square error is obtained by averaging both sides of Equation 4.11 with respect to the ensemble of input functions, i.e., (see Appendix D)

\[ \bar{e}^2(n) = \bar{v}(n)\bar{v}(n) - 2 \sum_{k=0}^{N} h(k,n)\bar{v}(n)x(n-k) \]

\[ + \sum_{k=0}^{N} \sum_{r=0}^{N} h(k,n)h(r,n)\bar{x}(n-k)\bar{x}(n-r) \]

In terms of the correlation functions previously defined, this mean-square error can be written as
4.13 \[ E(n) = e^2(n) = R_{vv}(0,n) - 2 \sum_{k=0}^{N} h(k,n)R_{vx}(k,n) \]
\[ + \sum_{k=0}^{N} \sum_{r=0}^{N} h(k,n)h(4,n)R_{xx}(r-k,n-k) \]

The mean-square error given by Equation 4.13 is the error resulting from use of the system with an impulse response function \( h(k,n) \). In order that this error be the minimum error, and thus \( h(k,n) \) be the impulse response of the optimum system, replacement of \( h(k,n) \) by \( h(k,n) + g(k,n) \), where \( g(k,n) \) is any impulse response, must result in a larger value for the mean-square error. Replacement of \( h(k,n) \) by \( h(k,n) + g(k,n) \) in Equation 4.13 results in

4.14 \[ E^+(n) = R_{vv}(0,n) - 2 \sum_{k=0}^{N} [h(k,n) + g(k,n)]R_{vx}(k,n) \]
\[ + \sum_{k=0}^{N} \sum_{r=0}^{N} [h(k,n) + g(k,n)][h(r,n) + g(r,n)]R_{xx}(r-k,n-k) \]

where \( E^+(n) \) denotes the mean-square error corresponding to an impulse response function \( h(k,n) + g(k,n) \). Expanding and rearranging terms in Equation 4.1 we have

4.15 \[ E^+(n) = R_{vv}(0,n) - 2 \sum_{k=0}^{N} g(k,n)R_{vx}(k,n) \]
\[ + \sum_{k=0}^{N} \sum_{r=0}^{N} h(k,n)h(r,n)R_{xx}(r-k,n-k) - 2 \sum_{k=0}^{N} g(k,n)R_{vx}(k,n) \]
\[ + 2 \sum_{k=0}^{N} \sum_{r=0}^{N} g(k,n)h(r,n)R_{xx}(r-k,n-k) \]
\[ + \sum_{k=0}^{N} \sum_{r=0}^{N} g(k,n)g(r,n)R_{xx}(r-k,n-k) \]
Reference to Equation 4.13 shows that the sum of the first three terms in Equation 4.15 is the mean-square error $E(n)$ corresponding to the impulse response function $h(k,n)$. The last term of Equation 4.15 can be written as the ensemble average of an expression which is nonnegative making the last term always positive, i.e.,

$$
4.16 \quad \sum_{k=0}^{N} \sum_{r=0}^{N} g(k,n)g(r,n)R_{xx}(r-k,n-k) = \left[ \sum_{k=0}^{N} g(k,n)x(n-k) \right]^2
$$

Thus, Equation 4.15 can be written as

$$
4.17 \quad E^+(n) = E(n) - 2 \sum_{k=0}^{N} g(k,n)R_{vx}(k,n)
$$

$$
+ 2 \sum_{k=0}^{N} \sum_{r=0}^{N} g(k,n)h(r,n)R_{xx}(r-k,n-k) + \left[ \sum_{k=0}^{N} g(k,n)x(n-k) \right]^2
$$

which yields the inequality

$$
4.18 \quad E^+(n) \geq E(n) - 2 \sum_{k=0}^{N} g(k,n)R_{vx}(k,n)
$$

$$
+ 2 \sum_{k=0}^{N} \sum_{r=0}^{N} g(k,n)h(r,n)R_{xx}(r-k,n-k)
$$

Therefore, the minimizing condition $E^+(n) \geq E(n)$ is satisfied if

$$
4.19 \quad -2 \sum_{k=0}^{N} g(k,n)R_{vx}(k,n)
$$

$$
+ 2 \sum_{k=0}^{N} \sum_{r=0}^{N} g(k,n)h(r,n)R_{xx}(r-k,n-k) = 0
$$
Since Equation 4.19 can be written in the form

\[
4.20 \quad 2 \sum_{k=0}^{N} g(k,n) \left[ - R_{vX}(k,n) + \sum_{r=0}^{N} h(r,n) R_{xx}(r-k,n-k) \right] = 0
\]

the minimizing condition is satisfied for all discrete impulse response functions \( g(k,n) \) if

\[
4.21 \quad \sum_{r=0}^{N} h(r,n) R_{xx}(r-k,n-k) = R_{vX}(k,n), \quad \text{for } k = 0, 1, \ldots, N
\]

This equation defines the impulse response of the optimum system. The development here proves the sufficiency of Equation 4.21 for a minimum. Calculus-of-variations methods can be used to establish its necessity.

Equation 4.21 is a system of \( N+1 \) linear equations in \( N+1 \) unknowns at each value of \( k \). Thus, the computations optimizing a time-variant filter are somewhat more formidable than the case of a constant coefficient, or time-invariant, filter. In constant coefficient filtering the statistics do not vary with time; therefore, the auto- and cross-correlation functions do not depend on the instant of observation. In other words, the filter is optimized for all time. It would appear that additional information could be extracted by the use of time-variant filters. Whether the amount of information gained by using time-variant filters is enough to warrant the extra computation
is a question that has to be investigated but will not be answered in this paper.

From the geophysicist's point of view there are some apparent difficulties in the optimization procedure. How does one obtain an ensemble average, that is, the autocorrelation function and the crosscorrelation function? Either the assumption must be made that the seismic trace available is representative of the ensemble, or an actual ensemble of input traces must be present. The former is a very precarious assumption to say the least and the latter is not a feasible field operation at present. In the future, however, such a procedure might be necessary to gain the extra information from the seismic data.
CHAPTER V

SUMMARY AND CONCLUSIONS

The objectives of this investigation were to introduce some concepts of linear time-variant digital filters as a method for analyzing seismic data.

In this context, then, the characterization of linear time-variant discrete systems were considered in general. It was found that there are three ways of representing the impulse response function: \( h'(n,j) \), the response to an impulse input at time \( j \) measured at time \( n \); \( h''(q,j) \), the response to an impulse input at time \( j \) measured at time \( n = j + q \); and \( h'''(k,n) \); the response measured at time \( n \) to a unit impulse input at time \( n - k \). \( h''(q,j) \) and \( h'''(k,n) \) have the advantage of having direct Fourier transforms. In addition, the realizability condition for \( h''(q,j) \) and \( h'''(k,n) \) involves only one variable, whereas it involves two variables in \( h'(n,j) \). The variable \( q \) measures elapsed time and \( k \) measures the age of the input. Picturing the linear time-variant system impulse response on an elapsed time-input time plane (Figure 4) was a useful conceptual aid in this study.

The input-output relationships were derived in the time domain as well as their corresponding relationships in the frequency domain. The frequency response functions.
H'(ω,n) and H''(ω,n) are just the response of the system to the sinusoidal input $e^{iωn}$ divided by $e^{iωn}$. The frequency domain variable $μ$ is associated with the system variation and $ν$ a variable corresponding to output frequencies.

The development of the linear time-variant digital delay filters is probably the most useful concept to emerge from this study. This is primarily because these models can actually be duplicated by a digital computer. Although the use of delay operators is not unique, insofar as time-invariant discrete systems are concerned, it is a new concept when applied to time-variant digital systems.

The output from a series arrangement of a time-variant filter and a delay operator depends on which one the input encounters first. A Type 2 time-variant delay filter distinguishes the filter in which the time-variant delay filter precedes the delay operator; in a Type 3 time-variant delay filter, the delay operator precedes the time-variant filter. It was found that the output from a Type 2 and a Type 3 filter is identical to the impulse response functions $h''(q,j)$ and $h''(k,n)$, respectively.

The convolution of a spike record with a time-varying Ricker wavelet gave a visual illustration of the time-variant digital delay filter (Figures 10 and 11).

In the frequency domain the amplitude and phase characteristics depend on the instant of observation as well as frequency. A simple example illustrated the fact
that, in general, a family of curves resulted when either the amplitude characteristic or the phase-lag characteristic were plotted against frequency, one curve for each value of n.

Since noise is inherent in a seismic record, and because the process of inverse convolution has to be approximated, an optimization problem exists. The optimization of a linear time-variant digital filter was carried out, in the ensemble mean-square error sense, under the assumption that the input was a nonstationary random process, and the autocorrelation of the input and the crosscorrelation of the input with the desired output is known. The results (Equation 4.21) is a system of \( N + 1 \) equations in \( N + 1 \) unknowns for each value of \( k \).

It is the solution of this equation which will require further investigation. Practical means of determining the autocorrelation function and crosscorrelation function will have to be developed.

It can be concluded that the use of linear time-variant digital filters will withdraw additional information from seismic data, but at an increase in computation time. Of course, the practicality of the extra computation weighted against the additional information gained will have to be considered.

It is hoped that this paper has served its purpose, that is, of introducing linear time-variant digital filters
as a method for processing seismic data. This thesis has not, by any means, developed the subject to its fullest extent. Many phases of making the development a practical and workable technique will have to be investigated.


APPENDIX A

RELATIONS BETWEEN THE VARIOUS FORMS OF IMPULSE RESPONSE AND FREQUENCY RESPONSE

These relations have been given in the last section of Chapter II, and here we explain how they are derived.

a. Transformations between impulse responses

These are almost evident from the relations

\[ q = n - j = k \]

Thus, let us consider a unit impulse input to the system at time \( j \). The response of the system after \( q \) seconds is identically the same as the response at time \( q + j \).

Therefore,

\[ h'(q,j) = h'(q+j,j) \]

The other relations are similarly obtained.

b. Input-output relations in the time domain

In the first section of Chapter II we derived

\[ y(n) = \sum_{j=-\infty}^{n} h'(n,j)x(j) \]

as Equation 2.4. Now let

\[ j' = n - j \]

and Equation A-3 becomes

\[ y(n) = \sum_{j'=0}^{\infty} h'(n,n-j')x(n-j') \]

Change \( j' \) to \( j \) and we have Equation 2.31. Equations 2.32 and 2.33 can be similarly derived, but they are most
conveniently obtained from Equation 2.31 by using the transformations in a. For example, substituting Equation 2.25, i.e.,

\[ h'(n,j) = h''(n-j,j) \]

into Equation 2.31, we have

\[ y(n) = \sum_{j=0}^{\infty} h''(j,n-j)x(n-j) \]

Changing \( j \) to \( q \) we have Equation 2.32.

c. Transformations between frequency responses

From Equation 2.11 we have

\[ H'(\omega,\mu) = \sum_{j} \sum_{n} h'(n,j)e^{-i\omega(n-j)}e^{-i\mu n} \]

\[ = \sum_{j} \sum_{n} h''(n-j,j)e^{i\omega j}e^{-i(\omega+\mu)n} \]

Let \( n - j = q \) and

\[ H'(\omega,\mu) = \sum_{j} \sum_{q} h''(q,j)e^{-i(\omega+\mu)q}e^{-i\mu j} \]

\[ H'(\omega,\mu) = H''(\omega+\mu,\mu) \]

which is Equation 2.34. Equations 2.35 and 2.36 are rearrangements of Equation 2.34.
APPENDIX B

MODIFICATION OF THE SYMMETRIC RICKER WAVELET

The symmetric Ricker wavelet can be written as (29)

\[ B-1 \quad W(t) = C \left( 6t^2/b^2 - 1/2 \right) e^{-6t^2/b^2} \]

where \( b \) is the wavelet breadth and \( C \) a constant. Writing Equation B-1 as the sum of two terms, and normalizing \( W(t) \) such that \( C = 1 \), we have

\[ B-2 \quad W(t) = 6t^2/b^2 e^{-6t^2/b^2} - 1/2 \quad e^{-6t^2/b^2} \]

A graph of the first term on the right of Equation B-2 is always positive and will be parabolic in shape near the origin, i.e., \( t = 0 \). As \( t \) increases, in the positive or negative direction, the exponential factor will dominate and asymptotically approach the \( t \) axis as \( t \) goes to infinity. It has a relative minimum at \( t = 0 \) and relative maxima at \( t = \pm b/\sqrt{6} \). The second term is just an inverted Gaussian error curve.

Now, if we translate the first term relative to the second term the wavelet will become asymmetrical. The first peak will become higher, the trough will be displaced in the direction of translation, and the second peak is reduced in amplitude. If we let the translation variable be \( u \), Equation B-2 can be written as a function of \( t \) and \( u \) as
B-3 \[ W(t,u) = 6(t+u)^2/b^2 \ e^{-6(t+u)^2/b^2} \]
\[ - \frac{1}{2} \ e^{-6t^2/b^2} \]

Figure 11(b) illustrates the results of sampling B-3 and convolving it with the spike record in Figure 11(a). The first wavelet is the asymmetrical wavelet with \( u \) at its maximum displacement. \( u \) is decreased in equal increments until it approaches the symmetric form again, as reference to Equation B-3 will indicate, and is depicted as the last wavelet in Figure 11(b).
APPENDIX C

A BRIEF DISCUSSION OF THE z-TRANSFORM

The z-transformation can be studied as a modification of the Laplace transformation, as a modification of the Fourier transformation, or approached directly as the operational calculus of number sequences. A brief note on the latter two will be included here.

Consider the impulse sequence (or sampled time function) \( h^*(t) \). This was defined in Equation 2.4 of Chapter II as

\[
C-1 \quad h^*(t) = h(t) \delta_T(t)
\]

where \( h(t) \) is the continuous time function and \( \delta_T(t) \) is a periodic train of unit impulses spaced \( T \) seconds apart. Equation C-1 may be written as

\[
C-2 \quad h^*(t) = \sum_n h(nT) \delta(t-nT)
\]

where we assume here that \( h^*(t) \) represents a sampled time function over positive and negative time.

Let us now take the Fourier transform of the sampled time function \( h^*(t) \) as given in Equation C-2. Denoting the result by \( H(\omega) \) we obtain

\[
H(\omega) = F[h^*(t)]
\]

\[
= F[\sum_n h(nT) \delta(t-nT)]
\]
\[ H(\omega) = \int_{-\infty}^{\infty} \sum_{n} h(nT) \delta(t-nT) e^{-i\omega t} dt \]

\[ = \sum_{n} h(nT) \int_{-\infty}^{\infty} \delta(t-nT) e^{-i\omega t} dt \]

\[ = \sum_{n} h(nT) e^{-i\omega nT} \]

In the discussion of strictly discrete time functions, Equation C-2 defines directly the Fourier transform of a discrete time function \( h(nT) \).

Now consider the z-transform from an operational calculus point of view. In the mathematical theory of discrete time functions, the z-transform of a discrete time function \( h(nT) \) is defined as

\[ H^*(z) = \sum_{n} h(nT) z^n \]

This definition has been used by Rice(38) and Trietel and Robinson(44) in connection with inverse filtering problems in seismic work. It makes use of Laplace's original definition. Some writers prefer to define the z-transform as

\[ H^*(z) = \sum_{n} h(nT) z^{-n} \]

For example, Ragazzini and Franklin(36) designate Equation C-4 as the two-sided z-transform since it describes the pulse sequence \( h(nT) \) for negative as well as positive time.

We return to Equation C-4 and consider the unit circle in the complex \( z \) plane. The equation of the unit circle
can be written as
\[ C-6 \quad z = e^{-i\omega T} \]

If Equation C-6 is substituted into Equation C-4 we obtain

\[ C-7 \quad H*(e^{-i\omega T}) = \sum_{n} h(nT)e^{-i\omega nT} \]

which is just the Fourier transform \( H(\omega) \) given in Equation C-2. Hence, the Fourier transform \( H(\omega) \) of the sampled time function \( h^*(t) \) is identical to the function \( H*(e^{-i\omega T}) \), the z-transform evaluated on the unit circle.

If, instead of a one dimensional sampled time function, we have a two dimensional sampled function \( h(kT,nT) \) we can define a z-transform with respect to one of the time variables \( kT \) or \( nT \). For example, the z-transform of \( h(kT,nT) \) with respect to \( kT \) is

\[ C-8 \quad H^*(z,n) = \sum_{k} h(kT,nT) z^{k} \]

If we restrict \( h(kT,nT) \) to be a finite, physically realizable, time-variant digital delay filter with a sampling interval of one unit of time, i.e., \( T = 1 \), we have Equation 3.4.

A thorough treatment of the z-transform theory can be found in a book by Jury(20).
APPENDIX D

THE ENSEMBLE AVERAGE

The mathematical description of a random process depends upon the concepts of a probability-distribution function and a probability-density function of various orders. The first-order probability-density function associated with an ensemble of a discrete process (or a discrete random function) \( x(t) \) is designated \( f_1(x_i, t) \), where \( x_i \) is the range variable of \( x(t) \) at time \( t \). This function is defined so that \( f_1(x_i, t) \) is the probability that the variable \( x(t) \) equals \( x_i \) at time \( t \). In Figure 14(a) (or Figure 14(b)) we have a graphical representation of an ensemble. In Figure 14(a) the random variable \( x(t) \) represents the amplitude of a time function at time \( t \).

For example, if the random amplitude of any one of the member functions at time \( t_1 \) is \( x(t_1) \), as depicted in Figure 14(a), with probability-density \( f_1(x_{i_1}, t_1) \), we express this symbolically as

\[
D-1 \quad P[x(t_1) = x_{i_1}; t_1] = f_1(x_{i_1}, t_1)
\]

where \( x_{i_1}, i = -n, \cdots, m, \) is a discrete range variable representing the finite or infinite range of \( x(t_1) \). More generally, the probability that \( x(t) \) lies in a finite range \( x_k \leq x(t) \leq x_m \) at time \( t \) is given by the summation

\[
D-2 \quad P[x_k \leq x(t) \leq x_m; t] = \sum_{i=k}^{m} f_1(x_i, t)
\]
Figure 14. Two Statistically Dependent Ensembles of Random Impulse Functions
The second-order probability-density function associated with a random variable \( x(t) \) is designated \( f_2(x_{1i}, t_1; x_{2j}, t_2) \) and is defined such that it is the probability that the variable \( x(t_1) \) equals \( x_{1i} \) at time \( t = t_1 \) and also \( x(t_2) \) equals \( x_{2j} \) at time \( t = t_2 \), i.e., (see Figure 14(a))

\[
D-3 \quad P[x(t_1)=x_{1i}; t_1: x(t_2)=x_{2j}; t_2] = f_2(x_{1i}, t_1; x_{2j}, t_2)
\]

Here \( x_{2j}, j = -n, \cdots, m, \) represents the discrete range variable of \( x(t_2) \) at time \( t_2 \). The second-order density function provides a more detailed description of the process than does the first-order function, and as higher order density are described more and more detail is available about \( x(t) \). Seldom, however, is one so fortunate as to know the probability-density functions of all orders which describe a random process, and in many cases one must settle for much less specific information.

In a typical situation we may know some of the moments of the distributions which describe the random process being considered. The first moment is the expected value or ensemble average or mean and is defined by the summation

\[
D-4 \quad \bar{x}(t) = \sum_{i} x_i f_1(x_i, t)
\]

Again, referring to Figure 14(a), the ensemble average of \( x(t_1) \) would be given by
D-5 \[ \overline{x(t_1)} = \sum_i x_{1i} f_1(x_{1i}, t_1) \]

This ensemble average is the expected value of \( x(t_1) \) per trial, and the probability that the empirical average differs from the ensemble average by a preassigned quantity, however small, can be made as close to 1 as we please by making the number of trials sufficiently large.

The generalized second moment of the random variable \( x(t) \) is the autocorrelation function, defined by

D-6 \[ R_{xx}(t_1, t_2) = \sum_j \sum_i x_{1i} x_{2j} f_2(x_{1i}, t_1; x_{2j}, t_2) = \overline{x(t_1)x(t_2)} \]

which is the ensemble average of the product of values of the random variable \( x(t_1) \) at time \( t_1 \) the values of the random variable \( x(t_2) \) at time \( t_2 \). For the special case when \( t_1 \) equals \( t_2 \), then \( x_1 = x_2 \), and

D-7 \[ R_{xx}(t_1, t_2) = \sum_j \sum_i x_{1i}^2 f_2(x_{1i}, t_1; x_{2j}, t_2) = \sum_i x_{1i}^2 f_1(x_{1i}, t_1) = \overline{x^2(t_1)} \]

which is the ensemble mean-square value of the random variable at \( t_1 \). In the second step of Equation D-7, use has been made of the fact that the summation of the second-order function \( f_2 \) over all values of \( x_2 \) is just the first-order density function \( f_1 \). That is, the probability that
the variable \( x(t) = x_1 i \) at time \( t_1 \) and has any value at all at time \( t_2 \) is simply \( f_1(x_1 i, t_1) \). From the symmetry of Equation D-6 it is clear that \( R_{xx}(t_1, t_2) = R_{xx}(t_2, t_1) \).

Where the analysis of two related random processes is involved, as, for example, in the consideration of the input to a filter \( x(t) \), and the output of the filter, \( y(t) \), one must define joint probability-density functions to describe the simultaneous characteristics of the two variables. The first joint density function is defined so that \( f_{11}(x_1 i, t_1; y_2 j, t_2) \) gives the probability that \( x(t_1) \) equals \( x_1 \) at time \( t = t_1 \) and \( y(t_2) \) equals \( y_2 j \) at time \( t = t_2 \). This is illustrated in Figure 14, where \( y_2 j \) is the range variable of \( y(t_2) \) at time \( t_2 \). Mathematically we have

\[
P[x(t_1) = x_1 i; t_1; y(t_2) = y_2 j; t_2] = f_{11}(x_1 i, t_1; y_2 j, t_2)
\]

The most elementary moment of the first-order joint density function is the crosscorrelation function, defined as

\[
R_{xy}(t_1, t_2) = \sum_{i} \sum_{j} x_1 i y_2 j f_{11}(x_1 i, t_1; y_2 j, t_2)
\]

\[
R_{xy}(t_1, t_2) = \frac{x(t_1) y(t_2)}{E(x(t_1)) E(y(t_2))}
\]

which expresses the crosscorrelation of \( x(t) \) and \( y(t) \) as an ensemble average.

Since this paper deals with digital systems we may replace the \( t \), in the above development, by an integer times the sampling interval. The sampling interval has been taken
to be one, hence, replacement of \( t \) by an integer completely specifies the time.

To make Equations D-6 and D-9 conform to the definition of the autocorrelation function and crosscorrelation function given by Equations 4.4 and 4.5 (Chapter IV), respectively, we make the substitution

\[
\begin{align*}
D-10 & \quad t_1 = n \\
& \quad t_2 = n - s
\end{align*}
\]

in Equations D-5 and D-8, i.e.,

\[
\begin{align*}
D-11 & \quad R_{xx}(s,n) = \sum_j \sum_i x_{1i} x_{2j} \Phi_2(x_{1i},n;x_{2j},n-s) \\
& \quad = \overline{x(n)x(n-s)}
\end{align*}
\]

and

\[
\begin{align*}
D-12 & \quad R_{vx}(s,n) = \sum_j \sum_i v_{1i} x_{2j} \Phi_{11}(v_{1i},n;x_{2j},n-s) \\
& \quad = \overline{v(n)x(n-s)}
\end{align*}
\]

A more comprehensive treatment of these concepts are given by Laning and Battin(29) and Lee(30).
VITA

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In September 1960 he moved to Rolla, Missouri, where he enrolled as a Ph.D. candidate at Missouri School of Mines and Metallurgy, at the same time accepting the title of Instructor of Mathematics. He was granted National Science Foundation scholarships for the summers of 1961 and 1963. He was employed by Pan American Petroleum Corporation as an office computer during the summer of 1964.

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