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An analytical study of a confined viscous vortex

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AN ANALYTICAL STUDY OF A CONFINED VISCOUS VORTEX

by

DAVID EDWARD CROW, 1944-

A DISSERTATION

Presented to the Faculty of the Graduate School of the

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ABSTRACT

A composite solution for the flow field of a three dimensional, incompressible, laminar, viscous vortex confined inside a conical converging nozzle is presented. This model is used to study the strong mainstream boundary layer interaction that occurs in confined vortex flows.

The confined vortex flow field is divided into mainstream and boundary layer regions. A particular class of solutions of the general equations of motion is used to represent the vortex flow in the mainstream region. That is, the tangential velocity in the mainstream region is assumed to be of the form $W = T(\theta)/R$. Velocity and pressure profiles from the mainstream region are used as boundary conditions to generate an integral momentum solution in the boundary layer region. The mainstream boundary layer interaction is modeled by iteratively matching the transverse velocity at the common edge of the two regions, while conserving the mass flow and radial momentum of the total system at all points along the length of the nozzle.

The results for the velocity profiles and the boundary layer growth obtained by separately considering each region of the flow field are presented and discussed. The velocity and pressure profiles obtained from the composite solution are compared with experimental data from previous investigations.
ACKNOWLEDGEMENT

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Dimensional Quantities

\[ B(\theta) = \text{First derivative of tangential velocity for (ft}^2/\text{sec)} \]

\[ B_m = \text{Boundary layer mass flow (lb}_m/\text{sec)} \]

\[ F(\theta) = \text{Transverse velocity form (ft}^2/\text{sec)} \]

\[ F_s = \text{Force at any nozzle station (lb}_f/\text{)} \]

\[ G(\theta) = \text{Radial velocity form (ft}^2/\text{sec)} \]

\[ \Theta(\theta) = \text{Second radial velocity form (ft}^3/\text{sec)} \]

\[ g_c = \text{Gravitational constant (lb}_m/\text{ft/lb}_f/\text{sec}^2) \]

\[ H(\theta) = \text{Velocity form grouping (ft}^2/\text{sec)} \]

\[ M_s = \text{Mainstream mass flow (lb}_m/\text{sec)} \]

\[ P = \text{Static pressure (lb}_f/\text{ft}^2) \]

\[ Q(\theta) = \text{First derivative of radial velocity form (ft}^2/\text{sec)} \]

\[ R = \text{Spherical radius (ft)} \]

\[ T(\theta) = \text{Tangential velocity form (ft}^2/\text{sec)} \]

\[ U = \text{Mainstream radial velocity (ft/sec)} \]

\[ u = \text{Boundary layer radial velocity (ft/sec)} \]

\[ V = \text{Mainstream transverse velocity (ft/sec)} \]

\[ v = \text{Boundary layer transverse velocity (ft/sec)} \]

\[ W = \text{Mainstream tangential velocity (ft/sec)} \]

\[ w = \text{Boundary layer tangential velocity (ft/sec)} \]

\[ W_f = \text{Momentum dissipated in the boundary layer (lb}_f) \]

\[ x = \text{Transverse angle at edge of the boundary layer (radians)} \]
$Y = \text{Product of the boundary layer thickness ratio and the square of the radial boundary layer thickness} \quad (Y = K\delta_R^2)$

$Z = \text{Product of the square of the radius and the square of the radial boundary layer thickness} \quad (Z = R^2\delta_R^2)$

$\delta_R = \text{Radial boundary layer thickness (ft)}$

$\delta_T = \text{Tangential boundary layer thickness (ft)}$

$\alpha = \text{Transverse angle at the cone wall (radians)}$

$\nu = \text{Fluid kinematic viscosity (ft}^2/\text{sec)}$

$\rho = \text{Fluid density (lb}_m/\text{ft}^3)$

$\mu = \text{Fluid viscosity (lb}_m/\text{ft-sec)}$

$\theta = \text{Spherical transverse angle (radians)}$

$\phi = \text{Spherical tangential angle (radians)}$

Nondimensional Quantities

$A = \text{Centerline value of the first derivative of the tangential velocity form}$

$a = \text{Boundary layer integral coefficient}$

$b = \text{Boundary layer integral coefficient}$

$C = \text{Constants of integration}$

$c = \text{Boundary layer integral coefficient}$

$D = \text{Centerline value of the second derivative of the radial velocity form}$

$d = \text{Boundary layer integral coefficient}$

$e = \text{Boundary layer integral coefficient}$

$f = \text{Boundary layer radial velocity profile}$

$g = \text{Boundary layer tangential velocity profile}$
h = Boundary layer integral coefficient

j = Boundary layer integral coefficient

K = Boundary layer thickness ratio (K = δR/δT)

m = Boundary layer integral coefficient

M = Radial nozzle station

N = Nondimensional radial boundary layer distance (N = \( \frac{R(\alpha - \theta)}{\delta_R} \))

N_1 = Nondimensional tangential boundary layer distance (N_1 = \( \frac{R(\alpha - \theta)}{\delta_T} \))

Re = Reynolds Number

Ro = Rossby Number

X = Value of N at the edge of the larger boundary layer thickness

Subscripts

c = Centerline quantities

i = Inlet conditions

o = Dimensionless quantities
I. INTRODUCTION

Few internal flow fields in fluid mechanics have received more attention than the confined vortex. Interest in flows of this type arises both naturally as in tornado type flows [1] and from the many devices which attempt to use particular properties of the vortex. A list of these devices includes such items as the cyclone separator [2], Ranque-Hilsch tube [3], swirl atomizer [4], vortex-contained nuclear rocket engine [5], gas turbine cyclone combustion chamber [6], rocket engine thrust modulation [7], magnetohydrodynamic power generation [8], and oil-water separator [9], to name only a few. However, with such obvious interest and extensive investigation, the flow characteristics of the confined vortex still defy complete explanation. In fact, today we are only a few steps removed from Leonardo Da Vinci [10], who said,

"of the eddies one is slower at the centre than on the sides; another is swifter at the centre than on the sides; others there are which turn back in the opposite direction to their first movement".

When a vortex is maintained inside a container, the wall boundary layers interact with the mainstream flow not only in the classical sense by displacing the mainstream flow from the wall, but also by limiting the very strength of the swirling or vortex motion.

* Numbers in brackets refer to references at the end of the paper
This mainstream-boundary layer interaction is important in such widely varying flow fields as rotating pumps, density separators, nuclear rocket engines, turbine inlets and diffusers, tornados and many others. Because of the general interest in rotating flow fields of this type, it was decided to attempt a numerical solution of both the mainstream vortex and the wall boundary layer for swirling air flow through a conical converging nozzle. (See Figure 1)

Experimental investigations [11, 12] have indicated that a number of factors that have not been included in previous analytical studies are important in the determination of the confined vortex flow field in the converging nozzle. For example, previous analytical investigations [4, 13, 14] of the cone wall boundary layer have not included the effects of the spherical radial velocity component on the growth of the cone wall boundary layer. Analytical investigations [15, 16] of the complete nozzle flow field have considered the flow to be inviscid and did not include the effects of the wall boundary layer on the confined vortex flow field. In the composite solution described in this dissertation, the effect of the fluid viscosity, the effects of the spherical radial velocity component on the cone wall boundary layer growth, and the effect of the mainstream-boundary layer interaction are considered in the determination of the confined vortex flow field in a converging nozzle.

The solution procedure consists of breaking the flow field into mainstream and boundary layer regions. In the mainstream region,
Figure 1. Nozzle Flow Field
where the dominant velocity is the tangential or swirl velocity, a solution was obtained by assuming a particular form for the tangential velocity distribution and numerically solving the resulting incompressible Navier-Stokes Equations. The boundary layer region, where the wall viscous shear stresses destroy the centrifugal force created by the mainstream tangential velocity, was solved with integral momentum techniques. The effects of the mainstream-boundary layer interaction were then obtained by iteratively matching the mainstream and boundary layer transverse velocities at the common edge of the two regions.

In the body of this dissertation, the solution for each individual region of the flow field is first presented and the composite solution formed by combining the solutions for the two regions is then revealed. The discussions of the individual regions are concerned with the method of solution and the accuracy of the results obtained by considering each region separate from the entire flow field. Discussions of the composite solution contain the philosophy and the results of the solution for the entire confined vortex flow field. In this latter section, the results from the composite solution are compared with data from a previous experimental investigation [12].
II. REVIEW OF LITERATURE

Probably since the beginning of time, mankind has been fascinated by the diversity of vortex motions and, out of this fascination, the ingenuity of man has created a wide variety of devices that make use of the properties of vortex motion. At the onset, many of these devices appear to be quite simple and to hold a great deal of promise. However, on further investigation, the vortex motion defies complete understanding and, therefore, many of the devices do not fulfill their earlier promise. Thus, the intense and unceasing struggle of man to understand and improve these devices has produced a vast array of investigations into the subject of the confined vortex.

This literature review is divided into two sections. In the first section, investigations concerned with the flow of the mainstream of the confined vortex are discussed, and in the second section the boundary layer produced by the walls of the vortex container is discussed. The reader should be aware that this division is artificial and there will be some overlap of the two areas.

A. Mainstream Literature Review

In the early 1900's, G. I Taylor [17] dominated the work in general vortex motion with a series of articles in the proceedings of the Royal Society of London. However, most of Taylor's work did not directly apply to the confined vortex flow field. The boom in confined
vortex studies began in 1931, when a French metallurgist, G. Ranque [18] noticed a temperature separation in the vortex of a cyclone separator. Ranque constructed a device (herein called the Ranque-Hilsch tube) which attempted to use the vortex temperature separation as a refrigeration process. A typical configuration of this device (See Figure 2) consisted of a cylindrical tube with tangential injection nozzles around the periphery of one end of the tube. The "cold" stream is ejected through an orifice near the vortex centerline at the injection end of the tube. The "hot" stream is ejected through a back pressure valve at the opposite end of the tube. After patenting the device in 1932, Ranque indicated after further investigation that the vortex temperature separation was too inefficient to compete with conventional refrigeration devices. At this point, Ranque apparently decided that the device was useless and ceased to work on his discovery.

From 1931 to 1946, the Ranque-Hilsch tube and, for that matter, the entire field of confined vortex flow lay dormant. In 1946, at the end of World War II, papers were found in R. Hilsch's [3] laboratory that showed the results of a parametric study to determine the effects of tube geometry and operating conditions on the efficiency of the Ranque-Hilsch tube. When these papers were brought back to the United States, they created such interest that by 1954 Westley [19] compiled a list of 116 papers dealing with the Ranque-Hilsch tube and in 1960 Donaldson [20] reviewed an additional 112 papers on
Figure 2. Typical Ranque-Hilsch Tube Geometry
the subject. The bulk of the work done during this time period was similar to Hilsch's, that is, experimental parametric studies attempting to improve the refrigeration efficiency of the Ranque-Hilsch tube. Two papers typical of this work are by Westley [21] and by Martynovskii and Alekseev [22]. Like Ranque and Hilsch, the consensus of all this work was that the vortex temperature separation was not competitive with the standard refrigeration cycles.

Perhaps more important than the large number of parametric studies of this time period was the small number of investigators who studied the basic flow characteristics of the confined vortex. Linden [11] experimentally measured the velocity and pressure distribution in a cyclone dust collector. This investigation gave the first indications of the vigorous three dimensional character of the confined vortex. That is, in the spherical radial direction, Linden found evidence of flow reversal near the vortex centerline. Unfortunately, the importance of this investigation was to go unnoticed.

In 1968 Lineberry [12] (discussed in detail in Chapter V) experimentally studied the behavior of swirling air flow in a conical converging nozzle. In this study it was observed that the spherical radial component of velocity approached the same magnitude as the tangential component as the apex of the cone was approached. As did Linden [11], Lineberry found areas of reversed radial flow in the core of the vortex.

Hartnett and Eckert [23] also experimentally studied the basic
flow characteristics of the Ranque-Hilsch tube. They reported velocity, pressure, and temperature profiles for a number of inlet conditions and different "hot" stream exit geometries. However, the two most important things found in this investigation were the areas of reversed axial flow and that the energy separation in a vortex seemed to be independent of the fluid Prandtl Number.

Lay [24], in the first of a two part paper, presented the measured values of the velocity, temperature, and pressure profiles at a number of axial stations along a Ranque-Hilsch tube. From these measurements, Lay recognized the three dimensional character of the flow field inside a Ranque-Hilsch tube. In part two, the analytical portion of his study, Lay [25] used an inviscid model to describe the flow field in the vortex tube. Solutions for the two dimensional form of this model were obtained with the use of a hodograph transformation of the inviscid general equations of motion. From the axial momentum equation, Lay then showed that the superposition of a uniform axial flow does not affect the two dimensional form of the solution. Thus, the solutions obtained are represented by superposition of a potential vortex and a radial sink flow on a uniform axial flow field. As will be shown later, the effects of the mainstream viscosity and the wall boundary layers, neglected by Lay, must be included to accurately describe the confined vortex flow field in the Ranque-Hilsch tube.

Einstein and Li, [26] after studying the "bath tub" vortex formed
by emptying a container, proposed one of the earliest vortex models applicable to the Ranque-Hilsch tube. In this model, it was assumed that the flow could be represented by a plane vortex which was divided into two areas; the area outside of the actual drain opening and the area inside the drain opening. Outside the drain opening, the mass flow across any concentric cylinder is considered to be constant and inside the drain opening the mass flow is assumed to vary inversely as the square of the cylindrical radius. With these assumptions, it is not surprising that the authors found solutions that are a function of two nondimensional parameters: the ratio of the radius to the drain radius and a Reynolds Number based on radial mass flow. The limiting forms of these solutions show that for very large radial Reynolds Numbers, the tangential velocity profile approaches that of a potential vortex (constant circulation) and for very small radial Reynolds Numbers the tangential velocity profile approaches that of a forced vortex (wheel flow). Lewellen [27] showed that the solutions of Einstein and Li are the "zeroth order" terms of a power series expansion solution for confined vortex flows. In order to apply their model to turbulent flows, Einstein and Li made a "Boussinesq" type assumption. That is, they assumed that the turbulent Reynolds Stresses could be represented by a constant eddy diffusivity (virtual viscosity) times the local rate of strain of the mean values of the flow velocity. Unfortunately, a number of investigators who followed Einstein and Li attempted to experimentally
measure this virtual viscosity without accounting for the effects of the wall boundary layers on the mainstream flow.

Long [1], in his attempt to explain the large vertical velocities in a tornado, assumed that the vortex circulation approaches a constant as the cylindrical radius approaches infinity. From an analysis of the kinetic momentum transfer and a dimensional argument, Long obtained a similarity transformation. He then combined the similarity transformation with the boundary layer assumptions to reduce the Navier-Stokes Equations to a set of three ordinary differential equations. Although no numerical results were given, a qualitative discussion of the ordinary differential equations indicates that as the vortex becomes more concentrated near the vortex centerline, the vertical velocity continually increases. Lewellen [27], commenting on this solution, notes that even at large cylindrical radii, the axial velocity does not disappear; therefore, he doubts the usefulness of this solution for the prediction of tornado velocity profiles.

Pengelley [28], Deissler and Perlmutter [29], and Mack [30] all presented one dimensional analytical investigations of the energy separation in vortex flows. Mack neglected the cylindrical radial mass flow in the vortex and found that the energy separation in the vortex is a function of the fluid Prandtl Number. The other two investigations neglected the thermal conductivity of the fluid in comparison with the transport of energy by the radial mass flow. As one would expect, the vortex energy separation in these cases was
found to be a function of the radial mass flow. Deissler and Perlmutter's results showed that vortex energy separation cannot occur if the Reynolds Number based on radial mass flow is greater than four. Therefore, they reached the conclusion that turbulence is of the utmost importance to the energy separation process.

Donaldson [20] in 1957, presented a particular class of vortex solutions to the incompressible Navier-Stokes Equations. These solutions were obtained in the following manner: Working in cylindrical coordinates, Donaldson assumed that the tangential (vortex) velocity was axisymmetric and furthermore a function of radius only. With this assumed form of the tangential velocity, an analysis of the general equations of motion showed that the radial velocity must be a function of the radius only, and that the axial velocity must be a function of the radius, times the axial distance, plus an arbitrary function of the cylindrical radius. Substitution of these forms of the velocity components into the general equations of motion transforms the equations to a set of nonlinear ordinary differential equations. Donaldson applied these transformed equations of motion to the viscous, incompressible flow field inside a rotating, porous cylinder. Closed form solutions for the very viscous and nonviscous cases were obtained. General cases were then solved by means of a power series expansion about these limiting solutions. Donaldson and Sullivan [31] extended this work by numerically obtaining the complete class of solutions for the transformed equations discussed
above. In this complete class of solutions, five distinct types of solutions, which depend upon the radial Reynolds Number and an axial pressure gradient parameter, were obtained. Multi-celled vortex motions, that is, areas of reversed axial flow, seen in previous experimental studies were exhibited analytically in these solutions. The solutions of Burgers [32] were shown to be a special case of this complete set of solutions. These two investigations represent the earliest analytical evidence of the important effect of the viscous stresses on the three dimensional character of confined vortex flows.

Lewellen [33] considered a flow model which is composed of a strong cylindrical tangential rotation and a radial sink flow which exhausts axially inside a finite radius. With this model, the incompressible Navier-Stokes Equations can be reduced to two coupled partial differential equations in terms of the stream function and circulation. These equations contain three nondimensional parameters, the radial Reynolds Number, the ratio of the characteristic mass flow per unit length to circulation, and a ratio of characteristic axial and radial dimensions. Lewellen showed that for strong vortex flows the characteristic ratio of mass flow per unit length to circulation is very small. Thus, a solution of the two partial differential equations is obtained by means of a power series expansion in terms of the last two dimensionless parameters. The zeroth order terms of this expansion are shown to represent the solutions obtained by the vortex flow model of Einstein and Li [26]. The difficulty with this
solution is that it cannot be carried into the side wall boundary layers. That is, inside the boundary layers the characteristic ratio of mass flow per unit length to circulation can no longer be considered small; thus, the series is invalid in the boundary layer region. A number of investigators (Rosenzweig, Lewellen, and Ross [34], Logan [35], and Linderstrom-Lang [36]) have circumvented this difficulty by matching the power series solution for the vortex mainstream to a boundary layer solution near the container walls. In all three cases, however, only the zeroth order terms of Lewellen's power series have been used in the vortex mainstream.

Mager [15] analytically studied the incompressible, viscous swirling flow field in a converging-diverging nozzle. In this study, the nozzle flow field was divided into a vortex core and a mainstream region. In the vortex core near the nozzle centerline, a boundary layer integral momentum solution was matched to a potential vortex in the mainstream region. Conservation of total system mass flow and momentum was the criteria used to change this solution as the flow progressed axially through the nozzle. King [16] considered an inviscid, compressible case of the same problem and found that the mass flow through the nozzle was strongly dependent on the ratio of inlet tangential to radial velocity. Above a critical value of this ratio, reversed flow occurred in the subsonic portion of the nozzle, but no reversed flows were found to occur in the supersonic portion of the nozzle. Both of these studies neglected the effects of the nozzle wall
boundary layers, which, as will be shown later, must be included if the analytical model is to describe confined vortex flows.

Around 1960, the vortex-contained nuclear rocket engine replaced the Ranque-Hilsch tube as the dominant device initiating confined vortex flow studies. However, because of the similarity of geometries, many studies pertaining to the nuclear rocket engine could also be related to the Ranque-Hilsch tube. A large number of experimental investigators in this time period (Savino [37], Ragsdale [38], Schowalter and Johnstone [39]) attempted to explain the differences between predicted and measured values of confined vortex flows by the phenomenological theories of turbulence. That is, the eddy diffusivity was either assumed constant or a function of an appropriate mixing length. Keyes [40], using the turbulent model proposed by Einstein and Li [26], calculated the eddy diffusivity necessary to match experimental values of the confined vortex flow. Ragsdale [38] performed the measurements necessary to evaluate the universal constant for the Prandtl and Von Karman mixing length theories. The essential weakness of all these studies was brought to light in two separate, very carefully controlled investigations. The first investigation by Kendall [41] showed that nearly the entire system radial mass flow was contained in the boundary layers of the end walls of a cylindrical vortex tube. The second investigation by Donaldson and Williamson [42] experimentally confirmed that the predominance of the radial mass flow was contained in the end wall boundary layers.
This investigation also measured very low turbulent intensities in the vortex mainstream. Thus, the one and two dimensional turbulent models used by the previous investigators could not possibly explain the actual flow field in the confined vortex. These two investigations also dramatically demonstrated, for the first time, the important effect of the solid wall boundary layers on the flow of a confined vortex.

Piviotto [43], working on the gaseous core nuclear rocket engine, observed that even a very small probe inserted radially across a cylindrical vortex drastically disturbed the side wall static pressure profiles (See Figure 3). The same physical reason explains the importance of both the wall boundary layers and the disturbance of the probe in vortex flows. The centrifugal forces created by the tangential vortex velocities are counter balanced by a radial static pressure gradient. That is, the static pressure in a vortex is higher in outer regions of the vortex than it is near the vortex centerline. When something such as a probe or the viscous shear of the wall locally destroys the centrifugal force, the radial pressure gradient imposed by the undestroyed portion of the vortex flow, pumps fluid radially inward toward the vortex centerline. Thus, it is seen that secondary flows are very important in the determination of the overall flow field in a confined vortex.

B. Boundary Layer Literature Review

The strong wall boundary layer-mainstream interaction that occurs
Figure 3. Effects of Sampling Probe on Vortex Flow

\[ \frac{P_{\text{wall}} - P_{\text{Q}}}{(P_{\text{wall}} - P_{\text{Q}})_{\text{no probe}}} \]

- PIVIROTTO [43]
In confined vortex flows can have a dominant role in determining the overall vortex flow field. In the paragraphs below, a historical discussion of the development of the analytical procedures necessary to analyze these boundary layers is presented.

In 1950, Taylor [4] presented an integral momentum solution for the cone wall boundary layer in a swirl atomizer. The mainstream of this investigation was represented by a potential vortex with no spherical radial flow. That is, Taylor neglected the effects of the mainstream radial velocity component in comparison with the tangential (swirl) velocity component. In this solution (called the T-Method hereafter) the spherical radial and tangential boundary layer integral momentum equations are solved for a boundary layer thickness and a boundary layer radial velocity weighting function. Since the mainstream has no radial velocity component, Taylor introduced a velocity weighting function to allow the magnitude of the radial component of velocity in the boundary layer to change as a function of spherical radial distance. The boundary layer thickness calculated by the T-Method monotonically increased with decreasing spherical radius until a maximum was reached at a nondimensionalized spherical radius of 0.85, then the boundary layer thickness slowly decreased with decreasing radius. Taylor gave no physical explanation for this behavior. However, by using the results obtained with the T-Method, Mack [44] found that the boundary layer mass flow increased monotonically throughout the complete range of radial distance. Thus, the
decrease in the T-Method boundary layer thickness is caused by the coupling of the boundary layer thickness and the boundary layer radial velocity weighting function through the boundary layer continuity equation. That is, Taylor's radial velocity weighting function increases so fast that the boundary layer thickness is required to decrease in order to maintain the boundary layer mass flow. Weber [13] numerically solved this same problem and, as would be expected, arrived at the same results obtained by Taylor.

Cooke [14] also used a momentum integral technique (called the C-Method hereafter) to solve the boundary layer on the conical wall of a swirl atomizer. Unlike Taylor [4], Cooke noted that when the boundary layer integral momentum technique is extended to three-dimensional boundary layers, the boundary layer velocity components do not asymptotically approach their respective mainstream values at the same distance from the wall. Therefore, the boundary layer tangential and radial momentum equations must be solved for two boundary layer thicknesses. However, since in the C-Method, both momentum equations were used to calculate boundary layer thicknesses, the radial boundary layer velocity weighting function had to be determined by a different method than that used in the T-Method. Cooke obtained this weighting function by forcing the assumed boundary layer radial velocity profile to satisfy the limiting values of the radial momentum equation as the cone apex was approached.

In a series of articles, Mack [45, 46, 47] compared the results
obtained by the application of the T-Method and the C-Method to a potential vortex flow above a finite stationary disk. Comparison of the results obtained with these two methods indicated that the behavior of the cylindrical radial mass flow in the boundary layer was vastly different. The boundary layer mass flow calculated with the T-Method increased monotonically with decreasing cylindrical radius until at the disk centerline a sudden eruption occurred. On the other hand, the boundary layer mass flow calculated with the C-Method increased and then decreased to zero at the disk center line. Thus, the boundary layer mass flow from the T-Method was returned to the mainstream by a sudden eruption near the disk centerline and the boundary layer mass flow from the C-Method was gradually returned to the mainstream. In order to determine which method was correct, Mack [47] derived a sixth order power series expansion solution for the boundary layer momentum equations. This series solution indicated that the T-Method produced the correct results. Mack was unable, however to explain why the results of the C-Method were inferior to those of the T-Method. Rott and Lewellen [48] reviewed the integral momentum solutions and concluded that the use of the limiting form of the radial momentum equation to specify the boundary layer radial velocity weighting function, leads to an over specification for the system of boundary layer equations.

Rott and Lewellen [48], Anderson [49], and Mack [45] have all developed simplified forms of the momentum integral solution for the
boundary layer of a vortex above a cylindrical disk. In all of these forms of the momentum integral solution, the radial integral momentum equation is replaced by the limiting value \((\theta \to \alpha)\) of the radial boundary layer momentum equation at the disk wall. As in the previous momentum integral methods, these simplified methods neglect the effect of the cylindrical radial velocity on the growth of the disk boundary layer. King [50] compared the results of the simplified methods with the results of the T-Method and found a relatively close agreement everywhere. The obvious advantage of the use of the simplified methods is its analytical simplicity.

Anderson [51] numerically solved the boundary layer momentum equations for a vortex flow above a disk. In this solution, the partial derivatives along the radius of the disk were replaced by finite difference relations while derivatives normal to the disk were retained. Thus, Anderson obtained a set of differential equations normal to the disk that were solved to yield boundary layer solutions. The boundary conditions for this solution, like all previous solutions, did not include the cylindrical radial velocity. This solution was extended to consider the effects of compressibility by making a small Mach number assumption. Bruggrof, Stewartson, and Belcher [52] extended Anderson's work to the centerline of the disk. The mass flux in the boundary layer for these solutions does not vanish as the disk centerline is approached, thus the boundary layer must erupt at the disk centerline.
Wilks [53] presented an integral momentum solution for the growth of the boundary layer due to swirling flow through a converging nozzle. Unlike the other boundary layer solutions discussed above, Wilks considered the mainstream flow to be composed of two velocity components: a uniform spherical radial flow and a potential vortex aligned with the vortex filament along the nozzle centerline. In this investigation Wilks assumed that the relationships between the spherical radial momentum thickness and the boundary layer thickness and between the mixed (radial and tangential) momentum thickness and the boundary layer thickness were linear. That is, the ratio of each respective momentum thickness to its derivative was assumed equal to the ratio of the boundary layer thickness to its derivative. With this basic assumption, Wilks used the two parameter method of Weighardt [54] to solve the integral momentum equations for the boundary layer radial velocity profile and a boundary layer thickness. Results presented from this solution indicated that the radial velocity within the boundary layer exceeded the corresponding mainstream component. Houlihan and Hornstra [55] extended Wilks [53] solution to account for the effect of the boundary layer growth upon the tangential and axial velocities in the mainstream flow. As a result of this extension no velocities within the boundary layer were found to exceed the corresponding mainstream values.

The reader should be aware that this literature review does not
include all articles on the subject of confined vortex motion. However, the articles listed herein are the most significant of the more than 375 articles reviewed by the author.
III. MAINSTREAM

Even in very simple container geometries, the confined vortex presents a bewildering variety of mainstream motions. In many cases, the axial flow near the centerline of a confined vortex exhibits a strong updraft, similar to those found in a tornado. In other cases, the centerline axial flow can be brought from a value of several times the vortex velocity to rest, just as if a stagnation point created by a solid body was present. In still other instances, the centerline axial flow reverses itself, forming multi-celled vortex motions similar to those described by Donaldson and Sullivan [31]. Perhaps the most striking and least understood feature of all these flows is their vigorous and highly responsive three dimensional character. Between the different flow components, there is a strong interaction which is associated with a marked responsiveness to changes in geometry and inlet conditions. The importance of viscous stresses in determining the nature of this interaction has been clearly demonstrated both analytically and experimentally by previous investigators (Donaldson [20], Rosenzweig [56], and Burgers [32]). Thus in this investigation, a class of solutions of the incompressible Navier-Stokes Equations is used to represent the mainstream vortex in a converging conical nozzle.

This Chapter is devoted to a discussion of the particular class of solutions to the Navier-Stokes Equations used in the vortex
mainstream region. In this class of solutions, the tangential velocity in spherical coordinates \((R, \theta, \phi)\) is of the form \(W = T(\theta)/R\). That is, the mainstream tangential velocity is assumed axisymmetric and to vary as the inverse of the spherical radius times a function of the transverse angle \((\theta)\). It has been shown (See Appendix A) that the most general conditions under which a solution of the type \(W = T(\theta)/R\), exists, is that the radial and transverse velocities functions are \(U = G(\theta)/R + \overline{G}(\theta)/R^2\) and \(V = F(\theta)/R\), respectively. An examination of these velocity functions shows that they do exhibit the essential characteristics of conical vortex flows. For example, if \(T(\theta)\) were equal to \(1/\sin(\theta)\) and if \(\overline{G}(\theta)\) were a constant, while \(F(\theta)\) and \(G(\theta)\) were equal to zero, then the velocity profiles would be those of a potential vortex superimposed on a conical sink flow.

Further examination of these velocity functions shows that the form of the velocity profile at any radial station is determined by the four functions of the transverse angle, \(T(\theta), G(\theta), \overline{G}(\theta),\) and \(F(\theta)\). Thus, these functions are refered to herein as velocity profile forms.

Unfortunately, the above four velocity profile forms, plus the mainstream static pressure represent five unknowns to be solved from the four incompressible equations of motion. Therefore, one of the radial velocity profile forms must be neglected to reduce the number of unknowns to a solvable system. If the \(G(\theta)\) profile form is neglected the equations of motion are decoupled (See Appendix A) and cannot produce the strong interaction necessary to describe the
mainstream vortex flow. For this reason, the \( \tilde{G}(\theta) \) profile form is neglected, thus the number of unknowns is reduced to a solvable set.

It should be noted, that all the velocity functions used in this investigation are functions of two dimensions. In the velocity functions of previous investigators, (for example, Donaldson [20]) only one velocity component was a function of two dimensions. Thus, the velocity functions of this investigation are more general than those of previous investigations. However, when applied to a confined vortex flow through a conical converging nozzle, the particular forms of the velocity profiles used in this investigation do not insure that mass and momentum of the total system are conserved at all points in the nozzle. Therefore, in the composite solution (See Chapter V) conservation of total system mass and momentum is assured by obtaining mainstream solutions at a number of stations along the conical nozzle.

A. Development of Equations

Assuming the tangential velocity in the mainstream vortex can be represented by \( W = T(\theta)/R \) and neglecting the \( \tilde{G}(\theta) \) radial velocity profile form, one can show that the mainstream vortex velocities are of the form:

\[
W = T(\theta)/R \quad U = G(\theta)/R \quad V = F(\theta)/R
\]

(1)

* See Appendix A for a detailed development
The complete axisymmetric Navier-Stokes Equations for an incompressible fluid with velocities of the form given in equation (1)** are:

\[
\frac{dF}{d\theta} + F \cot \theta = -G
\]  
(2)

\[
\frac{F}{\nu} + \frac{FT}{\nu} \cot \theta = \frac{d^2T}{d\theta^2} + \frac{dT}{d\theta} \cot \theta - T \csc^2 \theta
\]  
(3)

\[
F \frac{dF}{d\theta} - T^2 \cot \theta = \frac{R^2 \rho}{\rho} \frac{\partial P}{\partial \theta} + \nu \left[ \frac{d^2F}{d\theta^2} + \frac{dF}{d\theta} \cot \theta - F \csc^2 \theta + 2 \frac{dG}{d\theta} \right]
\]  
(4)

\[
-\frac{G^2}{d\theta} + \frac{dG}{d\theta} F - F^2 - T^2 = -\frac{\rho c R^3}{P} \frac{\partial P}{\partial R} + \nu \left[ \frac{d^2G}{d\theta^2} + \frac{dG}{d\theta} \cot \theta \right]
\]  
(5)

After removing the static pressure by cross differentiation and combining equations (4) and (5), one finds that the combined radial and transverse momentum equation becomes:

\[
2T^2 \cot \theta + 2T \frac{dT}{d\theta} + 2G \frac{dG}{d\theta} - F \frac{d^2G}{d\theta^2} - \frac{dF}{d\theta} \frac{dG}{d\theta}
\]  
(6)

\[
+ \nu \left[ 2 \frac{dG}{d\theta} - \frac{dG}{d\theta} \csc^2 \theta + \frac{d^2G}{d\theta^2} \cot \theta + \frac{d^3G}{d\theta^3} \right] = 0
\]

In order to nondimensionalize the equations of motion, the following nondimensional velocity forms are defined.

\[
U = \frac{U R_i}{R} C_o(\theta); \quad W = \frac{W_i R_i}{R} T_o(\theta); \quad V = \frac{U R_i F_o(\theta)}{R}
\]  
(7)

** Numbers in parentheses denote equation numbers.
Where the subscript "o" denotes nondimensional quantities, "i" denotes inlet quantities, and "c" denotes centerline quantities. After non-dimensionalization and rearrangement, the equations of motion become:

\[ \frac{dF_o}{d\theta} + F_o \cot \theta = -G_o \]  \hspace{1cm} (8)

\[ \frac{dT_o}{d\theta} = T_o \csc^2 \theta - \frac{dT_o}{d\theta} \cot \theta + Re_c F_o \left( \frac{dT_o}{d\theta} + T_o \cot \theta \right) \]  \hspace{1cm} (9)

\[ \frac{d^3G_o}{d\theta^3} - \frac{dG_o}{d\theta} \csc^2 \theta + \frac{d^2G_o}{d\theta^2} \cot \theta + 2 \frac{dG_o}{d\theta} + Re_c \left[ 2G_o \frac{dG_o}{d\theta} - \frac{dF_o}{d\theta} - \frac{dT_o}{d\theta} + 2Ro_i^2 \right] \]  \hspace{1cm} (10)

The parameters \( Re_c \) and \( Ro_i \) are the centerline radial Reynolds Number and the Rossby Number, which are respectively defined as

\[ Re_c = \frac{U_c R}{\nu} \hspace{1cm} Ro_i = \frac{W_i}{U_c} \]  \hspace{1cm} (11)

The class of physical flows to be investigated with equations (8), (9), and (10) is represented by the flow of an incompressible viscous vortex confined inside a conical converging nozzle. Since, in the overall solution of the nozzle flow field, the mainstream-boundary layer interaction is modeled by matching the mainstream and boundary layer solutions at the outer edge of the boundary layer, the first mainstream solution uses slip velocity boundary conditions at the outer edge of the vortex. The physical characteristics of the confined vortex flow require that, at the nozzle centerline, the
values of the tangential velocity and the transverse velocity and the partial derivative of the radial velocity with respect to the transverse angle, all be zero. The boundary conditions for equations (8), (9), and (10) are completed with the assumption that at the nozzle centerline, the derivatives of all the velocity components remain finite. The complete set of boundary conditions for the equations of motion are as follows:

\[
\begin{align*}
G_0(0) = 1.0 & \quad \frac{dG_0}{d\theta} \bigg|_{\theta=0} = 0 & \quad F_0(0) = 0 \\
T_0(0) = 0 & \quad \frac{dT_0}{d\theta} \bigg|_{\theta=0} = A & \quad \frac{d^2G_0}{d\theta^2} \bigg|_{\theta=0} = D
\end{align*}
\]  

(12)

Once again, the reader should be reminded that the form of the radial velocity profile used in this investigation does not insure the conservation of total system mass flow at all points along the length of the nozzle. Therefore, in the composite solution of the nozzle flow field, it is necessary that the mainstream solution obtained at each nozzle station conserve mass flow and momentum of the total system and also match the boundary layer transverse velocity at the outer edge of the boundary layer. Section C of this chapter shows that with the proper choice of the centerline boundary conditions, all the above conditions required of the mainstream solution are satisfied.
B. Limiting Solutions

Closed form solutions of equations (8), (9) and (10) have been obtained for both the very viscous \((\text{Re}_c \to 0)\) and the inviscid \((\text{Re}_c \to \infty)\) cases. The paragraphs below will first discuss the inviscid solution and then discuss the very viscous solution.

From an inviscid consideration of vortex flows inside a conical nozzle, one would expect a solution that would be represented by a potential vortex superimposed on radial sink flow. It will be shown below that the inviscid solution for the class of solutions discussed in this chapter compares well with these expectations.

If the flow is considered inviscid, that is \((\text{Re}_c \to \infty)\), then the limiting forms of equations (8), (9), and (10) are:

\[
\frac{dF_0}{d\theta} + F_0 \cot \theta = -G_0 \quad (13)
\]

\[
\frac{dT_0}{d\theta} + T_0 \cot \theta = 0 \quad (14)
\]

\[
2G_0 \frac{dG_0}{d\theta} - \frac{dF_0}{d\theta} \frac{dG_0}{d\theta} - F_0 \frac{d^2G_0}{d\theta^2} + 2 \text{Ro} \frac{2}{i} T_0 \left[ T_0 \cot \theta + \frac{dT_0}{d\theta} \right] = 0 \quad (15)
\]

Equations (13), (14), and (15) require boundary conditions which are two less than those for the complete set of transformed equations of motion. The exclusion of the boundary condition on the tangential velocity profile form reflects the fact that in inviscid flows there are no viscous shear stresses to require the vanishing of the tangential velocity at the nozzle centerline. Likewise, the exclusion of
the boundary condition on the second derivative of the radial velocity profile form reflects the neglect of the viscous shear stresses. From equation (14), one can see that in a nonviscous fluid, the transverse velocity profile form does not appear in the transformed tangential momentum equation. Thus, the tangential velocity profile form is independent of the profile forms of the other two velocity components. Substitution of the tangential velocity profile form from equation (14) into equation (15) also shows that the tangential velocity profile form does not effect the radial and transverse velocity profile forms. Therefore, only two coupled differential equations, equations (13) and (15), must be solved to yield an inviscid solution to the transformed equations of motion.

The general solutions of equations (13), (14), and (15) are (See Appendix C for details of the solution)

\[ T_0 = \frac{C}{\sin \theta} \quad F_0 = -\tan(\theta/2) \]

\[ G_0 = 1.0 \]

Therefore, with the assumed forms of the velocity profiles, the inviscid velocity solutions are:

\[ W = \frac{W_1 R_1 \sin \alpha}{R \sin \theta} \quad V = -\frac{U_c R_1}{R} \tan(\theta/2) \quad U = \frac{U_c R_1}{R} \]

(17)

As expected, the tangential velocity is found to be that of a potential vortex. On the other hand, the radial velocity varies as the inverse
of the radial distance and not as the square of radial distance. Thus, the radial component of velocity does not represent radial sink flow. Once again, in the overall solution (See Chapter V) this shortcoming is accounted for by the determination of mainstream solutions at various stations along the length of nozzle. This approach allows \( U_c \) to be a function of the radial distance.

As the centerline radial Reynolds Number approaches zero (very viscous), the limiting forms of equations (8), (9), and (10) are:

\[
\frac{dF_0}{d\theta} + F_0 \cot \theta = -G_0 \tag{18}
\]

\[
\frac{d^2T_0}{d\theta^2} + \frac{dT_0}{d\theta} \cot \theta - T_0 \csc^2 \theta = 0 \tag{19}
\]

\[
\frac{d^3G_0}{d\theta^3} - \frac{dG_0}{d\theta} \csc^2 \theta + \frac{d^2G_0}{d\theta^2} \cot \theta + 2 \frac{dG_0}{d\theta} = 0 \tag{20}
\]

In the very viscous flows, as in the nonviscous case, the tangential velocity form is decoupled from the radial and transverse velocity forms. However, the very viscous equations of motion require the same number of boundary conditions as the complete equations of motion. With the assumption that slip conditions exist at the nozzle wall, these boundary conditions are:

\[
G_0(\theta) = 1.0 \quad \frac{d^2G_0}{d\theta^2} \bigg|_{\theta=0} = D \quad \frac{dG_0}{d\theta} \bigg|_{\theta=0} = 0
\]

\[
F_0(0) = 0 \quad T_0(0) = 0 \quad \frac{dT_0}{d\theta} \bigg|_{\theta=0} = A
\]
With these boundary conditions, the solutions for the very viscous velocity profiles are:

\[ W = 2 \frac{Wi Ri}{R} A \tan(\theta/2) \]

\[ U = \frac{Uc Ri}{R} \left[ D(1 - \cos \theta) + 1 \right] \]

\[ V = \frac{Uc Ri}{R} \left[ \frac{D}{2} \sin \theta - (D + 1.) \tan \theta/2 \right] \]

A close examination of this solution indicates that the multi-celled vortex structure that is so prevalent in most vortex solutions is also contained in equation (22). If the centerline value of the second derivative of the radial velocity form is negative, the radial velocity component of equation (22) shows that a dual celled vortex can exist. On the other hand, if \( D \) is positive then only a single celled vortex is present. Physical reasons for the celled vortex structure in the very viscous limiting solution are not fully understood. However, as will be shown below in the general case a negative centerline value of the second derivative of the radial velocity profile form implies that the flow is advancing into an area of increased centerline static pressure which requires reversed radial flow to occur at the nozzle centerline.

C. General Solutions

In order to obtain general solutions of equations (8), (9), and (10), numerical techniques must be employed. Equations (8), (9), and (10) are ordinary differential equations, and are therefore easier to solve than the original Navier-Stokes Equations. Even with this
simplification, however, equations (8), (9), and (10) are still very
difficult to solve since they still contain all of the nonlinearities
of the Navier-Stokes Equations.

In order to facilitate numerical calculations, equations (8),
(9), and (10) are transformed into a system of six first order
differential equations.

\[
\frac{dF_0}{d\theta} + F_0 \cot \theta = -G_0 \tag{23}
\]

\[
\frac{dB_0}{d\theta} = Re_c F_0 [B_0 + T_0 \cot \theta] + T_0 \csc^2 \theta - B_0 \cot \theta \tag{24}
\]

\[
\frac{dT_0}{d\theta} = B_0 \tag{25}
\]

\[
\frac{dH_0}{d\theta} = -2 Re_c Ro_0^2 T_0^2 \cot \theta \tag{26}
\]

\[
\frac{dQ_0}{d\theta} = H_0 - Q_0 \cot \theta - 2G_0 - Re_c [G_0^2 - F_0 Q_0 + Ro_1^2 T_0^2] \tag{27}
\]

\[
\frac{dG_0}{d\theta} = Q_0 \tag{28}
\]

After experimentation with a number of numerical techniques, the
fourth order Runge-Kutta method was found to be the most convenient
technique for use in obtaining the solution of this system of equations.
The accuracy of the Runge-Kutta method is indicated in Figure 4. In
this figure, a comparison of the numerical and analytical solutions
for the very viscous case \((Re_c \rightarrow 0)\) with \(D\) equal to \(-5000.0\) is shown.
Figure 4. Comparison of Analytical and Numerical Solutions for the Radial Velocity Form
An examination of Figure 4 shows that the numerical and analytical solutions for the limiting very viscous case agree within one tenth of one per cent.

A typical general solution for the radial velocity profile form is shown in Figure 5. For this figure, the generating parameters are; \( \text{Re}_{c} = 100.0 \), \( \text{Ro}_{1} = 1.0 \), \( A = 2 \times 10^{4} \), and \( D = -4 \times 10^{7} \). On close examination, Figure 5 shows that the flow field contains two distinct regions. Near the nozzle centerline where the viscous shear stresses are the dominating influence, the radial velocity profile form approaches that of the very viscous solution. At some point away from the nozzle centerline, inertia forces begin to dominate and the radial velocity profile form reaches a maximum, after this point the general solution approaches the inviscid solution. Figure 6 indicates the behavior of the point of maximum velocity as a function of centerline Reynolds Number for \( \text{Ro}_{1} = 1.0 \), \( A = 1 \times 10^{4} \), and \( D = 2 \times 10^{7} \). That is, as the flow becomes less viscous the Reynolds Number increases, with a corresponding decrease in the size of the region where viscous forces are dominant.

With a variation of the ratio of angular to radial momentum, the behavior of the vortex motion undergoes a smooth transition from the single cell vortex shown in Figure 5 to the two and three celled vorticies shown in Figures 7 and 8, respectively. The parameters used to generate Figure 7 are; \( \text{Re}_{c} = -1000.0 \), \( \text{Ro}_{1} = 10.0 \), \( A = 1.5 \times 10^{4} \), and \( D = -2 \times 10^{7} \). The same values for the centerline
Figure 5. Radial Velocity Form
Figure 6. Effect of Centerline Reynolds Number on the Point of Maximum Radial Velocity
Figure 7. Typical Distribution of the Radial Velocity Form in a Two Celled Vortex

\[ R_e = -1000.0 \]
\[ R_{oi} = 10.0 \]
\[ A = 1.5 \times 10^4 \]
\[ D = -2 \times 10^7 \]
Figure 8. Typical Distribution of the Radial Velocity Form in a Three Celled Vortex. Note: Scale changes at zero for $G_o(\theta)$ and at .02 for transverse angle.
Reynolds Number and inlet Rossby Number as used in Figure 7 are used in Figure 8, however, in Figure 8, the values of A and D are $2.332 \times 10^4$ and $-4. \times 10^7$, respectively. The celled vortex motion shown in Figures 7 and 8 has been demonstrated by previous investigators such as Burgers [32] and Donaldson and Sullivan [31]. However, the profiles shown here differ in that all the velocity components are a function of two dimensions while in the previous investigations only one velocity component was a function of two dimensions.

Figure 9 shows the behavior of the calculated radial mass flow as a function of the ratio of tangential to radial momentum at a constant centerline Reynolds Number. That is, as the centerline value of the first derivative of the tangential velocity profile form is increased, the ratio of the tangential to the radial momentum of the system also increases. The generating parameters for Figure 9 are; $Re_c = 1000.0$, $Ro_I = 1.0$, and $D = 2. \times 10^7$. As shown by Figure 9 the radial mass flow in the negative radial direction decreases in magnitude and finally changes direction as the ratio of the tangential to radial momentum is increased. King [16] using an inviscid model for a confined vortex inside a converging nozzle reported similar trends. It should be noted, however, that this trend may not exist in the actual confined vortex flows. That is, neither the mainstream solution presented in this Chapter, nor the inviscid model of King's, includes the transport of radial mass flow by the wall boundary layers.
Figure 9. Effect of the Ratio of Tangential to Radial Momentum on System Mass Flow. Note: If the centerline Reynolds Number is held constant, an increase in $A$ causes a corresponding increase in the ratio of tangential to radial momentum.
Tangential and transverse velocity profile forms corresponding to the radial velocity form of Figure 8 are shown in Figures 10 and 11, respectively. The tangential velocity profile form for all cases considered has the characteristic shape shown in Figure 10. That is, the tangential velocity increases from a zero value at the vortex centerline to some maximum and then decreases as angular momentum is conserved in the outer regions of the vortex. This characteristic shape is once again explained by the fact that near the vortex centerline, the viscous shear stresses are the dominating influence and in the exterior regions of the vortex, inertia forces are dominant. It should also be noted that the sign of the tangential velocity profile form does not effect the magnitude or the sign of the other velocity profile forms. However, if the tangential velocity changes direction, then a solution of the type described herein does not exist. The physical reason for the above is that if the viscous shear stresses dissipate the total system angular momentum, the solutions of the class described in this Chapter contain no physical mechanisms whereby the generation of angular momentum can take place. The inward flowing transverse velocity profile form shown in Figure 11 is not typical of all general cases. In fact, the coupling of the transverse velocity and the radial velocity through the continuity equation requires that the transverse velocity decrease toward the area of increasing radial velocity. Thus, the behavior of the
Figure 10. Typical Distribution of the Tangential Velocity Form
Figure 11. Typical Distribution of the Transverse Velocity Form for a Three Celled Vortex

\[ \text{Re}_c = -1000.0 \]
\[ \text{Ro}_i = 10.0 \]
\[ A = 2.332 \times 10^4 \]
\[ D = -4.0 \times 10^7 \]
transverse velocity depends upon the magnitude of the radial velocity and the number of vortex cells.

A close examination of the radial momentum equation, equation (5), yields a physical interpretation of the celled vortex motions discussed above. The limiting nondimensional form of equation (5) as the nozzle centerline is approached is

$$\frac{\partial P}{\partial R} \bigg|_{\theta=0} = 1.0 + \frac{2D}{Re_c}$$

From equation (29), the change in mainstream static pressure along the nozzle centerline is seen to be only a function of the centerline Reynolds Number and the second derivative of the radial velocity profile form. Away from the nozzle centerline, however, the partial derivative of the static pressure with respect to the radius is a function of the interaction of all the velocity components. Thus, celled vortex motions occur when the interaction of the velocity components forces the partial derivative of the static pressure with respect to the radius to change sign. That is, at the nozzle centerline, the flow could be advancing into an area of reduced static pressure while at a distance away from the nozzle centerline the flow could encounter an area of increased static pressure.

A comparison of Figures 8 and 10 shows that the magnitude of the tangential velocity ranges from nearly the same value near the vortex centerline to almost an order of magnitude higher than the radial
velocity near the edge of the vortex. This evidence of strong tangential velocities coupled with the variety of types of motions possible, indicates that the particular class of solutions to the general equations of motion described in this Chapter possess the necessary characteristics to describe the motions of confined vortex flows.
IV. BOUNDARY LAYER

In confined vortex flows, the very definition of a boundary layer thickness is quite complicated. The entire flow field of a confined vortex must be considered viscous; thus, the traditional definition of the boundary layer must be replaced by a more general concept. The boundary layer thickness used herein is defined as the distance from the wall where the boundary layer velocities and shear stress asymptotically approach the corresponding mainstream values. Furthermore, confined vortex boundary layers are part of an internal flow field as opposed to an external flow field. The boundary layers in an external flow field only displace the mainstream from the walls, thereby, causing small perturbations to the mainstream flow. On the other hand, in confined vortex flow fields, the boundary layers not only displace the mainstream from the container walls, but, in fact, limit the very strength of the mainstream vortex. Finally, confined vortex boundary layers are in general three dimensional and thus are even more complicated.

Most investigators of the confined vortex boundary layer (Taylor [4], Cooke [14], Garbsch [57], Mack [45], Weber [13], Rott [48], Anderson [51], and Burrgrof [52]) have assumed that the effect of the mainstream velocity component along the container wall could be neglected in comparison with the effect of the mainstream vortex
component. That is, they assumed that the component of velocity along the wall was zero both at the wall and at the outer edge of the boundary layer. When the surface on which the boundary layer grows is perpendicular to the axis of the mainstream vortex, as in the case of the end walls of a cylindrical container, the above assumption has merit; however, when the surface is converging to or diverging from the swirl axis, as in the case of boundary layer growth along the walls of a conical nozzle, the above assumption is not valid. For example, consider the flow of a vortex inside a conical converging nozzle (See: Linden [11], Lineberry [12]). As the apex of the cone is approached, conservation of mass and angular momentum require that the mainstream radial velocity component approach the same order of magnitude as the tangential component, regardless of the ratio of inlet radial to tangential velocity.

Most of the boundary layer investigations mentioned above can be classified into two groups. The first group used a technique developed by Taylor [4] (the T-method), which allows the solution of the boundary layer integral momentum equations for a boundary layer thickness and a boundary layer velocity weighting function. Taylor introduced the boundary layer velocity weighting function in order to allow the magnitude of the boundary layer velocity component along the wall to vary as a function of distance along the wall. The second group composed of the work by Cooke [14] who noted that when the boundary
layer integral momentum technique is extended to three dimensional boundary layers, the boundary layer velocity components do not asymptotically approach their respective mainstream values at the same distance from the wall (See: Cooke [14] and Hartree [58]). Thus, Cooke solved the integral momentum equations for two boundary layer thicknesses (the C-method). In order to allow the magnitude of the boundary layer velocity to vary along the wall, Cooke defined a boundary layer velocity weighting function which satisfied the radial momentum equation at the vortex centerline.

In a very recent article, Houlihan and Hornstra [55] presented an integral momentum solution in which they considered effects that were not included in the boundary layer solutions discussed above. That is, the effect of the velocity component along the wall on the growth of the boundary layer and the effect of the boundary layer thickness on both the tangential and axial mainstream velocities are considered. In order to obtain a solution, Houlihan and Hornstra assumed linear relationships between the radial momentum thickness and the boundary layer thickness and between the mixed momentum thickness and the boundary thickness. With this assumption, the integral momentum equations were solved for single boundary layer thickness and the radial velocity profile in the boundary layer. In section B of this chapter, the effects of this assumed function between the two momentum thicknesses and the boundary layer thickness are discussed in detail.
In the following paragraphs, a boundary layer integral momentum method (E-method) that includes the effects of both the mainstream vortex component and the component of mainstream velocity along the wall is developed. Similar to Cooke's method, the E-method involves solution of the integral momentum equations for two boundary layer thicknesses; however, the component of mainstream velocity along the wall, not the radial compatibility condition, is used as the boundary layer velocity weighting function. The E-method is then used to predict the boundary layer growth for a vortex confined inside a converging conical nozzle (See Figure 12).

A. Development of the Boundary Layer Integral Momentum Equations*

The incompressible boundary layer equations are obtained from an order of magnitude analysis of the Navier-Stokes Equations. The usual assumption that the boundary layer thickness is very small compared with the characteristic length of the cone, is the only assumption necessary for this analysis.

The steady, incompressible equations of motion in spherical coordinates are:

Continuity equation

\[
\frac{1}{R^2} \frac{\partial}{\partial R} (R^2 u) + \frac{1}{R \sin \theta} \frac{\partial}{\partial \theta} (v \sin \theta) + \frac{1}{R \sin \theta} \frac{\partial w}{\partial \phi} = 0
\]  

(30)

* For detailed development, Appendix B
Figure 12. Boundary Layer Geometry
Radial Momentum Equation

\[ u \frac{\partial u}{\partial R} + \frac{v}{R} \frac{\partial u}{\partial \theta} + \frac{w}{R \sin \theta} \frac{\partial u}{\partial \phi} - \frac{w^2 + v^2}{R} = - \frac{1}{\rho} \frac{\partial P}{\partial R} \]

\[ + v \left( \nabla^2 u + \frac{2}{R^2} u - \frac{2}{R^2} \frac{\partial v}{\partial \theta} - \frac{2w}{R^2} \cot \theta - \frac{2}{R^2 \sin \theta} \frac{\partial w}{\partial \phi} \right) \]

Transverse Momentum Equation

\[ u \frac{\partial v}{\partial R} + \frac{v}{R} \frac{\partial v}{\partial \theta} + \frac{w}{R \sin \theta} \frac{\partial v}{\partial \phi} + \frac{uv}{R} - \frac{w^2 \cot \theta}{R} = - \frac{1}{\rho} \frac{\partial P}{\partial \theta} \]

\[ + v \left( \nabla^2 v + \frac{2}{R^2} \frac{\partial u}{\partial \theta} - \frac{v}{R^2 \sin^2 \theta} - \frac{2 \cos \theta}{R^2 \sin \theta} \frac{\partial w}{\partial \phi} \right) \]

Tangential Momentum Equation

\[ u \frac{\partial w}{\partial R} + \frac{v}{R} \frac{\partial w}{\partial R} + \frac{w}{R \sin \theta} \frac{\partial w}{\partial \phi} + \frac{wu}{R} + \frac{wv}{R} \cot \theta = \]

\[ - \frac{1}{\rho} \frac{\partial P}{\partial \phi} + v \left( \nabla^2 w - \frac{w}{R^2 \sin^2 \theta} + \frac{2}{R^2 \sin \theta} \frac{\partial u}{\partial \phi} + \frac{2 \cos \theta}{R^2 \sin \theta} \frac{\partial v}{\partial \phi} \right) \]

Where:

\[ \nabla^2 = \frac{1}{R^2} \frac{\partial}{\partial R} \left( R^2 \frac{\partial}{\partial R} \right) + \frac{1}{R^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{R^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \]

In the equations above, \( R \) is the spherical radius; \( \theta \) is the angle across the cone, the transverse angle; \( \phi \) is the angle around the cone centerline; and \( u, v, w \) are the respective velocity components (See Figure 12); \( \rho \) is the density, and \( \nu \) is the kinematic viscosity.
An order of magnitude analysis of the governing equations yields the following results: In the continuity equation (30), the term \( (v/R) \) is at least one order of magnitude smaller than the other terms in the equation. The term \( (v^2/R) \) and all viscous terms except
\[
\frac{1}{R^2} \frac{\partial^2 u}{\partial \theta^2}
\]
can be neglected in comparison with the other terms in the radial momentum equation (31). The entire transverse momentum equation (32) is found to be an order of magnitude smaller than the other governing equations, thus, the usual assumption of constant pressure across the boundary layer holds. In the tangential momentum equation (33), the term \( (vw \cot \Theta) \) and all viscous terms except
\[
\frac{1}{R^2} \frac{\partial^2 w}{\partial \theta^2}
\]
are an order of magnitude less than the other terms in the equation. A word of warning is necessary at this point. If the reader attempts to repeat the order of magnitude analysis, he should note that all trigonometric functions refer to the boundary layer position, not the boundary layer thickness and are therefore of order "one".

With the axisymmetric flow assumption, the order of magnitude analysis reduces the governing equations to the following boundary layer form.

Boundary Layer Continuity Equation
\[
\frac{2u}{R} + \frac{\partial u}{\partial R} + \frac{1}{R} \frac{\partial v}{\partial \theta} = 0
\]
Boundary Layer Radial Momentum Equation

\[
\frac{u}{R} \frac{\partial u}{\partial R} + \frac{v}{R} \frac{\partial u}{\partial \theta} - \frac{w^2}{R} = -\frac{1}{\rho} \frac{\partial P}{\partial R} + \frac{\nu}{R^2} \frac{\partial^2 u}{\partial \theta^2}
\]  (36)

Boundary Layer Tangential Momentum Equation

\[
u \frac{u}{R} \frac{\partial w}{\partial R} + \frac{v}{R} \frac{\partial u}{\partial \theta} + \frac{uw}{R} = \frac{\nu}{R^2} \frac{\partial^2 w}{\partial \theta^2}
\]  (37)

Boundary Layer Transverse Momentum Equation

\[
\frac{\partial P}{\partial \theta} = 0
\]  (38)

The boundary conditions to be satisfied in the E-method are:

At the cone wall, \( \theta = \alpha \): \( u(R, \alpha) = w(R, \alpha) = 0 \)  (39)

At the edge of the boundary layer, \( \theta = x \): \( u(R, x) = U \) and \( w(R, x) = W \). \( U \) and \( W \) denote the mainstream values of \( u \) and \( w \) at \( \theta = x \).

The radial and tangential boundary layer momentum equations (36) and (37) can now be integrated to form the integral boundary layer momentum equations.

\[
2 \int_{x}^{\alpha} u \frac{\partial u}{\partial R} \, d\theta + \int_{x}^{\alpha} u^2 \, d\theta - \frac{U}{R} \int_{x}^{\alpha} 2ud\theta - \frac{U}{R} \int_{x}^{\alpha} u \frac{\partial u}{\partial R} \, d\theta + \int_{x}^{\alpha} \left[ \frac{1}{\rho} \frac{\partial P}{\partial R} \right] \, d\theta
\]

\[
- \int_{x}^{\alpha} \frac{w^2}{R} \, d\theta = \frac{\nu}{R^2} \frac{\partial u}{\partial \theta} \bigg|_{\alpha}
\]  (40)
\[ \int_{x}^{\alpha} \frac{\partial}{\partial R} (uw) \, d\theta + 3 \int_{x}^{\alpha} \frac{uw}{R} \, d\theta = \frac{\nu}{R^2} \frac{\partial w}{\partial \theta} \bigg|_{x}^{\alpha} + \frac{1}{R} (vw) \bigg|_{x} \]  

(41)

where \( \alpha \) denotes the transverse angle at the cone wall and \( x \) is either the transverse angle at the edge of the radial or tangential boundary layer thickness. That is, equations (40) and (41) must be integrated across the entire boundary layer; therefore, the thicker component of the boundary layer defines the lower limit of integration.

The terms \( \left( \frac{\partial}{\partial R} (Q \cdot \bar{u}) \right) \) and \( \left( R \frac{\partial}{\partial R} (Q \cdot \bar{u}) \right) \) in equation (40) do not appear in the integral momentum equations used either in the T-method or the C-method. These two terms represent the net change in boundary layer. Thus, in the previous studies, these terms were set equal to zero, since the mainstream radial component of velocity was considered to be zero.

In order to carry out the indicated integration of equations (40) and (41), boundary layer velocity profile functions which satisfy the boundary conditions of equation (39) must be assumed. Thus, the boundary layer velocity profiles are assumed to be a function of the distance from the wall times the respective mainstream velocity component at the outer edge of the boundary layer (See Figure 13). That is, the radial boundary layer velocity profile is assumed to have the form \( u(R, \theta) = U(R) f(N) \) where \( N \) is defined as the ratio of the
Figure 13. Boundary Layer Nomenclature
distance from the wall to the radial boundary layer thickness, i.e. 
\[ N = \frac{R(\theta - \alpha)}{\delta_R} \]. Likewise, the tangential boundary layer velocity profile is assumed to have the form 
\[ w(R, \theta) = W(R) g(N_1) \] where \( N_1 \) is defined as the ratio of the distance from the wall to the tangential boundary layer thickness, i.e. 
\[ N_1 = \frac{R(\theta - \alpha)}{\delta_T} \]. With these assumed forms of the boundary layer velocity profiles, the integration of equations (40) and (41) yields the following results: The nondimensional radial boundary layer integral momentum equation is

\[ \frac{d(\Delta^2)}{dR_0} = \Lambda^2 \left[ (-4 - 2b) \frac{1}{U_o} \frac{dU_o}{dR_0} - \frac{2}{R_o} + \frac{2}{a-b} \frac{X}{\rho_o U_o^2} \frac{\partial \rho}{\partial R} \right] \]

\[ + \frac{2c}{a-b} (R_{o1})^2 \left( \frac{W_o}{U_o} \right)^2 \left( \frac{1}{R_o} \right) \]

\[ + 2d \frac{1}{a-b} \left( \frac{1}{R_{e1}} \right) \left( \frac{1}{U_o} \right) \] \( \tag{42} \)

The nondimensional tangential boundary layer integral momentum equation is

\[ \frac{dK}{dR_0} = \frac{1}{U_o} \frac{dU_o}{dR_0} \left[ \frac{K(e - b)}{m} \right] + \frac{1}{W_o} \frac{dW_o}{dR_0} \left[ \frac{Ke}{m} \right] + \frac{1}{R_o} \left[ \frac{K(h + m + 3e - b + X)}{m} \right] \]

\[ - \frac{1}{\Lambda} \frac{d\Lambda}{dR_0} \left[ \frac{K(m + b + h + X)}{m} \right] - \frac{1}{R_{e1}} \left[ \frac{1}{U_o \Lambda^2 m} \right] \] \( \tag{43} \)

where \( K = \frac{\delta_T}{\delta_R}, \quad \Lambda = \frac{\delta_R}{R_{e1}} \) and the subscripts "o" and "i" denote nondimensional and inlet quantities, respectively. The two nondimensional
parameters are the inlet radial Reynolds Number \( \text{Re}_i \) and the Rossby Number \( \text{Ro}_i \), the ratio of inlet tangential to radial velocity. The integral coefficients are defined by

\[
a = \int_X^\infty f^2(N)\,dN, \quad b = \int_X^\infty f(N)\,dN, \quad c = \int_X^\infty g^2(N_1)\,dN_1, \quad d = \frac{\partial f}{\partial N} \bigg|_{N=0}
\]

\[
e = \int_X^\infty f(N) \, g(N_1) \, dN_1, \quad h = \int_X^\infty g(N_1) \frac{\partial f}{\partial N} \bigg|_{N=0} \, dN_1, \quad m = \int_X^\infty f(N) \frac{\partial g}{\partial N_1} \, dN_1 \, dN_1, \quad j = \frac{\partial g}{\partial N_1} \bigg|_{N_1=0}
\]

(44)

where \( X \) denotes the value of \( N \) at the edge of the larger boundary layer.

It is important to note two things from equations (42) and (43); first, the only features of the arbitrary velocity functions \( f(N) \) and \( g(N_1) \) that have been used to obtain equations (42) and (43) are the boundary conditions at the wall and at the outer edge of the boundary layer. Thus, the functions \( f(N) \) and \( g(N_1) \) are still arbitrary except for the boundary conditions. Second, as soon as \( N \) reaches unity, \( f(N) \) reaches unity and remains unity for values of \( N \) greater than one. Likewise, \( g(N_1) \) has the value of unity for all values of \( N_1 \) equal to or greater than unity. The reader should be aware that the above statements imply that after reaching the boundary layer edge the mainstream velocity component in the corresponding direction remains
constant until the edge of the other boundary layer is reached. An error is introduced into the calculations by this assumption; however, it is very small since the difference in the thickness of two boundary layers is small and the change in magnitude of the mainstream velocity across this distance is small compared with the actual magnitude of the mainstream velocity.

The integral coefficients are taken across the entire boundary layer thickness. Consequently, they will have different values depending upon whether the boundary layer thickness ratio $K$ is greater than or less than unity.

To illustrate this point, we evaluate the integral $e$ with $K$ less than one, that is, with the radial boundary layer thickness greater than the tangential boundary layer thickness.

$$ e = \int_{N_1}^{N_2} f(N) g(N) \, dN = \int_{N_1}^{K} f(N) \, dN + \int_{K}^{N_2} f(N) g(N_1) \, dN \quad (45) $$

On the other hand, if $K$ is greater than one, the tangential boundary layer thickness is greater than the radial boundary layer thickness and the limits of the integration on $N$ must run from $K$ to zero.

$$ e = \int_{K}^{N_2} f(N) g(N_1) \, dN = \int_{K}^{1} g(N_1) \, dN + \int_{1}^{N_2} f(N) g(N_1) \, dN \quad (46) $$

Since the boundary layer thickness ratio $K$ is contained in the limits for the integral coefficients in equations (42) and (43), $K$
being greater than or less than unity changes not only the value of the coefficients, but also the degree of the nonlinear terms in equation (43). Thus, it is necessary to know whether $K$ is greater than or less than one before equations (42) and (43) can be solved.

The starting value of the boundary layer thickness ratio can be obtained by the following analysis of equations (42) and (43). At the inlet, the point of tangential fluid injection into the cone ($R_i$), the radial boundary layer thickness is zero. Therefore, by taking the limit of equation (42) as the inlet is approached, the slope of the radial boundary layer thickness is shown to be:

$$\frac{d(A^2)}{dR_o} = \frac{2d}{a-b} \frac{1}{\text{Re}_i U_0}$$  \hspace{1cm} (47)

Also, in the limit, equation (43) becomes:

$$- \frac{d(A^2)}{dR_o} \left[ \frac{K(m + b + h + f_X)}{2m} \right] = \frac{j}{m} \frac{1}{\text{Re}_i} \frac{1}{U_0} = 0$$  \hspace{1cm} (48)

After combining and rearranging equations (47) and (48), we obtain:

$$K(m + b + h + X) = \frac{j}{d} (a-b)$$  \hspace{1cm} (49)

The starting value for $K$ can now be obtained by the following procedure:

1) Choose the polynomial boundary layer velocity profiles $f(N)$ and $g(N_1)$. 
2) Assume \( K \) is greater than or less than unity.

3) Calculate the integral coefficients \( m, h, j, d, a, \) and \( b \). Solve for the roots of equation (49).

4) Check to see if the positive real roots agree with the assumed value of \( K \).

The results obtained after application of the above procedure to a number of assumed polynomial boundary layer velocity profiles are listed in Table I.

TABLE I

Starting Value of the Boundary Layer Thickness Ratio

<table>
<thead>
<tr>
<th>Velocity Profile</th>
<th>Roots for ( K ) assumed less than one</th>
<th>Roots for ( K ) assumed greater than one</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Cooke's ( f(N) = N - 2N^2 + N^3 ) ( g(N_1) = \frac{3}{2} N_1 - \frac{1}{2} N_1^3 )</td>
<td>1.7827</td>
<td>1.7242</td>
</tr>
<tr>
<td>2. Taylor's ( f(N) = N - 2N^2 + N^3 ) ( g(N_1) = 2N_1 - N_1^3 )</td>
<td>2.0648</td>
<td>1.78665</td>
</tr>
<tr>
<td>3. ( f(N) = 3/2N - 1/2N^3 ) ( g(N_1) = 3/2N_1 - 1/2N_1^3 )</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>4. ( f(N) = N - N^2 + N^3 ) ( g(N_1) = 3/2N_1 - 1/2N_1^3 )</td>
<td>1.45449</td>
<td>1.16</td>
</tr>
</tbody>
</table>

A number of things should be noted from Table I. First, in all cases the boundary layer thickness ratio is greater than or equal to unity. Cooke, on the other hand, found the starting value for the boundary layer thickness ratio to be 0.87933. This difference in the
starting value for the boundary layer thickness ratio arises from two reasons which are both related to the assumption of a nonzero mainstream radial velocity. The first reason pertains to the assumed form of the boundary layer velocity profile. Since Cooke considered zero radial mainstream velocities, he used the radial boundary layer thickness as a boundary layer velocity weighting function. That is, he assumed the boundary layer velocity to be of the form \( u(R, \theta) = C \delta_R^2 f(N) \). Where \( C \) is a constant, \( \delta_R^2 \) is the square of the radial boundary layer thickness and \( f(N) \) is a polynomial which satisfies the zero velocity boundary conditions both at the wall and at the outer edge of the boundary layer. Therefore, in Cooke's calculations, all of the boundary layer terms containing \( \partial u / \partial R \) produce different forms of the differentials of the radial boundary layer thickness than do boundary layer velocity profiles satisfying the nonzero radial mainstream velocity condition. The second reason for the difference in the starting value of the boundary layer thickness ratio arises from the integration of the boundary layer equations (40) and (41). If the radial component of the mainstream velocity is considered to be zero, equation (49) becomes:

\[
K(m+f+h) + \frac{a_j}{d} = 0
\]  

(50)

Substitution of the polynomial form of Cooke's velocity profile, without the boundary layer velocity weighting function, into equation
(50) and the use of the procedure for finding the starting value of the boundary layer thickness ratio previously described, yields a starting value greater than unity. Thus we see that Cooke's assumed weighting function forces the starting value of the boundary layer thickness ratio to be less than one. The reader should note that the only boundary layer velocity weighting function which can satisfy the nonzero mainstream radial velocity condition is the mainstream radial velocity component.

The boundary layer velocity profiles of Case 3 in Table I were used for the boundary layer calculations contained herein. In the previous investigations, the use of the standard definition of boundary layer thickness allowed the use of radial boundary layer velocity profiles that contained boundary layer velocities which were greater in magnitude than the mainstream radial component. That is, a boundary layer radial component of velocity was assumed even though no mainstream radial component was allowed. However, when the effects of the radial mainstream flow are included, a duality of points at which the definition of the boundary layer thickness is satisfied exists, if the boundary layer velocities are permitted to be greater than mainstream velocities. Therefore, boundary layer velocity profiles which satisfy the boundary conditions of equations (39) and (40), and contain no velocities greater than the mainstream velocity
were selected. The reader should be cautioned, however, that with the velocity profile of Case 3, the shear is zero at the outer edge of the boundary layer. But, since the mainstream must be considered viscous in a contained vortex flow, the shear is not necessarily zero at the boundary layer edges.

Case 3 in Table I shows that the starting value obtained from equation (49) is equal to unity no matter whether $K$ is assumed greater than or less than one. Two different methods were employed to insure the correct starting value of $K$. The first approach was to take the limit of equation (42) at the inlet to the cone (as $R \to R_i$). Since the radial boundary layer thickness is zero at this point, $dK/dR$ in equation (43) approaches minus infinity. Thus, as the fluid proceeds in the negative $R$ direction (from a larger spherical radius at the inlet to a smaller spherical radius at the exit) the boundary layer thickness ratio grows. The second approach was to numerically solve equations (42) and (43) with the assumption that the boundary layer thickness was less than unity. The results for $Re_i = -7.75 \times 10^4$ and $Ro_i^2 = 1087.0$ are shown in Figure 14. This figure shows a clear contradiction to this assumed value of the boundary layer thickness ratio. Thus, one must draw the conclusion that the boundary layer velocity profiles contained herein yield values of the boundary layer thickness ratio equal to or greater than unity.
B. Solution of the Boundary Layer Momentum Integral Equations

Equations (42) and (43) are rearranged into the following forms in order to facilitate numerical calculations.

\[
\frac{dZ_0}{d\rho_0} = Z_0 \left[ (-4 - \frac{2b}{a-b}) \frac{1}{U_0} \frac{dU_0}{d\rho_0} + \frac{2}{a-b} \frac{X}{P_0 U_0^2} \frac{dP_0}{d\rho_0} \right] + \frac{2e}{a-b} \frac{R_0^2}{R_0} \left( \frac{W_0}{U_0} \right)^2 + \frac{2d}{a-b} \frac{R_0^2}{Re_1} \frac{1}{U_0}
\]

where \( Z_0 = \Lambda^2 R_0^2 \)

\[
\frac{dY_0}{d\rho_0} = \frac{Y_0}{U_0} \frac{dU_0}{d\rho_0} \left[ \left( e-b \right) \frac{m}{m} \right] + \frac{Y_0}{W_0} \frac{dW_0}{d\rho_0} \left[ \frac{m}{m} \right] + \frac{Y_0}{R_0} \left[ \frac{(h+m+3e-b+X)}{m} \right] - \frac{d\Lambda^2}{d\rho_0} \left[ \frac{K(h+b+X-m)}{2m} \right] - \frac{1}{m} \frac{1}{Re_1} \frac{1}{U_0}
\]

where \( Y_0 = K\Lambda^2 \).

At first glance one would be tempted to try to obtain numerical solutions for equations (51) and (52) by the standard Runge-Kutta technique. However, it was found that for most mainstream flows of interest, the set of differential equations was very "stiff". That is, the magnitudes of the eigenvalues of the system are vastly different. Since the stability requirements of Runge-Kutta force the use of a very small integration step size (\( \Delta R = 1 \times 10^{-5} \)), it is not an acceptable technique to numerically solve this set of equations.

Investigation of numerical techniques to efficiently solve systems of stiff differential equations is currently a very active area of
Figure 14. Boundary Layer Thickness Ratio

B.L. THICKNESS RATIO (K)

NONDIMENSIONAL RADIUS (R/R_i)

Re_i = 7.75 \times 10^4

R_0i = 1087.0
numerical analysis; thus, numerous articles on this subject can be found in the literature. However, for the purpose of this presentation, it suffices to mention only two of these. The first, a survey article by Seinfeld, Lapidus and Hwang [59], points out that implicit techniques have the stability characteristics necessary to solve stiff differential equations. That is, implicit methods are always stable and accuracy is the only requirement that limits integration step size. The second article by Martinez [60] uses an implicit technique which approximates the exponential nature of first order differential equations by the use of Pade' approximants. With the use of the Pade' approximant method, equations (51) and (52) were solved with an integration step size of $\Delta R = 2 \times 10^{-3}$. A comparison of the boundary layer results obtained by the Runge-Kutta and the Pade' approximant method for $Re_1 = -1 \times 10^4$ and $Ro_1 = 130.0$ is shown in Figure 15. The mainstream conditions for this comparison are represented by a potential vortex superimposed on radial sink flow. The total Runge-Kutta computation time for the results shown in Figure 15 is 1.05 minutes, while the time for the Pade' approximant method was 20 seconds. Thus, for this set of equations (51) and (52) the Pade' approximant method obtains accurate numerical results much more efficiently than does Runge-Kutta. Therefore, the Pade' approximant method was used for the boundary layer calculations contained herein.
Figure 15. Comparison of Calculation Methods
A word of caution is necessary about the Pade' approximate method. The stability of this method is insured only as long as the eigenvalues of the system to be solved are all positive.

The results obtained with the E-method for a potential mainstream with \( \text{Re}_i = -7.75 \times 10^4 \) and \( \text{Ro}_i^2 = 1082.0 \) are plotted in Figure 16. That is, the mainstream is represented by a potential vortex with vortex filament at the conical centerline superimposed on a radial sink flow with the sink at the apex of the cone. As seen in the figure, the radial boundary layer thickness increases monotonically along the length of the cone and erupts as the apex of the cone is approached. Even though the boundary layer assumptions obviously do not apply close to the cone apex, the sudden eruption of the radial boundary layer near the cone apex has been shown by Mack [47] and Burggrof [52] to be correct. The implications of this behavior of the radial boundary layer thickness are investigated further in the boundary layer mass flow discussion. The boundary layer thickness ratio in Figure 16 increases from its inlet limiting value of unity and then remains nearly constant at a value of two along the region of interest. However, when the boundary layer calculation was extended to very small values of the nondimensional radius, the boundary layer thickness ratio tended toward unity.

Figures 17 and 18 show a comparison of the results calculated with the E-method, the T-method, and the C-method. The mainstream
Figure 16. Potential Vortex Boundary Layer Solution Calculated with the E-method
\[ \delta_0 = \delta_R \left( \frac{W_i}{R_i \nu} \right)^5 \]

\[ \text{Re}_i = 7.75 \times 10^4 \]

\[ \text{Ro}_i^2 = 1082 \cdot 0 \]

C-METHOD ---
E-METHOD --
T-METHOD ---

Figure 17. Comparison of Boundary Layer Solutions
Figure 18. Comparison of Boundary Layer Solutions
flow for the E-method in Figures 17 and 18 is represented by the potential mainstream described above with Reynolds Number equal to $-7.75 \times 10^4$ and the square of the Rossby Number equal to 1082.0 and 2704.0, respectively. On the other hand, to obtain results with the T-method and C-method, the mainstream flow is represented by a potential vortex with the vortex filament coinciding with the conical centerline and a zero radial mainstream velocity. From a comparison of Figures 17 and 18, it can be seen that as the strength of the mainstream potential vortex is increased relative to the radial sink flow, the radial boundary layer thickness obtained with the E-method approaches those obtained by the T-method and the C-method.

The large difference in the results $(\delta_0)$ obtained with the E-method and the C-method can be explained by two facts. First and most important, the effects of the mainstream radial velocity are included in the E-method, while they are not included in Cooke's calculations. Second, Cooke assumed a form of the boundary layer velocity weighting function that satisfies the radial compatibility condition. The radial compatibility condition is obtained by taking the limit of the radial boundary layer momentum equation (Eq. 36) as the cone wall is approached. Under these conditions, equation (36) becomes

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{R^2}{\rho v} \frac{\partial P}{\partial R}.$$  

Cooke used the radial compatibility condition to determine his boundary layer velocity weighting function.
Mack [44] transformed both the T-method and the C-method into cylindrical coordinates in order to study boundary layer growth produced by a swirling mainstream on a stationary finite disk. By comparing the results of the two methods, Mack observed that although the magnitudes of the radial boundary layer thicknesses agreed within 10 per cent, their behavior as a function of radius was quite different. Even more striking differences were observed if the radial boundary layer mass flow was used as a basis for comparison. The radial boundary layer mass flow computed with the T-method increases monotonically with decreasing radius, but the radial mass flow computed with the C-method increases to a maximum at a nondimensional radius of .45 and then decreases to zero as the apex of the cone is approached. Thus, Mack observed that the results of the T-method implied that the mass flow in the boundary layer was returned to the mainstream flow by a sudden eruption occurring at the centerline of the disk. The results of the C-method on the other hand implied that the boundary layer mass flow was returned to the mainstream flow smoothly until at the centerline no mass flow was contained in the boundary layer. After a detailed investigation of a Stewartson [61] type series solution of the boundary layer equations, Mack [47] concluded that the T-method yields the correct results for the boundary layer mass flow. Extending Mack's work, Rott and Lewellen [48] concluded that the use of the radial compatibility condition as a
boundary condition for the boundary layer velocity profile leads to an "over specification" for the system of integral momentum boundary layer equations. Thus, it is important to note that although the magnitude of the boundary layer mass flow does not agree with those of the C-method or the E-method, the boundary layer mass flow (See Figure 19) obtained with the E-method does exhibit a sudden eruption at the cone apex similar to that of the T-method.

Differences between the magnitude of the results of the T-method and the E-method are attributed to the following reasons: The T-method neglects the effect of the mainstream radial velocity component. The T-method also neglects the fact that the boundary layer velocity components do not reach the required percentage of the respective mainstream velocity components as the same distance from the wall (i.e. two boundary layer thicknesses). Taylor [4] gave no reason for his choice of a boundary layer velocity weighting function other than to provide the boundary layer with an upstream "memory".

Like Taylor [4], Houlihan and Hornstra[55] solved the integral momentum equations for a single boundary layer thickness and the radial velocity profile within the boundary layer. Unfortunately, Houlihan and Hornstra did not present the boundary layer thickness in their paper on the swirling flow though a conical convergent nozzle; therefore, a direct comparison of their results with those from the E-method is not possible. However, the following observation
Figure 19. Boundary Layer Mass Flow Calculated with the E-method
points out the differences between the method used by Houlihan and Hornstra and the E-method. In order to reduce the number of unknowns in the integral momentum equations to a solvable set, Houlihan and Hornstra assumed linear relationships between the radial momentum thickness and the boundary layer thickness and between the mixed momentum thickness and the boundary layer thickness. This assumption requires that the ratio of the tangential boundary layer thickness to the radial boundary layer thickness be a constant along the entire length of the nozzle. An examination of Figure 16 shows that for a potential vortex flow this assumption has merit over much of the nozzle length. However, as shown by Figure 24, if the mainstream is a viscous fluid, this assumption is not valid. Therefore, it is the author's opinion that the results obtained with the E-method correctly demonstrate the effect of the radial velocity on the growth of the boundary layer for swirling flow through a converging conical nozzle.
V. COMPOSITE SOLUTION

The purpose of this Chapter is to describe how the mainstream and boundary layer solutions, discussed in Chapters III and IV, respectively, are combined to form a composite solution of the fluid motion of an incompressible, viscous vortex confined inside a conical converging nozzle. The results obtained by the application of this model are compared with experimental values (Lineberry [12]) in section A of this Chapter.

Two features, the wall boundary layer mainstream interaction and the convergence of the vortex flowlines, have a dominating influence on this confined vortex flow field. The interaction between the wall boundary layer and the mainstream has been shown by many investigators (Rosenzweig [53], Kendall [41]) to limit the strength (circulation) of the mainstream vortex. Equally important, previous investigators (Benjamin [62], Burgers [32]) have shown that the change in area of the vortex due to the flow progressing through the nozzle causes many phenomena, such as vortex "burst" and celled vortex motions, to occur. The vortex model described below, iteratively includes the effects of both of these dominating influences.

A digital approach to the composite solution is necessary in order to insure that the continuum fluid model is maintained. This solution technique allows the undetermined boundary conditions at
each succeeding nozzle station to be determined. The digital approach is referred to herein as the digital model.

The digital model of the confined vortex flow field proceeds in the following manner (See Figure 20). A viscous mainstream solution is obtained for the entire flow field with the assumption that velocity slip conditions exist at the nozzle walls. The velocity and pressure profiles obtained from this first complete mainstream solution are used as boundary conditions to generate a nozzle wall boundary layer solution. A second complete mainstream solution which includes the mass flow and the loss of radial momentum in the nozzle wall boundary layer is then obtained. In this second solution, the boundary layer and mainstream solutions are matched at the common edge of the two regions. The velocity and pressure profiles of the second complete mainstream solution are used to generate a new boundary layer solution. In theory, this process is then repeated until the wall boundary layer-mainstream interaction produces no changes in the flow field. However, in section B of this Chapter, it will be shown that the improvement of the results obtained by repeating the calculation procedure of the digital model more than once, does not justify the additional computing time required. The paragraphs below will discuss each individual section of the digital model and then discuss the details of the combination of the mainstream and the boundary layer solutions.
First complete mainstream solution: use particular solutions of Chapter III, with slip velocity conditions on the nozzle wall; at each radial nozzle station.

Boundary layer solution: use velocity and pressure profiles from first complete mainstream solution as boundary conditions; calculate boundary layer mass flow and radial momentum loss in the boundary layer along the length of the nozzle.

Second complete mainstream solution: use particular solution of Chapter III; solve to the edge of the boundary layer; account for mass flow and loss of radial momentum in boundary layer; match the transverse velocity at the interface of mainstream and boundary layer regions; at each radial nozzle station.

Boundary layer solution: use velocity and pressure profiles from the second complete mainstream solution as boundary conditions; calculate boundary layer solution along length of the nozzle.

End

Figure 20. Flow Diagram
In the mainstream region, the tangential velocity is assumed to be of the form, \( W = W_i \frac{R_i T_0(\theta)}{R} \). That is, the tangential velocity is assumed to be axisymmetric and to vary as the inverse of the spherical radius times a function of the transverse angle. With this assumed form of the tangential velocity, the steady, incompressible Navier-Stokes Equations are transformed into a set of nonlinear ordinary differential equations (See Chapter III, equation (8), (9), and (10)). The mainstream solution at any radius in the nozzle is obtained by numerically integrating this transformed set of general equations of motion from the nozzle centerline to the edge of the mainstream region. A shortcoming of this particular class of mainstream solutions is that if the mainstream solution from one nozzle radius is extended to another radius, there is no assurance that mass and momentum of the total system are conserved at the second radius. To overcome this difficulty, the nozzle flow field is divided into a number of stations along the radial length of the nozzle (See Figure 21). At each nozzle station, there is obtained a separate mainstream solution that has the assumed form of the tangential velocity and also satisfies the mass and momentum conservation of the total system. That is, the boundary conditions for equations (8), (9), and (10) of Chapter III vary along the length of the nozzle in such a manner that mass and momentum of the total system are conserved at each nozzle station.
Figure 21. Radial Nozzle Stations
The complete mainstream solution consists of the separate mainstream solutions at all nozzle stations. In the digital model the complete mainstream solution is used twice for calculation of the nozzle flow field. The first time the complete mainstream solution is used, velocity slip conditions are assumed to exist at the nozzle wall. The second complete mainstream solution proceeds to the edge of the boundary layer and not to the nozzle wall. Thus, it is necessary to specify boundary conditions at the centerline of each nozzle station. Six boundary conditions are required for solving the transformed set of the general equations of motion. Vortex symmetry conditions require that both the transverse velocity and the partial derivative of the radial velocity with respect to the transverse angle are zero on the nozzle centerline. Viscous considerations specify that the tangential velocity also vanish on the nozzle centerline. Thus, three of the six boundary conditions needed at the centerline are specified by physical conditions. The three unspecified boundary conditions are the centerline radial velocity, the second partial derivative of the radial velocity with respect to the transverse angle, and the first partial derivative of the tangential velocity with respect to the transverse angle. As will be shown below, these last three unspecified boundary conditions can be chosen so that the mass and radial momentum requirements for the total system are satisfied and that the transverse velocity at the
edge of the mainstream region matches that at the edge of the boundary layer region.

The velocity and pressure profiles from the complete mainstream solution are used as boundary conditions to generate an integral momentum solution in the boundary layer region. Most analytical investigations of the confined vortex boundary layer have neglected the effects of the velocity component parallel to the wall on the boundary layer growth. However, experimental studies of vortex flows confined inside a converging nozzle (Lineberry [12] and Linden [11]) show that as the flow progresses through the nozzle, the radial velocity increases until it becomes significant in comparison with the tangential velocity. Thus, in the integral momentum method (See Chapter IV) used in the nozzle wall boundary layer region, the effects of the radial velocity component on the growth of the boundary layer is included. Four quantities, the radial mass flow in the boundary layer, the amount of radial momentum dissipated by the wall, the boundary layer thickness, and the boundary layer transverse velocity at the edge of the boundary layer, are required from the boundary layer solution at each radial nozzle station. These values are all used to generate the second complete mainstream solution.

The solution of the confined vortex flow field is initiated by reading the nozzle geometry, fluid properties, the centerline static pressure, and the unspecified boundary layer conditions at the first radial nozzle station into the digital model. With these input values,
a mainstream solution at the first nozzle station is obtained by numerical integration of the transformed equations of motion from the nozzle centerline to the nozzle wall. The mass flow and radial momentum of the total system are calculated (See Appendix D for details) from the velocity and pressure profiles of the mainstream calculation at the first station. The mainstream solution is then extended to nozzle station number two where the values for the three unspecified mainstream boundary conditions at the centerline of station two are assumed. One of the unspecified boundary conditions, the first partial derivative of the tangential velocity with respect to the transverse angle, is assumed in this first complete mainstream solution to vary as the inverse of the radial distance. The other two unspecified boundary conditions, the radial velocity and the second partial of the radial velocity with respect to the transverse angle, are iteratively varied until the mass flow and radial momentum at station two agree with that of station one. This process is then repeated at each succeeding station along the length of the nozzle. Thus, the complete mainstream solution is composed of a number of particular solutions of the general equations of motion which conserve mass flow and radial momentum of the total system along the length of the conical nozzle. Three points should be noted about this first complete mainstream solution. First, the solution contains the effects of the convergence of the vortex flow lines and does not
consider the effect of the mainstream boundary layer interaction. Second, the first partial derivative of the tangential velocity with respect to the transverse angle at the nozzle centerline, which dictates the tangential momentum of the system, is assumed to vary as the inverse of the radial distance throughout the nozzle. If the vortex flow were inviscid, this assumption would imply that tangential momentum was conserved throughout the nozzle. However, viscous dissipation of the tangential momentum does occur in this first complete solution. The only justification for this assumed variation of the partial derivative of the tangential velocity with respect to the transverse angle is that it appears to yield the correct tangential momentum distribution along the length of the nozzle. Third, in this digital model, the loss of radial momentum due to viscous dissipation in the mainstream flow is neglected. This neglect is justified by two reasons: First, in the actual flow, the radial viscous dissipation in the flow is expected to be small in comparison with the pressure and inertia forces. Second, in the radial momentum equation, equation (5), the higher order radial viscous dissipation terms do not appear because of the particular form of the mainstream solutions. Thus, it is consistent with the particular form of the solution used in the mainstream region, to neglect radial viscous dissipation in the mainstream.
The velocity and pressure profiles obtained from the first complete mainstream solution are now used as boundary conditions to generate an integral momentum boundary layer solution along the nozzle walls. In this boundary layer solution, the boundary layer thickness, radial momentum dissipated by the wall, mass flow in the boundary layer are calculated along the entire nozzle radius. These values are then used for the calculation of the second complete mainstream solution.

In the second complete mainstream solution, the transformed equations are numerically integrated from the nozzle centerline to the edge of the boundary layer. That is, once again at each radial station, the centerline values of the radial velocity and the second partial derivative of the radial velocity with respect to the transverse angle are iteratively varied until mass flow and radial momentum of the total system are accounted for in both the mainstream and the boundary layer. For the second complete mainstream solution, however, the centerline value of the first partial derivative of the tangential velocity with respect to the transverse angle does not vary as the inverse of the radial distance, but is varied iteratively until the transverse velocity of the mainstream matches that of the boundary layer at the interface of the two regions. Thus, the second complete mainstream solution includes the effects of both the convergence of
the vortex flow lines and the boundary layer mainstream interaction.

Finally, the velocity and pressure profiles from the second complete mainstream solution are used to generate a new boundary layer solution. The reader is once again reminded that the calculation procedure described above can be repeated until the mainstream boundary layer interaction produces no change in the confined vortex flow field. However, in practice, the computing time required is so large that the procedure is completed only once. The last section of this Chapter shows that the improvement of results obtained by repeating the calculation procedure more than once does not justify the additional computer time required.

A. Numerical Results

In this section, the velocity profiles in the mainstream and the boundary layer solutions calculated from two different forms of the composite solution are examined. The first form, the modified form, uses the very viscous velocity profiles of equation (22), Chapter III, to represent the mainstream velocity profiles. The second form, the general form, solves the transformed equations of motion (equations (8), (9), and (10) of Chapter III) for the velocity profiles of the mainstream region. The results from both of these forms of the composite solution are examined in detail and the results of the general form are compared with experimental data obtained by Lineberry [12].
1. Modified Composite Solution

Since the velocity profiles used in the modified composite solution could not exist in a fluid with a finite viscosity, the flow field of the modified composite solution is somewhat artificial. That is, a finite fluid viscosity is used in the static pressure and boundary layer calculations even though the mainstream velocity profiles represent those of a fluid with an infinite viscosity. The results of this model, however, do exhibit the proper trends and are helpful in explaining the confined vortex flow field in a conical nozzle.

The inlet conditions for the modified composite solution were chosen such that the total system mass flow, the inlet tangential momentum, and the inlet centerline static pressure matched those of Lineberry's [12] experimental study. That is, the total system mass flow for the modified composite solution is 0.2827 lb.\text{m}/\text{sec.}, the inlet tangential velocity has the profiles shown in Figure 22, and the inlet centerline static pressure is 15 psia.

The radial velocity profiles of the first complete mainstream solution from the modified model are shown in Figure 23. It can be seen from the figure that conservation of the system mass flow and radial momentum has required an increase in the magnitude of the radial velocity at each radial nozzle station. The tangential velocity profiles for this first complete solution are shown in
Figure 22. Limiting Tangential Velocity Profiles for a Very Viscous Fluid ($\text{Re}_c \to 0$)
Figure 23. Limiting Radial Velocity for a Very Viscous Fluid \((\text{Re}_c \to 0)\)
Figure 22. It is important to note that the total tangential momentum of the system is not conserved as the flow progresses through the nozzle. However, the magnitude of the tangential velocity does increase at each radial station along the nozzle. The physical basis for this behavior can be explained by two competing factors. That is, conservation of angular momentum in the converging nozzle tends to increase the magnitude of the tangential velocity at each radial station along the nozzle. On the other hand, viscous dissipation of the tangential momentum by the vortex tends to decrease the magnitude of the tangential velocity at each nozzle station.

The integral momentum boundary layer solution obtained from the velocity and pressure profiles of the first complete mainstream solution, is shown in Figure 24. A comparison of Figures 24 and 16 shows a marked difference between the boundary layer solution for a vortex with very viscous velocity profiles and the boundary layer solution for the potential vortex described in Chapter IV. This difference in the magnitude of the two solutions is caused by the differences in the radial variation of the mainstream velocity and pressure profiles between the two cases.

The effects of the mainstream boundary layer interaction on the radial and tangential velocity profiles at a typical nozzle station are shown in Figures 25 and 26, respectively. As seen from Figure 25, the deficit of radial mass flow in the boundary layer causes the radial
Figure 24. First Boundary Layer Solution, Modified Model \( (Re_c \to 0) \)
Figure 25. Comparison of the First and Second Solutions for the Radial Velocity Profiles, Modified Model \((\text{Re}_c \to 0)\)
Figure 26. Comparison of the First and Second Solutions for the Tangential Velocity Profiles, Modified Model \((Re_c \to 0)\)
velocity profiles of the second complete mainstream solution to increase over that of the first complete mainstream solution. The magnitude of the tangential velocity also shows an increase in the second complete mainstream calculation. Since the second complete mainstream solution includes the viscous dissipation of angular momentum both in the mainstream and at the nozzle wall, the increase in the magnitude of the tangential velocity in the second complete mainstream solution would appear to be incorrect. However, close examination of the composite solution shows that in the area occupied by the cone wall boundary layer, the first complete mainstream solution contains more angular momentum than is dissipated by the first boundary layer calculation. Thus, the tangential velocity profiles of the second complete mainstream solution should increase slightly.

In Figure 27, the radial boundary layer thickness obtained from the first complete mainstream solution is compared with the boundary layer thickness obtained from the second complete mainstream solution. The relative close agreement in the magnitudes and the behavior of these two boundary layer solutions suggests that the mainstream boundary layer interaction does not strongly effect the confined vortex flow field when the very viscous velocities are used in the mainstream region. In the results of the general model, described below, the importance of this mainstream boundary layer interaction is shown.
Figure 27 Comparison of the First and Second Boundary Layer Solutions, Modified Model
\((Re \rightarrow 0)\)
2. General Composite Solution

In the following paragraphs, a critical review of Lineberry's [12] experimental study of an air vortex confined in a conical converging nozzle is presented. Then the results of the composite solution obtained by application of the general digital model to the same confined vortex flow field are presented and compared with these experimental data.

In 1968 Lineberry [12] experimentally studied swirling air flow through a converging nozzle. The nozzle* (See Figures 28 and 29) was conical in shape with inlet and exit diameters of 11 15/16 in. and 3 15/16 in., respectively. The included half angle of the conical nozzle was 11.2°; thus, the total nozzle centerline length from inlet to exit was 20 1/16 in. Static and total pressure profiles were measured at four positions along the length of the nozzle.

A pitot probe constructed of hypodermic needles and mounted on two 1/4 in. O.D. tubes was used to measure the pressure profiles. The mounting mechanism allowed the probe three degrees of freedom, thus, the magnitude and direction of the fluid velocity were measured. The pitot probe was inserted across the vortex flow field in the cylindrical radial direction. The insertion of a probe in this manner can seriously affect the entire flow field of the confined vortex (See

* See Thompson [63] for complete nozzle specifications.
Figure 28. Experimental Apparatus Used by Lineberry [12]
Figure 29. Nozzle Dimensions  Note: This nozzle geometry was used both in the calculations herein and in Lineberry's [12] Experimental Study.
Figure 3). Unfortunately, no wall static pressure measurements were included in this investigation and the extent of the change of the vortex flow field due to the insertion of the probe is therefore, not known. However, the relative size of the probe and the nozzle diameters leads one to believe that the probe did not greatly affect the overall vortex flow field. Some of the other features of this experimental data that are due to the geometry of the nozzle construction will be discussed below in the comparison of the results of the composite solution and the experimental data.

The general digital model divided the nozzle flow field into five radial nozzle stations. The first nozzle station was at the inlet to the nozzle and the other four stations were located at spherical radii corresponding to Lineberry's four measurement stations. As in the modified model, the inlet conditions for the general digital model were chosen so that the total system mass flow, tangential momentum and inlet centerline static pressure matched those of Lineberry's as nearly as possible. However, as shown by Figure 29, Lineberry's first measurement station was not at the nozzle inlet. Therefore, the inlet system tangential momentum and inlet centerline static pressure for the digital model were estimated from the measured values at the first measurement station.

The radial velocity profiles for the first complete solution of the general digital model are presented in Figure 30. These velocity
Figure 30. Radial Velocity Profiles, General Model
profiles, like those of the modified model, increase at each radial station along the length of the nozzle. Close to the nozzle centerline, the vortex core, the radial velocity profiles approach the very viscous solutions of Chapter III. Further away from the nozzle centerline, the radial velocity profiles change until they approach the inviscid velocity profiles of Chapter III. The tangential velocity profiles for the first complete mainstream solution also exhibit the characteristic behavior discussed in Chapter III. That is, the tangential velocity increases from zero at the nozzle centerline to a maximum and then decreases as tangential momentum is conserved in the outer regions of the vortex. As in the modified model, viscous dissipation in the vortex decreases the tangential momentum of the system at each radial station along the nozzle.

The cone wall boundary layer resulting from the velocity and pressure profiles of the first complete mainstream solution is shown in Figure 31. This boundary layer solution exhibits a behavior and magnitude similar to the boundary layer solution from the potential vortex described in Chapter IV. The similarity of these two boundary layer solutions is attributed to the fact that, near the nozzle wall, the first complete mainstream solution strongly resembles a potential vortex flow field.

In Figure 32, the experimental radial velocity profiles from Lineberry's measurement stations 1, 2, 3, and 4 are compared with the
Figure 31. First Boundary Layer Solution, General Model
radial velocity profiles calculated at stations 2, 3, 4, and 5 of the second complete mainstream solution. Since, the experimental data were measured in a plane perpendicular to the nozzle centerline and the values of the digital model were calculated at constant spherical radii, the physical positions of the experimental and calculated values are not in complete agreement. That is, the spherical radii of the last four stations of the digital model correspond to the centerline radii of the four experimental measurement stations. Therefore, the positions of the calculated and experimental values agree only at the nozzle centerline. However, because the included half angle of the conical nozzle is small, the maximum misalignment of the positions of the measured and calculated values is only one half of an inch.

As shown in Figure 32, the calculated and measured radial velocity profiles at the first measurement station do not agree either in behavior or in magnitude. The two celled vortex motion exhibited by the experimental data is attributed to a combination of the following three physical factors; the sudden eruption of the end wall boundary layer of the vortex generator near the centerline of the cylindrical end wall (See Figure 29), the incomplete filling of the vortex generator near the inlet, and the step change in the flow area between the vortex generator and the conical nozzle. On the other hand, it was assumed in the digital model that the flow field at the first measurement
Figure 32. Comparison of the Calculated Radial Velocity Profiles with Lineberry's Experimental Data
The station was influenced only by the mainstream boundary layer interaction and the convergence of the vortex flow lines. Thus, the lack of agreement is not surprising since the physical flow field at the first measurement station is strongly influenced by the vortex generator and this effect is not accounted for in the digital model.

Further along the nozzle, as the effects of the mainstream boundary layer interaction begin to dominate the confined vortex flow field, the results obtained with the digital model begin to exhibit the same trends as the experimental data. At measurement station 3 both the experimental and calculated radial velocity profiles have a three celled vortex structure. Lineberry [12] noted in his experimental investigation that air from the atmosphere surrounding the nozzle exit flowed into the low pressure region near the nozzle centerline. That is, the static pressure at the centerline of the nozzle exit plane was lower than the atmospheric exhaust pressure; therefore, air flowed up the nozzle centerline. This additional air flow was then entrained in the vortex mainstream and exhausted near the nozzle walls. Thus, the comparison of the behavior of the calculated and experimental radial velocity profiles is much more important than the comparison of the magnitudes of the radial velocity profiles obtained from the two studies.

Figure 33 shows a comparison of the calculated and experimental tangential velocity profiles at measurement stations 2 and 3. The
agreement between the magnitudes of these tangential velocity profiles is very good. However, the point of maximum tangential velocity occurs nearer the nozzle centerline in the calculated profiles than in Lineberry's experimental data. As shown by Figure 6, Chapter III, the point of maximum tangential velocity moves away from the nozzle centerline as the fluid viscosity is increased. Thus, the difference in the points of maximum tangential velocity is attributed to the turbulence of the physical vortex flow.

The measured and calculated transverse velocity profiles at the third experimental measurement station are compared in Figure 34. In general, the magnitudes of the two velocity profiles agree quite well. However, since the digital model matches the transverse velocity of the mainstream and boundary layer regions at the edge of the boundary layer, the behavior of the transverse velocity near the nozzle wall is very important. The calculated transverse velocity near the nozzle wall indicates that mass is flowing into the boundary layer from the mainstream region. The behavior of the experimental data near the nozzle wall, on the other hand, is somewhat confusing. That is, the data point nearest the wall indicates that mass is flowing into the boundary layer. However, the magnitude of this data point leads one to question its validity. A comparison of the experimental radial velocity profiles at measurement stations 2 and 3 of Figure 32 shows that the radial velocity near the nozzle wall increases very little.
Figure 33. Comparison of Calculated Tangential Velocity Profiles with Lineberry's Experimental Data
Figure 34. Comparison of the Calculated Transverse Velocity Profiles with Lineberry's Experimental Data
Thus, one must conclude that mass is entering the wall boundary layer at a slower rate than indicated by the experimental data at measurement station 3.

Figure 35 shows both the experimental and the calculated static pressure profiles for the nozzle flow field. In general, both of the static pressure fields have the same characteristic vortex behavior. That is, they both show a general increase in the static pressure with increasing distance from the nozzle centerline. Since, the magnitude of the calculated static pressure is in all cases greater than that of the experimental static pressure, the inlet centerline static pressure input to the digital model must have been estimated higher than actually existed in the physical flow.

A comparison of the first and second boundary layer calculations for the composite solution is shown in Figure 36. The radial behavior of the second boundary layer solution is very much like that of the first boundary layer solution. However, the magnitude of the second boundary layer solution is greater than that of the first. This small difference in the two boundary layer solutions is attributed to three facts: the mainstream radial velocity near the boundary layer does not change directions in either solution; the mainstream tangential velocity is not greatly affected by the celled vortex motion of the radial velocity components; and the radial variation of mainstream static pressure near the nozzle wall is not vastly different
<table>
<thead>
<tr>
<th>(Lineberry)</th>
<th>(Digital Model)</th>
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<tbody>
<tr>
<td>14.608 + (4.952)</td>
<td>14.579 + (4.950)</td>
</tr>
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<td>14.204 + (4.929)</td>
</tr>
<tr>
<td>14.086 + (4.923)</td>
<td>14.097 + (4.930)</td>
</tr>
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</table>

\[ R = 2.18' \quad R = 1.78' \quad R = 1.38' \quad R = 0.98' \quad (\theta) \]

Figure 35. Comparison of Calculated Flow Field Static Pressures with Lineberry's Experimental Data
Figure 36. Second Boundary Layer Calculation. General Model
for the two boundary layer solutions. Because the boundary layer is relatively insensitive to changes in the mainstream near the nozzle centerline, it will be shown in section B of this Chapter that the calculation procedure of the general composite solution need not be repeated more than once.

Two general conclusions should be drawn from the discussion of the results of the general composite solution and Lineberry's experimental data. First, a comparison of the first and second complete composite solutions indicates that the effects of the mainstream boundary layer interaction are more important than the effects of the convergence of the vortex flow lines in the determination of the vortex flow field through a conical nozzle. Second, the results predicted with the general digital model contain the same general trends found in Lineberry's [12] experimental investigation. Thus, one must conclude that the digital model does account for the dominant effects which determine the flow field of a vortex confined in a conical converging nozzle.

B. Additional Mainstream Boundary Layer Iteration

In order to determine if the mainstream boundary layer interaction required that the calculation procedure in the general composite solution be repeated more than once, a third complete mainstream solution was obtained for the nozzle flow field. The radial velocity profiles
from the fifth radial nozzle station for the second and third complete mainstream solutions are shown in Figure 37. The very slight differences found in these two radial velocity profiles are typical of the agreement found between the second and third complete mainstream solutions. This good agreement between the two complete mainstream solutions is a direct result of the nozzle wall boundary layer being relatively insensitive to the changes in the vortex flow field near the nozzle centerline. Thus, for the flow field considered in this investigation, the improvement in the results obtained by repeating the calculation procedure in the composite solution does not justify the additional 10 to 15 minutes of computer time required for the third complete mainstream solution.
Figure 37. Comparison of the Second and Third Solutions for the Radial Velocity Profiles
VI. CONCLUSIONS AND RECOMMENDATIONS

The following conclusions are drawn from the present numerical study of the flow of a confined vortex through a conical, converging nozzle.

1. The particular form of the solutions of the general equations of motion used in this investigation can represent the mainstream of a confined vortex flow field.

2. The effects of the spherical radial velocity component on the growth of the nozzle wall boundary layer can not be neglected in comparison with the effects of the tangential velocity component.

3. The mainstream boundary layer interaction is of the utmost importance in the determination of the confined vortex flow field through the conical converging nozzle.

4. Results obtained with the composite solution of the nozzle flow field, presented in Chapter V, compare well with experimental data.

5. Repeating the digital model calculation procedure more than once is not required to model the mainstream boundary layer interaction.

The following is a list of recommendations for further work in the confined vortex area.
1. An experimental study of the radial and tangential velocity profiles in the nozzle wall boundary layer should be performed.

2. The complete class of solutions for the general equations of motion with the particular velocity functions used in the mainstream region needs to be numerically obtained.

3. The effects of the boundary layer at the vortex generator end wall should be included in the composite solution.

4. The composite solution should be extended to include the effects of compressibility in the vortex fluid.

5. An experimental study of the effect of the variation of the ratio of inlet tangential to radial momentum on the mass flow of the system should be undertaken.

6. An experimental study to map the turbulent intensity of the conical nozzle vortex flow field would be of value.

7. The composite solution should be extended to consider suction in the wall boundary layer as a mechanism to aid the separation of dust particles in the cyclone separator.

8. The optimum flow field and container shape for a cyclone separator should be determined.


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APPENDICES
Appendix A

Mainstream Equations

The purpose of this appendix is to develop the equations which govern the motion of a steady, incompressible, axisymmetric vortex. In this development, the vortex tangential velocity is assumed to be of the form:

\[ W(R, \theta) = \frac{T(\theta)}{R} \]  

(A-1)

Where \( R \) is the spherical radial distance measured from the cone apex and \( T \) is a function only of the conical transverse angle, \( \theta \). In the paragraphs below, it will be shown that with the assumed form of the vortex tangential velocity, the radial and transverse velocities are of particular forms. The spherical tangential momentum equation for the mainstream vortex is:

\[ U \frac{\partial W}{\partial R} + \frac{V}{R} \frac{\partial W}{\partial \theta} + \frac{WU}{R} + \frac{WW}{R} \cot \theta = \nu \left[ \frac{1}{R^2} \frac{\partial}{\partial R} \left( R^2 \frac{\partial W}{\partial R} \right) \right. \]

\[ \left. + \frac{1}{R^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial W}{\partial \theta} \right) - \frac{W}{R^2 \sin^2 \theta} \right] \]  

(A-2)

After introducing the assumed form of the tangential velocity into equation (A-2), one obtains:

\[ -\frac{UT}{R^2} + \frac{V}{R^2} \frac{dT}{d\theta} + \frac{UT}{R} + \frac{VT}{R} \cot \theta = \nu \left[ \frac{1}{R^3} \frac{d^2T}{d\theta^2} \right. \]

\[ \left. + \frac{1}{R^3} \frac{dT}{d\theta} \cot \theta - \frac{T}{R^3 \sin^2 \theta} \right] \]  

(A-3)
The solution of equation (A-3) for the tangential velocity is equation (A-4)

\[ v(R, \theta) = \frac{\nu}{R} \left( \frac{d^2 T}{d\theta^2} + \frac{dT}{d\theta} \cot \theta - T \csc^2 \theta \right) \]

(A-4)

If the fluid viscosity is assumed constant, the transverse velocity has the form \( V = F(\theta)/R \).

The steady, incompressible continuity equation in spherical coordinates is:

\[ \frac{\partial}{\partial R} [R^2 \sin(\theta) U] + \frac{\partial}{\partial \theta} [R \sin(\theta) V] = 0 \]

(A-5)

Substitution of the form for the transverse velocity obtained above into the continuity equation yields:

\[ \frac{\partial}{\partial R} [R^2 U] = -F \cot \theta - \frac{dF}{d\theta} \]

(A-6)

Therefore, the most general form that the radial velocity can take is

\[ U(R, \theta) = \frac{C(\theta)}{R} + \frac{G(\theta)}{R^2} \]

(A-7)

Equations (A-1), (A-4), and (A-7) show that four velocity profile forms must be found for the most general solution in which the tangential velocity is of the form of equation (A-1). Unfortunately, the four velocity profile forms and the static pressure cannot be
found from the four governing equations. Hence, it is necessary to neglect one of the radial velocity profile forms. At first glance, it would seem desirable to retain the radial velocity profile form, that contains the higher order of radial distance. However, substitution of this form into the continuity equation yields:

\[
\frac{2\frac{dV}{dR}}{R^3} - \frac{2\frac{dV}{dR}}{R^3} + \frac{1}{R \sin \theta} \frac{2}{\partial} \left[ V \sin(\theta) \right] = 0
\]  

(A-8)

Equation (A-8) reveals that if the higher order radial velocity profile form is retained the radial and transverse velocities are independent of each other. Since, the interaction of the velocity components is thought to be important in the description of the motion of the confined vortex, the radial velocity profile form which varies as the inverse of the radial distance is used in all mainstream calculations.

Substitution of the velocity profile forms obtained above into the axisymmetric incompressible Navier-Stokes Equations yields the following transformed equations of motion. The continuity equation in terms of the velocity profile functions is:

\[
\frac{dF}{d\theta} + F \cot \theta = -G
\]  

(A-9)

The tangential momentum equation in terms of the velocity profile function is:

\[
\frac{d^2T}{d\theta^2} + \left[ \cot \theta - \frac{F}{\nu} \right] \frac{dT}{d\theta} - \left[ \csc^2 \theta + \frac{F}{\nu} \cot \theta \right] T = 0
\]  

(A-10)
The transverse momentum equation, in terms of the velocity profile functions, is:

\[ F \frac{dF}{d\theta} - T^2 \cot \theta = - \frac{R^2}{\rho} \frac{dP}{d\theta} + \nu \left( \frac{d^2F}{d\theta^2} + \frac{dF}{d\theta} \cot \theta \right) + 2 \frac{dG}{d\theta} - \frac{F}{\sin^2 \theta} \]  

(A-11)

The radial momentum equation, in terms of the velocity profile functions, is:

\[ -G^2 + F \frac{dG}{d\theta} - F^2 - T^2 = - \frac{R^3}{\rho} \frac{dP}{dR} + \nu \left( \frac{d^2G}{d\theta^2} + \frac{dG}{d\theta} \cot \theta \right) \\
- 2G - 2 \frac{dF}{d\theta} - 2F \cot \theta \]  

(A-12)

The static pressure is removed from equations (A-11) and (A-12) by differentiating equation (A-11) with respect to \( R \) and equation (A-12) with respect to the transverse angle, \( \theta \). After the cross differentiation, the combined radial and transverse momentum equation is:

\[ 2T^2 \cot \theta + 2T \frac{dT}{d\theta} + 2G \frac{dG}{d\theta} - F \frac{d^2G}{d\theta^2} - \frac{dF}{d\theta} \frac{dG}{d\theta} \]  

\[ \nu \left[ \frac{dG}{d\theta} \left[ 2 \csc^2 \theta \right] + \frac{d^2G}{d\theta^2} \cot \theta + \frac{d^3G}{d\theta^3} \right] = 0 \]  

(A-13)

In order to aid numerical calculations, equations (A-9), (A-10) and (A-13) are nondimensionalized by the use of the following
Inlet conditions are denoted in equation (A-14) by the subscript "i" and nozzle centerline conditions are denoted by the subscript "c". In nondimensional form, equations (A-9), (A-10), and (A-13) become equations (A-15), (A-16), and (A-17), respectively.

\[
\frac{dF_o}{d\theta} + F_o \cot \theta = -G_o
\]

(A-15)

\[
\frac{d^2 T_o}{d\theta^2} + [\cot \theta - \text{Re}_c F_o] \frac{dT_o}{d\theta} - [\csc^2 \theta + \text{Re}_c F_o \cot \theta] T_o = 0
\]

(A-16)

\[
\frac{d^3 G_o}{d\theta^3} + \frac{d^2 G_o}{d\theta^2} \cot \theta - \frac{dG_o}{d\theta} \csc^2 \theta + 2 \frac{dG_o}{d\theta} + \text{Re}_c \left( 2G_o \frac{dG_o}{d\theta} \right)
\]

(A-17)

Where the two nondimensional parameters, the centerline radial Reynolds Number and the inlet Rossby Number, are defined by equation (A-18).

\[
\text{Re}_c = \frac{U_c R_i}{\nu} \quad \text{and} \quad \text{Ro}_i = \frac{W_i}{U_c}
\]

(A-18)

Since slip conditions are assumed to exist at the nozzle walls, the six boundary conditions necessary for solving equations (A-15), (A-16), and (A-17) are specified at the nozzle centerline. These boundary conditions are defined by equations (A-19).
\[ G_0(0) = 1.0 \quad \left. \frac{dG_0}{d\theta} \right|_{\theta=0} = 0 \quad \left. \frac{d^2G_0}{d\theta^2} \right|_{\theta=0} = D \]

\[ T_0(0) = 0 \quad \left. \frac{dT_0}{d\theta} \right|_{\theta=0} = A \quad F_0(0) = 0 \]  

(A-19)

The physical reasons for the use of these boundary conditions are described below: The nozzle centerline conditions for the transverse velocity profile form \( F_0(\theta) \) and the first derivative of the radial velocity profile form are provided by axisymmetric flow considerations. Viscous shear stresses dictate that the tangential velocity is zero at the nozzle centerline; therefore, the centerline value of the tangential velocity profile form must also be zero. The radial velocity profile form has the value of unity at the nozzle centerline because the inlet radial centerline velocity is used to nondimensionalize equations (A-9), (A-10), and (A-13). In the composite solution, the centerline values of the second derivative of the radial velocity profile form and the first derivative of the tangential velocity profile form are varied until mass flow and momentum of the total system are conserved. Thus, in the overall solution, the values of A and D from equations (A-19) are functions of the radial distance.

To obtain numerical solutions, equations (A-15), (A-16), and (A-17) are transformed into a set of six first order differential equations presented below.
\[ \frac{dF_0}{d\theta} = -G_0 - P_0 \cot \theta \]  

(A-20)

\[ \frac{dB_0}{d\theta} = [Re_c P_0 - \cot \theta] B_0 + [\csc^2 \theta + Re_c P_0 \cot \theta] T_0 \]  

(A-21)

\[ \frac{dT_0}{d\theta} = B_0 \]  

(A-22)

\[ \frac{dH_0}{d\theta} = -2 Re_c R_i^2 \cot \theta T_0^2 \]  

(A-23)

\[ \frac{dQ_0}{d\theta} = H_0 - Q_0 \cot \theta - 2 G_0 - Re_c [G_0^2 - P_0 Q_0 + R_i^2 T_0^2] \]  

(A-24)

\[ \frac{dG_0}{d\theta} = Q_0 \]  

(A-25)

The centerline boundary conditions for this set of transformed equations are:

\[ G_0(0) = 1.0 \quad Q_0(0) = 0 \quad H_0(0) = 2D + 2 + Re_c \]  

(A-26)

\[ T_0(0) = 0 \quad B_0(0) = A \quad F_0(0) = 0 \]

This set of six first order differential equations with corresponding boundary conditions are used for the vortex mainstream calculations.
Appendix B

Boundary Layer Equations

The purpose of this appendix is to develop the spherical integral momentum boundary layer equations from the general equations of motion. The following assumptions are necessary for this development:

1) Steady flow
2) Constant density and viscosity
3) Axially symmetric flow
4) Boundary layer assumptions (i.e. Flow property changes normal to the cone wall are much greater than flow property changes along the wall
5) Similar velocity profiles exist in the boundary layer
6) Laminar flow

The boundary layer equations are obtained from the general equations of motion by an order of magnitude analysis. The boundary layer assumptions imply that the boundary layer thickness (δ) is very small compared with the characteristic length of the cone. That is, δ = O(ε), where O(ε) denotes "the order of" and ε is very small. A term by term order of magnitude analysis of each of the equations of motion is presented below.

The continuity equation in spherical coordinates is:

\[ \frac{2u}{R} + \frac{\partial u}{\partial R} + \frac{1}{R} \frac{\partial v}{\partial \theta} + v \cot \theta = 0 \]  \hspace{1cm} (B-1)

The trigonometric functions in the equations of motion refer to the position of the boundary layer, not the boundary layer thickness, and are therefore of order one. With the characteristic length such
that $\frac{\partial u}{\partial R}$ is of order one, a termwise order of magnitude analysis of
the continuity equation yields:

$$0(1) + 0(1) + 0\left(\frac{\nu}{\varepsilon}\right) + 0\left(\frac{\nu}{R}\right) = 0$$

(B-2)

Thus, the transverse velocity $v$ must be of order $\varepsilon$ and the order of magnitude of the continuity equation is:

$$0(1) + 0(1) + 0(1) + 0(\varepsilon) = 0$$

(B-3)

Retaining terms of order one, one obtains the boundary layer continuity equation:

$$\frac{2u}{R} + \frac{\partial u}{\partial R} + \frac{1}{R} \frac{\partial v}{\partial \Theta} = 0$$

(B-4)

The radial momentum equation in spherical coordinates is:

$$u \frac{\partial u}{\partial R} + v \frac{\partial u}{\partial \Theta} - \frac{v^2}{R} - \frac{w^2}{R} = - \frac{\rho c}{\rho} \frac{\partial P}{\partial R} + \nu \left[ \frac{\partial^2 u}{\partial R^2} + \frac{2}{R^2} \frac{\partial u}{\partial R} \right]$$

$$+ \frac{1}{R^2} \frac{\partial^2 u}{\partial \Theta^2} + \frac{1}{R^2} \frac{\partial u}{\partial \Theta} \cot \Theta - \frac{2u}{R^2} - \frac{2}{R^2} \frac{\partial v}{\partial \Theta} - \frac{2v}{R^2} \cot \Theta$$

(B-5)

A termwise order of magnitude analysis of the radial momentum equation yields:

$$0(1) + 0(1) - 0(\varepsilon^2) - 0(1) = - \frac{0(\Delta R)}{\rho R} + \nu 0(1) + \nu 0(1)$$

$$+ \nu 0\left(\frac{1}{\varepsilon^2}\right) + \nu 0\left(\frac{1}{\varepsilon}\right) - \nu 0(1) - \nu 0(1) - \nu 0(\varepsilon)$$

(B-6)
The kinematic viscosity \( \nu \) is known to be very small; therefore, if any of the viscous terms are to be important, the smallest value which the kinematic viscosity can take is of order \( \epsilon^2 \). It will be shown from the transverse momentum equation that the pressure can be assumed constant across the boundary layer. Hence, \( 0(\Delta P_R) = 0(\rho u^2) \), where capital letters denote free stream conditions. Therefore, the termwise order of magnitude analysis of the radial momentum equation yields:

\[
0(1) + 0(1) - 0(\epsilon^2) - 0(1) = -0(1) + 0(\epsilon^2) + 0(\epsilon^2)
\]

\[
+ 0(1) + 0(\epsilon) - 0(\epsilon^2) - 0(\epsilon^2) - 0(\epsilon^3)
\]

Retaining terms of order one, one obtains the radial boundary layer momentum equation:

\[
\frac{u \partial u}{\partial R} + \frac{v \partial u}{R \partial \Theta} - \frac{w^2}{R} = -\frac{g \epsilon}{R \partial \Theta} + \frac{\nu}{R^2 \partial^2 \Theta}
\]

The tangential momentum equation in spherical coordinates is:

\[
\frac{u \partial w}{\partial R} + \frac{v \partial w}{R \partial \Theta} + \frac{w u}{R} + \frac{\nu w \cot \Theta}{R} = \nu \left[ \frac{\partial^2 w}{\partial R^2} \right]
\]

\[
+ \frac{2}{R} \frac{\partial w}{\partial R} + \frac{1}{R^2} \frac{\partial^2 w}{\partial \Theta^2} + \frac{1}{R^2} \frac{\partial w}{\partial \Theta} \cot \Theta - \frac{w}{R^2 \sin^2 \Theta}
\]

In all cases of interest in this investigation, the free stream values of the tangential velocity \( w \) are equal to or greater than the free
stream values of the radial velocity $u$: thus, the tangential velocity is assumed to be of the same order of magnitude as the radial velocity in the boundary layer. With this assumption, a termwise order of magnitude analysis of the tangential momentum equation yields:

$$0(1) + 0(1) + 0(1) + 0(\epsilon) = 0(\epsilon^2) + 0(\epsilon^2) + 0(\epsilon^2)$$

$$+ 0(1) + 0(\epsilon) - 0(\epsilon^2)$$

(B-10)

Retaining terms of order one, the tangential boundary layer momentum equation becomes:

$$u \frac{\partial w}{\partial R} + v \frac{\partial w}{\partial \theta} + wu = \frac{\nu}{\rho R^2 \partial^2 \theta^2}$$

(B-11)

The transverse momentum equation in spherical coordinates is:

$$u \frac{\partial v}{\partial R} + v \frac{\partial v}{\partial \theta} + uv \frac{w^2 \cot \theta}{R} = - \frac{\rho \frac{\partial P}{\partial \theta}}{\rho R^2 \partial \theta} + v \frac{\partial^2 v}{\partial R^2}$$

$$+ \frac{2}{R} \frac{\partial v}{\partial \theta} + \frac{1}{R^2} \frac{\partial^2 v}{\partial \theta^2} + \frac{1}{R^2} \frac{\partial v}{\partial \theta} \cot \theta + \frac{2}{R^2} \frac{\partial u}{\partial \theta} - \frac{v}{R^2 \sin^2 \theta}$$

(B-12)

A termwise order of magnitude analysis of the transverse momentum equations yields:

$$0(\epsilon) + 0(\epsilon) + 0(\epsilon) - 0(1) = - 0(\frac{\Delta P \theta}{\epsilon}) + 0(\epsilon^3) + 0(\epsilon^3)$$

$$+ 0(\epsilon) + 0(\epsilon^2) + 0(\epsilon) - 0(\epsilon^3)$$

(B-13)

From equation (B-13), the change in pressure across the boundary
layer ($\Delta P\theta$) is seen to be at most of order $\epsilon$. Therefore, a constant pressure can be assumed across the boundary layer.

After substitution of the boundary layer continuity equation into the boundary layer momentum equations and rearranging, one obtains equations (B-14) and (B-15), respectively.

\[
2u \frac{\partial u}{\partial R} + \frac{2}{R} u^2 + \frac{1}{R} \frac{\partial}{\partial \theta} (uv) + \frac{\rho c}{\rho} \frac{P}{R} \frac{\partial P}{\partial R} \bigg|_x - \frac{u^2}{R} = \frac{\nu}{R^2} \frac{\partial^2 u}{\partial \theta^2} \tag{B-14}
\]

\[
3 \frac{uw}{R} + \frac{\partial}{\partial R} (uw) + \frac{1}{R} \frac{\partial}{\partial \theta} (wv) = \frac{\nu}{R^2} \frac{\partial^2 w}{\partial \theta^2} \tag{B-15}
\]

The integral momentum equations (B-16) and (B-17) are formed by integrating equations (B-14) and (B-15) across the entire boundary layer thickness.

\[
2 \int_x^\alpha u \frac{\partial u}{\partial R} d\theta + \frac{2}{R} \int_x^\alpha u^2 d\theta - \frac{1}{R} \frac{\partial}{\partial \theta} (uv) \bigg|_x + \int_x^\alpha \left( \frac{\rho c}{\rho} \frac{P}{R} \frac{\partial P}{\partial R} \bigg|_x - \frac{u^2}{R} \right) d\theta = \frac{\nu}{R^2} \frac{\partial u}{\partial \theta} \bigg|_\alpha \tag{B-16}
\]

\[
3 \int_x^\alpha \frac{uw}{R} d\theta + \int_x^\alpha \frac{\partial}{\partial R} (uw) d\theta - \frac{1}{R} \frac{\partial}{\partial \theta} (wv) \bigg|_x = \frac{\nu}{R^2} \frac{\partial w}{\partial \theta} \bigg|_\alpha \tag{B-17}
\]

where $\alpha$ denotes the transverse angle at the cone wall. The value of the lower limit of integration $x$ of the integrals of equations (B-16) and (B-17) is the transverse angle at the outer edge of the larger boundary layer thickness.

The transverse velocity at the edge of the boundary layers
\( v \bigg|_x \) is evaluated from equation (B-18), which is the integrated form of the boundary layer continuity equation.

\[
v \bigg|_x = 2 \int_x^\alpha u \, d\theta + R \int_x^\alpha \frac{\partial u}{\partial R} \, d\theta
\]  

(B-18)

The boundary conditions to be satisfied by equations (B-16) and (B-17) are:

At the cone wall
\( u(R, \alpha) = w(R, \alpha) = v(R, \alpha) = 0 \)  

(B-19)

At the edges of the boundary layers
\( u(R, x) = U(R) \) and \( w(R, x) = W(R) \)  

(B-20)

After substitution of equation (B-18), into equations (B-16) and (B-17) and use of the cone wall boundary conditions, the integral momentum equations become:

\[
2 \int_x^\alpha u \frac{\partial u}{\partial R} \, d\theta + \frac{2}{R} \int_x^\alpha u^2 \, d\theta - \frac{2}{R} \int_x^\alpha u \, d\theta - \int_x^\alpha \frac{\partial u}{\partial R} \, d\theta \\
+ \int_x^\alpha \left( \frac{\partial \rho}{\partial R} \right) \frac{\partial u}{\partial R} \, d\theta = \frac{\rho v}{R^2} \frac{\partial u}{\partial \theta} \bigg|_\alpha
\]  

(B-21)

\[
3 \int_x^\alpha \frac{uw}{R} \, d\theta + \int_x^\alpha \frac{\partial w}{\partial R} (wu) \, d\theta - \frac{2}{R} \int_x^\alpha u \, d\theta \\
- W \int_x^\alpha \frac{\partial u}{\partial R} \, d\theta = \frac{\rho v}{R^2} \frac{\partial w}{\partial \theta} \bigg|_\alpha
\]  

(B-22)

In order to introduce the radial and tangential boundary layer thicknesses into equations (B-8) and (B-9), the following nondimensional
distances are defined:

\[ N = \frac{R (\alpha - \theta)}{\delta_R} \quad \text{and} \quad N_1 = \frac{R (\alpha - \theta)}{\delta_T} \]  

(B-23)

where \( \delta_R \) and \( \delta_T \) are the radial and tangential boundary layer thicknesses, respectively.

The boundary layer thickness ratio \( K \) is defined as the ratio of the tangential boundary layer thickness to the radial boundary layer thickness (i.e. \( K = \delta_T/\delta_R \)). With this definition, the nondimensional transverse distances are related by:

\[ N_1 K = N \]  

(B-24)

The boundary layer velocity profiles are assumed to be of the form \( u(R, \theta) = U(R) f(N) \) and \( w(R, \theta) = W(R) g(N_1) \), where \( f(N) \) and \( g(N_1) \) are equal to zero at the cone wall and one at the outer edge of the boundary layers and satisfy the boundary conditions of equations (B-19) and (B-20).

After substitution of the boundary layer velocity profiles into equations (B-21) and (B-22) and alteration of the variable of integration, the momentum integral equations become:

\[ - \frac{\delta_R}{R} \left[ \frac{dU}{dR} - \frac{U}{R} \right] (2 \int_0^\infty f^2 \, dN - 2 \int_0^\infty f dN) + U^2 \left( \frac{1}{R} - \frac{1}{\delta_R} \frac{d\delta_R}{dR} \right) \]

Equation continued on the next page
The value of the lower limit of integration $X$ of the integrals in equations (B-25) and (B-26) now depends upon the value of the boundary layer thickness ratio. If the boundary layer thickness ratio $K$ is greater than unity, the tangential boundary layer thickness is larger than the radial boundary layer thickness; thus, the limits of integration must go from the outer edge of the tangential boundary layer to the cone wall. On the other hand, if $K$ is less than one, the radial boundary layer is thicker than the tangential boundary layer and the limits of integration must go from the edge of the radial boundary layer ($X = 1.0$) to the cone wall.

Two of the integral coefficients of equation (B-25) are integrated by parts to yield:
After substitution of equations (B-27) and (B-28) into equation (B-25) and some rearrangement, the radial integral momentum equation becomes:

\[
\frac{d\delta_R^2}{dR} = 2 \left[ (-4 - \frac{2b}{a-b}) \frac{1}{U} \frac{dU}{dR} - \frac{2X}{R} \frac{gc}{a-b} \frac{\partial \Delta}{\partial R} \right] \\
\quad + \frac{2c}{a-b} \frac{1}{R} \left( \frac{W}{U} \right)^2 + \frac{2d}{a-b} \frac{\nu}{U}
\]

where the integral coefficients have been replaced by the following definitions.

\[
a = \int_X^O f^2 dN \quad b = \int_X^O f dN \\
c = \int_X^O N dN \quad d = \frac{df}{dN} \bigg|_{N=0}
\]

The boundary layer thickness ratio \( K \) is introduced into equation (B-26) by the use of the chain rule identity.

\[
\frac{d\delta_T}{dR} = \delta_R \frac{dK}{dR} + K \frac{d\delta_R}{dR}
\]
With this identity and some rearrangement, the tangential integral momentum equation becomes

$$\frac{dK}{dR} = \frac{K(e - b)}{m} \frac{1}{U} \frac{dU}{dR} + \frac{Ke}{m} \frac{1}{W} \frac{dW}{dR} + \frac{K(h + m + 3e - b + X)}{m} \frac{1}{R}$$

$$- \frac{K(m + b + h + X)}{m} \frac{1}{\varepsilon_R} \frac{d\varepsilon_R}{dR} - \frac{\nu}{\varepsilon_R^2} \frac{1}{mU}$$

where the integral coefficients in equation (B-32) have been replaced by the definitions below:

$$e = \int_X^0 f g dN \quad h = \int_X^0 g \frac{df}{dN} N dN$$

$$m = \int_X^0 f \frac{dg}{dN} N_1 dN \quad j = \frac{dg}{dN_1} \bigg|_{N_1 = 0}$$

For computational purposes, equations (B-29) and (B-32) are rearranged to yield different dependent variables. The radial momentum equation becomes

$$\frac{dZ}{dR} = z \left[ (-4 - \frac{2b}{a-b}) \frac{1}{U} \frac{dU}{dR} + \frac{2X}{a-b} \frac{ge}{U^2} \frac{d\phi}{dR} \right]$$

$$+ \frac{2c}{a-b} \frac{1}{R} \left( \frac{W}{U} \right)^2 + \frac{2d}{a-b} \frac{\nu R^2}{U}$$

and the tangential momentum equation becomes

$$\frac{dY}{dR} = \frac{d\delta R^2}{dR} \left[ \frac{K(m - b - h - X)}{2m} \right] + \frac{Y}{U} \frac{dU}{dR} \left[ \frac{(e - b)}{m} \right]$$

$$+ \frac{Y}{W} \frac{dW}{dR} \left[ \frac{g}{m} \right] + \frac{Y}{R} \left[ \frac{(h + m + 3e - b + X)}{m} \right] - \frac{\nu}{U} \frac{1}{m}$$

(B-35)
where the new dependent variables are defined by

\[ Z = R^2 \delta_R^2 \quad \text{and} \quad Y = K_\delta R^2 \quad (B-36) \]

The following nondimensional quantities are defined in order to nondimensionalize equations (B-34) and (B-35).

\[ Z_o = \frac{R^2 \delta R^2}{R_i^2} \quad U_o = \frac{U}{U_i} \quad W_o = \frac{W}{W_i} \quad P_o = \frac{P}{\rho_i U_i^2} \quad (B-37) \]

\[ R_o = R/R_i \quad Y_o = \frac{K_\delta R^2}{R_i^2} \quad \rho_o = \frac{\rho}{\rho_i} \quad \Lambda = \frac{\delta_R}{R_i} \]

where the subscript "o" denotes nondimensional quantities, and the subscript "i" denotes inlet conditions. After substitution of the nondimensional variables into equations (B-34) and (B-35), one obtains:

\[ \frac{dZ_o}{dR_o} = Z_o \left[ (-4 - \frac{2b}{a-b}) \frac{1}{U_o} \frac{dU_o}{dR} + \frac{2}{a-b} \frac{\rho_o}{\rho_i} U_o \frac{dP_o}{\delta R_o} \right] \quad (B-38) \]

\[ + \frac{2c}{a-b} \frac{R_o}{R_i^2} \left( \frac{W_o}{U_o} \right)^2 + \frac{2d}{a-b} \frac{1}{Re_i} \frac{R_o^2}{U_o} \]

\[ \frac{dY_o}{dR_o} = \frac{dR_R^2}{dR} \left[ \frac{K(m - b - h - \chi)}{2m} \right] + \frac{Y_o}{U_o} \frac{dU_o}{dR_o} \left[ \frac{(e - b)}{m} \right] \quad (B-39) \]

\[ + \frac{Y_o}{W_o} \frac{dW_o}{dR_o} \left[ \frac{e}{m} \right] + \frac{Y_o}{R_o} \left[ \frac{(h + m + 3e - b + \chi)}{m} \right] - \frac{1}{m} \frac{1}{Re_i} \frac{1}{U_o} \]

The two nondimensional parameters \( Re_i \) and \( R_o \) are the inlet radial
Reynolds Number and the Rossby Number (the ratio of inlet tangential to radial velocity).

\[
Re_i = \frac{U_i R_i}{\nu} \quad \text{and} \quad Ro_i = \frac{W_i}{U_i}
\]

(B-40)

Equations (B-38) and (B-39) are in the form used for all boundary layer calculations contained herein.
Appendix C

Limiting Mainstream Solutions

The purpose of this appendix is to develop the closed form limiting solutions (very viscous and nonviscous) for the transformed equations of motion developed in Appendix A.

In the very viscous limiting case, the inlet radial Reynolds Number approaches zero. For this case the limiting forms of equations (A-9), (A-10), and (A-13) from Appendix A are:

\[
\frac{dF_0}{d\theta} + F_0 \cot \theta = -G_0 \tag{C-1}
\]

\[
\frac{d^2T_0}{d\theta^2} + \frac{dT_0}{d\theta} \cot \theta - T_0 \csc^2 \theta = 0 \tag{C-2}
\]

\[
\frac{d^3G_0}{d\theta^3} - \frac{dG_0}{d\theta} \csc^2 \theta + \frac{d^2G_0}{d\theta^2} \cot \theta + 2 \frac{dG_0}{d\theta} = 0 \tag{C-3}
\]

The boundary conditions for this set of equations are the complete set from equations (A-19) of Appendix A.

Equation (C-2) is integrated twice to yield:

\[
T_0 = -C_1 \cot \theta + C^2/\sin \theta \tag{C-4}
\]

After applying the two tangential boundary conditions from equations (A-19) of Appendix A, equation (C-4) becomes equation (C-5).
The first integration of equation (C-3) yields:

\[ \frac{d^2 G_0}{d\theta^2} + \frac{dG_0}{d\theta} \cot \theta + 2G_0 = C_1 \]  \hspace{1cm} (C-6)

Equation (C-6) is transformed into a nonhomogeneous, first order, Legendre differential equation by letting \( z = \cos(\theta) \)

\[ (1 - z^2) \frac{d^2 G_0}{d\theta^2} - 2z \frac{dG_0}{d\theta} + 2G_0 = C_1 \]  \hspace{1cm} (C-7)

Thus, the general solution of equation (C-5) is:

\[ G_0 = C_2 \cos \theta + C_3 \left( \frac{\cos \theta}{2} \ln \left[ \frac{1 + \cos \theta}{1 - \cos \theta} \right] - 1 \right) + C_1 \]  \hspace{1cm} (C-8)

After application of the three radial velocity profile boundary conditions from equation (A-19) of Appendix A, equation (C-8) becomes equation (C-9).

\[ G_0 = D \left[ 1 - \cos \theta \right] + 1.0 \]  \hspace{1cm} (C-9)

Substitution of equation (C-9) into equation (C-1) and performance of the indicated integration yields:

\[ F_0 \sin \theta = \frac{D}{2} \sin^2 \theta + [D + 1.0] \cos \theta + C_1 \]  \hspace{1cm} (C-10)
After rearranging equation (C-10) and applying the transverse profile boundary condition from equation (A-19) of Appendix A, one obtains equation (C-11).

\[
F_o = \frac{D}{2} \sin \theta - [D + 1.0] \tan(\theta/2)
\]  

The radial Reynolds Number in the inviscid case approaches infinity. Therefore, the inviscid limiting forms of equation (A-9), (A-15), and (A-16) from Appendix A are as follows.

\[
\frac{dF_o}{d\theta} + F_o \cot \theta = -G_o
\]  

\[
\frac{dT_o}{d\theta} + T_o \cot \theta = 0
\]  

\[
2G_o \frac{dG_o}{d\theta} - \frac{dF_o}{d\theta} \frac{dG_o}{d\theta} - F_o \frac{d^2G_o}{d\theta^2} + 2R_o \cot^2 T_o \frac{dT_o}{d\theta}
\]

Equations (C-12), (C-13), and (C-14) require only four boundary conditions for a complete solution.

\[
T_o \left( \alpha \right) = 1.0 \quad F_o \left( 0 \right) = 0
\]

\[
\frac{dG_o}{d\theta} \bigg|_{\theta=0} = 0 \quad G_o \left( 0 \right) = 1.0
\]  

Equation (C-13) is integrated to produce equation (C-16).
Substitution of the tangential velocity boundary condition from equation (C-15) into equation (C-16) yields the following result.

\[ T_0 \sin \theta = C_1 \]  
(C-16)

Substitution of equation (C-17) into equation (C-14) uncouples the tangential and radial velocity profile forms, and equation (C-14) becomes:

\[ 2G_0 \frac{dG_0}{d\theta} - \frac{dF_0}{d\theta} \frac{dG_0}{d\theta} - F_0 \frac{d^2 G_0}{d\theta^2} = 0 \]  
(C-18)

The first integration of equation (C-18) yields equation (C-19).

\[ G_0^2 - F_0 \frac{dG_0}{d\theta} = C_1 \]  
(C-19)

Substitution of equation (C-12) into equation (C-19) results in equation (C-20).

\[ -F_0^2 + \left( \frac{dF_0}{d\theta} \right)^2 + F_0 \frac{d^2 F_0}{d\theta^2} + 3 F_0 \frac{dF_0}{d\theta} \cot \theta = C_1 \]  
(C-20)

After multiplying equation (C-20) by $\sin \theta$ and regrouping terms, one obtains:

\[ \frac{d}{d\theta} \left[ F_0^2 \cos \theta \right] + \frac{d}{d\theta} \left[ F_0 \frac{dF_0}{d\theta} \sin \theta \right] = C_1 \sin \theta \]  
(C-21)
Equation (C-21) can now be integrated to yield

\[ F_0^2 \cos \theta + F_0 \frac{dF_0}{d\theta} \sin \theta = -C_1 \cos \theta + C_2 \]  \hspace{1cm} (C-22)

Equation (C-23) is obtained after multiplication of equation (C-22) by \( \sin \theta' \) and performance of the indicated integration.

\[ F_0^2 \sin^2 \theta = -C_1 \sin^2 \theta - C_2 \cos \theta + C_3 \]  \hspace{1cm} (C-23)

The constants \( C_1, C_2, \) and \( C_3 \) are evaluated by applying the boundary conditions of equation (C-15) to equations (C-19) and (C-23). From equation (C-23), the transverse velocity profile form for the inviscid case is found to be

\[ F_0 = -\tan(\theta/2) \]  \hspace{1cm} (C-24)

The radial velocity profile form is found by direct substitution of equation (C-24) into equation (C-12).

\[ G_0 = 1.0 \]  \hspace{1cm} (C-25)
Appendix D

Static Pressure Calculation

The purpose of this appendix is to describe the calculation of the mainstream static pressure, total system momentum, and mass flow at each radial station along the nozzle. From the mainstream solution, Chapter III, equation (4), the partial derivative of the static pressure with respect to the spherical radius is:

\[
\frac{\partial p}{\partial R} = \frac{\rho}{R^3 g c} \left[ G^2 - \frac{dG}{d\theta} F + T^2 + \nu \left( \frac{d^2 G}{d\theta^2} + \frac{dG}{d\theta} \cot \theta \right) \right] \quad (D-1)
\]

Equation (5), of the same Chapter, gives the partial derivative of the static pressure with respect to the transverse angle as:

\[
\frac{\partial p}{\partial \theta} = \frac{\rho}{R^2 g c} \left[ -F - T^2 \cot \theta + \nu \left( \frac{d^2 F}{d\theta^2} + \frac{dF}{d\theta} \cot \theta \right) \right. \\
\left. - F \csc^2 \theta + 2 \frac{dG}{d\theta} \right] \quad (D-2)
\]

From Chapter III equation (8), the transformed continuity equation is:

\[
\frac{dF}{d\theta} + F \cot \theta = -G \quad (D-3)
\]

Substitution of the derivative of the transformed continuity equation, equation (D-3), into equation (D-2) yields:

\[
\frac{\partial p}{\partial \theta} = \frac{\rho}{R^2 g c} \left[ -F \frac{dF}{d\theta} + T^2 \cot \theta + \nu \left( \frac{dG}{d\theta} \right) \right] \quad (D-4)
\]
The static pressure can now be found at any point in the flow field by the following integration.

\[
P_2 - P_1 = \int_{R_1, \theta_1}^{R_2, \theta_2} \left( \frac{\partial P}{\partial R} \, dR + \frac{\partial P}{\partial \theta} \, d\theta \right)
\]

Both physical and mathematical reasoning show that the static pressure is independent of the path of integration. Therefore, the most convenient path of integration can be used to evaluate the static pressure.

In order to initiate the overall solution, the static pressure \( P_{c1} \) at the centerline of the radial nozzle station number 1 is specified. Every point on station 1 is at a constant spherical radius, thus the static pressure at any point of station 1 is:

\[
P(R_1, \theta) = \frac{\rho}{R_1^2 \rho_g} \left[ -\frac{F^2}{2} + \nu G + \int_0^\theta 2 \cot \theta \, d\theta - \nu G(0) \right] + P_{c1}
\]

If the centerline static pressure is known at any station, equation (D-6) can be generalized to find the static pressure at any point on the station.

The static pressure at the centerline of any nozzle station can be found in the following manner. The path of integration of the partial derivative of the static pressure with respect to the radius is chosen to be along the nozzle centerline \((\theta = 0)\). Along the nozzle centerline, the partial derivative of the static pressure with respect
to the radius is:

$$\frac{\partial P}{\partial R} \neq 0 = \frac{\rho}{R^3 g_c} [U_c^2 R^2 + 2\nu U_c R D]$$  \hspace{1cm} (D-7)

As seen from Chapter V, the values of $U_c$ and $D$ must change as a function of spherical radius in order to conserve system mass flow and momentum. However, even though the values of $U_c$ and $D$ are known at each station, the functional form of the relationship connecting these values between adjacent stations is not known. Since, the distance between radial stations is small, a linear relationship for both $U_c$ and $D$ is assumed to connect the values between adjacent stations. Thus, the static pressure at the centerline of any station ($M$) can be found by equation (D-8).

$$P_{c,M} = \frac{\rho}{g_c} \left[ C_1^2/2 \left( R_M^2 - R_{M-1}^2 \right) + \left( 2C_1 C_2 + 2\nu C_1 C_3 \right) \left( R_M - R_{M-1} \right) \right]$$

$$+ \left[ C_2^2 + 2\nu \left[ C_4 C_1 + C_2 C_3 \right] \right] \ln \left[ \frac{R_M}{R_{M-1}} \right] - 2\nu C_4 C_2 \left[ \frac{1}{R_M} - \frac{1}{R_{M-1}} \right]$$

$$+ P_{c,M-1}$$  \hspace{1cm} (D-8)

where $M$ denotes the radial station number and $C_1$, $C_2$, $C_3$, and $C_4$ are defined as

$$C_1 = \frac{(U_{i,M} - U_{i,M-1})}{(R_M - R_{M-1})}$$

$$C_2 = \frac{U_{i,M-1} - C_1 R_{M-1}}{R_{M-1}}$$
\[ C_3 = (D_M - D_{M-1}) / (R_M - R_{M-1}) \]

\[ C_4 = D_{M-1} - C^3_{3R_{M-1}} \]  \hspace{1cm} \text{(D-9)}

The static pressure at any point in the flow field can now be obtained by using equation (D-8) to find the centerline pressure corresponding to the radius of the point and then using equation (D-6) to find the static pressure at that point.

The total system mass flow in the mainstream at any radial station is found by integrating equation (D-10) to the nozzle wall in the first mainstream solution and to the edge of the nozzle wall boundary layer in the second mainstream solution.

\[ \frac{\text{d}M_s}{\text{d}\theta} = 2\pi R^2 \rho U_c G_0 \sin \theta \]  \hspace{1cm} \text{(D-10)}

The total force in the radial direction on any radial station is found by integrating equation (D-11) to the nozzle wall in the first mainstream solution and to the edge of the nozzle wall boundary layer in the second solution.

\[ \frac{\text{d}F_s}{\text{d}\theta} = \frac{2\pi R^2 \rho U_c^2 G_0^2 \sin \theta}{gc} + 2\pi R^2 p \sin \theta \]  \hspace{1cm} \text{(D-11)}
Appendix E

Boundary Layer Mass and Momentum Equations

The purpose of this appendix is to describe the calculation of the mass flow and the radial momentum dissipation in the cone wall boundary layer.

The mass flow in the boundary layer at any radial station along the nozzle is found by the following integration

$$d(Bm) = 2\pi R^2 \rho \sin \theta \, d\theta \, u$$  \hspace{1cm} (E-1)

From Chapter IV, the boundary layer radial velocity is:

$$u(R,\theta) = U \left[ \frac{3}{2} N - \frac{1}{2} N^3 \right]$$  \hspace{1cm} (E-2)

and

$$N = \frac{R}{\delta R} (\alpha - \theta)$$  \hspace{1cm} (E-3)

Substitution of equations (E-2) and (E-3) into equation (E-1) yields

$$d(Bm) = 2\pi \rho U R^2 \left[ \frac{3}{2} N - \frac{1}{2} N^3 \right] \sin (\alpha - N \frac{\delta R}{R}) \frac{\delta R}{R} dN$$  \hspace{1cm} (E-4)

The limits of integration for equation (E-4) are from the edge of the larger boundary layer thickness (X) to the cone wall (\alpha).

With the use of a trigonometric identity for the difference of...
two angles, the integration of equation (E-4) yields

\[ B_m = 2\pi R^2 \mu \rho \left[ - \frac{3}{2} \frac{R}{\delta R} \sin \alpha + \frac{3}{2} \frac{R}{\delta R} \sin \left( \alpha - \frac{\delta R}{R} \chi \right) + \frac{3}{2} \chi \cos \left( \alpha - \frac{\delta R}{R} \chi \right) \right. \]

\[ \left. - \frac{R}{\delta R} \frac{3}{2} \sin \alpha - \frac{R}{\delta R} \left( \frac{R}{\delta R} \right)^2 \sin \left( \alpha - \frac{\delta R}{R} \chi \right) + \left( \frac{R}{\delta R} \right)^3 \sin \left( \alpha - \frac{\delta R}{R} \chi \right) \right] \]

\[ \left. \cos \left( \alpha - \frac{\delta R}{R} \chi \right) + 3 \left( \frac{R}{\delta R} \right)^2 \cos \left( \alpha - \frac{\delta R}{R} \chi \right) \right] \]

(E-5)

The radial momentum dissipated by the wall is calculated by the integration of equation (E-6) from the inlet of the nozzle to any radial station.

\[ d(W_f) = 2\pi R \, dR \left. \frac{\partial u}{\partial \theta} \right|_{\alpha} \]

(E-6)

Substitution of the boundary layer radial velocity profile into equation (E-6) yields

\[ W_f = - \frac{3\pi \mu}{\rho c} \int_{R_i}^{R} \frac{UR}{\delta R} \, dR \]

(E-7)