Integrated guidance and control of missiles with $\theta$-D method

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Abstract—A new suboptimal control method is proposed in this study to effectively design an integrated guidance and control system for missiles. Optimal formulations allow designers to bring together concerns about guidance law performance and autopilot responses under one unified framework. They lead to a natural integration of these different functions. By modifying the appropriate cost functions, different responses, control saturations (autopilot related), miss distance (guidance related), etc., which are of primary concern to a missile system designer, can be easily studied. A new suboptimal control method, called the $\theta$-$D$ method, is employed to obtain an approximate closed-form solution to this nonlinear guidance problem based on approximations to the Hamilton–Jacobi–Bellman equation. Missile guidance law and autopilot design are formulated into a single unified state space framework. The cost function is chosen to reflect both guidance and control concerns. The ultimate control input is the missile fin deflections. A nonlinear six-degree-of-freedom (6-DOF) missile simulation is used to demonstrate the potential of this new integrated guidance and control approach.

Index Terms—Missile integrated guidance and control, non-linear systems, optimal control.

I. INTRODUCTION

INTEGRATED guidance and control (IGC) design is an emerging trend in missile technology. This is a response to the need for improving the accuracy of interceptors and extending their kill envelope. Current and past practices in industry have been to design guidance and control systems separately and then integrate them into the missile. These subsystems typically had different bandwidths. Despite the fact that this paradigm has been applied successfully on many systems, it can be argued that it is not truly optimized; therefore, the overall system performance can be improved. Hit-to-kill capabilities required in the next generation missile system will demand an integrated approach in order to exploit synergism between various missile subsystems and thereby improve the total system performance. An IGC design can be formulated as a single optimization problem, thus providing a unified approach to intercept performance optimization.

Lin and Yueh [1] first addressed the application of an IGC scheme to a homing missile. An optimal controller was designed to combine the conventionally separated guidance law and autopilot design into one framework by minimizing a quadratic cost functional subject to intercept dynamics. The advantages gained in this optimal control law were minimization of the root-mean-square (rms) miss, the terminal angle of attack, the pitch rate, and the control surface “flapping” rate in the presence of unmodeled errors. However, this paper only dealt with a nonmaneuvering target. Evers et al. [2] extended the concepts presented in [1] to include a target acceleration model as a first-order Markov process. The resulting IGC law was expected to be less sensitive to the errors in estimating the current target acceleration. Menon and Ohlmeyer [3] employed the feedback linearization method in conjunction with the linear quadratic regulator (LQR) technique to design a nonlinear integrated guidance and control laws for homing missiles. The IGC design was presented in three formulations which were based upon three different guidance objectives. A six-degrees-of-freedom (6-DOF) nonlinear dynamic model of an air-to-air homing missile was simulated and each of the three IGC schemes achieved a similar favorable performance. Menon et al. [4] employed the feedback linearization technique to the IGC of a moving-mass actuated kinetic warhead. A 9-DOF simulation demonstrated good results for interception of nonmaneuvering and weaving targets in both endo-atmospheric and exo–atmospheric conditions. Although feedback linearization is a powerful tool, it could cancel beneficial nonlinearities and result in a large control. Also, it is only applicable to systems which satisfy some conditions of feedback linearizability [5].

Other IGC schemes that have been developed incorporate various control theories. Shkolnikov et al. [6] developed an IGC design using sliding-mode control. They divide their controller development into inner and outer loop objectives. Menon and Ohlmeyer [7] employed the state dependent Riccati equation (SDRE) technique [8] to deal with a more comprehensive model that is nonlinear with motion in three dimensions. The design was evaluated based on a 6-DOF nonlinear missile model with two types of target models, nonmaneuvering targets and weaving maneuvering targets. The numerical results demonstrated the feasibility of designing integrated guidance/control systems for the next generation high-performance missile systems. Palumbo and Jackson [9] formulated the IGC problem as a single nonlinear minmax optimization problem, which is to find a controller that minimizes the final miss distance and control energy under worst case target maneuver and worst case process and measurement disturbances. The state dependent Riccati difference equation (SDRDE) technique was employed to handle this finite-time horizon nonlinear problem. The sim-
ulation results compared favorably with the benchmark system using Dynamic Inversion and Optimal Guidance. However, as mentioned in that paper, solving state dependent Riccati equation online is time consuming. Particularly for a 6-DOF missile with an integrated guidance/control design, the system order grows much higher and the SDRE approach requires significant computational capability for online implementation that is sometimes not feasible.

In this paper, an IGC scheme is formulated as an infinite-horizon optimal control problem. The terminal guidance problem is a finite-horizon problem by nature. Time-to-go \( t_{\text{to}} \) is required in a traditional guidance law derived from a finite-time linear quadratic optimal control formulation or similar formulations. However, an accurate estimate of \( t_{\text{to}} \) is difficult to obtain in scenarios involving a maneuvering target whose maneuvers are unknown to the missile. The IGC does not need \( t_{\text{to}} \). Nevertheless, the infinite time formulations tend to compensate for the error in proportion to the magnitude and may cause a large control at the initial stage or large oscillations in the states. In order to account for the finite-time nature of the guidance problem, an innovative approach of state-dependent weight on the position error to avoid large control effort. As penalize the relative range inversely proportional to its magnitude and to compensate for the error in proportion to the magnitude and sometimes not feasible.

The nonlinear infinite-horizon IGC problem is solved by utilizing the \( \theta \)-D technique [10]. This method is developed from optimal control theory and gives an approximate closed-form suboptimal feedback controller with no iterative solutions as in the case of the SDRE approach. The technique has already been successfully employed in a 6-DOF nonlinear missile autopilot design in [11] and has demonstrated great potential in the missile control problem.

The paper is organized as follows. Section II summarizes the \( \theta \)-D technique. The 6-DOF nonlinear missile model is given in Section III and the IGC formulation using the \( \theta \)-D technique is presented in Section IV. Section V discusses the simulation results and conclusions are given in Section VI.

II. SUMMARY OF \( \theta \)-D SUBOPTIMAL CONTROL METHOD

Consider a class of nonlinear time-invariant systems described by

\[ \dot{x} = f(x) + B(x)u. \]  

(1)

The objective is to find a stabilizing controller that minimizes the cost functional

\[ J = \frac{1}{2} \int_0^\infty (x^T Q x + u^T R u) dt \]  

(2)

where \( x \in R^n, f \in R^n, R \in R^{n \times n}, u \in R^m, Q \in R^{n \times n}, R \in R^{m \times m} \). Assume that \( x \in \Omega \) and \( \Omega \) is a compact set in \( R^n \); \( Q(x) \) is positive semidefinite and \( R \) is a positive definite constant matrix. It is assumed that \( f(x) \) is of class \( C^1 \) in \( x \) on \( \Omega \) and \( f(0) = 0 \).

The optimal solution of this infinite-horizon nonlinear regulator problem can be obtained by solving the Hamilton–Jacobi–Bellman (HJB) partial differential equation [12]

\[ \frac{\partial V^T}{\partial x} f(x) - \frac{1}{2} \frac{\partial V^T}{\partial x} B(x) R^{-1} B^T(x) \frac{\partial V}{\partial x} + \frac{1}{2} x^T Q(x) x = 0 \]  

(3)

where \( V(x) \) is the optimal cost, i.e.,

\[ V(x) = \min_u \frac{1}{2} \int_0^\infty [x^T Q(x) x + u^T R u] dt. \]  

(4)

Assume that \( V(x) \) is continuously differentiable and \( V(x) > 0 \) with \( V(0) = 0 \). Optimal control is given by

\[ u = -R^{-1} B^T(x) \frac{\partial V}{\partial x}. \]  

(5)

The HJB equation is extremely difficult to solve in general, so an approximate solution is attempted by considering perturbations to the cost function

\[ J = \frac{1}{2} \int_0^\infty \left\{ x^T \left[ Q(x) + \sum_{i=1}^\infty D_i \theta^i \right] x + u^T R u \right\} dt \]  

(6)

where \( \theta \) and \( D_i \) are chosen such that \( Q(x) + \sum_{i=1}^\infty D_i \theta^i \) is positive semidefinite.

For later manipulation, the state equation is rewritten as

\[ \dot{x} = f(x) + B(x)u \]

\[ = \left\{ A_0 + \theta \left[ A(x) \theta \right] \right\} x + \left\{ g_0 + \theta \left[ g(x) \theta \right] \right\} u \]  

(7)

where \( A_0 \) is a constant matrix, such that \( (A_0, g_0) \) is a stabilizable pair and \( \left\{ [A_0 + A(x)], [g_0 + g(x)] \right\} \) is pointwise controllable. \( \theta \) is an intermediate variable for the purpose of power series expansion.

Furthermore, write the perturbed cost function as

\[ J = \frac{1}{2} \int_0^\infty \left\{ x^T \left[ Q_0 + \theta \frac{Q_0(x)}{\theta} + \sum_{i=1}^\infty D_i \theta^i \right] x + u^T R u \right\} dt \]  

(8)

such that \( Q(x) = Q_0 + Q_0(x) \) and \( Q_0 \) is a constant matrix. Define

\[ \lambda = \frac{\partial V}{\partial x}. \]  

(9)

By using (8) and (9) in the HJB (3), the perturbed HJB equation becomes

\[ x^T f(x) - \frac{1}{2} \lambda^T B(x) R^{-1} B^T(x) \lambda \]

\[ + \frac{1}{2} x^T \left[ Q_0 + \frac{Q_0(x)}{\theta} + \sum_{i=1}^\infty D_i \theta^i \right] x = 0. \]  

(10)
The objective is to solve for $\lambda$ in (10). Therefore, a power series solution for $\lambda$ is assumed in terms of $\theta$ as

$$\lambda = \frac{\partial V}{\partial x} = \sum_{i=0}^{\infty} T_i \theta^i x$$

(11)

where $T_i$ are assumed to be symmetric and to be determined.

Substitute (11) into the perturbed HJB (10) and equate the coefficients of the powers of $\theta$ to zero to get the following equations:

$$T_0 A_0 + A_0^T T_0 - T_0 g_0 R^{-1} g_0^T T_0 + Q_0 = 0$$

(12)

$$T_1 A_0 - A_0^T T_0 + T_0 g_0 R^{-1} g_0^T T_0 + T_0 g(x) \frac{\partial}{\partial x} R^{-1} g(x) T_0 + T_0 g(x) \frac{\partial}{\partial x} T_0 = 0$$

(13)

$$T_n A_0 - A_0^T T_{n-1} + T_0 g_0 R^{-1} g_0^T T_n + T_n g(x) \frac{\partial}{\partial x} R^{-1} g(x) T_{n-1} + T_n g(x) \frac{\partial}{\partial x} T_{n-1} = 0$$

(14)

$$T_n A_0 - A_0^T T_{n-1} + T_0 g_0 R^{-1} g_0^T T_n + T_n g(x) \frac{\partial}{\partial x} R^{-1} g(x) T_{n-1} + T_n g(x) \frac{\partial}{\partial x} T_{n-1} = 0$$

(15)

Since the right-hand side of (13)–(15) involve $x$ and $\theta, T_i$ would be a function of $x$ and $\theta$. Thus, we denote it as $T_i(x, \theta)$. The expression for control can be obtained in terms of a power series as

$$u = -R^{-1} B^T(x) \frac{\partial V}{\partial x} = -R^{-1} B^T(x) \sum_{i=0}^{\infty} T_i(x, \theta) \theta^i x$$

(16)

It is easy to see that (12) is an algebraic Riccati equation. The rest of the equations are Lyapunov equations that are linear in terms of $T_i(i = 1, \ldots, n)$.

We construct the following expression for $D_i, i = 1, \ldots, n$:

$$D_1 = k_1 e^{l_1 t} \left[ T_1 A(x) - A^T(x) T_1 + T_0 g_0 R^{-1} g_0^T(x) T_1 + T_0 g(x) \frac{\partial}{\partial x} R^{-1} g(x) T_0 + T_0 \frac{\partial}{\partial x} Q(x) \right]$$

(17)

$$D_2 = k_2 e^{l_2 t} \left[ T_1 A(x) - A^T(x) T_1 + T_0 g_0 R^{-1} g_0^T(x) T_1 + T_0 g(x) \frac{\partial}{\partial x} R^{-1} g(x) T_0 + T_0 \frac{\partial}{\partial x} Q(x) \right]$$

(18)

$$D_n = k_n e^{l_n t} \left[ T_{n-1} A(x) - A^T(x) T_{n-1} + \sum_{j=0}^{n-1} T_j + \sum_{j=0}^{n-2} T_j g(x) \frac{\partial}{\partial x} R^{-1} g(x) T_{n-2-j} + \sum_{j=1}^{n-1} T_j g(x) \frac{\partial}{\partial x} T_{n-1-j} \right]$$

(19)

where $k_i$ and $l_i > 0, i = 1, \ldots, n$ are adjustable design parameters.

The $D_i$ are chosen such that

$$-T_{i-1} A(x) - A^T(x) T_{i-1} + \sum_{j=0}^{i-1} T_j g(x) \frac{\partial}{\partial x} R^{-1} g(x) T_{i-2-j} + \sum_{j=1}^{i-1} T_j g(x) \frac{\partial}{\partial x} T_{i-1-j}$$

$$= \varepsilon_i(t) \left[ -T_{i-1} A(x) - A^T(x) T_{i-1} + \sum_{j=0}^{i-1} T_j g(x) \frac{\partial}{\partial x} R^{-1} g(x) T_{i-2-j} + \sum_{j=1}^{i-1} T_j g(x) \frac{\partial}{\partial x} T_{i-1-j} \right]$$

(20)

where

$$\varepsilon_i(t) = 1 - k_i e^{-l_i t}$$

(21)

is a small number. The $\varepsilon_i$ serve three functions. The first is to suppress large control magnitudes. To see this, for example, when $A(x)$ includes a cubic term, its magnitude could be large if $x$ is large. This large value will be reflected into the solution for $T_i$, i.e., the left-hand side of (13)–(15). Since $T_i$ will be used in the next equation to solve for $T_{i+1}$, this large value will be
propagated and amplified and, consequently, cause higher control or even instability. ε can be used to prevent the large value from propagating in (13)–(15). The second function is to satisfy some conditions required in the proof of convergence and stability of the above algorithm [10]. The third usage is to modulate the system transient performance. The exponential term e−lt with li > 0 is used to let the perturbation terms in the cost function and HJB equation diminish as time evolves.

Remark: θ is just an intermediate variable. Its value can be kept as unity.

The steps of applying the method are summarized as follows.
1) Solve the algebraic Riccati equation (12) to get $T_0$ once $A_0, g_0, Q_0$, and $R$ are determined. Note that the resulting $T_0$ is a positive-definite constant matrix.
2) Solve Lyapunov equation (13) to get $T_1(x, \theta)$. Note that it is a linear equation in terms of $T_1$ and an interesting property of this and the rest of the equations is that the coefficient matrices $A_0 - g_0R^{-1}g_0^T T_0$ and $A_0^T - T_0g_0R^{-1}g_0^T$ are constant. Let $A_0 = A_0 - g_0R^{-1}g_0^T T_0$. Through linear algebra, (13) can be brought into a form like $\dot{A}_0 = \dot{A}_0 - g_0R^{-1}g_0^T$ in which $\dot{A}_0$ is the right-hand side of the (13); $\text{Vec}(M)$ denotes stacking the elements of matrix $M$ by rows in a vector form; $\dot{A}_0 = I \otimes A_0 + A_0^T \otimes I$ is a constant matrix and the symbol $\otimes$ denotes the Kronecker product. The resulting solution of $T_1$ can be written in closed form as $\dot{A}_0 = \dot{A}_0 \text{Vec}[Q_1(x, \theta, \ell)]$.
3) Solve (14) and (15) by following the same procedure as in step 2. The number of $T_i$s needed depends on the problem. The simulation results show that $T_0, T_1$, and $T_2$ are usually sufficient to achieve satisfactory performance for this class of IGC problems.

As can be seen, closed-form solutions for $T_0, \ldots, T_n$ can be obtained with just one matrix inverse operation. The expression $Q_i(x, \theta, \ell)$ on the right-hand side of the equations is already known and needs only simple matrix multiplications and additions. If finite terms in (16) are taken (three terms have been found to be sufficient for this problem and some others), the resulting control law has a closed-form expression and allows for easy online implementation.

III. 6-DOF MISSILE MODEL

The translational and rotational dynamics of a missile in the missile body coordinate can be described by the following six nonlinear differential equations [7]:

$$\dot{U} = Vr - Wq - \frac{7S}{m} \left[ a_{0c_r} + a_{1c_r} \alpha + a_{2c_r} \beta \right]$$

$$+ a_{3c_r} \alpha \beta \right] + \frac{F_{2g}}{m} \right] \right)$$

$$\dot{V} = Wp - Ur + \frac{7S}{m} \left[ a_{1c_v} \alpha + a_{2c_v} \beta + a_{3c_v} \alpha^3 \right]$$

$$+ a_{4c_v} \beta \delta p + a_{6c_v} \delta q$$

$$+ a_{7c_v} \delta \beta \right] + \frac{F_{2g}}{m} \right) \right)$$

$$\dot{W} = Uq - Vp - \frac{7S}{m} \left[ a_{1c_s} \alpha + a_{2c_s} \beta + a_{3c_s} \alpha^3 \right]$$

$$+ a_{4c_s} \beta \delta p + a_{6c_s} \delta q$$

$$+ a_{7c_s} \delta \beta \right] + \frac{F_{2g}}{m} \right) \right)$$

$$\dot{\theta} = \frac{7S}{I_x} \left[ a_{1c_\theta} \alpha + a_{2c_\theta} \beta + a_{3c_\theta} \alpha^3 + a_{4c_\theta} \beta \right]$$

$$+ a_{5c_\theta} \delta p + a_{6c_\theta} \delta q + a_{7c_\theta} \delta \beta \right] \right)$$

(24)

(25)

(26)

(27)

where $U, V, W$ are the velocity components in the missile body axis system; $p, q, r$ are the body rotational rate; $F_{2g}, F_{y}, F_{z}$ are the gravitational forces acting along the body axes; $\dot{q}$ is the dynamic pressure; $S$ is the reference area; $l$ is the reference length; $I_x, I_y, I_z$ are moment of inertia about the body $X-Y-Z$ axis. Assume that the missile configuration is symmetrical about the $XZ$ and $XY$ plane, i.e., $I_{y} = I_{z} = I_{xx} = 0$. However, use of this method is not restricted by this assumption. The aerodynamic force and moment coefficients are described in a polynomial form with respect to angle of attack $\alpha$, angle of sideslip $\beta$, roll fin deflection $\delta p$, pitch fin deflection $\delta q$, and yaw fin deflection $\delta r$. Coefficients of the polynomials describing the aerodynamic coefficients were derived by carrying out least squares fits on the aerodynamic data from [13].

The missile speed $V_t$, dynamic pressure $q$, angle of attack $\alpha$, and the angle of sideslip $\beta$ are defined as

$$V_t = \sqrt{U^2 + V^2 + W^2}, \quad q = \frac{1}{2} \rho V_t^2, \quad \alpha = \tan^{-1} \frac{W}{U}, \quad \beta = \tan^{-1} \frac{V}{U}$$

(28)

where $\rho$ is the air density.

IV. IGC DESIGN WITH $\theta$-D TECHNIQUE

The advantage of adopting $U, V, W$ as states is that the dynamic pressure appears linearly in all the aerodynamic force and moment equations and, consequently, the missile velocity components can be extracted from the equations of motion to yield a linear-like structure [7].

The IGC design in this paper, commands the missile to track the position and the velocity of the target. Since the missile seeker defines the target position relative to the missile body coordinates, it is desirable to define the missile and target position in the missile body frame. Denote the positions of the target and the missile in the missile body coordinate system as $\bar{r}_t = [x_{t}^n, y_{t}^n, z_{t}^n]^T$ and $\bar{r}_m = [x_{m}^n, y_{m}^n, z_{m}^n]^T$, respectively.
The kinematic equations of the missile position can be described by

\[ \begin{align*}
\dot{x}_m^m &= U + y_b^m r - z_b^m q \\
\dot{y}_b^m &= V - x_b^m r + z_b^m p \\
\dot{z}_b^m &= W + x_b^m q - y_b^m p.
\end{align*} \tag{29-31} \]

The advantage of describing the target and missile position in the rotating coordinate system is that it circumvents the need for computing the Euler angles required in the transformation matrix when implementing the IGC law.

Note that the missile and target positions in the missile body coordinates can be related to their respective positions in the inertial coordinates through the transformation matrix (as shown in (32) at the bottom of the page), where \(\psi, \theta, \phi\) are yaw, pitch, and roll Euler angles. The Euler angle rates with respect to body rotational rates are given by the expressions in (33)-(35)

\[ \begin{align*}
\dot{\psi} &= p + q \sin \phi \tan \theta + r \cos \phi \tan \theta \\
\dot{\phi} &= (q \sin \phi + r \cos \phi) \sec \theta.
\end{align*} \tag{33-35} \]

The guidance objective is to minimize the relative distance between the missile and the target where

\[ r_t - r_m = [x_t^m - x_b^m \quad y_t^m - y_b^m \quad z_t^m - z_b^m]^T. \tag{36} \]

Besides the guidance objective, the IGC design must also stabilize all the states of the missile. It must meet the position and rate limits on the fin deflections and be able to adapt to maneuverable targets. In this study, the actuator dynamics are assumed sufficiently fast and are not modeled in the IGC development. Also, all the states in the IGC feedback law are assumed available as measurements.

Note that integral states are added to the state space to improve the steady-state performance. The final state space is chosen to be

\[ \begin{align*}
x &= [p \quad q \quad r \quad U \quad V \quad W \quad y_b^m \quad z_b^m \quad \int y_b^m dt \quad \int z_b^m dt]^T \\
u &= [\delta p \quad \delta q \quad \delta r]^T.
\end{align*} \tag{37-38} \]

The cost function is chosen to be a quadratic type of function as shown in (2).

In this paper, the \(\theta-D\) design of the IGC is employed as an integral servomechanism [14] as described in (39) at the bottom of the page, where \((y_b^m, z_b^m)\) serves as a command for the missile guidance to track; \(V^t\) and \(W^t\) are the target inertial velocity components projected along the missile body \(y\)- and \(z\)-axis. Although it is formulated as a control problem, the guidance objective is achieved by driving the position error to zero while keeping the control effort small. So both guidance and autopilot design objectives are formulated as a unified optimal control problem through a single appropriate cost function.

The \(\theta-D\) design technique requires that the nonlinear system dynamics be written in a linear-like structure for use in the controller development

\[ \dot{x} = F(x) + B(x)u = \left( A_0 + \theta \left( A(x) \over \theta \right) \right) x + \left( g_0 + \theta \left( g(x) \over \theta \right) \right) u. \tag{40} \]

The gravitational force contributions to the equations of motion are dropped from consideration to follow the standard practice in missile guidance design. The elements of the coefficient matrix in (40) are given to be

\[ \begin{align*}
F(1,1:3) &= 0 \\
F(1,7:10) &= 0 \\
F(1,4) &= \frac{\nu S_I}{2 I_x} \left[ a_{1 \epsilon \epsilon} \alpha + a_{2 \epsilon \epsilon} \beta + a_{3 \epsilon \epsilon} \alpha^3 + a_{4 \epsilon \epsilon} \beta^3 \right] U \\
F(1,5) &= \frac{\nu S_I}{2 I_x} \left[ a_{1 \epsilon \epsilon} \alpha + a_{2 \epsilon \epsilon} \beta + a_{3 \epsilon \epsilon} \alpha^3 + a_{4 \epsilon \epsilon} \beta^3 \right] V \\
F(1,6) &= \frac{\nu S_I}{2 I_x} \left[ a_{1 \epsilon \epsilon} \alpha + a_{2 \epsilon \epsilon} \beta + a_{3 \epsilon \epsilon} \alpha^3 + a_{4 \epsilon \epsilon} \beta^3 \right] W \\
F(2,1) &= -\frac{I_x - I_y}{I_y} r \\
F(2,2:3) &= 0 \\
F(2,7:10) &= 0 \\
F(2,4) &= \frac{\nu S_I}{2 I_y} \left[ a_{1 \epsilon \epsilon m} \alpha + a_{2 \epsilon \epsilon m} \beta + a_{3 \epsilon \epsilon m} \alpha^3 + a_{4 \epsilon \epsilon m} \beta^3 \right] U \\
F(2,5) &= \frac{\nu S_I}{2 I_y} \left[ a_{1 \epsilon \epsilon m} \alpha + a_{2 \epsilon \epsilon m} \beta + a_{3 \epsilon \epsilon m} \alpha^3 + a_{4 \epsilon \epsilon m} \beta^3 \right] V \\
F(2,6) &= \frac{\nu S_I}{2 I_y} \left[ a_{1 \epsilon \epsilon m} \alpha + a_{2 \epsilon \epsilon m} \beta + a_{3 \epsilon \epsilon m} \alpha^3 + a_{4 \epsilon \epsilon m} \beta^3 \right] W
\end{align*} \]

The feedback control law is

\[ u = -R^{-1}B(x) \sum_{i=0}^{\infty} T_i(x, \theta) \theta \begin{bmatrix} p \quad q \quad r \quad U \quad V - V_t \quad W - W_t \quad y_b^m - y_b^t \quad z_b^m - z_b^t \quad \int y_b^m dt - \int y_b^t dt \quad \int z_b^m dt - \int z_b^t dt \end{bmatrix}^T \tag{39} \]

\[ T_B^f(\phi, \theta, \psi) = \begin{bmatrix}
\cos \theta \cos \psi & \sin \theta \sin \phi \cos \psi & -\cos \phi \sin \psi & \cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi & \cos \phi \sin \theta \sin \psi & \sin \phi \cos \psi & \cos \phi \sin \psi & \sin \phi \cos \psi \\
\cos \theta \sin \psi & \sin \theta \sin \phi \cos \psi & -\cos \phi \sin \psi & \cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi & \cos \phi \sin \theta \sin \psi & \sin \phi \cos \psi & \cos \phi \sin \psi & \sin \phi \cos \psi \\
-\sin \theta & \sin \phi \sin \psi & \cos \phi \cos \psi & \sin \phi \cos \psi & \cos \phi \sin \psi & \sin \phi \cos \psi & \cos \phi \sin \psi & \sin \phi \cos \psi \\
\end{bmatrix} \tag{32} \]
\[ F(3,1) = -\frac{I_y - I_x}{I_z} q \]
\[ F(3,2 : 3) = 0 \]
\[ F(3,7 : 10) = 0 \]
\[ F(3,4) = \frac{\rho S}{2I_z} \left[ a_1 c_n \alpha + a_2 c_n \beta + a_3 c_n \alpha^3 + a_4 c_n \beta^3 \right] U \]
\[ F(3,5) = \frac{\rho S}{2I_z} \left[ a_1 c_n \alpha + a_2 c_n \beta + a_3 c_n \alpha^3 + a_4 c_n \beta^3 \right] V \]
\[ F(3,6) = \frac{\rho S}{2I_z} \left[ a_1 c_n \alpha + a_2 c_n \beta + a_3 c_n \alpha^3 + a_4 c_n \beta^3 \right] W \]
\[ F(4,1) = 0 \]
\[ F(4,2) = -W \]
\[ F(4,3) = V \]
\[ F(4,7 : 10) = 0 \]
\[ F(4,4) = \frac{\rho S}{2m} \left[ a_0 c_A + a_1 c_A \alpha + a_2 c_A \beta + a_3 c_A \alpha \beta + a_4 c_A \alpha \beta^2 \right] U \]
\[ F(4,5) = \frac{\rho S}{2m} \left[ a_0 c_A + a_1 c_A \alpha + a_2 c_A \beta + a_3 c_A \alpha \beta + a_4 c_A \alpha \beta^2 \right] V \]
\[ F(4,6) = \frac{\rho S}{2m} \left[ a_0 c_A + a_1 c_A \alpha + a_2 c_A \beta + a_3 c_A \alpha \beta + a_4 c_A \alpha \beta^2 \right] W \]
\[ F(5,1) = W \]
\[ F(5,2) = 0 \]
\[ F(5,3) = -U \]
\[ F(5,7 : 10) = 0 \]
\[ F(5,4) = \frac{\rho S}{2m} \left[ a_1 c_Y \alpha + a_2 c_Y \beta + a_3 c_Y \alpha^3 + a_4 c_Y \beta^3 \right] U \]
\[ F(5,5) = \frac{\rho S}{2m} \left[ a_1 c_Y \alpha + a_2 c_Y \beta + a_3 c_Y \alpha^3 + a_4 c_Y \beta^3 \right] V \]
\[ F(5,6) = \frac{\rho S}{2m} \left[ a_1 c_Y \alpha + a_2 c_Y \beta + a_3 c_Y \alpha^3 + a_4 c_Y \beta^3 \right] W \]
\[ F(6,1) = -V \]
\[ F(6,2) = U \]
\[ F(6,3) = 0 \]
\[ F(6,7 : 10) = 0 \]
\[ F(6,4) = -\frac{\rho S}{2m} \left[ a_1 c_N \alpha + a_2 c_N \beta + a_3 c_N \alpha^3 + a_4 c_N \beta^3 \right] U \]
\[ F(6,5) = -\frac{\rho S}{2m} \left[ a_1 c_N \alpha + a_2 c_N \beta + a_3 c_N \alpha^3 + a_4 c_N \beta^3 \right] V \]
\[ F(6,6) = -\frac{\rho S}{2m} \left[ a_1 c_N \alpha + a_2 c_N \beta + a_3 c_N \alpha^3 + a_4 c_N \beta^3 \right] W \]
\[ F(7,1) = \frac{\rho S}{2m} \]
\[ F(7,2) = 0 \]
\[ F(7,3) = -X_b^m \]
\[ F(7,4) = 0 \]
\[ F(7,5) = 1 \]
\[ F(7,6 : 10) = 0 \]
\[ F(8,1) = -Y_b^m \]
\[ F(8,2) = X_b^m \]
\[ F(8,3 : 5) = 0, F(8,6) = 1 \]
\[ F(8,7 : 10) = 0 \]
\[ F(9,1 : 6) = 0 \]
\[ F(9,7) = 1 \]
\[ F(9,8 : 10) = 0 \]
\[ F(10,1 : 7) = 0 \]
\[ F(9,9 : 10) = 0. \]

The control coefficient matrix is linear in \( u \) and is a function of the states

\[ B(1,1) = \frac{\rho S}{I_x} a_{\alpha c} \]
\[ B(1,2) = \frac{\rho S}{I_x} a_{\alpha c} \]
\[ B(1,3) = \frac{\rho S}{I_y} a_{\alpha c} \]
\[ B(2,1) = \frac{\rho S}{I_y} a_{\alpha c} \]
\[ B(2,2) = \frac{\rho S}{I_y} a_{\alpha c} \]
\[ B(2,3) = \frac{\rho S}{I_y} a_{\alpha c} \]
\[ B(3,1) = \frac{\rho S}{I_z} a_{\alpha c} \]
\[ B(3,2) = \frac{\rho S}{I_z} a_{\alpha c} \]
\[ B(3,3) = \frac{\rho S}{I_z} a_{\alpha c} \]
\[ B(4,1 : 3) = 0 \]
\[ B(5,1) = \frac{\rho S}{m} a_{\alpha c} \]
\[ B(5,2) = \frac{\rho S}{m} a_{\alpha c} \]
\[ B(5,3) = \frac{\rho S}{m} a_{\alpha c} \]
\[ B(6,1) = \frac{\rho S}{m} a_{\alpha c} \]
\[ B(6,2) = \frac{\rho S}{m} a_{\alpha c} \]
\[ B(6,3) = \frac{\rho S}{m} a_{\alpha c} \]
\[ B(7 : 10, 1 : 3) = 0. \]

In the \( \theta-D \) formulation, we choose the factorization of nonlinear (7) as

\[
\dot{x} = \left\{ F(x_0) + \theta \left[ \frac{F(x) - F(x_0)}{\theta} \right] \right\} x + \left\{ B(x_0) + \theta \left[ \frac{B(x) - B(x_0)}{\theta} \right] \right\} u. \tag{41}
\]

The advantage of choosing this factorization is that in the \( \theta-D \) formulation, \( T_0 \) is solved from \( A_0 \) and \( g_0 \) in (12). If we select \( A_0 = F(x_0) \) and \( g_0 = B(x_0) \), we have a good starting point for \( T_0 \) because \( F(x_0) \) and \( B(x_0) \) retain much more system information than an arbitrary choice of \( A_0 \) and \( g_0 \) would have.

Note that the IGC design is an infinite-horizon regulator problem eliminating the need to know time-to-go. The finite-time nature of the guidance problem is equivalently treated
though proper selection of the range dependent weighting function \( Q \).

After some numerical experiments, the \( Q \) and \( R \) matrices are chosen to be

\[
Q = \text{diag}(3 \times 10^6 r_d, 2 \times 10^5 r_d, 11 \times 10^2 r_d, 0, 2 \times 10^4, \\
100, 9500, 100, 0.1, 0.01)
\]

(42)

\[
R = \text{diag}(3 \times 10^{14}/r_d, 1.8 \times 10^{14}/r_d, 1.45 \times 10^{13}/r_d)
\]

(43)

where \( r_d = |r_d| = \sqrt{(x_b^2 - x_0^2)^2 + (y_b^2 - y_0^2)^2 + (z_b^2 - z_0^2)^2} \)

is the range between the missile and the target. When the relative range is large, we put a small weight on the position error to avoid large control effort. As the missile approaches the target, a large weight is imposed to ensure small miss distances.

Once \( A_0, g_0, A(x), g(x), Q \), and \( R \) are selected, algorithm (12)–(15) can be applied to get the optimal control (39).

As mentioned in Section II, the \( D_i \) matrices play an important role in the \( \theta-D \)-method design. The \( k_i \) and \( l_i \) in the \( D_i \) (17)–(19) are design parameters. They can be used to adjust the system transient performance. \( D_1 \) and \( D_2 \) are chosen to be

\[
D_1 = \text{diag}(k_{11}e^{-ht_1t}, k_{12}e^{-lt_1t}, k_{13}e^{-lt_3t}, \\
k_{14}e^{-lt_4t}, k_{15}e^{-lt_5t}, k_{16}e^{-lt_6t}, k_{17}e^{-lt_7t}, \\
k_{18}e^{-lt_8t}, k_{19}e^{-lt_9t}, k_{110}e^{-lt_10t}) M_1(x, \theta)
\]

(44)

\[
D_2 = \text{diag}(k_{21}e^{-lt_1t}, k_{22}e^{-lt_2t}, k_{23}e^{-lt_3t}, k_{24}e^{-lt_4t}, \\
k_{25}e^{-lt_5t}, k_{26}e^{-lt_6t}, k_{27}e^{-lt_7t}, k_{28}e^{-lt_8t}, k_{29}e^{-lt_9t}, k_{210}e^{-lt_10t}) M_2(x, \theta)
\]

(45)

where \( M_1(x, \theta) \) and \( M_2(x, \theta) \) are the state dependent terms on the right-hand side of (13) and (14) (except for \( D_1 \) and \( D_2 \)).

The selection of \( (k_i, l_i) \) is done systematically. Note that the \( \theta-D \)-method is similar to the SDRE approach. To see this, assume that

\[
\frac{\partial V}{\partial x} = P(x)x
\]

(46)

and \( P(x) \) is symmetric. Substituting (46) into the HJB (3) and writing nonlinear \( f(x) \) in a linear like structure

\[
f(x) = F(x)x = [A_0 + A(x)]x
\]

leads to the state dependent Riccati equation

\[
F^T(x)P(x) + P(x)F(x) - P(x)B(x)R^{-1}B^T(x)P(x) + Q(x) = 0
\]

(47)

This is, in fact, the idea behind the state dependent Riccati equation (SDRE) technique [8]. Compared with the SDRE approach, the \( \theta-D \) method solves a perturbed HJB equation

\[
\frac{\partial V}{\partial x} f(x) - \frac{1}{2} \frac{\partial V^T}{\partial x} B(x)R^{-1}B^T(x) \frac{\partial V}{\partial x} \\
+ \frac{1}{2} x^T Q(x) + \sum_{i=1}^{\infty} D_i \theta^i \right] x = 0
\]

(48)
and assumes that

$$\frac{\partial V}{\partial x} = \sum_{i=0}^{\infty} T_i(x, \theta) \theta^i x.$$  (49)

As can be seen, the $\theta$-$D$ method is similar to the SDRE approach in the sense that both bring the nonlinear equation into a linear-like structure and solve a quadratic optimal control problem. The former gives an approximate closed-form solution, while the latter solves the algebraic Riccati equation online. So, it is assumed in this development that the $\theta$-$D$ solution is close to the SDRE solution, i.e., $P(\theta-D) \approx P(SDRE)$, where $P(\theta-D) = \sum_{i=0}^{\infty} T_i(x, \theta) \theta^i$ is obtained from solving (12)–(15) offline and $P(SDRE)$ is obtained by solving (47) online.

An effective procedure for finding the $(k_i, l_i)$ is as follows. An SDRE controller is used to generate a state trajectory and then the maximum singular value of $P(SDRE)$, i.e., $\sigma_{\text{max}}[P(SDRE)]$, is computed at each state point. Similarly, the $(k_i, l_i)$ parameters determine $P(\theta-D)$ and its associated $\sigma_{\text{max}}[P(\theta-D)]$. Curve fits are then applied to $\sigma_{\text{max}}[P(SDRE)]$ and $\sigma_{\text{max}}[P(\theta-D)]$, and the $(k_i, l_i)$ are selected to minimize the difference between these singular value histories in a least squares sense. Hence, all the $k_i$ and $l_i$ parameters can be determined in one offline least squares run.
V. NUMERICAL RESULTS AND ANALYSIS

A 6-DOF missile simulation was used to evaluate the performance of the IGC design. The aerodynamic coefficients were obtained from [13].

The first engagement scenario chosen was the interception of a crossing target, which is a nonmaneuvering target flying along a straight line path. The missile was assumed to be flying at an altitude of 50,000 feet with a Mach number of 5. The airframe was initially trimmed at an angle of attack and angle of sideslip of 0.1°. The missile down-range and cross-range positions in the inertial frame at the initial time instant were assumed to be zero. The target initial position was assumed to be at 50,300 ft in altitude, 10,000 ft in down range, and 300 ft in cross range. The velocity of the target was assumed to be a Mach number of 1 along both the cross range and altitude directions.

Trajectories of the missile and target are presented in Figs. 1–3. Figs. 1 and 2 show the horizontal plane trajectory and the vertical plane trajectory, respectively. The missile flight is smooth since the target cross-range velocity and descent velocity are constant. For a better perspective of the complete motion, the three-dimensional (3-D) trajectory is given in Fig. 3. Fig. 4 shows the fin deflections produced by the θ-D controller. Observe that the maximum control effort is less than 10°. Figs. 5 and 6 show the histories of the body velocity component $U, V,$ and $W$, aerodynamic angles $\alpha, \beta$, and body angular rate $p, q,$ and $r$. All the variables show good transient responses. In [7], the SDRE results for the same missile, the same initial conditions and constant weighting matrix $Q$, showed relatively large initial oscillations in both the cross-range and altitude trajectories. That is due to the lightly damped body rates which result from trying to achieve fast response with an infinite horizon regulator formulation. In this paper, however, range dependent weights are used. Body rates are penalized more initially and are relaxed later as the range
The second scenario chosen was the interception of a weaving target. See [15]–[18] for further discussion of this problem. The target was assumed to be located at 10 000 ft down range, 300 ft cross range with respect to the missile and at 50 300 ft in altitude. It has 3000 ft/s down-range velocity, 40 ft/s cross-range velocity, and 40 ft/s descent velocity. The weaving target model follows the formulation of [15]; the equations of motion of the target are given by

\[ \dot{X}_t = -10g \sin 2t, \quad \dot{Y}_t = -10gt \cos 2t, \quad \dot{Z}_t = 0. \]
Trajectories of the missile and the target are presented in Figs. 8–10. The miss distance in this case is 1.4 ft that occurs at 5.88 s. The horizontal plane trajectory is shown in Fig. 8. The missile flies nearly straight initially and when approaching the target, it adopts a weaving maneuver in response to the weaving target. However, the missile does not maneuver much in the vertical plane as shown in Fig. 9 since the target moves in an almost straight line in that plane. This is because the target is assumed to have a constant descent velocity and the down-range weaving acceleration is small compared with the large down range velocity. As before, the 3-D trajectory is given in Fig. 10 for a clearer perspective. Note from Fig. 11 that the maximum fin deflection during the entire flight is less than 3°. The corresponding missile body velocity components $\dot{U}$, $\dot{V}$, and $\dot{W}$ are given in Fig. 12. The longitudinal velocity $\dot{U}$ falls continuously due to the axial drag. The velocity components $\dot{V}$ and $\dot{W}$ oscillate continuously due to the natural response to pursuing a weaving target. The missile aerodynamic angles $\alpha$ and $\beta$ and body rate histories are presented in Fig. 13. They are all well-behaved. As in the nonmaneuvering target case, the body rates and other states are also well damped due to the use of range dependent weights. The missile accelerations along the missile body $y$- and $z$-axes are shown in Fig. 14. It can be seen that the missile only requires small levels of acceleration against a weaving target. Since the target has a large weaving motion in the cross range, the missile also exhibits a large weaving response accordingly as can be observed from the cross range related variables such as yaw rate, yaw control, and the acceleration along the body $y$-axis.

In the above feedback controller design, it is assumed that all the states can be accurately obtained. In reality, this is not true due to uncertainties such as sensor noise, radome aberration, and unmodeled dynamics, etc. In order to demonstrate the robustness of the $\theta$-$D$ method in the IGC design, noises are added to the states used in the control calculations. Every noise
process is assumed as white and Gaussian with a standard deviation chosen to be 10% of the respective state. Figs. 15–20 show the results when this controller is employed to intercept the same weaving target. The miss distance only rises to about 8 ft due to the measurement noises. The only discernable difference from Figs. 8–13 is a little larger miss in the altitude response. The overall performance does not change much.

VI. CONCLUSION

An integrated guidance and control design for missiles was developed in this paper through an optimal control formulation. It shows that both guidance objectives and autopilot design concerns can be addressed in a unified framework. A new nonlinear optimal control technique, the $\theta$-$D$ method, was employed to solve this integrated guidance and control problem. This method produced an approximate closed-form feedback controller, which does not require online computation of the state dependent Riccati equation as with the SDRE technique. Range dependent weights were adopted to address the finite time nature of the guidance problem. A nonlinear 6-DOF missile simulation was used to evaluate performance against a nonmaneuvering target and a weaing target. Based on the numerical results, it can be concluded that the $\theta$-$D$ technique shows good potential for implementing the IGC concept. Monte Carlo simulation based on more realistic missile models including sensor noises, actuator dynamics, aerodynamic uncertainties, and disturbances, etc., can help validate the usefulness of the IGC design concept to the missile community.

REFERENCES


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