A numerical, time domain solution for the response of a Gimbal supported gyro

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A NUMERICAL, TIME DOMAIN SOLUTION FOR THE RESPONSE OF A GIMBAL SUPPORTED GYRO

by

FLOYD STANLEY HALL, 1941-

A DISSERTATION

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ABSTRACT

This paper develops an algorithm to generate a numerical solution for the response of a gimbal supported gyro to arbitrary forcing functions and base motion, and includes the normal parasitic effects of bearing friction, viscous pivot damping, and pivot spring constants. There are relatively few restrictions on gyro and gimbal structure geometry. Solution accuracy is limited only by the computing machine accuracy; the algorithm being an explicit function of the input data. The algorithm is fast and requires only a moderate amount of computer memory.
The subject matter of this paper was developed to fulfill a specific requirement in the design of electro-optically guided missiles. In attempts to obtain longer guidance ranges, higher and higher resolution telescopes are mounted on inertially stabilized platforms. This requires an increase in precision of the performance of the stabilized platform to prevent blurring of the image. Gyro stabilized platforms have been one popular stabilization technique and empirical design methods have prevailed in the past. Such methods are no longer adequate for new designs for two reasons; first, an order of magnitude better performance is required over past designs; second, the experience of the pioneers is largely lost, most of the skilled designers have migrated to other endeavors. The net result is that an accurate simulation of each proposed tracker is desirable to optimize new designs before expensive prototype hardware is produced.

The time variant nature of the gyro equations has defied a workable classical solution and even good simulation on analog computers. A literature search provides very little assistance in this area for several reasons; the problem is a relatively specialized one, and theoretical interest is gyros died fifty years ago - long before the advent of digital computers. A significant portion of this dissertation is devoted to reformulating familiar vector mechanics into matrix mechanics so that the newer numerical techniques may be applied. A numerical solution such as developed herein, proves to be a practical, extremely versatile, and economical method to simulate not only the stabilized platform, but the
total missile system as well.

The writer wishes to express appreciation to Messrs. F. Knemeyer and C. Smith of the Naval Weapons Center for their support of this work, and to Dr. H. Zenor for his advice and patience.
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I INTRODUCTION

The free gyroscope is one of the simplest possible mechanical systems, consisting merely of a rigid, spinning rotor. Nevertheless, due to the six degrees of freedom and the complicated interrelationship between these degrees of freedom, the task of obtaining mathematical solutions to even simple gyro systems has engaged a number of notable mathematicians over the last two centuries; Newton, Euler, Poinsot, Routh, Kelvin, and Klein have each contributed to the solution of various problems concerning the free gyro.¹

A. EARLY HISTORY

The basic mathematical foundation was laid by Euler (1707-1783) in the middle 18th century. Euler formulated the "Euler equations" which became the starting point for many gyro solutions by succeeding mathematicians. The usefulness of the Euler equations is limited by several factors:

1. Angular rotations are not commutative thus a closed expression for the position of the gyro is a nonlinear expression,
2. For the most general case, either the product of inertia coefficients are time dependent or the reference coordinate system is rotating with the gyro, therefore the coefficients are time varying,
3. The equations are not amenable to accepting constraints; solutions for systems including a gyro have been obtained only for very special, simple cases.
Of the limitations cited above, the last is the most serious. The gyro is merely an element in a larger system and is subject to constraint forces imposed by the remainder of the system. In engineering applications, it is necessary to predict the effect of various constraint forces. In attempts to fulfill this requirement solutions have been obtained for many special cases involving simple constraint mechanisms (gravity on a top, flexible rotor shafts etc.) but few solutions have been obtained when multiple constraint mechanisms are present.

B. ENGINEERING DEVELOPMENT

An important problem in the practical application of Euler's equations was to determine the effects of the gimbal inertias. Euler's equations were formulated for a free gyro, and until recently free gyros were physically impossible; a gimbal was necessary for mechanical support. The first order effects of gimbal inertias were a painless modification. Until the 1950's the modified Euler equations were the most practical analytical approach and the result of the analyses was sufficiently accurate for engineering purposes. Typical applications for gyros during this period were artificial horizon indicators for aircraft, roll stabilizers for ships, lead computers for antiaircraft guns, and aircraft autopilots. The technical developments of the 50's and 60's opened new, more demanding applications for gyros; inertial reference systems for longe range missile guidance systems, gyro stabilized telescopes for anti-aircraft missiles, and attitude stabilization of artificial satellites.
The varied nature of these applications made different demands upon the gyro and hence upon the type of analysis required. Engineers designing inertial references found that the modified Euler equations were no longer accurate enough; Plymale & Goodstein\(^2\) used Lagrangian dynamics to compute the second order effects of gimbal inertias. The results demonstrated that the gimbal inertias created a system operating at a point of unstable equilibrium. Rotor vibrations, which always exist because of rotor dynamic balancing tolerances, result in undesirable precession of the rotor. Significant moneys have been expended to develop new gyros for inertial reference systems that do not require gimbals. Four such inventions\(^4,5\) are:

1. The cryogenic gyro - the rotor being a superconductor supported by a magnetic field.
2. The electrostatic gyro - the rotor being a metallic sphere supported by an electric field.
3. The particle gyro - a solid state device utilizing the angular and magnetic moments of a molecular structure as the sensitive element.
4. The laser gyro - an optical interference instrument of rather large dimensions.

Only the first two devices have achieved any degree of usefulness.

In the area of spacecraft stabilization, Dr. Wernher von Braun and the associated Army team received a quick, painfull lesson in gyro-dynamics with the launching of Explorer I, the first satellite launched by the United States.\(^6,7\) Explorer I was intended to be spin stabilized
about the rocket roll axis, the axis of minimum moment of inertia. Although this is a point of stable equilibrium for a rigid body,\textsuperscript{6} it is easily shown that it is an unstable equilibrium point for real, nonrigid bodies:\textsuperscript{8,9} a fact known by makers of dynamic balancing machines for decades.\textsuperscript{10,11} The fact that Explorer I tumbled end over end is seldom mentioned in historical reviews.

C. GOALS OF PRESENT WORK

This paper is primarily concerned with the gyrodynamic problems encountered in gyro stabilized trackers in tactical, short range guided missiles. Typically such missiles consist of the following subsystems: A tracker, to continuously monitor the location of the target, a guidance unit to compute the desired flight vector, an autopilot/airframe to provide the commanded flight vector, proximity fuze, and a warhead/payload. The target is acquired by the tracker prior to missile launch. The tracker must maintain optical contact with the target throughout the launch transients and flight maneuvers - a difficult task due to the fact that launch accelerations may be greater than 150 meters per second per second, and flight maneuvers will involve accelerations of around 50 meters per second per second at supersonic linear velocities and angular velocities approaching 10 radians per second. These in conjunction with unavoidable parasitic effects such as bearing friction, gimbal inertias, sensor imbalances, structural elasticity etc., produce torques which perturb the line of sight of the tracker.\textsuperscript{12} It is desirable therefore that the tracker be mounted on a platform having a low sensitivity to
these disturbing torques. A platform stabilized by a gyro often appears
to offer a practical solution, particularly when the gyro can be used
simultaneously for other functions.

To predict the performance of a gyro stabilized tracker a com-prehen-
sive simulation is required which not only includes the tracker signal
processing, the tracker/platform stabilization electronics and the major
gyro parameters, but also includes the parasitic effects mentioned pre-
viously. This is a very complicated task; to date a comprehensive sim-
ulation suitable for programming on a general purpose digital computer
has not been completed. This paper presents the derivation of a mathema-
tical model for the electro-mechanical components of a gyro stabilized
platform and an algorithm for obtaining a numerical solution for the
motion of the gyro components as a function of time. It is intended that
this algorithm be included as a portion of a larger algorithm which will
provide solution to the total missile flight dynamics.
II IDEALIZED MECHANICAL MODEL

The first step in any simulation program is to establish a model, the choice of an appropriate model being dictated in part by prevailing design practices. It is desirable that the model be versatile enough to include as many of the currently active tracker configurations as is consistent with reasonable solution times. Fortunately the majority of missile trackers bear a striking resemblance to one another thus making the formulation of a "Universal" model easier than might be expected.

A. FUNCTIONAL DESCRIPTION OF GYRO STABILIZED TRACKER

A gyro stabilized tracker involves the following mechanical functions: A mounting base (the missile airframe), a telescope, a gyro, a gimbal set, a platform (upon which the telescope and gyro are mounted), a spin motor, torquers, and a nutation damper. The basic design philosophy centers about the principle that if no external torques are present, the angular momentum vector of a gyro defines a non-rotating reference vector (relative to an inertial frame of reference). With the proper gyro geometry and a nutation damper the gyro spin axis may be forced to be coincident with the angular momentum vector. It is then relatively easy to mount the gyro and the telescope on a common platform such that the gyro spin axis and the optical axis of the telescope are parallel. By isolating this platform from external torques with a gimbal set, the telescope line of sight is effectively space stabilized. The final requirements are a spin motor to maintain the gyro angular momentum, and torquers to process the gyro spin axis to the desired heading.
B. BASIC MECHANICAL SIMPLIFICATIONS

Few models of mechanical systems include all the known physical parameters, to do so would result in an unworkably complex model. A major simplification results by modeling each structural member as a perfectly rigid member. Next, bearings and joints are assumed to be dimensionally perfect with no flexing or play. And since the location of the target is a known quantity in a simulation program, there is no need to include the electro-optical parameters in this portion of the simulation. The mass and moments of inertia of the telescope is included as a portion of the platform.

A schematic diagram of the adopted model is presented in Figure (2-1). The gyro element includes all elements rigidly attached to the gyro proper such as spin motor rotor and some optical elements in certain designs. The platform includes the platform proper, the non-spinning portions of the telescope, spin motor stator, the gimbal pitch torquer rotor, a portion of the push-rod masses (an item clarified later) and any additional parts rigidly attached to the platform. The gimbal or "spider" (a terminology adopted from reference 1) includes the gimbal proper, the gimbal pitch torquer stator, the yaw gimbal torquer rotor, etc. The mass and moments of inertias of the mounting base are assumed to be sufficiently large that the base motion is unaffected by the gimbal forces. The push-rods are perfectly rigid but have zero mass; the mass of the push rods are included as part of the platform and push-rod torquer rotors. Each pivot may exhibit friction forces, viscous damping, and even spring forces.
FIGURE (2-1) SCHEMATIC DIAGRAM OF MECHANICAL MODEL
The gyro and platform portions are rather straightforward, the gimbal set and push rod arrangements require additional explanation. A two degree of freedom gimbal set was selected because it is in universal use even though it exhibits two major shortcomings. First, there are two unuseable areas where the platform has only one degree of freedom. This condition occurs whenever the pitch pivot angle is $90^\circ \pm 90^\circ$ degrees; the gyro spin axis is then parallel to the yaw pivot axis. This condition can be eliminated by the use of an additional roll gimbal. This solution has not yet been introduced in any operational missile for the following reasons. If the roll gimbal is positioned by a servo-mechanism, the cost and complexity is increased by the servo. If a freely rotating roll gimbal is used, the relationship between the target line of sight and the three gimbal angles becomes a nonsingular matrix; hardware to perform a coordinate conversion/reduction is then necessary. Additional problems are also encountered in providing spin torques to the gyros.

The second shortcoming of a two axis gimbal set is caused by the moments of inertia of the gimbal and platform structures. When the missile airframe is rolling and the target line of sight is not dead ahead, the gimbal must be in constant motion in order to maintain the desired gyro orientation. Torques are necessary to produce this motion and part of this torque is supplied by the gyro with the result that the telescope line of sight is disturbed slightly. For precision tracking even slight perturbations may be too much. One way to minimize the moments of inertia of the gimbal structure is to use an
inverted gimbal. Most trackers use the classical gimbal similar to that of Figures (2-2), (2-3) and (2-4). The physical dimensions, hence moments of inertia of the gimbal can be drastically reduced by the use of an inside-out arrangement where the support is a central post and the gimbal is nested within the platform. An example of this type of construction is illustrated in Figure (2-5) and a close up of a gimbal designed for another, similar missile is illustrated in Figure (2-6).

The mathematical equations describing both types of gimbals are identical, thus for the present model it makes no difference which type gimbal is selected. The more traditional arrangement of Figure (2-2) has been arbitrarily selected.

The pitch pivot axis and the yaw pivot axis are constrained to be orthogonal and coplanar. These conditions are always met with high precision in all operational designs because a tracker mounted within such a gimbal set has the minimum moments of inertia hence may use the smallest torquers, or alternatively, may achieve higher angular accelerations with the same torquers. The high precision with which the pivot axes meet the orthogonal, coplanar conditions is because the mechanical tolerances which define the pivot axes also determine the stability of the dynamic balance.

The restrictions imposed upon the push-rod configuration are indulged in for mathematical convenience although deviation from actual practice is minimal. The mathematical equations relating the platform
FIGURE (2-2) Gyro/Gimbal Arrangement for an Optical Tracker
FIGURE (2-3) GYRO/GIMBAL ARRANGEMENT FOR AN INFRARED TRACKER
FIGURE (2-4) GIMBAL ARRANGEMENT FOR AN INFRARED TRACKER
FIGURE (2-5) INVERTED GIMBAL ARRANGEMENT FOR AN INFRARED TRACKER
FIGURE (2-6) CLOSE-UP VIEW OF INVERTED GIMBAL
motion and the push-rod torquer rotor motions are very complicated. For reasons of mathematical simplicity only the z-axis motion is considered. The z-axis motion of the platform push-rod joint is required to be equal to the z-axis motion of the push-rod torquer rotor joint. Physically, this is equivalent to requiring the yaw push rod geometry defined by the push-rods and torque arms in Figures (2-7) and (2-8) to either form a parallelogram or the ratio of torque arm to push-rod lengths be infinitesimally small.

In actual designs, the above conditions are the design goals because the center of mass of the entire system is not a function of the tracker line of sight vector if they are met. It is possible to make the yaw push-rod geometry a parallelogram, thus no errors are introduced by the yaw push-rod model. The pitch push-rod geometry can form a parallelogram only for a given magnitude of yaw gimbal angle. One can never attain infinite push-rod length thus some error is inevitable in the pitch push-rod model. The pitch push-rod geometry is generally designed so as to form a parallelogram for a yaw gimbal angle of zero degrees, and also maximize the ratio of push-rod to torque arm lengths to the extent physically possible. In view of this, the errors introduced by the mathematical constraint of the preceding paragraph are judged to be tolerable.
FIGURE (2-7) PUSH-ROD ARRANGEMENT FOR AN OPTICAL TRACKER
FIGURE (2-8) PUSH-ROD ARRANGEMENTS FOR AN INFRARED TRACKER
III MATHEMATICAL METHODOLOGY

Having established an idealized mechanical model, the next step is to decide upon a suitable mathematical approach. Recall that the primary objective is to enable one to accurately simulate the flight characteristics of various missile configurations. Characteristics that are desirable in a solution technique are discussed in the following paragraphs.

A. DESIRABLE OPERATIONAL CHARACTERISTICS

In order to permit rapid modification of pertinent parameters and also to allow the programmer to develop intuitive insight, it is desirable that the input data to the algorithm be readily measured or commonly known physical parameters. A minimum of manipulation of raw data by the programmer is desired, ideally limited to converting the usual "hybrid" measurements (such as torque measurements expressed in inch-grams) into a consistent system of units.

For data analyses purposes it must be possible to "observe" the operating conditions and responses of any desired component in the system.

It must be practical to incorporate this algorithm as a subsystem in a total missile simulation. As stated previously, the present effort applies only to the electro-mechanical components of the tracker. The complete missile simulation must also include the tracker electronics, the guidance system, and airframe dynamics. Obviously, the ability to interface between the mathematical simulations of each of these subsystems is crucial.
In view of the complexity to be expected in the total missile simulation, it is desirable to minimize the computational time associated with a practical implementation of the algorithm. A typical missile flight may be of thirty seconds duration; data may be required at one millisecond time increments thus requiring the computation of thirty thousand data points. With a typical computer charging rate of $600 per hour it becomes clear that the computation involved is destined to be relatively expensive; every effort should be made to minimize the time to compute each data point.

B. BASIC ALGORITHM AND CHARACTERISTICS

The solution algorithm selected as best satisfying the desired requirement is based upon the application of Newton's Equation (in matrix form) and a finite difference prediction scheme. Figure (3-1) presents a simple flow chart of the algorithm. The critical step is, of course, the computation of the desired data items and is the focal point of most of the present work.
A. Receive System Configuration Data
B. Receive Time Boundary Values/Finite Difference Mesh Dimension
C. Receive Initial Conditions
D. Receive (Compute) Base (Airframe) Forcing Functions
E. Receive (Compute) Tracker Forcing Functions
F. Compute Desired Data
G. Compare Present Time Value Against Time Boundary Values
H. Predict (Compute) Base (Airframe) State Vector For Next Iteration
I. Predict (Compute) Tracker State Vector For Next Iteration
T. Terminate

FIGURE (3-1) Basic Flow Chart of Selected Algorithm
The selected approach meets all of the desired characteristics to a high degree. The mass configuration specification parameters are simply the moments and products of inertia plus the location of the center of mass of each component. An unusual feature of this algorithm is that there are no restrictions on the geometry of each element with the exception of the push-rod arrangement. The initial conditions (state vectors) are commonly understood parameters such as gimbal angles, linear and angular velocity vectors etc., thus the requirement for easily obtained input data is fulfilled.

Because the algorithm computes the forces involved at each pivot location it is simple to monitor any desired component.

The algorithm as presently configures, computes the acceleration dependent forces internal to the tracker but assumes that all external tracker forcing functions are independent of tracker accelerations. Certainly one could generate configurations which violate this constraint (particularly in the area of tracker stabilization electronics) but such configurations are generally avoided because of the large perturbations caused by random noise inputs. The interface data between missile subsystems is therefore restricted to position and velocity dependent parameters. Thus there are no problems involved in interfacing with any number of subsystems. The interface data is in commonly understood terms such as position and velocity vectors - there are no eigenvectors or similar mathematical quantities that are difficult to relate with the physical system.
Finally, and somewhat unexpectedly, this approach results in a very fast, totally explicit algorithm. These two characteristics stem from the fact that, with the proper choice of coordinate systems and solution variables, the number of unknown variables may be reduced from ninety-plus to only three. And due to the occurrence of a large number of zero elements in the matrix equations the above reduction involves only a fraction of the number of computations normally expected. The fact that there are no converging, iterative series involved permits the accuracy of the calculations to be limited only by the number of significant figures carried in the computations not upon an error criterion, and thus requires no value judgements from the operator concerning when iterations should terminate.

Before proceeding to the mechanics of solution it may be appropriate to point out a few facts concerning the development of this approach.

Newtons equations for a micro-particle are so simple that, at first glance it appears deceptively easy to formulate equations for a larger system. But the matrix expression of Newtons equation for a macro-body (that is, a body possessing non-negligible moments of inertia) suddenly changes to the form

\[ F = MA - FW \]  

Where F is a 6 element "force" vector,
A is a 6 element "acceleration" vector,
M is a 6 x 6 "mass" matrix,
and FW is a 6 element "gyro" vector.
The "gyro" vector is created by the fact that angular velocities are absolute quantities, unlike linear velocities. As will be shown, the matrix M is not in general diagonal matrix but can be diagonal for certain geometric shapes when referenced to a specific coordinate system. It is these special cases to which solutions are typically advanced. These conditions are often the design goals for design purposes, but in reality all hardware deviates from the ideal. One objective of simulations is to investigate the sensitivity of a design to these deviations, therefore it is not acceptable to confine the algorithm to be useable only on components having a diagonal M matrix.

A second point of interest concerns the application of matrix mathematics to problems of mechanical dynamics. More precisely, matrix methods are conspicuous in published mechanics by their absence. Either classical vector analyses or Lagrangian formulations are used in all the texts to date. Admittedly some matrix notations have been introduced but these occasions are generally merely simplification in notation of results obtained by vector analyses methods. For example, the writer has been unable to find an expression for Newtons equation in matrix format similar to the expression of (3-1) in any publication. Although vector notation may well be the classical language of mechanics, matrix notation is certainly the language of computer programming. It is much simpler in practice to introduce matrices at the beginning of a derivation rather than to pursue a vector solution and then attempt to mold the algorithm into matrix format. In view of the fact that no rigorous development of matrix methods in mechanics has been published, the next few pages will be devoted to deriving a few basic matrix equations.
IV BASIC MATRIX EQUATIONS

A. VECTOR/MATRIX NOTATION

When dealing with multiple body systems one of the first problems encountered is to define a workable notation system. Most of the quantities of interest are vector quantities. The following five elements of information are required to assign a numerical value to a vector quantity:

1. Quantity Q (i.e. position, velocity, momentum etc.)
2. of point A
3. relative to point B
4. has the components Qx, Qy, Qz
5. as measured by reference C

In the work to follow, a general vector quantity will be represented in one of the two forms illustrated below.

\[
Q_{A,B,C} = \begin{bmatrix}
Q_x \\
Q_y \\
Q_z
\end{bmatrix}
\] (4-1)

where \(Q_{A,B,C}\) is the vector quantity,
\(Q_x\) is a scaler describing the x component,
\(Q_y\) is a scaler describing the y component,
\(Q_z\) is a scaler describing the z component
and \(A, B, C\) are as defined above.

Note that since a vector quantity may be expressed as a three element column matrix, such matrices will often be called vector matrices.
The various coordinate systems (references) are identified by arabic numerals, thus the subscript C will be a numeral designating a particular coordinate system. The subscripts A and B may be either alphabetic letters or arabic numerals. Alphabetic letters are used to designate various mechanical elements; arabic numerals indicate the origin of the designated coordinate system. The following example illustrate the notation.

If $Q \rightarrow V$ designates linear velocity,

$A \rightarrow R$ designates the gyro rotor,

$B \rightarrow B$ designates the base,

and $C \rightarrow 3$ designates coordinate system 3,

then $v_{R,B,3}$, is the linear velocity of the gyro rotor relative to the base as measured by coordinate system 3.

Similarly $v_{R,4,3}$, is the linear velocity of the gyro rotor relative to the origin of coordinate system 4 as measured by coordinate system 3.

This is an awkward number of subscripts and usually a more compact expression may be used which contains all the desired information. The following three expressions are used liberally throughout the remainder of this work.

When the point B is the origin of the coordinate system defined by C, the subscript C is dropped.
When the type of vector quantity is not of interest the abstract vector is represented simply as

\[
X_c = \begin{bmatrix}
X_c \\
Y_c \\
Z_c
\end{bmatrix}
\] (4-2)

the subscripts A and B being deleted.

Lastly, when there is no ambiguity introduced the commas separating the subscripts will be deleted.

B. TRANSLATION AND ROTATION OF VECTOR MEASUREMENTS

Perhaps the most important advantage of matrix notation is that information identifying the coordinate reference is implicit in each measurement, information that is not present in classical vector notation. One of the most fundamental matrix operations is the transformation of measurements by one coordinate system to those of another coordinate system. If a vector quantity \( x_9 \) has been measured by cartesian coordinate system \( 9 \), and the value of the same quantity \( x_3 \) as measured by cartesian coordinate system \( 3 \) is desired, the transformation is easily expressed as

\[
x_3 = L_{3,9} x_9 + o_{3,9}
\] (4-3)

where \( x_3 \) is a 3 element vector matrix,

\( x_9 \) is a 3 element vector matrix,

\( L_{3,9} \) is a 3 x 3 matrix which performs rotational transformation

and \( o_{3,9} \) is a 3 element vector matrix.
Although it is not implied by the above expression, coordinate systems introduced in this paper are restricted to cartesian coordinate system. Furthermore in order to facilitate cross checking matrix equations with classical vector equations, only right handed cartesian coordinate systems are allowed. It is also assumed that all quantities are expressed in a consistent system of units, preferably the MKS system.

The vector matrix $O_{39}$ is a measure of the contribution of the coordinate system $9$ to the vector $X_3$ and is dependent upon the type of vector under consideration. For example if $X_9$ is a position vector, $O_{39}$ is the location of the origin of coordinate system $9$ relative to the origin of coordinate system $3$ as measured by coordinate system $3$. If $X_9$ is an angular velocity vector then $O_{39}$ is $\omega_{93}$, the angular rotation rate of coordinate system $9$ relative to coordinate system $3$, as measured by coordinate system $3$.

The matrix $L_{3,9}$ performs a rotational transformation. It can be shown that the constraints imposed upon the coordinate systems restrict the matrix $L_{3,9}$ to be a real, orthogonal matrix. Thus

$$L_{9,3} = L_{3,9}^{-1} = L_{3,9}^T$$

all eigenvalues are real and equal to unity, and the three eigenvectors are orthogonal.

The numerical value of the matrix $L$ is easily established for any particular orientation of the two coordinate systems. During the course of mathematical derivations however, the numerical values are unknown and one is forced to express the elements of $L$ symbolically. Since a set of
simultaneous equations must be solved which must uniquely determine each of the nine elements, each L matrix could conceivably introduce nine variables into the solution. Fortunately this need not be the case, the nine elements of L are not independent. In fact the interdependence between elements is such that at most three independent angular position variables are required to completely define the entire matrix, and in many cases of practical interest only a single variable is required. Even when three independent angular position variables are necessary they are not unique; there are an infinite number of ways one may define the three variables. Due to the non-commutative nature of addition of finite angular rotations, there is no best way to define the three angular position variables that is useful in all problems, hence there has been little effort to establish a preferred system. In view of the tradition definitions for the angular velocity components, it seems desirable to choose the three angular position variables such that for infinitesimal angular displacements between the two coordinate systems, the time rate of change of each angular position variable is equal to a corresponding component of the angular velocity vector.

That is, if three angular position variables are designated $\phi_x$, $\phi_y$, and $\phi_z$,

then \[
\begin{align*}
\dot{\phi}_x &= \omega_x, \\
\dot{\phi}_y &= \omega_y, \\
and \quad \dot{\phi}_z &= \omega_z,
\end{align*}
\] (4-5)

for the condition $\phi_x = \phi_y = \phi_z = 0$. 
Even this requirement does not uniquely determine a relationship between $\varphi_x$, $\varphi_y$, $\varphi_z$ and the matrix $L$.

The necessity of reducing the number of variables should be clear at this point, and the relative freedom of choice of the exact definitions of $\varphi_x$, $\varphi_y$, and $\varphi_z$ should be suggested. Rather than pursue a relatively insignificant derivation, it should suffice to state that $\varphi_x$, $\varphi_y$, and $\varphi_z$ may be chosen to be the gimbal angles of a system similar to that of Fig. (2-1) having the base oriented with coordinate system 3 and the gyro rotor oriented with coordinate system 9. It can be shown that the matrix $L_{3,9}$ may be expressed

$$L_{3,9} = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix}_{3,9}$$

$$= \begin{bmatrix} (C_x C_z + S_x S_y S_z) & (S_x S_y C_z - C_x S_z) & (C_x S_y) \\ (C_x S_z) & (C_x C_z) & (-S_x) \\ (S_y S_z - S_x C_z) & (S_x C_y S_z + C_y S_z) & (C_x C_y) \end{bmatrix}_{3,9}$$

where

$$C_x = \cos \varphi_x,$$
$$S_x = \sin \varphi_x,$$
$$C_y = \cos \varphi_y,$$
$$S_y = \sin \varphi_y,$$
$$C_z = \cos \varphi_z,$$
$$S_z = \sin \varphi_z,$$
and $\phi_y$ is the angular rotation of the gimbal (spider) relative to the base, measured by the base.

$\phi_x$ is the angular rotation of the platform relative to the gimbal (spider), measured by the gimbal.

$\phi_z$ is the angular rotation of the gyro rotor relative to the platform, measured by the platform.

The rationale of these definitions for $\phi_x$, $\phi_y$, and $\phi_z$ is to choose mathematical variables that are commonly measured physical quantities whenever possible.

If the two coordinate systems differ only in a single axis of rotation, then only a single angular position variable is needed to specify the appropriate $L$ matrix. The following three cases illustrate the forms of the matrix $L$ for rotation about each of the coordinate axes. (Note: the numerical value of the subscripts presented have no significance at this point. The reasons for these choices will become clear later.)

For rotation about the $x$ axis,

$$x_5 = L_{5,7} x_7 + \phi_{5,7}$$  \hspace{1cm} (4-7)

and

$$L_{5,7} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_x & -s_x \\ 0 & s_x & c_x \end{bmatrix}$$

For rotation about the $y$ axis,

$$x_3 = L_{3,5} x_5 + \phi_{3,5}$$  \hspace{1cm} (4-8)
and

\[
L_{3,5} = \begin{bmatrix}
  c_y & 0 & s_y \\
  0 & 1 & 0 \\
  -s_y & 0 & c_y
\end{bmatrix}
\]

For rotation about the z axis,

\[
x_7 = L_{7,9} x_9 + O_{7,9}
\]

Refering again to Fig. (2-1) it is clear that the transformation from \(x_9\) to \(x_3\) need not be performed in a single operation, and that the matrix \(L_{3,9}\) may be obtained in the following manner.

\[
x_3 = L_{3,5} x_5 + O_{3,5},
\]

\[
= L_{3,5} L_{5,7} x_7 + L_{3,5} O_{5,7} + O_{3,5}
\]

\[
= L_{3,5} L_{5,7} L_{7,9} x_9 + L_{3,5} L_{5,7} O_{7,9} + L_{3,5} O_{5,7} + O_{3,5}
\]

Setting the last expression of (4-10) equal to (4-6) yields

\[
L_{3,9} = L_{5,5} L_{5,7} L_{7,9}
\]

and

\[
O_{3,9} = L_{3,5} L_{5,7} O_{7,9} + L_{3,5} O_{5,7} + O_{3,5}
\]
C. PRODUCTS AND MOMENTS OF INERTIA

Products and moments of inertia are commonly accepted physical parameters thus they are usually known input data. In most derivations, these parameters are arranged to form a momental dyadic, this being a convenient arrangement for certain solutions. It is not a very convenient arrangement when performing transformations of these parameters between coordinate systems, therefore a slightly different arrangement will be developed.

First, let us define the following terms:

\[
\begin{align*}
P_{xx} &= \rho \rho x^2 \, dv \\
P_{xy} &= \rho \rho xy \, dv \\
P_{xz} &= \rho \rho xz \, dv \\
P_{xy} &= \rho \rho y^2 \, dv \\
P_{yz} &= \rho \rho yz \, dv \\
P_{zz} &= \rho \rho z^2 \, dv
\end{align*}
\]

where \( \rho \) is the density of the material and \( \rho \, dv \) denotes that the integration encompasses a volume totally containing the body.

The cross-product terms \( P_{xy}, P_{xz}, P_{yz} \) etc. are the conventional products of inertia. The auto-product terms \( P_{xx}, P_{yy}, P_{zz} \) are new terms and
are related to the conventional moments of inertia by the relationship

\[
\begin{bmatrix}
I_x \\
I_y \\
I_z
\end{bmatrix} =
\begin{bmatrix}
P_{yy} + P_{zz} \\
P_{xx} + P_{zz} \\
P_{xx} + P_{yy}
\end{bmatrix}
\] (4-14)

Next let us define a product of inertia matrix

\[
P_{A,C,C} =
\begin{bmatrix}
P_{xx} & P_{xy} & P_{xz} \\
P_{yx} & P_{yy} & P_{yz} \\
P_{zx} & P_{zy} & P_{zz}
\end{bmatrix}
\] (4-15)

This matrix may also be expressed

\[
P_{A,C,C} = \oint \rho_A \, X_c \, X_{c}^T \, dv
\] (4-16)

where A and C are defined by (4-1)

Now, assuming that the product of inertia matrix is known as measured by one coordinate system, for example coordinate system 3, and it is desired to compute what the product of inertia matrix measured by a different coordinate system, say coordinate system 1.

Recall that

\[
X_1 = L_{1,3} \, X_3 + O_{1,3}
\] (4-17)

then

\[
P_{A,1} = \oint \rho_A \, X_1 \, X_{1}^T \, dv
\] (4-18)
\[
\begin{align*}
\mathbf{P}_{A,1} &= \oint_{v} \rho_{A} [L_{1,3} x_{3} + 0_{1,3}] [L_{1,3} x_{3} + 0_{1,3}]^{T} dv \\
&= \oint_{v} \rho_{A} [L_{1,3} x_{3} + 0_{1,3}] [x_{3}^{T} L_{13}^{T} + 0_{1,3}^{T}] dv \\
&= \oint_{v} \rho_{A} L_{1,3} x_{3} x_{3} L_{13}^{T} dv \\
&\quad + \oint_{v} \rho_{A} L_{1,3} x_{3} 0_{1,3}^{T} dv + \oint_{v} \rho_{A} 0_{1,3} x_{3} L_{1,3}^{T} dv \\
&\quad + \oint_{v} \rho_{A} 0_{1,3}^{T} 0_{1,3} dv
\end{align*}
\]

The matrices \(L_{1,3}\) and \(0_{1,3}\) are independent of the integration hence may be removed from the integral giving

\[
\begin{align*}
\mathbf{P}_{A,1} &= L_{13} \oint_{v} \rho_{A} x_{3} x_{3}^{T} dv L_{13}^{T} \\
&\quad + L_{13} \oint_{v} \rho_{A} x_{3}^{T} 0_{13} dv + 0_{13} \oint_{v} \rho_{A} x_{3} dv L_{13}^{T} \\
&\quad + 0_{13}^{T} \oint_{v} \rho_{A} dv.
\end{align*}
\]

Now each of the integral terms may be identified:

\[
\begin{align*}
\oint_{v} \rho_{A} x_{3} x_{3}^{T} dv &= \mathbf{P}_{A,3} \\
\oint_{v} \rho_{A} dv &= m_{A} \\
\oint_{v} \rho_{A} x_{3} dv &= G_{A3}
\end{align*}
\]
Where \( m_A \) is the mass of the body A,

And \( G_{A3} \) is a vector matrix locating the center of mass of A.

Substituting the above identities into equation (4-20) yields.

\[
P_{A1} = L_{13} P_{A3} L_{13}^T + m_A L_{13} G_{A3} O_{13}^T
\]

\[
+ m_A [L_{13} G_{A3} O_{13}^T]^T + m_A O_{13} O_{13}^T.
\]

Not only does the product of inertia matrix permit the transformation of measurements from one coordinate system to another; it also possesses the interesting characteristic that the eigenvectors of \( P_A \) are the principal axes of the Poinosot momental ellipsoid, and the geometry of the ellipsoid is defined by the eigenvalues. These last characteristics are not necessary for the work to follow hence the proofs shall not be included here.

Once the product of inertia matrix \( P \) has been obtained referenced to the desired coordinate system it is a simple matter to define the momental matrix \( P_I \),

\[
P_{I1} = \begin{bmatrix}
(P_{yy} + P_{zz}) & -P_{xy} & -P_{xz} \\
-P_{yx} & (P_{xx} + P_{zz}) & -P_{yz} \\
-P_{zx} & -P_{zy} & (P_{xx} + P_{yy})
\end{bmatrix}_{A1}
\]

D. TRANSFORMATION OF DERIVATIVES

The preceding pages were concerned only with the transformation of a vector \( Q \) from one coordinate system to another. The ability to perform similar transformations for the vectors \( Q \) and \( \ddot{Q} \) shall be necessary also.
Repeating equation (4-3) and performing the first time derivative,

\[ x_3 = L_{39} x_9 + \omega_{39} \]  
\[ \dot{x}_3 = \dot{L}_{39} x_9 + L_{39} \dot{x}_9 + \dot{\omega}_{39} \]  

(4-26)

The new vector matrices \( \dot{x}_9 \) and \( \dot{\omega}_{39} \) pose no problem; the matrix \( \dot{L}_{39} \) reintroduces the problem of too many variables; there are nine elements in \( \dot{L}_{39} \). This time however, the conventional, angular velocity variables \( \omega_{x93}, \omega_{y93}, \) and \( \omega_{z93} \) may be used to define \( L_{39} \). The functional relationship between \( \omega \) and \( L_{39} \) can easily be determined by equating the matrix equation to a classical vector equation. For simplicity let \( \dot{\omega}_{39} = 0 \), and \( \dot{x}_9 = 0 \). The vector equation is

\[ \dot{x}_3 = \omega_{93} x(L_{39} x_9) \]  
\[ \dot{L}_{39} x_9 = \omega_{93} x(L_{39} x_9) \]  

(4-27)

(4-28)

thus we may conclude

\[ \dot{L}_{39} x_9 = \omega_{93} x(L_{39} x_9) \]  

and therefore

\[ \dot{L}_{39} = \begin{bmatrix} 0 & -\omega_z & \omega_x \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} L_{39} \]  

(4-29)

where \( \omega_x = \omega_{x93}, \) \( \omega_y = \omega_{y93} \), \( \omega_z = \omega_{z93} \).
\[ \omega_z = \omega_{z93}. \]

The procedure to define the transformation of second time derivatives is very similar to the one just followed. Again repeating equations \((4-3)\) and \((4-26)\) and performing the time derivative of \((4-26)\)

\[ X_3 = L_{39} x_{9} + o_{39} \quad (4-3) \]

\[ \dot{X}_3 = \dot{L}_{39} x_{9} + L_{39} \dot{x}_{9} + \dot{o}_{39} \quad (4-26) \]

\[ \ddot{X}_3 = \ddot{L}_{39} x_{9} + 2L_{39} \dot{x}_{9} + L_{39} \dot{x}_{9} + \ddot{o}_{39} \quad (4-30) \]

If the constraints \(o_{39} = 0, x_{9} = \dot{x}_{9} = 0\) are imposed the vector equation becomes

\[ \ddot{X}_3 = \omega_{93} x (\omega_{93} x L_{39} x_{9}) + \omega_{93} x L_{39} \dot{x}_{9} \quad (4-31) \]

therefore

\[ \ddot{L}_{39} x_{9} = \omega_{93} x (\omega_{93} x L_{39} x_{9}) + \omega_{93} x L_{39} \dot{x}_{9} \quad (4-32) \]

giving

\[ \dddot{L}_{39} = \begin{bmatrix} -\left(\omega_{y}^2 + \omega_{z}^2\right) & \omega_{x} \omega_{y} & \omega_{x} \omega_{z} \\ \omega_{x} \omega_{y} & -\left(\omega_{x}^2 + \omega_{z}^2\right) & \omega_{y} \omega_{z} \\ \omega_{x} \omega_{z} & \omega_{y} \omega_{z} & -\left(\omega_{x}^2 + \omega_{y}^2\right) \end{bmatrix} L_{39} \]
\[
\begin{bmatrix}
0 & -\omega_z & \omega_y \\
\omega_z & 0 & -\omega_x \\
-\omega_y & \omega_x & 0 \\
\end{bmatrix}
\]
\[L_{39}\]

where

\[\omega_x = \omega_{x93},\]
\[\omega_y = \omega_{y93},\]
\[\omega_z = \omega_{z93},\]
\[.\]
\[\omega_x = \omega_{x93},\]
\[.\]
\[\omega_y = \omega_{y93},\]
\[.\]
\[\omega_z = \omega_{y93}.\]

It should be noted that the coordinate system 3 is not constrained to be an inertial system.

E. ANGULAR MOMENTUM

The angular momentum of a body A relative to a coordinate system C is simply

\[H_{AC} = \mathbf{P}_{|AC} \omega_{AC}\]

It is traditional to define angular momentum referenced to an inertial reference, hence coordinate reference C must be an inertial reference.
F. KINETIC ENERGY

The kinetic energy of body A relative to inertial coordinate system C may be expressed

\[ KE_{AC} = \frac{1}{2} m_A v_{AC}^T v_{AC} + \frac{1}{2} \omega_{AC}^T P_{IAC} \omega_{AC} \]  \hspace{1cm} (4-35)

where \( KE_{AC} \) is the kinetic energy scaler, of A relative to C, measured by C,
\( v_{AC} \) is the linear velocity of A relative to C, measured by C,
\( \omega_{AC} \) is the angular velocity vector of A relative to C, measured by C,
and \( P_{IAC} \) is the momental matrix of A relative to the origin of C, measured by C.

It should be noted that if either \( v_{AC} \) or \( \omega_{AC} \) is non-zero the momental matrix \( P_{IAC} \) is time dependent, indeed this time dependency accounts in large part, for the difficulty in attaining interpretable solutions to gyro problems.
V NEWTONS EQUATION FOR A FINITE BODY

The matrix equation derived in the next few paragraphs is the central equation of the complete algorithm; the preceding coordinate transformation equations merely obtain measurements of all parameters by a single, inertial reference so that Newtons equation is applicable.

A. MOTION SPECIFICATION VARIABLES FOR A FINITE BODY

When specifying the position and motion of a body of finite dimensions, it is generally accepted practice to specify only the position and motion of a specific point fixed to (but not necessarily within) the body. The position and motion of any other point may be computed if the linear position, velocity, and acceleration of the specified point plus the angular position, velocity and acceleration of the body relative to this point is known. In Figure (5-1), the body A is a rigid (but not necessarily homogeneous) mass and \((\rho \, dv)\) is an infinitesimal portion of the body. Also illustrated are two coordinate systems. Coordinate system 1 is an inertial system with which the motion shall be specified. Coordinate system 3 is also an inertial system but is a temporary tool, the origin of coordinate system 3 is momentarily coincident with point A, the specification point for body A. The motion of the body will be specified by the linear position terms \(X_{A1}, Y_{A1}, Z_{A1}\), the linear velocity terms \(V_{XA1}, V_{YA1}, V_{ZA1}\), and the linear acceleration terms \(A_{XA1}, A_{YA1}, A_{ZA1}\) of the point A, plus the angular position terms \(\phi_{XA1}, \phi_{YA1}, \phi_{ZA1}\), the angular velocity terms \(\omega_{XA1}, \omega_{YA1}, \omega_{ZA1}\), and the angular acceleration terms \(\omega_{XA1}, \omega_{YA1}, \omega_{ZA1}\) of the body relative to the inertial coordinate system 3.
FIGURE (5-1) PARTICLE MOTION AND ASSOCIATED FORCES
B. PARTICLE DYNAMICS

Recall that both force and acceleration may be represented by vectors and that the rules of linear vector addition apply. Now, instead of determining the motion resulting from an arbitrary forcing function of linear forces and angular torques, the procedure will be reversed: for an arbitrary specified motion, the torques and forces necessary to produce this motion will be determined. Specifically, the forces necessary to produce each independent component of the specified motion, shall be determined then the force components will be linearly summed.

Let \((\rho dv)\) be an infinitesimal particle and as such it possesses negligible rotational kinetic energy. The linear motion of this particle is governed simply by Newton's second law of motion (for constant mass systems), which may be expressed

\[
\begin{bmatrix}
F_{xP3} \\
F_{yP3} \\
F_{zP3}
\end{bmatrix} = m_p
\begin{bmatrix}
A_{xP3} \\
A_{yP3} \\
A_{zP3}
\end{bmatrix}
\]  

where \(m_p\) is the particle mass \((\rho dv)\).

The force \(F_{P3}\) and acceleration \(A_{P3}\) act through the particle. The force \(F_{P3}\) may be shifted to act through point \(A\) if a torque \(T_{P3}\) is introduced satisfying the equation
The resultant force and torque acting on the total body may be computed by integration of the above equations over the entire volume of the body.

As stated previously, both linear and angular accelerations obey the rules of linear addition, hence the next step in the derivation consists of determining the acceleration of equation (5-1) in terms of the independent variables of motion specified earlier.

It is clear that linear and angular positions do not create accelerations; likewise linear velocity does not imply accelerations. The only components of interest are linear acceleration, angular velocity, and angular accelerations.

1. **Linear Acceleration Term.** This is the simplest component. The acceleration of every particle is simply

\[
\begin{bmatrix}
    A_{xP3} \\
    A_{yP3} \\
    A_{zP3}
\end{bmatrix}
= m_p \begin{bmatrix}
    0 & -Z_{P3} & Y_{P3} \\
    Z_{P3} & 0 & -X_{P3} \\
    -Y_{P3} & X_{P3} & 0
\end{bmatrix}
\begin{bmatrix}
    A_{xP3} \\
    A_{yP3} \\
    A_{zP3}
\end{bmatrix}
\] (5-2)

(Note that point A is arbitrary and need not be the center of mass.)
where $A_{p3}$ is the acceleration of the particle
and $A_{A3}$ is the acceleration of point $A$

2. **Angular Acceleration Term.** The linear acceleration of a particle
due to angular acceleration of the body is given by the vector equation

$$A_{p3} = \omega_{A3} \times r_{p3}$$  \hspace{1cm} (5-4)

which may also be expressed

$$
\begin{bmatrix}
A_{xp3} \\
A_{yp3} \\
A_{zp3}
\end{bmatrix} =
\begin{bmatrix}
0 & Z_{p3} & -Y_{p3} \\
-Z_{p3} & 0 & X_{p3} \\
+Y_{p3} & -X_{p3} & 0
\end{bmatrix}
\begin{bmatrix}
\cdot \\
\omega_{xA3} \\
\omega_{yA3} \\
\omega_{zA3}
\end{bmatrix}
$$  \hspace{1cm} (5-5)

3. **Angular Velocity Terms.** The third term to be considered is the
acceleration of each particle by an angular velocity component. This is
both the prize and the curse of the gyro. It is this term which produces
the gyro effect, but because the acceleration experienced by each particle
is proportional to the square of the angular velocity, it is also the most
difficult to handle mathematically.

In classical vector notation

$$A_{p3} = \omega_{A3} \times (\omega_{A3} \times r_{p3})$$  \hspace{1cm} (5-6)

where $A_{p3}$ is the acceleration of the particle
$\omega_{A3}$ is the angular velocity of the body $A$
and $r_{p3}$ is the location of the particle relative to point $A$
as measured by coordinate system 3.
The vector equation may be expressed in matrix form as

\[
\begin{align*}
A_x P_3 &= \begin{bmatrix} A_{x3} & 0 & 0 \\ 0 & A_{y3} & 0 \\ 0 & 0 & A_{z3} \end{bmatrix} \begin{bmatrix} \omega_{x3}^2 + \omega_{z3}^2 \\ -\omega_{y3} \omega_{z3} \\ \omega_{y3} \end{bmatrix} \begin{bmatrix} x P_3 \\ y P_3 \\ z P_3 \end{bmatrix} \\
A_y P_3 &= \begin{bmatrix} 0 & A_{y3} & 0 \\ A_{x3} & 0 & 0 \\ 0 & 0 & A_{z3} \end{bmatrix} \begin{bmatrix} \omega_{x3}^2 + \omega_{z3}^2 \\ -\omega_{x3} \omega_{z3} \\ -\omega_{x3} \omega_{y3} \end{bmatrix} \begin{bmatrix} x P_3 \\ y P_3 \\ z P_3 \end{bmatrix} \\
A_z P_3 &= \begin{bmatrix} 0 & 0 & A_{z3} \\ 0 & A_{y3} & 0 \\ A_{x3} & 0 & 0 \end{bmatrix} \begin{bmatrix} \omega_{x3}^2 + \omega_{y3}^2 \\ -\omega_{y3} \omega_{z3} \\ -\omega_{x3} \omega_{y3} \end{bmatrix} \begin{bmatrix} x P_3 \\ y P_3 \\ z P_3 \end{bmatrix}
\end{align*}
\]

The resultant acceleration of a particle is the sum of the three terms above, and therefore is

\[
\begin{align*}
A_x P_3 &= A_{x3} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \omega_{x3}^2 + \omega_{z3}^2 \\ -\omega_{y3} \omega_{z3} \\ \omega_{y3} \end{bmatrix} \begin{bmatrix} x P_3 \\ y P_3 \\ z P_3 \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \omega_{x3}^2 + \omega_{y3}^2 \\ -\omega_{y3} \omega_{z3} \\ \omega_{x3} \omega_{y3} \end{bmatrix} \begin{bmatrix} x P_3 \\ y P_3 \\ z P_3 \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{align*}
\]

C. INTEGRATION OF PARTICLES TO FORM FINITE RIGID BODY

The force \( F_{P3} \) and torque \( T_{P3} \) required to produce the acceleration of equation (5-8) for the particle are obtained by substitution of equation (5-8) into equations (5-1), and (5-2). The total resultant force \( F_{A3} \) and torque \( T_{A3} \) are obtained by integration over the volume of body A.
The substitution of equation (5-8) into equations (5-9) and (5-10) results in too bulky an equation to manipulate conveniently, therefore each term will be examined separately.

The first term of $F_{A3}$ is

\[
\phi \frac{v}{v} \begin{bmatrix} A_{xA3} \\ A_{yA3} \\ A_{zA3} \end{bmatrix} \rho dv = \phi \frac{v}{v} \rho dv \begin{bmatrix} A_{xA3} \\ A_{yA3} \\ A_{zA3} \end{bmatrix}
\]

\[
= m_A \begin{bmatrix} A_{xA3} \\ A_{yA3} \\ A_{zA3} \end{bmatrix}
\]
where \( m_A \) is the scalar mass of the body.

The second term becomes

\[
\begin{bmatrix}
0 & Z_{P3} & -Y_{P3} \\
-Z_{P3} & 0 & X_{P3} \\
Y_{P3} & -X_{P3} & 0
\end{bmatrix}
\begin{bmatrix}
\omega_{xA3} \\
\omega_{yA3} \\
\omega_{zA3}
\end{bmatrix}
\rho \, dv = \begin{bmatrix}
0 & Z_{P3} & -Y_{P3} \\
-Z_{P3} & 0 & X_{P3} \\
Y_{P3} & -X_{P3} & 0
\end{bmatrix}
\begin{bmatrix}
\omega_{xA3} \\
\omega_{yA3} \\
\omega_{zA3}
\end{bmatrix}
\rho \, dv
\]

The integral terms of the last expression are the first moment of mass and may be rewritten

\[
\begin{bmatrix}
0 & 0 & -\delta Z_{P3} \rho \, dv \\
0 & 0 & -\delta Y_{P3} \rho \, dv \\
0 & 0 & -\delta X_{P3} \rho \, dv
\end{bmatrix}
= \begin{bmatrix}
\omega_{xA3} \\
\omega_{yA3} \\
\omega_{zA3}
\end{bmatrix}
\rho \, dv
\]

(5-12)

Substituting the applicable elements of equation (5-13) into equation (5-12) produces

\[
\begin{bmatrix}
\delta X_{P3} \rho \, dv \\
\delta Y_{P3} \rho \, dv \\
\delta Z_{P3} \rho \, dv
\end{bmatrix}
= \begin{bmatrix}
m_A G_{xA3} \\
m_A G_{yA3} \\
m_A G_{zA3}
\end{bmatrix}
\]

where \( G_{A3} \) is a vector matrix denoting the location of the center of mass of body A.
The third force term is

\[
\delta_v \begin{bmatrix}
0 & x_{P3} & y_{P3} \\
-x_{P3} & 0 & z_{P3} \\
y_{P3} & -x_{P3} & 0
\end{bmatrix}
\begin{bmatrix}
\omega_{xA3} \\
\omega_{yA3} \\
\omega_{zA3}
\end{bmatrix}
= m_A \begin{bmatrix}
0 & G_{zA3} & -G_{yA3} \\
-G_{zA3} & 0 & +G_{xA3} \\
+G_{yA3} & -G_{xA3} & 0
\end{bmatrix}
\begin{bmatrix}
\omega_{xA3} \\
\omega_{yA3} \\
\omega_{zA3}
\end{bmatrix}
\]
The first torque term is

\[
\begin{bmatrix}
0 & -Z_{P3} & Y_{P3} \\
Z_{P3} & 0 & -X_{P3} \\
-Y_{P3} & X_{P3} & 0
\end{bmatrix}
\begin{bmatrix}
A_{xA3} \\
A_{yA3} \\
A_{zA3}
\end{bmatrix}
\rho \, dv
\]

\[
= \begin{bmatrix}
0 & -\dot{Z}_{P3} \rho \, dv & \dot{Y}_{P3} \rho \, dv \\
\dot{Z}_{P3} & 0 & -\dot{X}_{P3} \rho \, dv \\
-\dot{Y}_{P3} & -\dot{X}_{P3} \rho \, dv & 0
\end{bmatrix}
\begin{bmatrix}
A_{xA3} \\
A_{yA3} \\
A_{zA3}
\end{bmatrix}
\]

\[
= m_{A}
\begin{bmatrix}
0 & -G_{zA3} & G_{yA3} \\
G_{zA3} & 0 & -G_{xA3} \\
-G_{yA3} & G_{xA3} & 0
\end{bmatrix}
\begin{bmatrix}
A_{xA3} \\
A_{yA3} \\
A_{zA3}
\end{bmatrix}
\]

The second torque term is

\[
\begin{bmatrix}
0 & -Z_{P3} & Y_{P3} \\
Z_{P3} & 0 & -X_{P3} \\
-Y_{P3} & X_{P3} & 0
\end{bmatrix}
\begin{bmatrix}
0 & Z_{P3} & -Y_{P3} \\
-Z_{P3} & 0 & X_{P3} \\
Y_{P3} & -X_{P3} & 0
\end{bmatrix}
\begin{bmatrix}
\omega_{xA3} \\
\omega_{yA3} \\
\omega_{zA3}
\end{bmatrix}
\rho \, dv
\]

\[
= \begin{bmatrix}
0 & -\dot{Z}_{P3} & \dot{Y}_{P3} \\
\dot{Z}_{P3} & 0 & -\dot{X}_{P3} \\
-\dot{Y}_{P3} & -\dot{X}_{P3} & 0
\end{bmatrix}
\begin{bmatrix}
\dot{\omega}_{xA3} \\
\dot{\omega}_{yA3} \\
\dot{\omega}_{zA3}
\end{bmatrix}
\]
Each of the integral terms may be identified from equation (4-13) and the matrix is of the form of equation (4-25) thus the above expression reduces to

$$
\begin{bmatrix}
\phi(Y_{p3} + Z_{p3}) \rho dv \\
-\phi X_{p3} Y_{p3} \rho dv \\
-\phi X_{p3} Z_{p3} \rho dv \\
\end{bmatrix}
\begin{bmatrix}
\omega_{x3} \\
\omega_{y3} \\
\omega_{z3} \\
\end{bmatrix}
$$

or simply

$$
\begin{bmatrix}
(P_{yy3} + P_{zz3}) \\
-P_{xy3} \\
-P_{xz3} \\
\end{bmatrix}
\begin{bmatrix}
\omega_{x3} \\
\omega_{y3} \\
\omega_{z3} \\
\end{bmatrix}
$$

or simply

$$
\begin{bmatrix}
P_{xy3} \\
(P_{xx3} + P_{zz3}) \\
P_{xz3} \\
-P_{yz3} \\
(P_{xx3} + P_{yz3}) \\
\omega_{x3} \\
\omega_{y3} \\
\omega_{z3} \\
\end{bmatrix}
$$

The third and last torque term is
\[
\begin{bmatrix}
0 & -Z_{p3} & Y_{p3} \\
Z_{p3} & 0 & -X_{p3} \\
-Y_{p3} & X_{p3} & 0
\end{bmatrix}
\begin{bmatrix}
-(\omega^2_{xA3} + \omega^2_{zA3}) x_{A3} y_{A3} \\
x_{A3} y_{A3} - (\omega^2_{xA3} + \omega^2_{zA3}) y_{A3} y_{A3} \\
x_{A3} z_{A3} - (\omega^2_{xA3} + \omega^2_{yA3}) x_{A3} z_{A3}
\end{bmatrix}
\begin{bmatrix}
X_{p3} \\
Y_{p3} \\
Z_{p3}
\end{bmatrix}
\]
\[
(5-20)
\]
\[
\begin{bmatrix}
-\omega x_{A3} y_{A3} \omega x_{A3} Z_{p3} P_{3}^{3} P_{3}^{3} \rho d\nu + (\omega^2_{xA3} + \omega^2_{zA3}) \omega y_{A3} Z_{p3} P_{3}^{3} P_{3}^{3} \rho d\nu \\
-\omega y_{A3} z_{A3} \omega x_{A3} Z_{p3} P_{3}^{3} P_{3}^{3} \rho d\nu + \omega x_{A3} \omega z_{A3} \omega x_{A3} Z_{p3} P_{3}^{3} P_{3}^{3} \rho d\nu \\
+\omega y_{A3} z_{A3} \omega x_{A3} Z_{p3} P_{3}^{3} P_{3}^{3} \rho d\nu - (\omega^2_{xA3} + \omega^2_{yA3}) \omega y_{A3} Z_{p3} P_{3}^{3} P_{3}^{3} \rho d\nu \\
-\omega x_{A3} \omega z_{A3} \omega x_{A3} X_{p3} P_{3}^{3} P_{3}^{3} \rho d\nu + (\omega^2_{xA3} + \omega^2_{yA3}) \omega x_{A3} X_{p3} P_{3}^{3} P_{3}^{3} \rho d\nu \\
+\omega x_{A3} \omega z_{A3} \omega x_{A3} X_{p3} P_{3}^{3} P_{3}^{3} \rho d\nu - \omega x_{A3} \omega z_{A3} X_{p3} P_{3}^{3} P_{3}^{3} \rho d\nu \\
(\omega^2_{yA3} + \omega^2_{zA3}) \omega x_{A3} Y_{p3} P_{3}^{3} P_{3}^{3} \rho d\nu - \omega x_{A3} \omega y_{A3} X_{p3} Y_{p3} P_{3}^{3} P_{3}^{3} \rho d\nu \\
-\omega x_{A3} \omega z_{A3} \omega y_{A3} X_{p3} Z_{p3} P_{3}^{3} P_{3}^{3} \rho d\nu + (\omega^2_{xA3} + \omega^2_{zA3}) \omega y_{A3} X_{p3} Z_{p3} P_{3}^{3} P_{3}^{3} \rho d\nu \\
-\omega x_{A3} \omega z_{A3} \omega y_{A3} X_{p3} Z_{p3} P_{3}^{3} P_{3}^{3} \rho d\nu - \omega x_{A3} \omega y_{A3} X_{p3} Z_{p3} P_{3}^{3} P_{3}^{3} \rho d\nu
\end{bmatrix}
\]

As was true before, each of the integrals may be identified by comparison with equation (4-13). Substitution of the product of inertia terms into the above matrix and rearranging, produces
This last expression appears to be as simple as can be obtained.

D. COORDINATE TRANSFORMATION

At this point the total equations for the resultant force and torque could be written in simple form. Before performing this task however, two changes will be made at this time which will allow easier use of the equations in work to follow.

All parameters and variables have thus far been measured by coordinate system 3, a temporary coordinate system constructed so as to have its origin coincident with point A at this particular instant in time. At some point it becomes imperative in the work to follow that all parameters and variables of the entire multi-bodied system be measured by a common, inertial coordinate system such as coordinate system 1. The transformation of the various quantities is easily performed if we require coordinate system 3 to be oriented in the same manner as coordinate system 1, so that

\[
\begin{pmatrix}
-\left(\omega^2 y_{A3}^2 - \omega^2 z_{A3}^2\right) + \omega y_{A3} \omega z_{A3} (P_{yA3}^2 - P_{zA3}^2) \\
-\omega x_{A3} \omega y_{A3} (P_{yA3}^2) + \omega x_{A3} \omega z_{A3} (P_{xA3}^2) \\
+\left(\omega^2 x_{A3}^2 - \omega^2 z_{A3}^2\right) - \omega y_{A3} \omega z_{A3} (P_{xA3}^2 - P_{zA3}^2) \\
+\omega x_{A3} \omega y_{A3} (P_{yA3}^2) - \omega y_{A3} \omega z_{A3} (P_{yA3}^2) \\
-\left(\omega^2 x_{A3}^2 - \omega^2 y_{A3}^2\right) + \omega y_{A3} \omega x_{A3} (P_{yA3}^2) \\
-\omega x_{A3} \omega z_{A3} (P_{zA3}^2) + \omega y_{A3} \omega z_{A3} (P_{xA3}^2)
\end{pmatrix}
\]
\[ L_{13} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]  \hspace{1cm} (5-22)

(This alignment is for convenience and is neither necessary nor the usual orientation in practical situations)

With this constraint the following equalities exist,

\[ \begin{bmatrix} A_{x1} \\ A_{y1} \\ A_{z1} \end{bmatrix} = \begin{bmatrix} A_{x3} \\ A_{y3} \\ A_{z3} \end{bmatrix} \]  \hspace{1cm} (5-23)

\[ \begin{bmatrix} \omega_{x1} \\ \omega_{y1} \\ \omega_{z1} \end{bmatrix} = \begin{bmatrix} \omega_{x3} \\ \omega_{y3} \\ \omega_{z3} \end{bmatrix} \]  \hspace{1cm} (5-24)

\[ \begin{bmatrix} \omega_{x1} \\ \omega_{y1} \\ \omega_{z1} \end{bmatrix} = \begin{bmatrix} \omega_{x3} \\ \omega_{y3} \\ \omega_{z3} \end{bmatrix} \]  \hspace{1cm} (5-25)
Obviously the above quantities may be substituted into previous equations. And in retrospect it is clear that coordinate system 3 need not have been oriented in the specified manner since the left hand side of equations (5-23) through (5-27) are independent of the orientation of coordinate system 3.

The second change is simply a matter of literary convenience, it is customary to write the unknown variables on the left hand side of equations and the known forcing functions and dependent variables on the right hand side. Examination of the previous equations reveal that the unknown variables are \( A_{A1}, \omega_{A1}, F_{A1}, \) and \( T_{A1} \).

The position and velocity vectors are considered to be known, dependent variables. The equations (5-9) and (5-10) thus become
\[
\begin{align*}
\mathbf{m}_A \begin{bmatrix}
A_{xA1} \\
A_{yA1} \\
A_{zA1}
\end{bmatrix} + \mathbf{m}_A \begin{bmatrix}
0 & G_{zA1} & -G_{yA1} \\
-G_{zA1} & 0 & G_{xA1} \\
G_{yA1} & -G_{xA1} & 0
\end{bmatrix} \begin{bmatrix}
\omega_{xA1} \\
\omega_{yA1} \\
\omega_{zA1}
\end{bmatrix} - \begin{bmatrix}
F_{xA1} \\
F_{yA1} \\
F_{zA1}
\end{bmatrix} &= \\
\mathbf{m}_A \begin{bmatrix}
(\omega_{yA1}^2 + \omega_{zA1}^2) & -\omega_{xA1}^2 \omega_{yA1} & -\omega_{xA1} \omega_{zA1} \\
-\omega_{xA1} \omega_{yA1} & +(\omega_{xA1}^2 + \omega_{zA1}^2) & -\omega_{yA1} \omega_{zA1} \\
-\omega_{xA1} \omega_{zA1} & -\omega_{yA1} \omega_{zA1} & +(\omega_{xA1}^2 + \omega_{yA1}^2)
\end{bmatrix} \begin{bmatrix}
G_{xA1} \\
G_{yA1} \\
G_{zA1}
\end{bmatrix}
\end{align*}
\]
and

\[
\begin{bmatrix}
0 & -G_{zA1} & G_{yA1} \\
G_{zA1} & 0 & -G_{xA1} \\
-G_{yA1} & G_{xA1} & 0
\end{bmatrix}
\begin{bmatrix} A_{xA1} \\ A_{yA1} \\ A_{zA1} \end{bmatrix}
\]

\[ (5-29) \]

\[
\begin{bmatrix}
(P_{yyA1} + P_{zzA1}) & -P_{xyA1} & -P_{xzA1} \\
-P_{xyA1} & (P_{xxA1} + P_{zzA1}) & -P_{yzA1} \\
-P_{xyA1} & -P_{yzA1} & (P_{xxA1} + P_{yyA1})
\end{bmatrix}
\begin{bmatrix} \omega_{xA1} \\ \omega_{yA1} \\ \omega_{zA1} \end{bmatrix}
= \begin{bmatrix} T_{xA1} \\ T_{yA1} \\ T_{zA1} \end{bmatrix}
\]

\[
\begin{bmatrix}
\omega_{yA1}^2 - \omega_{zA1}^2 & P_{yzA1} - \omega_{yA1} \omega_{zA1} P_{yyA1} - P_{zzA1} \\
\omega_{xA1} P_{yzA1} - \omega_{yA1} P_{xzA1} & -\omega_{xA1} \omega_{yA1} P_{xyA1} \\
\omega_{xA1} P_{xyA1} - \omega_{yA1} P_{yzA1} & \omega_{xA1} P_{xzA1} - \omega_{zA1} \omega_{yA1} P_{yyA1} - P_{zzA1} \\
\omega_{xA1} P_{xzA1} - \omega_{zA1} \omega_{yA1} P_{yyA1} - P_{zzA1} & \omega_{xA1} P_{yzA1} - \omega_{yA1} \omega_{zA1} P_{yyA1} - P_{zzA1} \\
\omega_{xA1} P_{yzA1} - \omega_{yA1} \omega_{zA1} P_{yyA1} - P_{zzA1} & \omega_{xA1} P_{xzA1} - \omega_{zA1} \omega_{yA1} P_{yyA1} - P_{zzA1} \\
\omega_{xA1} P_{xzA1} - \omega_{zA1} \omega_{yA1} P_{yyA1} - P_{zzA1} & \omega_{xA1} P_{yzA1} - \omega_{yA1} \omega_{zA1} P_{yyA1} - P_{zzA1}
\end{bmatrix}
\]

\[ \epsilon. \ \text{NEWTONS EQUATION} \]

The two equations (5-28) and (5-29) may be combined to form equation (5-30).
The above matrix equation is the finite body equivalent to Newton's second law equation for a particle. Hereafter it will be referenced as Newton's equation for a finite body.
As a point of interest, the inverse of the 6 x 6 matrix may be found in terms of smaller matrices. If the original matrix is partitioned as indicated then

\[
\begin{align*}
\begin{bmatrix}
I & -G \\
-G & I \\
+G & P
\end{bmatrix}
\begin{bmatrix}
I-G(G^2+P)^{-1}G & +G(G^2+P)^{-1} \\
-(G^2+P)^{-1} & (G^2+P)^{-1}
\end{bmatrix}
= I
\end{align*}
\] (5-31)

where the first matrix is the 6 x 6 matrix in equation (5-30).
VI KINEMATICS

The previous sections have presented the idealized mechanical model and the matrix equations necessary to construct the desired solution algorithm. In this section the motion of each component will be described in equation form and the constraints between components will be delineated.

A. FUNDAMENTAL EQUATIONS

As was noted previously, the algorithm rests upon Newton's equation as expressed in equation (5-30). There are six components in the system, (the push rods are omitted for reasons discussed later). Equations (6-1) through (6-6) are simply equation (5-30) written in the nomenclature of each component. Coordinate system 1 is the master inertial reference which may be regarded as a fixed, ground-based observer.

\[ M_{B_1}^{A_1} - F_{B_1} = F_{W_1} \]  \hspace{1cm} (6-1)
\[ M_{S_1}^{A_1} - F_{S_1} = F_{W_1} \]  \hspace{1cm} (6-2)
\[ M_{P_1}^{A_1} - F_{P_1} = F_{W_1} \]  \hspace{1cm} (6-3)
\[ M_{R_1}^{A_1} - F_{R_1} = F_{W_1} \]  \hspace{1cm} (6-4)
\[ M_{A_1}^{A_1} - F_{A_1} = F_{W_1} \]  \hspace{1cm} (6-5)
\[ M_{E_1}^{A_1} - F_{E_1} = F_{W_1} \]  \hspace{1cm} (6-6)
where

- $B$ designates a base parameter,
- $S$ designates a spider (gimbal) parameter,
- $P$ designates a platform parameter,
- $R$ designates a rotor (gyro) parameter,
- $A$ designates an azimuth (yaw) push rod torque parameter,
- $E$ designates an elevation (pitch) push rod torquer parameter.

$M$ is a $6 \times 6$ "mass" matrix,

$A$ is a 6 element "acceleration" vector matrix

and $FW$ is a 6 element "gyro" matrix.

Individual elements of each matrix may be easily identified by examination of equation (5-30).

The usefulness of the master inertial reference is primarily limited to measuring external characteristics, such as the position, attitude, and flight vector of the missile and similar characteristics. Newton's equation may of course be referenced to any inertial frame of reference. Due to the nature of the tracker constraint forces, it is much more convenient if the equations (6-1) through (6-6) are referenced to a different inertial reference, coordinate system 3, (to be described later). It is also helpful to separate and identify the source of the forces on each structural component. The equation (6-1) through (6-6) may be expressed,

$$M_{B3}A_{B3} + F_{BS3} + F_{BA3} + F_{BE3} = FW_{B3} \quad (6-7)$$

$$M_{S3}A_{S3} - F_{BS3} + F_{SP3} = FW_{S3} \quad (6-8)$$
\[ M_{P3} A_{P3} - F_{SP3} + F_{PR3} - F_{AP3} - F_{EP3} = F_{WP3} \]  \hspace{1cm} (6-9)

\[ M_{R3} A_{R3} - F_{PR3} = F_{WR3} \]  \hspace{1cm} (6-10)

\[ M_{A3} A_{A3} - F_{BA3} - F_{PA3} = F_{WA3} \]  \hspace{1cm} (6-11)

\[ M_{E3} A_{E3} - F_{BE3} - F_{PE3} = F_{WE3} \]  \hspace{1cm} (6-12)

The new force/torque matrices are

- \( F_{BS} \) - the force/torque of the base upon the spider (gimbal)
- \( F_{BA} \) - the force/torque of the base upon the yaw push rod torquer rotor,
- \( F_{BE} \) - the force/torque of the base upon the pitch push rod torquer rotor,
- \( F_{SP} \) - the force/torque of the spider (gimbal) upon the platform,
- \( F_{PR} \) - the force/torque of the platform upon the rotor (gyro),
- \( F_{AP} \) - the force/torque of the yaw push rod upon the platform,
- \( F_{PA} \) - the force/torque of the yaw push rod upon the yaw torquer rotor,
- \( F_{EP} \) - the force/torque of the pitch push rod upon the platform,
- \( F_{PE} \) - the force/torque of the pitch push rod upon the pitch torquer rotor.

The equations (6-65) through (6-70) identify all the forces, torques and accelerations necessary to define the motion of every component, a total of ninety-six scalar matrix elements.
B. COORDINATE SYSTEMS

The expressions for inter-component contraints are most simply formulated by the introduction of a number of auxiliary coordinate systems. Instead of introducing these coordinate systems one at a time, table (VI - I) lists all the coordinate systems that will be required. As a mnemonic aid, coordinate systems designated by an odd numbered subscript are inertial references; those designated by an even numbered subscript are body-fixed coordinate references which move and rotate with the assigned component.

Coordinate system 1 is the master inertial reference and corresponds to a fixed, ground based observer. The remaining inertial references, coordinate systems 3, 5, 7, 9, 11, and 13 are temporary in nature; they are constructed anew at the beginning of each iteration such that they are momentarily coincident with the corresponding body-fixed coordinate system.
<table>
<thead>
<tr>
<th>SUBSCRIPT</th>
<th>DESCRIPTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Master inertial reference</td>
</tr>
<tr>
<td>3</td>
<td>Inertial reference, constructed to be momentarily coincident with coordinate system 4.</td>
</tr>
<tr>
<td>4</td>
<td>Base (airframe), body-fixed coordinate reference</td>
</tr>
<tr>
<td>5</td>
<td>Inertial reference, constructed to be momentarily coincident with coordinate system 6.</td>
</tr>
<tr>
<td>6</td>
<td>Spider (gimbal), body-fixed coordinate reference.</td>
</tr>
<tr>
<td>7</td>
<td>Inertial reference, constructed to be momentarily coincident with coordinate system 8.</td>
</tr>
<tr>
<td>8</td>
<td>Platform, body-fixed coordinate reference.</td>
</tr>
<tr>
<td>9</td>
<td>Inertial reference, constructed to be momentarily coincident with coordinate system 10.</td>
</tr>
<tr>
<td>10</td>
<td>Rotor (gyro), body-fixed reference.</td>
</tr>
<tr>
<td>11</td>
<td>Inertial reference, constructed to be momentarily coincident with coordinate system 12.</td>
</tr>
<tr>
<td>12</td>
<td>Yaw push-rod torque motor rotor, body-fixed reference.</td>
</tr>
<tr>
<td>13</td>
<td>Inertial reference, constructed to be momentarily coincident with coordinate system 14.</td>
</tr>
<tr>
<td>14</td>
<td>Pitch push-rod torque motor rotor, body-fixed reference.</td>
</tr>
</tbody>
</table>
A logical orientation of the body-fixed coordinate systems is suggested by the geometry of the overall system (Reference Figure (2-1)). The gyro rotor generally approximates a figure of revolution about the spin axis and since the angular spin velocity is traditionally assigned the variable $\omega_z$, the $Z$ axis of coordinate system 10 is oriented colinear with the gyro spin axis. The pivot axes suggest that the $X$ axes of coordinate system 8 and 6 be colinear with the inner (pitch) pivot axis, and the $Y$ axes of coordinate systems 6 and 4 be colinear with the outer (yaw) pivot axis. Next, the origin of coordinate systems 4, 6, 8 and 10 will be located at the intersection of the gyro spin axis, the pitch pivot axis, and the yaw pivot axis. (These three axes do not quite intersect in physical hardware due to mechanical tolerances, however it has always been a design goal that they intersect for reasons discussed in Chapter II.) The gimbal angles $\phi_x$, $\phi_y$, and $\phi_z$ are defined to be zero when the coordinate systems 4, 6, 8 and 10 are all coincident. Finally, the coordinate systems 12 and 14 are oriented so that the $Z$ axis of coordinate systems 4, 12 and 14 are colinear and the corresponding $X$ and $Y$ axes are parallel, for the condition $\phi_x = \phi_y = 0$. 
C. COORDINATE TRANSFORMATION MATRICES

Now that the various coordinate systems have been defined it is possible to determine many of the rotational and translational coordinate transformation matrices. It is not desirable at this point to perform the transformations to the master reference, coordinate system 1. Instead the transformation matrices from one component to an adjacent component is given, all other transformation matrices may be easily obtained from these.

The transformation of measurements made by the base inertial coordinate system 3 to the master inertial coordinate system 1 is governed by the following matrices.

\[
L_{13} = \begin{bmatrix}
L_{1311} & L_{1312} & L_{1313} \\
L_{1321} & L_{1322} & L_{1323} \\
L_{1331} & L_{1332} & L_{1333}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
(C_xB C_zB + S_xB S_yB S_zB) & (S_xB S_yB C_zB - C_yB S_zB) & (C_xB S_yB) \\
(S_xB S_zB - C_yB S_zB) & (C_xB S_zB) & (-S_xB) \\
(S_xB C_yB S_zB - S_yB C_zB) & (S_xB C_yB C_zB + S_yB S_zB) & (C_xB C_yB)
\end{bmatrix}
\]

(6-13)
where

\[
\begin{bmatrix}
S_{xB} \\
C_{xB} \\
S_{yB} \\
C_{yB} \\
S_{zB} \\
C_{zB}
\end{bmatrix}
= 
\begin{bmatrix}
\sin \phi_{xB1} \\
\cos \phi_{xB1} \\
\sin \phi_{yB1} \\
\cos \phi_{yB1} \\
\sin \phi_{zB1} \\
\cos \phi_{zB1}
\end{bmatrix}
\]  \hspace{1cm} (6-14)

The angles \(\phi_{xB1}, \phi_{yB1}\) and \(\phi_{zB1}\) are defined by the relationship

\[
\begin{bmatrix}
\phi_{xB1} \\
\phi_{yB1} \\
\phi_{zB1}
\end{bmatrix}
= 
\begin{bmatrix}
-\arcsin (L_{1323}) \\
\arctan (L_{1315}/L_{1333}) \\
\arctan (L_{1321}/L_{1322})
\end{bmatrix}
\]  \hspace{1cm} (6-15)

The equations (6-13) through (6-15) appear at first glance to be trivial since equation (6-13) defines the elements of \(L_{13}\) in terms of \(\phi_{B1}\) but \(\phi_{B1}\) is defined only in terms of the elements of \(L_{13}\), hence the angle vector \(\phi_{B1}\) appears superfluous. The purpose of defining \(\phi_{B1}\) is to prevent computational truncation errors from modifying the orthogonal matrix properties of \(L_{13}\). (Recall from chapter IV that the rotational transformation matrix is a real, orthogonal matrix.) The value of \(L_{13}\) in the algorithm is changed at the end of every iteration by means of a
truncated Taylor's series, on an element by element basis. Due to the finite accuracy of computer calculations, errors will accumulate which would slowly perturb the properties of $L_{13}$ away from the conditions of an orthogonal matrix. To prevent this from occurring, at the beginning of each iteration the vector matrix $\theta_{B1}$ is computed. The original matrix $L_{13}$ is next deleted and replaced by a new matrix computed by equation (6-13).

Since coordinate systems 1 and 3 are both inertial references,

\[
\dot{L}_{13} = \emptyset, \text{ a null matrix}, \quad (6-16)
\]

and $L_{13} = \emptyset$, a null matrix.

The coordinate system 3 was constructed to satisfy the equation

\[
L_{34} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \quad (6-18)
\]

If the base has a finite angular velocity then the time derivatives of $L_{34}$ are non-zero and may be computed by equations (4-29) and (4-33) to be,
The rotational transformation between inertial coordinate systems 3 and 5 is easily obtained in terms of the outer gimbal angle using equation (4-8).

\[
L_{34} = \begin{bmatrix}
0 & -\omega z_{3} & \omega y_{3} \\
\omega z_{3} & 0 & -\omega x_{3} \\
-\omega y_{3} & \omega x_{3} & 0
\end{bmatrix}
\]  
(6-19)

and

\[
L_{34} = \begin{bmatrix}
-\left(\omega_{y_{3}}^{2} + \omega_{z_{3}}^{2}\right) & \omega_{x_{3}}\omega_{y_{3}} & \omega_{x_{3}}\omega_{z_{3}} \\
\omega_{x_{3}}\omega_{y_{3}} & -\left(\omega_{x_{3}}^{2} + \omega_{z_{3}}^{2}\right) & \omega_{y_{3}}\omega_{z_{3}} \\
\omega_{x_{3}}\omega_{z_{3}} & \omega_{y_{3}}\omega_{z_{3}} & -\left(\omega_{x_{3}}^{2} + \omega_{y_{3}}^{2}\right)
\end{bmatrix}
\]  
(6-20)

The rotational transformation between inertial coordinate systems 3 and 5 is easily obtained in terms of the outer gimbal angle using equation (4-8).

\[
L_{35} = \begin{bmatrix}
C_{y_{4}} & 0 & S_{y_{4}} \\
0 & 1 & 0 \\
-S_{y_{4}} & 0 & C_{y_{4}}
\end{bmatrix}
\]  
(6-21)
where \[
\begin{bmatrix}
S_{y4} \\
C_{y4}
\end{bmatrix} = \begin{bmatrix}
\sin (\phi_{ySB3}) \\
\cos (\phi_{ySB3})
\end{bmatrix}
\] (6-22)

Again, since both coordinate systems 3 and 5 are inertial references,

\[ L_{35} = L_{35} = [\emptyset] \] (6-23)

Following the same reasoning as before

\[
L_{56} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\] (6-24)

\[
L_{56} = \begin{bmatrix}
0 & -\omega_{zS5} & \omega_{yS5} \\
\omega_{zS5} & 0 & -\omega_{xS5} \\
-\omega_{yS5} & \omega_{xS5} & 0
\end{bmatrix}
\] (6-25)

\[
L_{56} = \begin{bmatrix}
-(\omega_{yS5}^2 + \omega_{zS5}^2) & \omega_{xS5}\omega_{yS5} & \omega_{xS5}\omega_{zS5} \\
\omega_{xS5}\omega_{yS5} & -(\omega_{xS5}^2 + \omega_{zS5}^2) & \omega_{yS5}\omega_{yS5} \\
\omega_{xS5}\omega_{zS5} & \omega_{yS5}\omega_{zS5} & -(\omega_{xS5}^2 + \omega_{yS5}^2)
\end{bmatrix}
\] (6-26)
The manner in which the remaining matrices may be obtained is clear at this point therefore they are presented below, for reference purposes, with no further comments.

\[
L_{57} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_{x6} & -S_{x6} \\ 0 & S_{x6} & C_{x6} \end{bmatrix} \tag{6-27}
\]

where

\[
\begin{bmatrix} S_{x6} \\ C_{x6} \end{bmatrix} = \begin{bmatrix} \sin (\phi_{xPS5}) \\ \cos (\phi_{xPS5}) \end{bmatrix} \tag{6-28}
\]

\[
L_{57} = L_{57} = [\phi] \tag{6-29}
\]

\[
L_{78} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6-30}
\]

\[
L_{78} = \begin{bmatrix} 0 & -\omega_{zP7} & \omega_{yP7} \\ \omega_{zP7} & 0 & -\omega_{xP7} \\ -\omega_{yP7} & \omega_{xP7} & 0 \end{bmatrix} \tag{6-31}
\]
\[ L_{78} = \begin{bmatrix}
-\omega_x^2 yP7 - \omega_z^2 zP7 & \omega_x \omega_x yP7 & \omega_x \omega_z zP7 \\
\omega_x \omega_x yP7 & -\omega_x^2 yP7 - \omega_z^2 zP7 & \omega_y \omega_z zP7 \\
\omega_x \omega_z zP7 & \omega_y \omega_z zP7 & -\omega_x^2 yP7 - \omega_z^2 zP7 
\end{bmatrix} \] (6-32)

\[ L_{79} = \begin{bmatrix}
C_{z8} & -S_{z8} & 0 \\
S_{z8} & C_{z8} & 0 \\
0 & 0 & 1 
\end{bmatrix} \] (6-33)

where \[ \begin{bmatrix}
S_{z8} \\
C_{z8}
\end{bmatrix} = \begin{bmatrix}
\sin(\theta_{zRP7}) \\
\cos(\theta_{zRP7})
\end{bmatrix} \] (6-34)

\[ L_{79} = L_{79} = [\theta] \] (6-35)

\[ L_{9,10} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 
\end{bmatrix} \] (6-36)
\[ L_{9,10} = \begin{bmatrix} 0 & -\omega_{zR9} & \omega_{yR9} \\ \omega_{zR9} & 0 & -\omega_{xR9} \\ -\omega_{yR9} & \omega_{xR9} & 0 \end{bmatrix} \] (6-37)

\[ L'_{9,10} = \begin{bmatrix} -\left(\omega_{yR9}^2 + \omega_{zR9}^2\right) & \omega_{xR9}\omega_{yR9} & \omega_{xR9}\omega_{zR9} \\ \omega_{xR9}\omega_{yR9} & -\left(\omega_{xR9}^2 + \omega_{zR9}^2\right) & \omega_{yR9}\omega_{zR9} \\ \omega_{xR9}\omega_{zR9} & \omega_{yR9}\omega_{zR9} & -\left(\omega_{xR9}^2 + \omega_{yR9}^2\right) \end{bmatrix} \] (6-38)

\[ L_{9,10} = \begin{bmatrix} 0 & -\omega_{zR9} & \omega_{yR9} \\ \omega_{zR9} & 0 & -\omega_{xR9} \\ -\omega_{yR9} & \omega_{xR9} & 0 \end{bmatrix} \]

\[ L_{3,11} = \begin{bmatrix} C_{A4} & 0 & S_{A4} \\ 0 & 1 & 0 \\ -S_{A4} & 0 & C_{A4} \end{bmatrix} \] (6-39)

where \[ \begin{bmatrix} S_{A4} \\ C_{A4} \end{bmatrix} = \begin{bmatrix} \sin(\varphi_{yA4}) \\ \cos(\varphi_{yA4}) \end{bmatrix} \] (6-40)

\[ L'_{3,11} = L_{3,11} = [\varnothing] \] (6-41)
\[
L_{11,12} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\] (6-42)

\[
L_{11,12} = \begin{bmatrix}
0 & -\omega_{z11} & \omega_{y11} \\
\omega_{z11} & 0 & -\omega_{x11} \\
-\omega_{y11} & \omega_{x11} & 0
\end{bmatrix}
\] (6-43)

\[
L_{11,12} = \begin{bmatrix}
-\left(\omega_{y11}^2 + \omega_{z11}^2\right) & \omega_{x11} & \omega_{x11} \omega_{z11} \\
\omega_{x11} \omega_{y11} & -\left(\omega_{x11}^2 + \omega_{z11}^2\right) & \omega_{y11} \omega_{z11} \\
\omega_{x11} \omega_{z11} & \omega_{y11} \omega_{z11} & -\left(\omega_{x11}^2 + \omega_{y11}^2\right)
\end{bmatrix}
\] (6-44)

\[
L_{11,12} = \begin{bmatrix}
0 & -\omega_{z11} & \omega_{y11} \\
\omega_{z11} & 0 & -\omega_{x11} \\
-\omega_{y11} & \omega_{x11} & 0
\end{bmatrix}
\] (6-45)

\[
L_{3,13} = \begin{bmatrix}
1 & 0 & 0 \\
0 & c_{E4} & -s_{E4} \\
0 & s_{E4} & c_{E4}
\end{bmatrix}
\]

where
\[
\begin{bmatrix}
S_{E4} \\
C_{E4}
\end{bmatrix} =
\begin{bmatrix}
\sin (\phi_{xE4}) \\
\cos (\phi_{xE4})
\end{bmatrix}
\] (6-46)

\[
\dot{L}_{3,13} = \ddot{L}_{3,13} = [\emptyset]
\] (6-47)

\[
L_{13,14} =
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\] (6-48)

\[
\dot{L}_{13,14} =
\begin{bmatrix}
0 & -\omega_{zE13} & \omega_{yE13} \\
\omega_{zE13} & 0 & -\omega_{xE13} \\
-\omega_{yE13} & \omega_{xE13} & 0
\end{bmatrix}
\] (6-49)

\[
L_{13,14} =
\begin{bmatrix}
0 & -\omega_{zE13} & \omega_{yE13} \\
\omega_{xE13} & 0 & -\omega_{zE13} \\
-\omega_{yE13} & \omega_{xE13} & 0
\end{bmatrix}
\] (6-50)

\[
\begin{bmatrix}
0 & -\omega_{zE13} & \omega_{yE13} \\
\omega_{zE13} & 0 & -\omega_{xE13} \\
-\omega_{yE13} & \omega_{xE13} & 0
\end{bmatrix}
\]
One more transformation should be mentioned, $L_{39}$. This matrix is manipulated in the same manner as described for $L_{13}$; at the end of every iteration $L_{39}$ is modified by a truncated Taylor's series. At the beginning of each new iteration the gimbal angles $\phi_{XPSE}$, $\phi_{YSB3}$, and $\phi_{zRP7}$ are determined from the equation

\[
\begin{bmatrix}
\phi_{XPSS} \\
\phi_{YSB3} \\
\phi_{zRP7}
\end{bmatrix} = \begin{bmatrix}
-arcsin (L_{3923}) \\
arctan (L_{3913}/L_{3933}) \\
arctan (L_{3921}/L_{1322})
\end{bmatrix}
\]

The numerical value of each transformation matrix is then recomputed using the new gimbal angle values. There are two orientations, ($\phi_{XPSS} = \pm 90$ deg), where this procedure fails. These are the same points at which gimbal lock occurs and thus the mathematical limitation does not affect the usefulness of the procedure.

D. COMPUTATION OF THE MASS MATRIX

The numerical computation of the mass matrix, is easily obtained from a knowledge of the mass, center of mass, and product of inertia terms. The elements of these lower order matrices are the elements of the mass matrix too, as indicated in equation (5-30).

The mass of a body is independent of its position hence the mass may be represented by a scalar constant $m$. 
The center of mass terms are

\[ G_{B3} = L_{34} G_{B4} \] (6-52)

\[ G_{S3} = L_{36} G_{S6} \] (6-53)

\[ G_{P3} = L_{38} G_{P8} \] (6-54)

\[ G_{R3} = L_{3,10} G_{P10} \] (6-55)

\[ G_{A3} = L_{3,12} G_{A12} \] (6-56)

\[ G_{E3} = L_{3,14} G_{E14} \] (6-57)

Note that

\[ L_{34} = L_{56} = L_{78} = L_{9,10} = L_{11,12} = L_{13,14} = [I] \] (6-58)

hence

\[ L_{34} = [I] \] (6-59)

\[ L_{36} = L_{35} \] (6-60)

\[ L_{38} = L_{37} \] (6-61)

\[ L_{3,10} = L_{3,9} \] (6-62)

\[ L_{3,12} = L_{3,11} \] (6-63)
\[ L_{3,14} = L_{3,13} \]  

(6-64)

The moment of inertia matrices are

\[ P_{I_{B3}} = P_{I_{B4}} \]  

(6-65)

\[ P_{I_{S3}} = L_{35}P_{I_{S6}} \]  

(6-66)

\[ P_{I_{P3}} = L_{37}P_{I_{P8}} \]  

(6-67)

\[ P_{I_{R3}} = L_{39}P_{I_{R10}} \]  

(6-68)

\[ P_{I_{A3}} = L_{3,11}P_{I_{A12}} \]  

(6-69)

\[ P_{I_{E3}} = L_{3,13}P_{I_{E14}} \]  

(6-70)

The mass \( m \), the center of mass vector \( G \), and the moment of inertia matrix \( P_I \), as measured by the body-fixed coordinate system indicated in the above equations, are fixed, physical constants and are required input data for each component.

E. SYSTEM FORCING FUNCTIONS

The algorithm being developed is not intended to be an end in itself, but merely a subroutine in a larger algorithm which simulates the entire missile aerodynamic behavior. It is therefore appropriate at this point to delineate the known forcing functions.
There are two basic sources of forcing functions: the missile airframe motion, and the tracker servomechanism. The airframe motion perturbs the tracker telescope by virtue of the mass of the platform and spider (gimbal), imperfect pivot bearings, cable effects, and in some designs by gears and/or nutation dampers. The tracker servomechanism affects the tracker telescopes by means of electric torque motors.

Clearly, the force/torques of the tracker structure upon the airframe produce unmeasureably small perturbations in the airframe motion, hence the airframe motion is independent of the mechanical torques mentioned above. The position, velocities, and accelerations of the base (airframe) may be regarded as known forcing functions. Symbolically these quantities are \( x_B, v_B, \phi_B, \omega_B \), and \( \omega_B \).

Some of the mechanisms by which base motion perturbs the tracker telescope is inherent in the mass distribution of the structural components, this distribution being defined by the "mass" matrix defined previously. The remaining mechanisms may be modelled by a spring, viscous damper, and friction brake on every pivot bearing. The operational constants of each of these parameters must be known system definition data. The force/torques produced by the last three effects may be considered to be known forcing functions for the present effort since they are explicit functions of known variables.

The force/torques of the electric torque motors is a little more complicated. The tracker sensor, the stabilization electronics, the torque motors, and the gimbal structure form a feedback control servo-
mechanism. Since the "plant" to be controlled (the gimbal structure) exhibits a finite gain-bandwidth, the feedback signal may be approximated by discrete steps provided the time increments are sufficiently small. The net effect upon the present effort is that the force/torques of the electric torque motors may be considered to be known forcing functions but only if the time increment between iterations is made sufficiently small. It will become apparent that the time increments required by the gyro rotor spin velocity are much smaller than those required by the above constraint, thus there is no drawbacks to considering the torques produced by torque motors to be known forcing functions.

The known pivot torques are thus,

\[ T_{yBS4} = T_{FYB4} - K_{DYB4}(\dot{y}_{SB4} - \dot{y}_{OSB4}) \] (6-71)

\[ - K_{VYB4} y_{SB4} - K_{CYB4}(\dot{y}_{SB4} / y_{SB4}) \]

\[ T_{xSP6} = T_{FXSP6} - K_{DXSP6}(\dot{x}_{PS6} - \dot{x}_{OPS6}) \] (6-72)

\[ - K_{VXSP6} x_{PS6} - K_{CXSP6}(\dot{x}_{PS6} / x_{PS6}) \]

\[ T_{zPR8} = -K_{MZPR8}(\dot{z}_{RP8} - \dot{z}_{ORP8}) - K_{VZPR8}(\dot{z}_{RP8}) \] (6-73)

\[ - K_{CZPR8}(\dot{z}_{RP8} / z_{RP8}) \]

\[ T_{yBA4} = T_{FYBA4} - K_{DYBA4}(\dot{y}_{AB4} - \dot{y}_{OAB4}) \] (6-74)

\[ - K_{VYBA4} y_{AB4} - K_{CYBA4}(\dot{y}_{AB4} / y_{AB4}) \]
\[ T_{xBE4} = TF_{BE4} - KD_{BE4}(\dot{x}_{EB4} - \dot{x}_{0EB4}) \]
\[ - KV_{BE4}\ddot{x}_{EB4} - KC_{xEB4}(\ddot{x}_{EB4}/\dot{x}_{EB4}) \]

where

TF designates the torque motor output,

KD is a spring constant,

KV is a viscous damping constant,

KC is a coulomb friction constant,

and KM is the gyro spin motor constant.

F. GIMBAL STRUCTURE CONSTRAINTS

The matrix equations (6-7) through (6-12) may be written as thirty-six algebraic equations of seventy-nine unknown variables (eleven of the original ninety unknown matrix elements have been identified as known forcing functions). A minimum of forty-three additional constraint equations are necessary to define all the unknown variables. In this section the constraint equations imposed by the gimbal structure will be formulated. These equations will be grouped into six categories; linear position, linear acceleration, angular position, angular velocity, and angular acceleration constraints.

1. Linear Position Constraints. By choosing the body-fixed coordinate systems such that all origins are coincident, and by defining the
origin of the appropriate body-fixed coordinate system to be the specification point for each component, the linear position constraint is very simple: All gimbal structure components are mathematically defined to exist at the same point. Therefore

\[
\begin{bmatrix}
X_B \\
Y_B \\
Z_B
\end{bmatrix} =
\begin{bmatrix}
X_S \\
Y_S \\
Z_S
\end{bmatrix} =
\begin{bmatrix}
X_P \\
Y_P \\
Z_P
\end{bmatrix} =
\begin{bmatrix}
X_R \\
Y_R \\
Z_R
\end{bmatrix}
\] (6-76)

2. Linear Velocity Constraints. For the position constraint to be true at all times it is obvious that both the linear velocity and linear acceleration terms must be the same for all components. Thus it must be true that

\[
\begin{bmatrix}
V_{xB} \\
V_{yB} \\
V_{zB}
\end{bmatrix} =
\begin{bmatrix}
V_{xS} \\
V_{yS} \\
V_{zS}
\end{bmatrix} =
\begin{bmatrix}
V_{xP} \\
V_{yP} \\
V_{zP}
\end{bmatrix} =
\begin{bmatrix}
V_{xR} \\
V_{yR} \\
V_{zR}
\end{bmatrix}
\] (6-77)

3. Linear Acceleration Constraints. From the argument presented above,

\[
\begin{bmatrix}
A_{xB} \\
A_{yB} \\
A_{zB}
\end{bmatrix} =
\begin{bmatrix}
A_{xS} \\
A_{yS} \\
A_{zS}
\end{bmatrix} =
\begin{bmatrix}
A_{xP} \\
A_{yP} \\
A_{zP}
\end{bmatrix} =
\begin{bmatrix}
A_{xR} \\
A_{yR} \\
A_{zR}
\end{bmatrix}
\] (6-78)
Before formulating the angular constraints, it might be pointed out that the extreme simplicity of the linear constraints is made possible by the requirement that the gyro spin axis and the two gimbal pivot axes intersect at a point. This requirement however was not generated to facilitate mathematical manipulation but was justified by the optimal performance characteristics exhibited by this arrangement.

4. Angular Position Constraints. The angular position constraints have already been specified, by the definitions of $\phi_{xPS}$, $\phi_{ySB}$, $\phi_{zRP}$ and the rotational transformation matrices $L_{35}$, $L_{57}$, and $L_{79}$ defined by equations (6-21), (6-27) and (6-33).

5. Angular Velocity Constraints. The angular velocity constraints are easily obtained since each component is only allowed one degree of freedom from the neighboring components. The first pivot is the yaw pivot, where

\[
\begin{bmatrix}
\omega_{xSB4} \\
\omega_{ySB4} \\
\omega_{zSB4}
\end{bmatrix} =
\begin{bmatrix}
0 \\
\phi_{ySB4} \\
0
\end{bmatrix}
\]  

(6-79)

and since angular velocity obeys the laws of vector addition

\[
\omega_{SB4} = \omega_{S4} - \omega_{B4}
\]  

(6-80)
Now

\[ \omega_{S3} = L_{34} \omega_{S4} + \omega_{B3} \]  \hspace{1cm} (6-81)

and

\[ \omega_{B4} = 0 \]  \hspace{1cm} (6-82)

Combining the above equations yield

\[
\begin{bmatrix}
\omega_{xS3} \\
\omega_{yS3} \\
\omega_{zS3}
\end{bmatrix} = \begin{bmatrix}
\omega_{xB3} \\
\omega_{yB3} \\
\omega_{zB3}
\end{bmatrix} + \begin{bmatrix}
0 \\
\phi_{yS84} \\
0
\end{bmatrix}
\]  \hspace{1cm} (6-83)

A similar analysis of the spider-platform pivot yields

\[
\begin{bmatrix}
\omega_{xP5} \\
\omega_{yP5} \\
\omega_{zP5}
\end{bmatrix} = \begin{bmatrix}
\omega_{xS5} \\
\omega_{yS5} \\
\omega_{zS5}
\end{bmatrix} + \begin{bmatrix}
\phi_{xPS6} \\
0 \\
0
\end{bmatrix}
\]  \hspace{1cm} (6-84)

The platform-rotor spin bearing produces

\[
\begin{bmatrix}
\omega_{xR7} \\
\omega_{yR7} \\
\omega_{zR7}
\end{bmatrix} = \begin{bmatrix}
\omega_{xP7} \\
\omega_{yP7} \\
\omega_{zP7}
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
\phi_{zRP8}
\end{bmatrix}
\]  \hspace{1cm} (6-85)
6. Angular Acceleration Constraints. The acceleration constraints are obtained in a manner similar to that of the angular velocity constraints. For the yaw pivot the mechanical constraint requires

\[
\dot{\omega}_{SB4} = \dot{\phi}_{ySB4}
\]  \hspace{1cm} (6-86)

and from the vector properties of angular velocity

\[
\dot{\omega}_{SB4} = \dot{\omega}_S - \dot{\omega}_B
\]  \hspace{1cm} (6-87)

Taking the time derivative of equation (6-81) gives

\[
\dot{\omega}_S = L_{34} \omega_4 + L_{34} \omega_4 + \omega_B
\]  \hspace{1cm} (6-88)

Substituting equation (6-81) into (6-88) and rearranging produces

\[
\dot{\omega}_S = \dot{\omega}_S - \omega_B - L_{34} (\omega_S - \omega_B)
\]  \hspace{1cm} (6-89)

By definition

\[
\omega_B = \omega_B = 0
\]  \hspace{1cm} (6-90)

Now, proper manipulation of the above equations will yield

\[
\begin{pmatrix}
\dot{\omega}_xS3 \\
\dot{\omega}_yS3 \\
\dot{\omega}_zS3
\end{pmatrix}
= \begin{pmatrix}
\dot{\omega}_xB3 \\
\dot{\omega}_yB3 \\
\dot{\omega}_zB3
\end{pmatrix}
+ \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
+ \begin{pmatrix}
- \omega_B \\
- \omega_B \\
- \omega_B
\end{pmatrix}
+ \begin{pmatrix}
- zB3 \dot{\phi}_{ySB4} \\
0 \\
0 + xB3 \dot{\phi}_{ySB4}
\end{pmatrix}
\]  \hspace{1cm} (6-91)
Applying the same method to the spider-platform pivot results in

\[
\begin{bmatrix}
\dot{\omega}_{xPS} \\
\dot{\omega}_{yPS} \\
\dot{\omega}_{zPS}
\end{bmatrix} =
\begin{bmatrix}
\dot{\omega}_{xS5} \\
\dot{\omega}_{yS5} \\
\dot{\omega}_{zS5}
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
+ \begin{bmatrix}
\phi_{xPS6} \\
-w_{yS5}\phi_{xPS6} \\
-w_{zS5}\phi_{xSP6}
\end{bmatrix}
\]

Likewise the platform-rotor spin axis yields

\[
\begin{bmatrix}
\dot{\omega}_{xR7} \\
\dot{\omega}_{yR7} \\
\dot{\omega}_{zR7}
\end{bmatrix} =
\begin{bmatrix}
\dot{\omega}_{xP7} \\
\dot{\omega}_{yP7} \\
\dot{\omega}_{zP7}
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
\phi_{zRP8}
\end{bmatrix}
+ \begin{bmatrix}
\phi_{zRP8} \\
-w_{xP7}\phi_{zRP8} \\
-w_{yP7}\phi_{zRP8}
\end{bmatrix}
\]

G. PUSH-ROD CONSTRAINTS

The push-rod constraints are a bit more complicated than those of the gimbal structure. The mathematical model for the physical push-rod system is very complex due to the fact that some joints have two degrees of freedom. The equations become much simpler if certain simplifications or approximations are made. The approximations to be made (some of which were mentioned in Chapter II) introduce a finite but tolerable deviation in the algorithm.

1. Push-Rod Model. To be technically correct the two push-rods should be treated just as another component, that is, the mass, center of mass, and product of inertia matrix should be required input data.
In conventional push-rod designs the angular acceleration forces are negligible; the push-rod and torque arm geometries are carefully designed to form parallelograms, a condition that does not produce angular accelerations in the push-rods. The purpose of these geometries is not to minimize angular rotation per se however, it is to preserve static balance of the entire assembly. The center of mass shift of a true parallelogram push-rod system as the tracker line of sight rotates, can be exactly balanced by a fixed mass attached to the platform. In short, the angular acceleration forces on push-rods are negligible, the static unbalance forces are not. Therefore the push-rod model will be simplified by the following approximation:

a. Each push-rod shall be modelled as a straight, rigid rod having all mass concentrated at the ball and socket bearings. The mass at each bearing shall be such that the total mass of both bearings equals the true push-rod mass and the center of mass of the model is as close to the center of mass of the physical push-rod as possible.

b. The mass is then "transferred" across the bearing to become a part of the platform and torque rotor, leaving a massless, rigid push-rod.

A massless, rigid push-rod held by ball and socket joints cannot transmit torques and can transmit linear force by only pure compression or tensile components. The force vector exerted by the push-rod upon the socket bearings is therefore colinear with the axis of the push-rod, this leads to a second set of approximations. If the ratio of push-rod length to torquer arm length is sufficiently large two factors emerge;
first, the push-rod force vector becomes parallel to the Z axis of coordinate system 4 (the body-fixed coordinate reference assigned to the base); second, the push-rod net effect is to force the $z_4$ axis motion of the platform socket bearing to be equal to the $z_4$ axis motion of the torque arm socket bearing (both motions being measured by coordinate system 4). If the geometry of the push-rod and torque arm form a parallelogram the above statements are true even for finite length push-rods. Unfortunately, the pitch push-rod arrangement cannot form a parallelogram for non-zero azimuth angles because the pitch platform pivot axis and the pitch push-rod torque arm pivot axis are parallel only if $\phi_{ySB4} = 0$. In view of the fact that practical designs minimize the deviation from the example, the torque rod shall be modeled as constraining the $z_4$ motion of the platform socket bearing to be equal to the $z_4$ motion of the torque arm socket bearing. Also, since the push-rods no longer possess mass or moments of inertia, it is pointless to attempt to apply Newton's equation to them, other forms of constraint equations must be formulated, (which is why the push rods were omitted from the set of equations (6-1) through (6-6)).

2. **Yaw Push-Rod Constraint Equations.** The yaw push-rod as modelled in the previous paragraph has only two functions; one, to transmit torque from the torque motor to the platform, second, to constrain the $z_4$ motion of the platform socket bearing and the torque arm bearing to be equal.

The first function can easily be expressed in equation form. First assume that the push-rod is under a compressive force $f_A$, (the base must then be experiencing a tensile force equal to $f_A$). Because
of the single degree of freedom of the yaw push-rod torque rotor
bearing, only the $\varphi_{y4}$ component of the torque $T_{PA4}$ is of interest.
If the length of the torque arm of the torque motor is $R_{A12}$, and the
torque arm is rotated by the angle $\varphi_{yA4}$, then the functional relationship is

$$f_A = + \frac{T_{yPA4}}{R_{A12} \cos (\varphi_{yA4})} \quad (6-94)$$

Now if the yaw push-rod socket bearing is located at $R_{A8}$ on the platform,
then the location of the socket bearing as measured by the base is

$$\begin{bmatrix} R_{xA4} \\ R_{yA4} \\ R_{zA4} \end{bmatrix} = L_{37} \begin{bmatrix} R_{xA8} \\ R_{yA8} \\ R_{zA8} \end{bmatrix} \quad (6-95)$$

The torque exerted by the push-rod upon the platform is

$$\begin{bmatrix} T_{xAP4} \\ T_{yAP4} \\ T_{zAP4} \end{bmatrix} = \begin{bmatrix} R_{yA4} \\ -R_{xA4} \\ 0 \end{bmatrix} f_A \quad (6-96)$$

Solving all equations for $f_A$ and equating terms, the first constraint
equation is

$$\frac{T_{xAP4}}{R_{yA4}} = - \frac{T_{yAP4}}{R_{xAP4}} = + \frac{T_{yPA4}}{R_{A12} \cos (\varphi_{yA4})} \quad (6-97)$$
The second push-rod function generates several constraint equations. In order to simplify some of the algebra the length of the push-rod is required to be of a length such that

\[ R_{zA4} = -R_{A12} \sin (\phi_{yA4}) \] (6-98)

A parallelogram push-rod system will satisfy this requirement, and so will many other arrangements of interest, hence this restriction does not noticeably reduce the number of practical systems that may be modelled.

Taking successive time derivatives of the above equations produce

\[ \ddot{R}_{zA4} = -R_{A12} \cos (\phi_{yA4}) \dot{\phi}_{yA4} \] (6-99)

and

\[ \dddot{R}_{zA4} = -R_{A12} [\cos (\phi_{yA4}) \ddot{\phi}_{yA4} - \sin (\phi_{yA4}) \dot{\phi}_{yA4}] \] (6-100)

It is appropriate at this time to develop expressions for \( R_{zA4} \), \( \dot{R}_{zA4} \), and \( \ddot{R}_{zA4} \) in terms of \( \phi_{xyPS63}, \phi_{ySB4}, \phi_{zRP8} \), and the other gimbal variables.

If the location of the platform yaw socket bearing as measured by coordinate system 8 is defined to be \( R_{A8} \), then

\[ R_{A4} = L_{48} R_{A8} \] (6-101)

\[ \dot{R}_{A4} = \dot{L}_{48} R_{A8} + L_{48} \dot{R}_{A8} \] (6-102)

and

\[ \dddot{R}_{A4} = \dddot{L}_{48} R_{A8} + 2 \dddot{L}_{48} \dot{R}_{A8} + \dddot{L}_{48} \dot{R}_{AB} \] (6-103)
The socket bearing is rigidly attached to the platform thus

\[ \begin{align*}
\dot{\mathbf{R}}_{A8} &= \ddot{\mathbf{R}}_{A8} = 0
\end{align*} \] (6-104)

The first derivative term becomes

\[ \dot{\mathbf{R}}_{A4} = \begin{bmatrix}
0 & -\omega_{zPB4} & \omega_{yPB4} \\
\omega_{zPB4} & 0 & -\omega_{xPB4} \\
-\omega_{yPB4} & \omega_{xPB4} & 0
\end{bmatrix} L_{48} \mathbf{R}_{A8} \] (6-105)

with the \( \dot{R}_{zA4} \) component

\[ \dot{R}_{zA4} = -\omega_{yPB4} R_{xA4} + \omega_{xPB4} R_{yA4} \] (6-106)

In order to obtain this quantity in terms of inertial measurements, the \( \omega_{PB4} \) will be replaced. First

\[ \omega_{PB4} = L_{43} \omega_{PB3} \] (6-107)

\[ = L_{43} (\omega_{P3} - \omega_{B3}) \]

and

\[ \dot{\omega}_{PB4} = L_{43} (\dot{\omega}_{P3} - \omega_{PB3}) + L_{43} (\omega_{P3} - \dot{\omega}_{B3}) \] (6-108)
Recalling that

\[ L_{43} = L_{34}^T \] (6-109)

and substituting equations (6-18), and (6-19) into the above equations produce the simplified equations,

\[ \omega_{PB4} = \omega_{P3} - \omega_{B3} \] (6-10)

and

\[
\begin{bmatrix}
0 & \omega_{ZB3} & -\omega_{YB3} \\
-\omega_{ZB3} & 0 & \omega_{XB3} \\
\omega_{YB3} & -\omega_{XB3} & 0
\end{bmatrix}
\] \[ (\omega_{P3} - \omega_{B3}) + (\omega_{P3} - \omega_{B3}) \] (6-111)

Next, inserting equation (6-111) into (6-106), equating the result to equation (6-99) and solving for \( \dot{\varphi}_{yA4} \) results in

\[ \dot{\varphi}_{yA4} = - (\omega_{XP3} - \omega_{XB3})FYA + (\omega_{YP3} - \omega_{YB3})FXA \] (6-112)

where

\[ FXA = \frac{R_{xa4}}{R_{a12} \cos (\varphi_{yA4})} \] (6-113)

and

\[ FYA = \frac{R_{ya4}}{R_{a12} \cos (\varphi_{yA4})} \] (6-114)

To complete the yaw push-rod motion constraint equation, the \( R_{zA4} \) component must be evaluated. Evaluating \( L_{48} \) by analogy with equation (4-33),
Substituting in equations (6-110) and (6-111) and extracting the $R_{zA4}$ component,

\[
R_{zA4} = \begin{bmatrix}
-\left(\omega_x^2 + \omega_z^2\right) & \omega_x \omega_y & \omega_x \omega_z \\
\omega_x \omega_y & -\left(\omega_y^2 + \omega_z^2\right) & \omega_y \omega_z \\
\omega_x \omega_z & \omega_y \omega_z & -\left(\omega_x^2 + \omega_y^2\right)
\end{bmatrix} L_{48}^{R_{A4}}
\]

Equating (6-116) to (6-100) and solving for $\dot{\phi}_{yA4}$.
\[
\phi_{yA4} = \frac{\sin (\phi_{yA4})}{\cos (\phi_{yA4})} \phi_{yA4} - (\omega_{xP3^2} \omega_{zP3^2} + \omega_{xB3} \omega_{zB3} - 2 \omega_{zP3} \omega_{xB3}) \text{FXA} \\
- (\omega_{yP3} \omega_{zP3} + \omega_{yB3} \omega_{zB3}) \text{FYA} \\
+ (\omega_{xP3} \omega_{xB3})^2 + (\omega_{yP3} \omega_{yB3})^2 \text{FZA} \\
- (\omega_{xP3} \omega_{yB3}) \text{FXA} \\
+ \omega_{xB3} \omega_{yB3} \text{FXA}
\]

where

\[
\text{FXA} = \frac{R_{xA4}}{R_{A12} \cos \phi_{yA4}} \tag{6-113}
\]

\[
\text{FYA} = \frac{R_{yA4}}{R_{A12} \cos \phi_{yA4}} \tag{6-114}
\]

\[
\text{FZA} = \frac{R_{zA4}}{R_{A12} \cos \phi_{yA4}} \tag{6-118}
\]

In summary, the yaw push-rod motion constraints are specified by equations (6-98), (6-112), and (6-117).

3. **Pitch Push-Rod Constraint Equations.** The pitch push-rod analysis follows exactly the same procedure performed for the yaw push-rod. Rather than pursue the entire derivation, the results are simply presented below.

\[
\frac{T_{xEP4}}{R_{yE4}} = - \frac{T_{yEP4}}{R_{xE4}} = - \frac{T_{xPE4}}{R_{E14} \cos \phi_{xE4}} \tag{6-119}
\]
\[ R_{zE4} = R_{E4} \sin \phi_{xE4} \]  \hspace{1cm} (6-120)

\[ \phi_{xE4} = (\omega_{xP3} - \omega_{xB3}) FYE - (\omega_{yP3} - \omega_{yB3}) FXE \]  \hspace{1cm} (6-121)

\[ \phi_{xE4} = \frac{\sin (\phi_{xE4})}{\cos (\phi_{xE4})} \phi_{xE4} \]  \hspace{1cm} (6-122)

\[ + (\omega_{xP3} \omega_{xP3} + \omega_{xB3} \omega_{xB3} - 2 \omega_{xP3} \omega_{xB3}) FXE \]

\[ + (\omega_{yP3} \omega_{xP3} + \omega_{yB3} \omega_{xB3} - 2 \omega_{yP3} \omega_{yB3}) FYE \]

\[ - (\omega_{xP3} - \omega_{xB3})^2 + (\omega_{yP3} - \omega_{yB3})^2) FZE \]

\[ + \omega_{xP3} FYE - \omega_{yP3} FXE \]

\[ - \omega_{xB3} FYE + \omega_{yB3} FXE \]

where

\[ FXE = \frac{R_{xE4}}{R_{E4} \cos \phi_{xE4}} \]  \hspace{1cm} (6-123)

\[ FYE = \frac{R_{yE4}}{R_{E4} \cos \phi_{xE4}} \]  \hspace{1cm} (6-124)

\[ FZE = \frac{R_{zE4}}{R_{E4} \cos \phi_{xE4}} \]  \hspace{1cm} (6-125)

4. **Yaw Push-Rod Torquer.** The push-rod torquers are included as part of the push-rod system and are implicitly included in equations (6-98) and (6-120), and all equations derived from these two. The
additional constraints desired in this section is the relationship between \( \omega_{A3} \) and \( \omega_{B3} \). Similar constraints were formulated earlier for the spider-base pivot in equations (6-83), and (6-91). Instead of repeating the derivation, equations (6-126) and (6-127) are presented below,

\[
\begin{bmatrix}
\omega_{xA3} \\
\omega_{yA3} \\
\omega_{zA3}
\end{bmatrix} = \begin{bmatrix}
\omega_{xB3} \\
\omega_{yB3} \\
\omega_{zB3}
\end{bmatrix} + \begin{bmatrix}
0 \\
\phi_{yA4} \\
0
\end{bmatrix}
\] (6-126)

\[
\begin{bmatrix}
\omega_{xA3} \\
\omega_{yA3} \\
\omega_{zA3}
\end{bmatrix} = \begin{bmatrix}
\omega_{xB3} \\
\omega_{yB3} \\
\omega_{zB3}
\end{bmatrix} + \begin{bmatrix}
0 \\
\phi_{yA4} \\
0
\end{bmatrix} + \begin{bmatrix}
-\omega_{zB3}\phi_{yA4} \\
0 \\
\omega_{xB3}\phi_{yA4}
\end{bmatrix}
\] (6-127)

5. **Pitch Push-Rod Torquer.** Following the same argument as before, the pitch push-rod torquer introduces the constraint equations,

\[
\begin{bmatrix}
\omega_{xE3} \\
\omega_{yE3} \\
\omega_{zE3}
\end{bmatrix} = \begin{bmatrix}
\omega_{xB3} \\
\omega_{yB3} \\
\omega_{zB3}
\end{bmatrix} + \begin{bmatrix}
\phi_{xE4} \\
0 \\
0
\end{bmatrix}
\] (6-128)
6. Torquer Coordinate System. Before leaving the push-rod system, one more approximation will be introduced for mathematical convenience. The linear acceleration of the origins of coordinate systems 12, and 14 are approximated as being equal to the linear accelerations of the base. Thus

\[
\begin{bmatrix}
\dot{\omega}_{xE3} \\
\dot{\omega}_{yE3} \\
\dot{\omega}_{zE3}
\end{bmatrix} =
\begin{bmatrix}
\dot{\omega}_{xB3} \\
\dot{\omega}_{yB3} \\
\dot{\omega}_{zB3}
\end{bmatrix} +
\begin{bmatrix}
\dot{\theta}_{xE4} \\
0 \\
0
\end{bmatrix} +
\begin{bmatrix}
+\omega_{xB3}\dot{\theta}_{xE4} \\
-\omega_{yB3}\dot{\theta}_{xE4}
\end{bmatrix}
\]

The error introduced by this approximation is a function of \(\omega_{xB3}\), \(\omega_{yB3}\) and the static imbalance of the push-rod assembly. The average values of \(\omega_{xB3}\), and \(\omega_{yB3}\) are very close to zero. Likewise the push-rod assemblies are very nearly balanced in practice, hence the error introduced by this approximation is unmeasureably insignificant.
VII MATHEMATIC REDUCTION

The previous chapter has formulated all the necessary equations to specify the motion of all the components, but in terms of close to one hundred different variables. In theory it is possible to generate a single matrix equation in these variables and "turn the crank" to produce a solution. The number of computations to reduce a matrix is proportional to the factorial of the order of the matrix; the solution of a 100 x 100 matrix is economically infeasible. In this chapter, the equations of chapter VI will be manipulated to form a 15 x 15 matrix. This matrix is then reduced to a 3 x 3 matrix which is then solved by standard determinant techniques.

A. LINEAR ACCELERATION REDUCTION

The first reduction is very simple; by substituting equations (6-78) and (6-130) into equations (6-7) through (6-12), a new set of equations may be generated. The only linear acceleration variables in this new set are $A_{XB3}$, $A_{YB3}$, and $A_{ZB3}$, known forcing functions. This step eliminates 36 variables and 18 equations.

Recall that Newton's equation (5-30) was generated by combining two other matrix equations (5-28) and (5-29). If the linear acceleration terms are known forcing functions, then for any component which has the center of mass at the origin of the body-fixed coordinate system, equation (5-28) becomes indeterminate. Since most tracker assemblies will closely approximate these conditions, equation (5-28) will normally be ill-conditioned. Eliminating equation (5-28) from (5-30) reduces Newton's equation
to equation (5-29). The equation (6-7) through (6-12) thus reduce to,

\[ P_{B3} \omega_{B3} + T_{BS3} + T_{BA3} + T_{B3E} = T_{W3} - m_{B} G_{B3} A_{B3} \] (7-1)

\[ P_{S3} \omega_{S3} - T_{BS3} + T_{SP3} = T_{W3} - m_{S} G_{S3} A_{B3} \] (7-2)

\[ P_{P3} \omega_{P3} - T_{SP3} + T_{PR3} - T_{AP3} - T_{EP3} = T_{W3} - m_{P} G_{P3} A_{B3} \] (7-3)

\[ P_{R3} \omega_{R3} - T_{PR3} = T_{W3} - m_{R} G_{R3} A_{B3} \] (7-4)

\[ P_{A3} \omega_{A3} - T_{BA3} - T_{PA3} = T_{W3} - m_{A} G_{A3} A_{B3} \] (7-5)

\[ P_{E3} \omega_{E3} - T_{BE3} - T_{PE3} = T_{W3} - m_{E} G_{E3} A_{B3} \] (7-6)

where

- \( P \) is a 3 x 3 moment of inertia matrix
- \( T \) is a 3 element gyro matrix
- \( m \) is a scalar mass
- \( G \) is a 3 x 3 center of mass matrix
- \( A_{B} \) is the linear acceleration of the base
- \( T \) is the 3 torque components elements of the corresponding 6 element force/torque vector defined in equations (6-7) through (6-12)
B. ANGULAR ACCELERATION REDUCTIONS FOR GIMBAL STRUCTURE

One of the known forcing functions is the angular acceleration of the base. This acceleration is produced by aerodynamic surfaces of the airframe thus the logical coordinate system to specify this acceleration is coordinate system 3. The known angular acceleration forcing function is thus chosen to be \( \dot{\omega}_{x3} \), \( \dot{\omega}_{y3} \), and \( \dot{\omega}_{z3} \).

Twelve variables can be eliminated if the angular accelerations of all components can be expressed in terms of the three variables above and \( \dot{\omega}_{x7} \), \( \dot{\omega}_{y7} \), and \( \dot{\omega}_{z7} \).

The matrix equations (6-92), (6-93), (6-94) and coordinate transformation equations based upon the derivations of chapter IV are presented below, in algebraic form.

\[
\begin{align*}
\omega_{x3} &= \omega_{x3} - \omega_{z3} \Phi y_{3B4} \\
\omega_{y3} &= \omega_{y3} + \Phi y_{3B4} \\
\omega_{z3} &= \omega_{z3} + \omega_{x3} \Phi y_{3B4} \\
\omega_{x3} &= C_{y4} \omega_{x5} - S_{y4} \omega_{z5} \\
\omega_{y3} &= \omega_{y5} \\
\omega_{z3} &= S_{y4} \omega_{x5} + C_{y4} \omega_{z5} \\
\omega_{x5} &= \omega_{x5} + \Phi x_{5B6}
\end{align*}
\]
\begin{align*}
\omega_{yP5} &= \omega_{yS5} + \omega_{zS5} \phi_{xPS6} \tag{7-14} \\
\omega_{zP5} &= \omega_{zS5} - \omega_{yS5} \phi_{xPS6} \tag{7-15} \\
\omega_{xP5} &= \omega_{xP7} \tag{7-16} \\
\omega_{yP5} &= S_{x6} \omega_{yP7} - S_{x6} \omega_{zP7} \tag{7-17} \\
\omega_{zP5} &= S_{x6} \omega_{yP7} + C_{x6} \omega_{zP7} \tag{7-18} \\
\omega_{xR7} &= \omega_{xP7} + \omega_{yP7} \phi_{zRP8} \tag{7-19} \\
\omega_{yR7} &= \omega_{yP7} - \omega_{xP7} \phi_{zRP8} \tag{7-20} \\
\omega_{zR7} &= \omega_{zP7} + \phi_{zRP8} \tag{7-21} \\
\end{align*}

This group of equations may be manipulated to produce the following results

\begin{align*}
\omega_{xS3} &= \omega_{xB3} - \omega_{zB3} \phi_{ySB4} \tag{7-7} \\
\omega_{yS3} &= \frac{1}{C_{x6}} \left[ \omega_{yR7} - S_{x6} (S_{y4} \omega_{xB3} + C_{y4} \omega_{zB3}) \right] \\
&\quad - \omega_{zS5} \phi_{xPS6} \frac{S_{x6}}{C_{x6}} (\omega_{xS5} \phi_{ySB4} - \omega_{yS5} \phi_{xPS6}) \\
&\quad + \frac{1}{C_{x6}} \omega_{xP7} \phi_{zRP8} \tag{7-22} \\
\end{align*}
\[ \omega_{S3} = \omega_{B3} + \omega_{B3} \dot{\theta}_{SB4} \]  
\text{(7-9)}

and

\[ \omega_{P5} = \omega_{R7} - \omega_{P7} \dot{\theta}_{RP8} \]  
\text{(7-23)}

\[ \omega_{P5} = \frac{1}{C_{x6}} \left[ \omega_{R7} - S_{x6} (S_{y4} \omega_{B3} + C_{y4} \omega_{z3}) \right] \]  
\text{(7-24)}

\[ \omega_{P5} = \frac{S_{x6}}{C_{x6}} (\omega_{S5} \dot{\theta}_{SB4} - \omega_{S5} \dot{\theta}_{PS6}) + \frac{1}{C_{x6}} \omega_{P7} \dot{\theta}_{RP8} \]  
\text{(7-25)}

These six equations specify the angular accelerations of the spider and platform in terms of the angular accelerations of the base and rotor plus various angular velocity terms. It is easy to modify equations (7-1) through (7-4) to accept the above expressions.

Reviewing equations (7-1) through (7-6) in preparation for the above modifications several other modifications are suggested. First, equation (7-1) is irrelevant to the present effort and therefore may be eliminated. It was postulated previously that the torques of the tracker have an insignificant effect upon the base motion. This means that equation (7-1) is illconditioned relative to the torques \( T_{BS}, T_{BA}, \) and \( T_{BE}, \) and since all other variables are known, equation (7-1) contains no useful information.
The second modification is to allow easier manipulation of the known torque forcing functions $T_{yBS3}$, $T_{xSP5}$, $T_{zPR7}$, $T_{yBA3}$ and $T_{xBE3}$. A coordinate transformation shall be introduced to allow the above torques to appear in the equation.

1. Spider Equation. Performing the coordinate transformations upon equation (7-2) produces

$$P_{IS3} \cdot \omega_{S3} - T_{BS3} + L_{35} T_{SP5} = T_{WS3} - m_g A_{S3B3}$$

(7-26)

The known inter-component torques are $T_{ySB3}$ and $T_{xPS5}$.

Now, substituting the angular acceleration and torque quantities into equation (7-26), and moving the known variables to the right of the equality,

$$P_{SL} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{C_{x6}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \omega_{R7} \\ 0 \\ T_{zBS3} \end{bmatrix} + L_{35} \begin{bmatrix} T_{xBS3} \\ T_{ySP5} \\ T_{zSP5} \end{bmatrix} = \begin{bmatrix} T_{FA} \\ T_{FB} \\ T_{FC} \end{bmatrix}$$

(7-27)

where

$$\begin{bmatrix} T_{FA} \\ T_{FB} \\ T_{FC} \end{bmatrix} = \begin{bmatrix} T_{DWBSA} \\ T_{DWBSB}+T_{WWSB}+T_{yBS3}-L_{35} \\ T_{DWBSC}+T_{WWSC} \end{bmatrix}$$

(7-28)
The second modification is to allow easier manipulation of the known torque forcing functions $T_{yBS3}$, $T_{xSPS}$, $T_{zPR7}$, $T_{yBA3}$ and $T_{xBE3}$. A coordinate transformation shall be introduced to allow the above torques to appear in the equation.

1. **Spider Equation.** Performing the coordinate transformations upon equation (7-2) produces

$$ P_{S3} \omega_{S3} - T_{BS3} + L_{3S} T_{SP5} = T_{WS3} - m_{S} G_{S3} A_{B3} $$

(7-26)

The known inter-component torques are $T_{ySB3}$, and $T_{xPS5}$.

Now, substituting the angular acceleration and torque quantities into equation (7-26), and moving the known variables to the right of the equality,

$$ P_{SL} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{C_{x6}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \omega_{R7} \\ T_{zBS3} \end{bmatrix} + L_{3S} \begin{bmatrix} T_{ySP5} \end{bmatrix} = \begin{bmatrix} T_{FA} \\ T_{FB} \\ T_{FC} \end{bmatrix} $$

(7-27)

where

$$ T_{FA} = T_{DWBSA} + T_{WWSA} + \begin{bmatrix} 0 \\ T_{yBS3} \end{bmatrix} - L_{3S} \begin{bmatrix} 0 \end{bmatrix} $$

$$ T_{FB} = T_{DWBSB} + T_{WWSB} + \begin{bmatrix} 0 \end{bmatrix} $$

$$ T_{FC} = T_{DWBS} + T_{WWSC} + \begin{bmatrix} 0 \end{bmatrix} $$

(7-28)
\[
\begin{bmatrix}
TWSA \\
TWSB \\
TWSC \\
\end{bmatrix} + \begin{bmatrix}
TMGASA \\
TMGASB \\
TMGASC \\
\end{bmatrix}
\]

and

\[ P_{SL} = P_{I_{S3}} \] (7-29)

\[
\begin{bmatrix}
TDWBBSA \\
TDWBBSB \\
TDWBSC \\
\end{bmatrix} = -P_{SL} \begin{bmatrix}
1 & 0 & 0 \\
\frac{-S}{C}x_6S_y & 0 & \frac{-S}{C}x_6C_y \\
0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
\dot{ω}_{x3} \\
\dot{ω}_{y3} \\
\dot{ω}_{z3} \\
\end{bmatrix} \] (7-30)

\[
\begin{bmatrix}
TWWSA \\
TWWSB \\
TWWSC \\
\end{bmatrix} = -P_{SL} \begin{bmatrix}
\dot{ω}_{z3} & 0 & 0 \\
0 & 0 & 0 \\
0 & -\frac{X}{C}x_6 & 0 \\
\end{bmatrix} \begin{bmatrix}
\dot{ϕ}_{y3} \\
\dot{ϕ}_{y3} \\
\dot{ϕ}_{y3} \\
\end{bmatrix} \] (7-31)

\[
\begin{bmatrix}
TWSA \\
TWSB \\
TWSC \\
\end{bmatrix} = TW_{S3} = \begin{bmatrix}
m_A &= m_S \\
ω_{A1} &= ω_{S3} \\
P_{A1} &= P_{S3} \\
\end{bmatrix} \] (7-32)

See Equation (5-29)
The nomenclature of these equations differs markedly from that introduced in chapter IV primarily to allow the reader to identify these terms in the computer listings in Appendix A. The use of upper case letters for scalar quantities is due to the fact that keypunch machines have no lower case letters.

Only the quantities of equation (7-27) are of interest in the remaining steps. The quantities in equations (7-28) through (7-33) are known at this point; only the sums TFA, TFB, and TFC are of interest in further work.

2. Platform Equation. Performing a coordinate transformation on equation (7-3) produces

\[
\mathbf{P} = \mathbf{L}_{35} \mathbf{\omega}_{5} \mathbf{L}_{35} \mathbf{T}_{SPS} + \mathbf{L}_{37} \mathbf{T}_{PR7} - \mathbf{T}_{AP3} - \mathbf{T}_{EP3}
\]

(7-34)

\[
= \mathbf{Tw}_{p3} - m_{p} \mathbf{G}_{P3} \mathbf{A}_{B3}
\]

The known inter-component torques are \( T_{xSP5}, T_{zPR7} \). Substituting equations (7-23) through (7-25) into equation (7-34) and moving the known quantities to the right yields
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & \frac{1}{C_{x6}} & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\omega_{R7} - L_{35} \begin{bmatrix}
T_{ySP5} \\
L_{37} \\
T_{zSP5} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
T_{xAP3} \\
T_{yAP3} \\
0 \\
\end{bmatrix}
- \begin{bmatrix}
T_{xEP3} \\
T_{yEP3} \\
0 \\
\end{bmatrix} = \begin{bmatrix}
T_{FD} \\
T_{FE} \\
T_{FF} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
T_{FD} \\
T_{FE} \\
T_{FF} \\
\end{bmatrix} = \begin{bmatrix}
T_{DWBPA} \\
T_{DWBPB} \\
T_{DWBPC} \\
\end{bmatrix}
+ \begin{bmatrix}
T_{WWPA} \\
T_{WWPB} \\
T_{WWPC} \\
\end{bmatrix}
+ L_{35} \begin{bmatrix}
T_{xSP5} \\
0 \\
0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
+ L_{37} \begin{bmatrix}
T_{zPR7} \\
TWPC \\
\end{bmatrix}
\end{bmatrix}
+ \begin{bmatrix}
TWPA \\
TWPB \\
\end{bmatrix}
+ \begin{bmatrix}
TMGAPA \\
TMGAPB \\
TMGAPC \\
\end{bmatrix}
\]

\[
\text{with}
\]

\[
\begin{bmatrix}
T_{DWBPA} \\
T_{DWBPB} \\
T_{DWBPC} \\
\end{bmatrix} = -P_{PL}
\begin{bmatrix}
0 & 0 & 0 \\
\frac{S_{x6} S_{y4}}{C_{x6}} & 0 & -\frac{S_{x6}}{C_{x6}} C_{y4} \\
\frac{S_{y4}}{C_{x6}} & 0 & C_{y4} \\
\end{bmatrix}
\omega_{B3}
\]
\[
\begin{bmatrix}
TWPA \\
TWPB \\
TWPC
\end{bmatrix}
= \begin{bmatrix}
-\omega_{yP7}z_{RP8} \\
-\frac{S}{C_x}x_6(\omega_{xS5}y_{SB4} - \omega_{yS5}x_{PS6}) + \frac{1}{C_x}x_6 \omega_{xP7}z_{RP8} \\
(\omega_{xS5}y_{SB4} - \omega_{yS5}x_{PS6})
\end{bmatrix}
\]  \hspace{1cm} (7-38)

\[
\begin{bmatrix}
TWPA \\
TWPB \\
TWPC
\end{bmatrix}
= \begin{bmatrix}
m_A = m_p \\
\omega_{A1} = \omega_{P3} \\
P_{A1} = P_{P3}
\end{bmatrix}
\]  \hspace{1cm} (7-39)

\[
\begin{bmatrix}
TMGAPA \\
TMGAPB \\
TMGAPC
\end{bmatrix}
= \begin{bmatrix}
0 & -G_{zP3} & G_{yP3} \\
G_{zP3} & 0 & -G_{xP3} \\
-G_{yP3} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
A_{xB3} \\
A_{yB3} \\
A_{zB3}
\end{bmatrix}
\]  \hspace{1cm} (7-40)

and

\[P_{PL} = P_{L35} \]  \hspace{1cm} (7-41)

As before, all quantities of equation (7-36) are known, therefore only equation (7-35) need be carried forward.

One more reduction of equation (7-35) is possible at this time.

The push-rod constraint equations (6-91) and (6-113) permit the replacement of \(T_{xAP3}, T_{yAB3}, T_{xEP3}, \) and \(T_{yEP3} \) by the equations
\[
\begin{bmatrix}
T_{xAP3} \\
T_{yAP3} \\
0
\end{bmatrix} = \begin{bmatrix}
0 \\
-FXA \\
0
\end{bmatrix} T_{yPA3}
\] (7-42)

and
\[
\begin{bmatrix}
T_{xEP3} \\
T_{yEP3} \\
0
\end{bmatrix} = \begin{bmatrix}
-FYE \\
FXE \\
0
\end{bmatrix} T_{xPE3}
\] (7-43)

The scalar quantities \(FXA, FYA, FXE, FYE\) are defined by equations (6-113), (6-114), (6-118), (6-123), (6-124) and (6-125). Substituting equations (7-42) and (7-43) into equation (7-35) produces,
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1/C_{x6} & 0 \\
0 & 0 & 0
\end{bmatrix} \cdot \begin{bmatrix}
\omega_{R7} \\
-L_{35} \\
T_{ySP5}
\end{bmatrix} + \begin{bmatrix}
0 \\
+L_{37} \\
T_{zSP5}
\end{bmatrix} = \begin{bmatrix}
T_{xPR7} \\
T_{yPR7} \\
T_{xPE3}
\end{bmatrix}
\] (7-44)

\[
\begin{bmatrix}
-FYA \\
+FXA \\
0
\end{bmatrix} + \begin{bmatrix}
+FYE \\
-FXE \\
0
\end{bmatrix} = \begin{bmatrix}
TFD \\
TFE \\
TFF
\end{bmatrix}
\]
3. Rotor Equations. The rotor equations are relatively simple since there is interaction with only one other component, the platform. Performing a coordinate transformation on equation (7-4), for reasons described previously

\[
P_{\text{RL}}^{\text{R3}} L_{37} \omega_{R7} - L_{37} T_{\text{PR7}} = T_{W_{R3}} - m_{R} G_{R3} A_{3}
\]  

(7-45)

Substituting in the known forcing functions and moving the known variables to the right produces,

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\omega_{R7} \\
- L_{37} \\
0
\end{bmatrix}
= \begin{bmatrix}
T_{x\text{PR7}} \\
T_{y\text{PR7}} \\
0
\end{bmatrix} = \begin{bmatrix}
T_{\text{FG}} \\
T_{\text{FH}} \\
T_{\text{FI}}
\end{bmatrix}
\]  

(7-46)

\[
\begin{bmatrix}
T_{\text{FG}} \\
T_{\text{FH}} \\
T_{\text{FI}}
\end{bmatrix} = L_{37}
\begin{bmatrix}
0 \\
0 \\
T_{z\text{PR7}}
\end{bmatrix}
+ \begin{bmatrix}
T_{\text{WRA}} \\
T_{\text{WRB}} \\
T_{\text{WRC}}
\end{bmatrix} + \begin{bmatrix}
T_{\text{MGARA}} \\
T_{\text{MGARB}} \\
T_{\text{MGARC}}
\end{bmatrix}
\]  

(7-47)

\[
\begin{bmatrix}
T_{\text{WRA}} \\
T_{\text{WRB}} \\
T_{\text{WRC}}
\end{bmatrix} = \text{See Equation (5-29)}
\]  

(7-48)

\[
\begin{align*}
m_{A} &= m_{R} \\
\omega_{A1} &= \omega_{R3} \\
P_{A1} &= P_{R3}
\end{align*}
\]
This concludes the angular acceleration reductions for the gimbal structure components.

C. ANGULAR ACCELERATION REDUCTIONS IN THE PUSH-ROD SYSTEM

The first step will be to determine the angular accelerations of each torque motor rotor in terms of \( \omega_{R7} \) and other, known variables. This is easily done by substituting the expression for \( \dot{\phi}_{yA4} \) (equation (6-117)) into equation (6-127), and the expression for \( \dot{\phi}_{xE4} \) (equation (6-123)) into equation (6-129).

1. Yaw Push-Rod System. The substitution of equation (6-117) into equation (6-127) produces,

\[
\begin{align*}
\dot{\omega}_{xA3} &= \omega_{xB3} \dot{\omega}_{xB3} \dot{y}_{A4} \\
\dot{w}_{yA3} &= +F_{yA} \omega_{xB3} + (1-F_{xA}) \omega_{yB3} \\
&\quad -F_{yA} \omega_{xP3} + F_{xA} \omega_{yP3} \\
&\quad \sin \left( \dot{\phi}_{yA4} \right) \frac{\dot{\phi}_{yA4}^2}{\cos \left( \phi_{yA4} \right)}
\end{align*}
\]
There is a snag in equation (7-51). The variables \( \omega_{xP3} \) and \( \omega_{yP3} \) are not the desired angular acceleration variables, hence another expression must be developed. From the coordinate transformation

\[
\omega_{zA3} = \omega_{zB3} + \omega_{xB3} \dot{\theta}_{A4} \tag{7-52}
\]

It may be determined that

\[
\dot{\omega}_{xP3} = C_{y4} \omega_{xP5} + S_{y4} \omega_{yP5} \tag{7-54}
\]

and

\[
\dot{\omega}_{yP3} = \omega_{yP5} \tag{7-55}
\]

The appropriate expressions for \( \omega_{xP5} \), \( \omega_{yP5} \) and \( \omega_{xP5} \) are given by equation (7-23), (7-24) and (7-25). Performing the appropriate substitutions equation (7-51) becomes

\[
\dot{\omega}_{yA3} = -C_{y4} FYA \omega_{xR7} - S_{y4} (S_{y4} \omega_{xB3} + C_{y4} \omega_{yB3}) FYA
\]

\[
+ FXA \frac{1}{C_{x6}} \omega_{yR7} - FXA S_{x6} (S_{y4} \omega_{xB3} + C_{y4} \omega_{yB3})
\]

\[
+ FYA \omega_{xB3} + (1 - FXA) \omega_{yB3}
\]
Grouping of similar variables condenses equation (7-56) to

\[ \dot{\omega}_{yA3} = -C_y 4 FYA_{xR7} + \frac{1}{C_x 6} FXA_{yR7} \]  

\[ + [FYA - S_y 4 (C_{x6} FXA + S_{y4} FYA)]_{xB3} \]  

\[ + (1-FXA)_{yB3} \]

\[ -C_y 4 (C_{x6} FXA + S_{y4} FYA)_{zB3} \]  

\[ + (FXA - \frac{1}{C_x 6} xP7 + FYAC_{y4} yP7)_{zRP8} \]  

\[ - (FXA - \frac{S_{x6} + FYAS_{y4}}{C_x 6} (xS5 ySB4 - yS5 xPS6)) \]
\[ \frac{\sin (\phi_{yA4})}{\cos (\phi_{yA4})} \theta_{yA4}^2 \]
\[- (\omega_{xP3} \omega_{zP3} + \omega_{xB3} \omega_{zB3} - 2 \omega_{zP3} \omega_{xB3}) FXA \]
\[- (\omega_{yP3} \omega_{zP3} + \omega_{yB3} \omega_{zB3} - 2 \omega_{zP3} \omega_{yB3}) FYA \]
\[+ \left( (\omega_{xP3} - \omega_{xB3})^2 + (\omega_{yP3} - \omega_{yB3})^2 \right) FZA \]

The equations (7-50), (7-52) and (7-57) may be substituted into equation (7-5) to produce,

\[
P_{AL} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -C_{y4} FYA & \frac{1}{C_{x6}} FXA & 0 & \omega_{R7} \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

\[
\begin{bmatrix} T_{xBA3} \\ 0 \\ T_{zBA3} \end{bmatrix} - \begin{bmatrix} 0 \\ T_{yPA3} \\ 0 \end{bmatrix} = \begin{bmatrix} TFJ \\ TFK \\ TFL \end{bmatrix} \]

\[
\begin{bmatrix} TFJ \\ TFK \\ TFL \end{bmatrix} = \begin{bmatrix} TDWBAA \\ TDWBAB \\ TDWBAC \end{bmatrix} + \begin{bmatrix} TWWAA \\ TWWAB \\ TWWAC \end{bmatrix} + \begin{bmatrix} 0 \\ T_{yBA3} \\ 0 \end{bmatrix} \]
\[
\begin{bmatrix}
TWAA \\
TWAB \\
TWAC
\end{bmatrix}
+ 
\begin{bmatrix}
TMGAAA \\
TMGAAB \\
TMGAAC
\end{bmatrix}
\]

where

\[
P_{AL} = P_{I_{A3}}
\]

\[
TDWBA

= 
\begin{bmatrix}
1 & 0 & 0 & 0 \\
(FYA - S_{y4}\left(\frac{S_{\times 6FXA + S_{y4}}}{C_{\times 6}}\right)) \ (1-FXA) & (-C_{y4}\left(\frac{S_{\times 6FXA + S_{y4}}}{C_{\times 6}}\right))
\end{bmatrix}_{\omega_{B3}}
\]
\[
\begin{align*}
\text{TWWAA} & = \begin{bmatrix}
\cdot \\
-\omega_{zB3} \dot{\theta}_{yA4} \\
\frac{1}{C_x6} x6 \dot{y}_{P7} + FYAC y_4 \omega_{yP7} \dot{\theta}_{zR8} \\
-(FXA \frac{S_x6 + FYAS_y4}{C_x6}) (\omega_{xS5} \dot{\theta}_{ySB4} - \omega_{yS5} \dot{\theta}_{xPS6}) \\
+ \frac{\sin (\phi_{yA4})}{\cos (\phi_{yA4})} \dot{\phi}_{yA4} \\
-(\omega_{xP3} \omega_{wP3} - \omega_{xB3} \omega_{wB3} - 2 \omega_{zP3} \omega_{xB3}) \nonumber \\
-(\omega_{yP3} \omega_{wP3} + \omega_{yB3} \omega_{wB3} - 2 \omega_{zP3} \omega_{yB3}) \nonumber \\
+ ((\omega_{xP3} - \omega_{wB3})^2 + (\omega_{yP3} - \omega_{wP3})^2) \nonumber \\
+ \omega_{xB3} \dot{\theta}_{yA4}
\end{bmatrix} \\
(7-62)
\end{align*}
\]

\[
\text{TWWAB} = -P_{A1}
\]

\[
\begin{align*}
\text{TWWAC} & = \begin{bmatrix}
\cdot \\
-(\omega_{xP3} \omega_{wP3} + \omega_{xB3} \omega_{wB3} - 2 \omega_{zP3} \omega_{xB3}) \nonumber \\
-(\omega_{yP3} \omega_{wP3} + \omega_{yB3} \omega_{wB3} - 2 \omega_{zP3} \omega_{yB3}) \nonumber \\
+ ((\omega_{xP3} - \omega_{wB3})^2 + (\omega_{yP3} - \omega_{wP3})^2) \nonumber \\
+ \omega_{xB3} \dot{\theta}_{yA4}
\end{bmatrix}
\end{align*}
\]

\[
\text{TWAA} = \begin{bmatrix}
\cdot \\
\nonumber \\
\text{See Equation (5-29)}
\end{bmatrix}
(7-63)
\]

\[
\text{TWAB} = \begin{bmatrix}
m_A = m_A \\
\omega_{A1} = \omega_{A3} \\
P_{A1} = P_{A3}
\end{bmatrix}
\]

\[
\text{TMGAAA} = \begin{bmatrix}
0 & -G_{zA3} & G_{yA3} \\
G_{zA3} & 0 & -G_{xA3} \\
-G_{yA3} & G_{xA3} & 0
\end{bmatrix} \begin{bmatrix}
A_{xB3} \\
A_{yB3} \\
A_{zB3}
\end{bmatrix}
(7-64)
\]
This completes the reduction of the yaw push-rod equations.

2. Pitch Push-Rod System. The reduction of the pitch push-rod system is executed by exactly the same procedure as demonstrated in the yaw push-rod case. Substituting the expression for $\dot{\phi}_{x E 4}$ (equation (6-123) into equation (6-129) produces)

\[
\dot{\omega}_{x E 3} = (1-FY E)\dot{\omega}_{x B 3} + F X E \dot{\omega}_{y B 3}
+ FY E \dot{\omega}_{x P 3} - F X E \dot{\omega}_{y P 3}
\]

\[
+ \frac{\sin (\phi_{x E 4}) \dot{\phi}^2}{\cos (\phi_{x E 4}) x E 4}
\]

\[
+ (\omega_{x P 3} \omega_{y P 3} + \omega_{x B 3} \omega_{z B 3} - 2 \omega_{z P 3} \omega_{x B 3}) F X E
\]

\[
+ (\omega_{y P 3} \omega_{z P 3} + \omega_{y B 3} \omega_{x B 3} - 2 \omega_{z P 3} \omega_{y B 3}) F Y E
\]

\[
- ((\omega_{x P 3} - \omega_{x B 3})^2 + (\omega_{y P 3} - \omega_{y B 3})^2) F Z E
\]

\[
\dot{\omega}_{y E 3} = \omega_{y B 3} + \omega_{z B 3} \phi_{x E 4}
\]

\[
\dot{\omega}_{z E 3} = \omega_{z B 3} - \omega_{y B 3} \phi_{x E 4}
\]

As before, the terms $\dot{\omega}_{x P 3}$ and $\dot{\omega}_{y P 3}$ may be eliminated using equations (7-54), (7-55), (7-23), (7-24) and (7-25). Equation (7-65) thus becomes,
\begin{equation}
\omega_{XE3} = ((1 - FYE) + S_{y4} (S_{y4} FYE + S_{\frac{S x6FXE}{C x6}})) \omega_{xB3} \tag{7-68}
\end{equation}

+FXE_\omega y_{B3}

+(S_{y4} FYE + S_{\frac{S x6FXE}{C x6}}) C_{y4} \omega z_{B3}

+C_{y4} FYE_\omega x_{R7}

-FXE_\omega \frac{C_{x6} \omega}{y_{R7}}

-(FXE_\omega \frac{C_{x6} \omega}{x_{P7}} + C_{y4} FYE_\omega y_{P7}) \dot{\phi}_{z_{R8}}

+(S_{\frac{S x6FXE}{C x6}} + S_{y4} FYE) (\omega x_{S5} \dot{\phi}_{y_{B4}} - \omega y_{S5} \dot{\phi}_{x_{P5} 6})

+\frac{\sin (\phi_{x_{E4}})^2}{\cos (\phi_{x_{E4}})} x_{E4}

+(\omega x_{P5} \omega z_{P3} + \omega x_{B3} \omega z_{B3} - 2 \omega z_{P3} \omega x_{B3}) FXE

+(\omega y_{P3} \omega z_{P3} + \omega y_{B3} \omega z_{B3} - 2 \omega z_{P3} \omega y_{B3}) FYE

-((\omega x_{P5} - \omega x_{B3})^2 + (\omega y_{P3} - \omega y_{B3})^2) FZE

The equations (7-66) through (7-68) are next substituted into equation (7-6), and after separating the unknown variables, the result is
\[
\begin{bmatrix}
C_{y4} F_{YE} & \frac{F_{XE}}{C_{x6}} & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \cdot \begin{bmatrix}
T_{xPE3} \\
\omega_{R7} - 0 \\
0
\end{bmatrix}
\]

where

\[
P_{EL} = P_{1E3}
\]
\[
\begin{align*}
\begin{bmatrix}
TDWBEA \\
TDWBEB \\
TDWBE\varepsilon C
\end{bmatrix}
&= \\
\begin{bmatrix}
((1-FYE) + S y_4 (S y_4 FYE + S C x_6 FXE)) \\
(FXE) \\
(C y_4 (S y_4 FYE + S C x_6 FXE))
\end{bmatrix} \cdot -P_{EL}
\end{align*}
\]

(7-72)

\[
\begin{align*}
\begin{bmatrix}
0 \\
1 \\
0 \\
0 \\
1
\end{bmatrix}
&= \\
\begin{bmatrix}
\frac{FXE}{C x_6} x P_7 + C y_4 FYE w y p_7 \phi z R P_8 \\
+C y_4 FPE \phi x E_4 \\
\sin (\phi x E_4) \phi x E_4 \\
\cos (\phi x E_4)
\end{bmatrix} \\
&+ (S x_6 FXE + S y_4 FYE) \left( \omega x_{S 5} \phi y S B_4 - \omega y S S \phi x P S_6 \right)
\end{align*}
\]

(7-73)

\[
\begin{align*}
\begin{bmatrix}
\frac{FXE}{C x_6} x P_7 + C y_4 FYE w y p_7 \phi z R P_8 \\
+C y_4 FPE \phi x E_4 \\
\sin (\phi x E_4) \phi x E_4 \\
\cos (\phi x E_4)
\end{bmatrix} \\
&+ (S x_6 FXE + S y_4 FYE) \left( \omega x_{S 5} \phi y S B_4 - \omega y S S \phi x P S_6 \right)
\end{align*}
\]

(7-73)
**D. TOTAL SYSTEM MATRIX**

The five individual matrix equations of motion may be combined to form a single, fifteenth order matrix equation.
\[
\begin{bmatrix}
0 & P_{SL12}/C_{x6} & 0 & -1 & 0 \\
0 & P_{SL22}/C_{x6} & 0 & 0 & 0 \\
0 & P_{SL32}/C_{x6} & 0 & 0 & -1 \\
P_{P111} & P_{PL12}/C_{x6} & 0 & 0 & 0 \\
P_{PL21} & P_{PL22}/C_{x6} & 0 & 0 & 0 \\
P_{PL31} & P_{PL32}/C_{x6} & 0 & 0 & 0 \\
P_{RL11} & P_{RL12} & P_{RL13} & 0 & 0 \\
P_{RL21} & P_{RL22} & P_{RL23} & 0 & 0 \\
P_{RL31} & P_{RL32} & P_{RL33} & 0 & 0 \\
-P_{AL12} C_{y4} FYA & +P_{AL12} FXA/C_{x6} & 0 & 0 & 0 \\
-P_{AL22} C_{y4} FYA & +P_{AL22} FXA/C_{x6} & 0 & 0 & 0 \\
-P_{AL32} C_{y4} FYA & +P_{AL32} FXA/C_{x6} & 0 & 0 & 0 \\
+P_{EL11} C_{y4} FYE & -P_{EL11} FXE/C_{x6} & 0 & 0 & 0 \\
+P_{EL21} C_{y4} FYE & -P_{EL21} FXE/C_{x6} & 0 & 0 & 0 \\
+P_{EL31} C_{y4} FYE & -P_{EL31} FXE/C_{x6} & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
(7-76)
\]

\[
\begin{bmatrix}
\omega_{xR7} \\
\omega_{yR7} \\
\omega_{zR7} \\
T_{xBS3} \\
T_{zBS3}
\end{bmatrix}
\]
\[
\begin{bmatrix}
0 & S_{y4} & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & C_{y4} & 0 & 0 & 0 & 0 \\
0 & -S_{y4} & C_{y4} & +S_{y4}S_{x6} & -FYA & +FYE \\
-1 & 0 & 0 & C_{x6} & +FXA & -FXE \\
0 & -C_{y4} & -S_{y4} & C_{y4}S_{x6} & 0 & 0 \\
0 & 0 & -C_{y4} & -S_{y4}S_{x6} & 0 & 0 \\
0 & 0 & 0 & -C_{x6} & 0 & 0 \\
0 & 0 & S_{y4} & -C_{y4}S_{x6} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
$$
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
\end{bmatrix}
\begin{bmatrix}
T_{xBA3} \\
T_{zBA3} \\
T_{yBE3} \\
T_{zBE3} \\
\end{bmatrix}
= 
\begin{bmatrix}
\text{TFA} \\
\text{TFB} \\
\text{TFC} \\
\text{TFD} \\
\text{TFE} \\
\text{TFF} \\
\text{TFG} \\
\text{TFH} \\
\text{TFI} \\
\text{TFJ} \\
\text{TFK} \\
\text{TFL} \\
\text{TFM} \\
\text{TFN} \\
\text{TFO} \\
\end{bmatrix}
$$
The advantage of defining the auxiliary coordinate system becomes clear at this point; the solution of equation (7-76) is greatly simplified by the additional null elements introduced.

Equation (7-76) may be reduced to a fifth order equation in the variables $\omega_{xR7}$, $\omega_{yR7}$, $\omega_{zR7}$, $T_{xBS3}$, and $T_{zBS3}$ by the following steps:

a. discard rows 10, 12, 14, and 15
b. eliminate the variable $T_{yPA3}$ by means of row 11
c. eliminate the variable $T_{xPE3}$ by means of row 13
d. add rows 7 to 4, 8 to 5, and 9 to 6 to eliminate the variables $T_{xPR7}$ and $T_{yPR7}$ from rows 4, 5, and 6.
e. add row 1 to 4, 2 to 5, and 3 to 6 to eliminate the variables $T_{ySP5}$ and $T_{zSP5}$ from rows 4, 5, and 6.
f. eliminate the variables $T_{xPR7}$ and $T_{yPR7}$ from row 9 by means of rows 7 and 8.
g. eliminate the variable $T_{zSP5}$ from row 1 by means of row 3.

The fifth order matrix equation (7-77) may then be formed by extracting rows 1, 4, 5, 6 and 9.

$$
\begin{bmatrix}
A1WX & A1WY & A1WZ & A1TX & 0 \\
A2WX & A2WY & A2WZ & 0 & 0 \\
A3WX & A3WY & A3WZ & 0 & A3TZ \\
A4WX & A4WY & A4WZ & 0 & 0 \\
0 & A5WY & 0 & A5TX & A5TZ
\end{bmatrix}
\begin{bmatrix}
\omega_{xR7} \\
\omega_{yR7} \\
\omega_{zR7} \\
T_{xBS3} \\
T_{zBS3}
\end{bmatrix}
= \begin{bmatrix}
TA1 \\
TA2 \\
TA3 \\
TA4 \\
TA5
\end{bmatrix}
$$

(7-77)
where

\[ A_{1WX} = P_{PL11} + P_{RL11} + P_{AL22} C_{y4} FYA^2 + P_{EL11} C_{y4} FYE^2, \]  
\[ (7-78) \]

\[ A_{2WX} = P_{PL21} + P_{RL21} - P_{AL22} C_{y4} FYA FXA - P_{EL11} C_{y4} FYEFXE, \]  
\[ (7-79) \]

\[ A_{3WX} = P_{PL31} + P_{RL31}, \]  
\[ (7-80) \]

\[ A_{4WX} = P_{RL31} + (S_{y4} P_{RL11} - S_{x6} P_{RL21}) / C_{y4}, \]  
\[ (7-81) \]

\[ A_{1WY} = (P_{SL12} + P_{PL12} - P_{AL22} FXA FYA - P_{EL11} FXE FYE) / C_{x6} + P_{RL12}, \]  
\[ (7-82) \]

\[ A_{2WY} = (P_{SL22} + P_{PL22} + P_{AL22} FXA^2 + P_{EL11} FXE^2) / C_{x6} + P_{RL22}, \]  
\[ (7-83) \]

\[ A_{3WY} = (P_{SL32} + P_{PL32}) / C_{x6} + P_{RL32}, \]  
\[ (7-84) \]

\[ A_{4WY} = P_{RL32} + (S_{y4} P_{RL12} - S_{x6} P_{RL22}) / C_{y4}, \]  
\[ (7-85) \]

\[ A_{5WY} = (P_{SL12} - S_{y4} P_{SL32}) / C_{y4}, \]  
\[ (7-86) \]

\[ A_{1WZ} = P_{RL13}, \]  
\[ (7-87) \]

\[ A_{2WZ} = P_{RL23}, \]  
\[ (7-88) \]

\[ A_{3WZ} = P_{RL33}, \]  
\[ (7-89) \]

\[ A_{4WZ} = P_{RL33} + (S_{y4} P_{RL13} - S_{x6} P_{RL23}) / C_{y4}, \]  
\[ (7-90) \]
\[ A_{1TX} = -1, \quad (7-91) \]
\[ A_{5TX} = -1, \quad (7-92) \]
\[ A_{3TZ} = -1, \quad (7-93) \]
\[ A_{5TZ} = \frac{S \chi y^4}{C y^4}, \quad (7-94) \]
\[ TA_1 = TFA + TFD + TFG - FYA \cdot TFK + FYE \cdot TFM, \quad (7-95) \]
\[ TA_2 = TFB + TFE + TFH + FXA \cdot TFK - FXE \cdot TFM, \quad (7-96) \]
\[ TA_3 = TFC + TFF + TFI \quad (7-97) \]
\[ TA_4 = TFI + \left( \frac{S \chi y^4 \cdot TFG - S \cdot x6 \cdot TFH}{C \chi 6} \right) / C y^4, \quad (7-98) \]
and
\[ TA_5 = TFA - \frac{S \chi y^4 \cdot TFC}{C y^4}. \quad (7-99) \]

This fifth order matrix equation is quickly reduced to a third order equation by two steps; first, eliminate the variable \( T \) from row 5 by means of row 3, then subtract row 5 from row 1. The third order matrix obtained is
\[
\begin{bmatrix}
B_{1WX} & B_{1WY} & B_{1WZ} \\
B_{2WX} & B_{2WY} & B_{2WZ} \\
B_{3WX} & B_{3WY} & B_{3WZ}
\end{bmatrix}
\begin{bmatrix}
\omega_{xR7} \\
\omega_{yR7} \\
\omega_{zR7}
\end{bmatrix}
= \begin{bmatrix}
TB_1 \\
TB_2 \\
TB_3
\end{bmatrix}
\quad (7-100)
where

\[ \begin{align*}
B_{1W} &= A_{1W} - A_{3W}S_y/C_y, \\
B_{2W} &= A_{2W}, \\
B_{3W} &= A_{4W}, \\
B_{1W} &= A_{1W} - A_{3W}S_y/C_y - A_{5W}, \\
B_{2W} &= A_{2W}, \\
B_{3W} &= A_{4W}, \\
B_{1W} &= A_{1W} - A_{3W}S_y/C_y, \\
B_{2W} &= A_{2W}, \\
B_{3W} &= A_{4W}, \\
TB_1 &= TA_1 - TA_3S_y/C_y - TA_5, \\
TB_2 &= TA_2, \\
TB_3 &= TA_4. 
\end{align*} \]

The third order matrix equation is then easily solved by simple determinants.

Any desired forces and torques may now be obtained by back substitution or by various equations that have been presented. Usually however,
the only information desired is the motion of the various components, the force/torques being immaterial in predicting the flight vector. For the present effort it shall be assumed that the force/torque vectors are not required.

Recalling that coordinate system 7 is a temporary reference, a coordinate transformation should be performed upon the solution variables \( \omega_{x7}, \omega_{y7}, \) and \( \omega_{z7} \) to obtain \( \omega_{x1}, \omega_{y1}, \) and \( \omega_{z1}. \) For the same reason a transformation should be performed on all vector quantities which define the motion of either tracker or missile, such as \( x_{B3}, v_{B1}, \omega_{B1}, \omega_{R1} \) and \( \omega_{R1}. \)

E. COMPUTATION OF COMPONENT ANGULAR VELOCITIES

In the solution above, knowledge of the angular velocity of every component is required. In chapter VI the angular velocity constraints between components were formulated but were not clear what form they should assume. At the present point it is clear from the previous paragraphs that the known angular velocity vectors are \( \omega_{B1}, \) and \( \omega_{R1} \) and that expressions for \( \omega_{B3}, \omega_{S3}, \omega_{P3}, \omega_{R3}, \omega_{A3}, \) and \( \omega_{E3} \) must be developed, as well as expressions for \( \dot{\phi}_{ySB4}, \dot{\phi}_{xPS6}, \dot{\phi}_{zPR8}, \dot{\phi}_{yAB4}, \) and \( \dot{\phi}_{xEB4}. \)

The previous work in reducing the angular acceleration constraint equations may be put to good use here. The various angular velocity constraints are repeated below:

\[
\begin{align*}
\omega_{x3} &= \omega_{xB3} \\
\omega_{y3} &= \omega_{yB3} + \dot{\phi}_{ySB4}
\end{align*}
\]
The above equations are very similar to equations (7-7) through (7-21), hence with a few dummy variable substitutions, the solution is easily obtained from examination of equations (7-7), (7-9) and (7-22) through (7-25).
The above expressions may be operated upon by appropriate coordinate transformation matrices to obtain \( \omega_{S3} \), \( \omega_{P3} \) etc. Once all the above transformations have been performed it is a simple matter to evaluate \( \dot{\phi}_{XPS6} \), \( \dot{\phi}_{YSB4} \), and \( \dot{\phi}_{ZRP8} \) by the equations

\[
\dot{\phi}_{YSB4} = \omega_{YS3} - \omega_{YB3} \tag{7-134}
\]

\[
\dot{\phi}_{XPS6} = \omega_{XP5} - \omega_{XS5} \tag{7-135}
\]

\[
\dot{\phi}_{ZRP8} = \omega_{ZR7} - \omega_{ZP7} \tag{7-136}
\]

The push-rod torque motors require a different, but just as simple, approach. Again the work has already been performed in the process of reducing the acceleration constraints. Equations (6-112) and (6-121) established \( \dot{\phi}_{XE_4B4} \) and \( \dot{\phi}_{YAB4} \) in terms of \( \omega_{P3} \) and \( \omega_{B3} \). These expressions may simply be inserted into equations (6-126) and (6-128) to evaluate \( \omega_{A3} \) and \( \omega_{E3} \).
VIII THE COMPLETE ALGORITHM

All the concepts and equations necessary to implement the algorithm outlined in Figure (3-1), are now available. In this chapter each step will be described in detail.

A. RECEIVE SYSTEM CONFIGURATION DATA

Certain aspects of the system configuration are specified in Figure (2-1) and in the development of certain constraint equations: the push-rod equations in particular. The remaining configuration variables may be grouped into three general categories:

1. mass distribution variables.
2. push-rod socket bearing location.
3. pivot constants.

The first group consists of the following parameters,

\( m_S, m_P, m_R, m_A, m_E \), the scalar mass of each component,

\( G_{S6}, G_{P8}, G_{R10}, G_{A12}, G_{E14} \), the center of mass vectors,

and \( P_{S6}, P_{P8}, P_{R10}, P_{A12}, P_{E14} \), the product of inertia matrices.

The second group consists simply of the location of the push-rod socket bearings which are mounted on the platform, \( R_{AP8} \) and \( R_{EP8} \).

The third group consists of a spring constant, viscous drag constant, and coulomb friction constant for each pivot. The gyro spin axis differs from the other pivot axes in that a spin motor constant is substituted for
the spring constant. The specification variables are then

\[ k_{DS4}, k_{DXS6}, k_{DYBA}, k_{DXBE}, \] the spring constants

\[ \theta_{0S4}, \theta_{0XP6}, \theta_{0YA}, \theta_{0XE}, \] the equilibrium spring angles

and \( k_{MZPR}, \) and \( \omega_{ZORB}, \) the spin motor constant and synchronous spin velocity.

\[ k_{VYS4}, k_{VXS6}, k_{VZPR}, k_{VYA}, k_{VXE}, \] the viscous drag constants,

\[ k_{CYS4}, k_{CXS6}, k_{CZPR}, k_{CYA}, k_{CXE}, \] the coulomb friction component.

The versatility of the mathematical model is suggested by the fact that seventy-six scalar constants are required to specify the system configuration.

B. RECEIVE TIME BOUNDARY VALUES/FINITE DIFFERENCE MESH DIMENSION

This first step is obvious, every numerical, time-domain solution must have a definite time limit. Often, this time limit is expressed in other terms which are time dependent. For example, in air to surface missile trajectory problems the solution is from missile launch to impact. The programmer must formulate an evaluation criteria that establishes a corresponding time boundary. For the effort at hand, a simple, numerical statement of the time boundary values is assumed.
Steps 8 and 9 of the original flow chart involve the prediction of state vectors at a future point in time. The accuracy with which this prediction may be performed is dependent upon the time increment involved, among other things. The programmer must therefore choose a time increment that is a compromise between solution accuracy and computation time and consequently cost.

The required input data for the first step may consist of three numerical values:

\[ \text{to}, \text{ the initial, start time}, \]
\[ \text{tf}, \text{ the final, stop time}, \]
\[ \Delta t, \text{ the incremental, delta time}. \]

C. RECEIVE INITIAL CONDITIONS

Because of the mechanical constraints between components, the initial conditions can be expressed by the state vectors of the base and rotor. For the present algorithm the following vector quantities will be required input data,

\[ x_{B_1}, \text{ the location of the airframe (base)}, \]
\[ v_{B_1}, \text{ the linear velocity of the airframe}, \]
\[ \phi_{B_1}, \text{ the attitude of the airframe}, \]
\[ \omega_{B_1} \text{ or } \omega_{B_3}, \text{ the angular velocity of the airframe}, \]
\[ \phi_{YSB4}, \phi_{xPS6}, \phi_{zRP8}, \text{ the tracker gimbal angles}, \]
\( \omega_{R1} \) or \( \omega_{R7} \), the gyro rotor angular velocity.

D. RECEIVE (COMPUTE) BASE (AIRFRAME) FORCING FUNCTIONS

In a total missile simulation the motion of the airframe is dependent upon various aerodynamic forces, some of which are dependent upon information derived from the tracker motion. For the present algorithm the airframe will follow a known, "canned" flight path hence the airframe forcing functions are known in the form of \( \dot{\omega}_B1 \) or \( \dot{\omega}_B3 \) and \( A_B1 \) or \( A_B3 \).

E. RECEIVE (COMPUTE) TRACKER FORCING FUNCTIONS

In a total system simulation the tracker not only experiences forces due to airframe motion but also from the torque motors associated with the tracker stabilization electronics. The forces induced by the airframe motion are not known explicitly but have already been specified by the airframe motion and system configuration variables. For the present, the torques introduced by the torque motors will be known, predetermined functions.

These define the variables;

\[
\begin{align*}
\text{TFY}_{BS4}, & \text{ the yaw gimbal torque,} \\
\text{TFX}_{SP6}, & \text{ the pitch gimbal torque,} \\
\text{TFY}_{BA4}, & \text{ the yaw push-rod torque,} \\
\text{TFX}_{BE4}, & \text{ the pitch push-rod torque.}
\end{align*}
\]

F. COMPUTE DESIRED DATA

This is the most time consuming step because of the large number
of calculations involved. Figure (8-1) presents an expanded flow-chart for this procedure. Each of the referenced quantities has been evaluated in either chapter VI or VII. Rather than cross-reference each computation to the applicable equation, it is easier to locate the desired equation in the computer program listing of Appendix A. The comment cards should provide adequate information to quickly locate any particular step of the flow-chart.

G. COMPARE PRESENT TIME VALUE AGAINST TIME BOUNDARY VALUE

As mentioned previously, a simulation of a trajectory path would normally be terminated when the missile has impacted. For the present effort the time boundary values have been explicitly defined by the constants $t_0$ and $t_f$, thus this step consists of the comparison of $t$ against $t_f$. If it is greater than or equal to $t_f$ the algorithm is terminated.

In computer programs this step would also include determining output printings; in general a read-out of every iteration is not necessary.
F-1 Compute \( P_{SL}, P_{PL}, P_{RL}, P_{AL}, \) and \( P_{EL} \).

F-2 Compute Gyroscopic Torques.

F-3 Compute Known Pivot Torques, \( T_{yBS4}, T_{xSP6}, T_{zPR8}, T_{yBA4}, \) and \( T_{xBE4} \).

F-4 Compute Torques Due To Linear Acceleration of The Airframe.

F-5 Compute Torques Due To Angular Acceleration of The Airframe.

F-6 Compute Torques Due To Tracker Motion Relative To The Airframe.

F-7 Generate Fifth Order Matrix Equation.

F-8 Reduce Fifth Order Equation For \( \omega_{R7} \).

F-9 Compute Additional Data As Required.

TABLE (VIII - I) Expanded Flow Chart Of Data Computation Procedure
H. PREDICT BASE STATE VECTOR FOR NEXT ITERATION

Both the airframe state vector and the airframe forcing functions are known from earlier steps. The state vector at the next data point is easily obtained by the use of truncated Taylor's series. Refering to Step C, the following equations are required.

\[ X_{B1} = X_{B1} + V_{B1} \Delta t + \frac{1}{2} A_{B1} \Delta t^2 \] (8-1)

\[ V_{B1} = V_{B1} + A_{B1} \Delta t \] (8-2)

\[ L_{14} = L_{14} + \dot{L}_{14} \Delta t + \frac{1}{2} \ddot{L}_{14} \Delta t^2 \] (8-3)

\[ \phi_{xB} = \arcsin (-L_{1423}) \] (8-4)

\[ \phi_{yB} = \arctan \left( \frac{L_{1413}}{L_{1433}} \right) \]

\[ \phi_{zB} = \arctan \left( \frac{L_{1421}}{L_{1422}} \right) \]
\[ \omega_{B1} = \omega_{B1} + \omega_{B1} \Delta t \quad (8-5) \]

where the left-hand side of each equation is the predicted value for the next iteration, and the right side of the equation contains the current values (with the exception of equation (8-4) which uses the results of equation (8-3)). The expressions for \( \dot{L}_{14} \) and \( \ddot{L}_{14} \) are easily determined by analogy to equations (4-29) and (4-33).

I. PREDICT TRACKER STATE VECTOR FOR NEXT ITERATION

This is done in the same manner as the airframe predictions, there are no linear degrees of freedom for the tracker however, hence the linear position and linear velocity variables are already known. This leaves the equations

\[ L_{3,9} = L_{4,10} + L_{4,10} \Delta t + \frac{1}{2} L_{4,10} \Delta t^2 \quad (8-6) \]

\[
\begin{bmatrix}
\phi_{xPS6} \\
\phi_{ySB4} \\
\phi_{zRP8}
\end{bmatrix} =
\begin{bmatrix}
\text{arcsin} (-L_{4,10,23}) \\
\text{arctan} (L_{4,10,13}/L_{4,10,33}) \\
\text{arctan} (L_{4,10,21}/L_{4,10,22})
\end{bmatrix} \quad (8-7)
\]

\[ \omega_{R1} = \omega_{R1} + \omega_{R1} \Delta t \quad (8-8) \]

The matrices \( \dot{L}_{4,10} \) and \( \ddot{L}_{4,10} \) may again be determined by equations (4-29) and (4-33), but first the relative angular velocity and acceleration terms must be evaluated. These terms are easily derived to be
\[ \omega_{RB4} = \omega_{R3} - \omega_{B3} \] 

and

\[ \omega_{RB4} = \omega_{R3} - \omega_{B3} + \begin{bmatrix} 
\omega_{zB3} \omega_{yRB4} - \omega_{yB3} \omega_{zRB4} \\
-\omega_{zB3} \omega_{xRB4} + \omega_{xB3} \omega_{zRB4} \\
\omega_{yB3} \omega_{xRB4} - \omega_{xB3} \omega_{yRB4} 
\end{bmatrix} \] 

At this point the next iteration may begin.
The chances of deriving and programming a long algorithm with no errors is very small thus a period of test and evaluation exercises is indicated. Also, some experimentation is required to establish a time increment value that is an acceptable compromise between solution accuracy and solution cost.

A listing of a computer program (in FORTRAN language) based upon the preceding work is contained in Appendix A. The results to follow were obtained by the use of this program.

A. SOLUTION ACCURACY AS A FUNCTION OF THE TIME INCREMENT

The position of the spinning gyro rotor is the most rapidly changing variable. The accuracy of the computed rotor position as a function of the time increment value should therefore provide a meaningful guideline for the selection of an appropriate time increment for further testing. In order to measure the solution error a test exercise must be selected to which the correct solution is known. Also, it is desirable that test exercises be formulated in such a manner that program errors can be detected and localized as easily as possible. The first test exercise is to compute the impulse response of a disc rotor. The solution to this exercise was obtained by Euler over two hundred years ago for the case of infinitesimal perturbations; the rotor will nutate in a closed circle with a period of one half the spin period. By making the rotor center of mass coincide with the origin of coordinate system 10, and making the mass of all remaining components zero, the gimbal structure influence on the disc rotor is negligible.
Figures (9-1) through (9-7) present data generated by a unit impulse using time increments ranging from a maximum of $0.03125 \, T$ down to $0.00048828125 \, T$, which represents a minimum of one iteration for every 11.25 degrees of gyro rotation to a maximum of one iteration for every 0.17578125 degrees of gyro rotation. The graphs represent the locus of the tip of a unit vector rigidly attached to the platform, projected upon the X-Y plane. Figure (9-8) graphically presents the magnitude of the error at $t = \frac{T}{2}$, the end of the first cycle of nutation. The slope of the curve indicates that round-off errors of the Univac 1108 computations is not yet the limiting error contributor even at the smallest value of time increment chosen and that an even smaller solution error can be obtained if one is willing to pay the higher computer costs. The transition point where greatest solution accuracy is obtained was not determined because it represents an unacceptably expensive solution for the application at hand. A time increment value of $0.00390625 \, T$ was selected as a reasonable value for the remaining test exercises.

B. VERIFICATION OF INTER-COMPONENT ACCELERATION DEPENDENT CONSTRAINTS

This series of test exercises is to check for gross errors in the constraint equation reduction. For the first test the gyro rotor is assumed to be six equal point masses attached to three orthogonal shafts, similar to a child's playing jack. The three shafts are colinear with the X, Y, and Z axes of coordinate system 10. The remaining components have zero mass. Figure (9-9) presents the computed response of this gyro to a unit impulse. The exact solution for infinitesimal perturbations is a circular nutation having the same period as the gyro
$\frac{\Delta t}{T} = 0.03125$

FIGURE (9-1) ACCURACY TEST-RUN 1
\[ \frac{\Delta t}{T} = 0.015625 \]

**FIGURE (9-2) ACCURACY TEST-RUN 2**
\[
\frac{\Delta t}{T} = 0.0078125
\]

FIGURE (9-3) ACCURACY TEST-RUN 3
\[ \frac{\Delta t}{T} = 0.00390625 \]

FIGURE (9-4) ACCURACY TEST-RUN 4
\[
\frac{\Delta t}{T} = 0.001953125
\]

FIGURE (9-5) ACCURACY TEST-RUN 5
\[ \frac{\Delta t}{T} = 0.0009765625 \]

FIGURE (9-6) ACCURACY TEST-RUN 6
\[ \frac{\Delta t}{T} = 0.004882825 \]

**FIGURE (9-7) ACCURACY TEST-RUN 7**
Relative Error - $\frac{\Delta L}{L}$

Time Increment - $\frac{\Delta t}{T}$

FIGURE (9-8) RELATIVE ERROR VRS TIME INCREMENT VALUE
rotation period, and the computed response agrees quite closely.

The second test consists of transferring the rotor masses on the \( z_{10} \) axis to the platform and \( z_8 \) axis. There should be no difference in the response of this system to the response of the previous system, and Figure (9-10) demonstrates that indeed there is no difference.

For the remaining exercises the exact solutions are not known, but intuition suggest that transferring the platform masses from the \( z_8 \) axis to the \( x_8 \) and \( y_8 \) axis should not significantly change the system impulse response. Figure (9-11) verifies this assumption.

Next, the platform masses on the \( x_8 \) axis are transferred to the spider gimbal \( x_6 \) axis. Again, there is no change in response as verified by Figure (9-12).

For the fifth test, starting with the conditions of run 12, the platform masses on the \( x_8 \) axis are transferred to the yaw push-rod torque motor rotor. The response of Figure (9-13) shows no difference from that of run 12.

Finally, the remaining platform masses (on the \( y_8 \) axis) are transferred to the pitch push-rod torque motor rotor. Figure (9-14) demonstrates that for a parallelogram push-rod arrangement, this mass transfer produces little change in the impulse response.

The above six test exercises do not verify all aspects of the constraint equations but were useful primarily in correcting algebraic sign
FIGURE (9-9) CONSTRAINT TEST EXERCISE-RUN 10
FIGURE (9-10) CONSTRAINT TEST EXERCISE-RUN 11
FIGURE (9-11) CONSTRAINT TEST EXERCISE-RUN 12
FIGURE (9-12) CONSTRAINT TEST EXERCISE-RUN 13
FIGURE (9-13) CONSTRAINT TEST EXERCISE-RUN 15
FIGURE (9-14) CONSTRAINT TEST EXERCISE-RUN 16
errors and key-punch errors, of which there were several.

The previous tests assumed that the base was non-rotating. The next two tests are performed on a rolling airframe. In Test Run 22 the platform is aimed dead ahead, no perturbing torques should be generated by a rolling airframe, and none are apparent in Figure (9-15). In Test Run 23 the tracker is aimed at +45 degrees to the left. The gimbal and platform inertias perturb the gyro as shown in Figure (9-16). There are no known solutions to this test so the results of Test Run 23 cannot be checked at this time.

The last series of tests to be performed introduce the pivot forces due to springs, viscous drag and friction. Test Run 25 illustrates the effect of viscous drag upon the gyro impulse response (Figure (9-17)). Test Run 26 illustrates the effect of spring constants (Figure (9-18)); Test Run 27, the effect of bearing friction (Figure (9-19)). Tests Runs 28, 29, 30 and 31 (Figures (9-20), (9-21), (9-22) and (9-23)) illustrate the effects of various combinations of these forces upon the gyro impulse response. Again, the exactness of these solutions is not known but the curves produced are credible.

Although the above series of tests do not exercise every possible configuration, the credibility of the computed solutions greatly enhance the probability that the computer program of Appendix A is error free. Also, it is unlikely that further testing would detect them - if there are errors, since there are no cataloged solutions with which to compare the computed solution. In conclusion it is the author's opinion that the
algorithm derived and the computer program based upon that algorithm are error free, having withstood numerous checking procedures.
FIGURE (9-15) CONSTRAINT TEST EXERCISE-RUN 22
FIGURE (9-16) CONSTRAINT TEST EXERCISE-RUN 23
FIGURE (9-17) EFFECT OF VISCOUS PIVOT DRAG ON GYRO IMPULSE RESPONSE
FIGURE (9-18) EFFECT OF PIVOT SPRINGS UPON GYRO IMPULSE RESPONSE
FIGURE (9-19) EFFECT OF BEARING FRICTION UPON GYRO IMPULSE RESPONSE
Figure (9-21) Effect of combined bearing friction and pivot springs upon gyro impulse response.
FIGURE (9-22) EFFECT OF COMBINED VISCOS DRAG AND BEARING FRICTION UPON GYRO IMPULSE RESPONSE
FIGURE (9-23) EFFECT OF COMBINED VISCOUS DRAG, PIVOT SPRINGS, AND BEARING FRICTION UPON GYRO IMPULSE RESPONSE
The preceding chapters have culminated in an algorithm to compute the response of a gimballed gyro to arbitrary forcing functions. The algorithm as demonstrated in the computer program of Appendix A, produces a numerical description of the gyro position as function of time and may also provide a numerical description of all intercomponent forces, component positions, velocities, and accelerations if desired. The data requirements of the algorithm are in a form that allows this work to be easily expanded to simulate an entire missile system including airframe aerodynamics and target maneuvers. The algorithm is an explicit function of the input data, there are no converging series hence the accuracy is limited only the computation accuracy of the computing machine or economic limitations. Due partly to the explicit nature of the algorithm, and the introduction of auxiliary coordinate systems the algorithm is extremely fast, the Univac 1108 computer executes approximately 1000 iterations in twenty seconds.
XI APPENDICES
APPENDIX A

COMPUTER LISTING OF ALGORITHM
INTEGER FLAGB, FLAGC, FLAGD
REAL KDYBS4, KKVYBS4, KCYBS4, MS
REAL KDXSP6, KKVXSP6, KCXSP6, MP
REAL KMZPR8, KVZPR8, KCZPR8, MR
REAL KDYBA4, KKVYBA4, KCYBA4, MA
REAL KDXBE4, KKVXBE4, K CXBE4, ME
REAL L1311, L1312, L1313, L1321, L1322, L1323, L1331, L1332, L1333
C
REAL L1511, L1512, L1513, L1521, L1522, L1523, L1531, L1532, L1533
C
REAL L1711, L1712, L1713, L1721, L1722, L1723, L1731, L1732, L1733
REAL L1111, L1112, L1113, L1121, L1122, L1123, L1131, L1132, L1133,
C
4 L1131, L1132, L1133, L1131, L1132, L1133
C
REAL L1911, L1912, L1913, L1921, L1922, L1923, L1931, L1932, L1933
REAL L3511, L3512, L3513, L3521, L3522, L3523, L3531, L3532, L3533
REAL L3711, L3712, L3713, L3721, L3722, L3723, L3731, L3732, L3733
REAL L3911, L3912, L3913, L3921, L3922, L3923, L3931, L3932, L3933
C
P1 = 3.14159265
DEG = 57.2957795
RAD = 1.74532925E-02
C - A DEFINE SYSTEM CONFIGURATION

100 READ 10, KDYBS4, FIYOS4, KKVYBS4, KCYBS4,
1 PXXS5, PXYS5, PXZS5, PYY5, PYZS5, PZZS5, MS, GXS5, GYS5, GZS5,
2 KDXSP6, FIXOP6, KKVXSP6, KCXSP6.
3 PXXP7, PXYP7, PXZP7, PYP7, PYZP7, PZZP7, MP, GXP7, GYP7, GZP7.
4 KMZPR8, WZOR8, KVZPR8, KCZPR8.
5 PXXR9, PXYR9, PXZR9, PYYR9, PYZR9, PRR9, MR, GXR9, GYR9, GZR9.
6 KDYBA4, FIYOA4, KKVYBA4, KCYBA4.
7 PXXA11, PXYA11, PXZA11, PYYA11, PYZA11, PZZA11, MA, GXA11, GYA11, GZA11.
8 KDXBE4, FIXOE4, KKVXE4, KCXBE4.
9 PXXE13, PXYE13, PXZE13, PYYE13, PYZE13, PZZE13, ME, GXE13, GYE13, GZE13.
9 RXAP8, RYP8, RZAP8, RXEP8, RZEP8
10 FORMAT ( )
C NOTE - THE ACTUAL INPUT PARAMETERS ARE PS6,PP8,PR10, ETC,
C THE SOLUTION REQUIRES PS5,PP7,PR9, BUT SINCE L56=L78=L910=I
C THE INERTIAL SUBSCRIPTS ARE USED WHEN READING IN THE DATA
RAP8=SQRT(RXAP8*RXAP8+RYAP8*RYAP8+RZAP8*RZAP8)
REP8=SQRT(RXEP8*RXEP8+RYEP8*RYEP8+RZEP8*RZEP8)

C - B DEFINE TIME BOUNDARIES, MESH DIMENSION
C ESTABLISH RUN IDENTIFICATION AND OUTPUT CONTROLS

200 READ 20, TO,DT,TF, RUN,
    1 TOPRNT,XNPRNT,XPUNCH
20 FORMAT ( )
NRUN=RUN+0.5
NPRINT=XNPRNT+0.5
NPUNCH=XPUNCH+0.5
NCARD=0
DT2=DT/2.
DT22=DT*DT/2.

FLAGB=1
FLAGC=1
FLAGD=1
C FLAGB - CONTROLS THE ITERATIONS PER PRINT OUT
C FLAGC - CONTROLS THE PRINT OUTS PER PAGE
C FLAGD - CONTROLS THE ITERATIONS PER PUNCH OUT

C COMPUTE REQUIRED NUMBER OF ITERATIONS
ITER=(TF-TO)/DT + 1
ITER=(ITER/NPRINT)+1
ITER=ITER*NPRINT+1

C - C ESTABLISH INITIAL CONDITIONS
300 READ 30, XOBl,YOBl,ZOB1, XOBl,YOB3,ZOB3,
1  VXOBl,YYOB1,YZOB1, VXOB3,VYOB3,VZOB3,
2  FIYB,FIYB,FIZB,
3  WXBl,WYBl,WZBl, WXBl,WYB3,WZB3,
4  FIY4,FIX6,FIZ8,
5  WXRl,WYRl,WZRl, WXR7,WYR7,WZR7
30 FORMAT( )

C PROCESS INPUT DATA TO FORMAT REQUIRED BY ALGORITHM
C PROVIDE DOCUMENTATION OF INPUT DATA
C CONVERT AIRFRAME ANGLES FROM DEGREES TO RADIANS

FIXB=FIXB* RAD
FIYB=FIYB* RAD
FIZB=FIZB* RAD

C COMPUTE L13

SXB=SIN(FIXB)
CXB=COS(FIXB)
SYB=SIN(FIYB)
CYB=COS(FIYB)
SZB=SIN(FIZB)
CZB=COS(FIZB)
L1311 = CYB*CZB+SXB*SYB*SZB
L1312 = -CYB*SZB+SXB*SYB*CZB
L1313 = CXB*SYB
L1321 = CXB*SZB
L1322 = CXB*CZB
L1323 = -SXB
L1331 = -SYB*CZB+SXB*CYB*SZB
L1332 = SYB*SZB+SXB*CYB*CZB
L1333 = CXB*CYB
310 PRINT 31, T0, DT, TF, NRUN,
1 XOB1, XOB3, VXOB1, VXOB3, WXB1, WXB3,
1 L1311, L1312, L1313, FIY4, WXR1, WXR7,
2 YOB1, YOB3, VYOB1, VYOB3, WYB1, WYB3,
2 L1321, L1322, L1323, FIX6, WYR1, WYR7,
3 ZOB1, ZOB3, VZOB1, VZOB3, WZB1, WZB3,
3 L1331, L1332, L1333, FIZ8, WZR1, WZR7,
4 PX5S5, PXYS5, PXZS5, PYY5S5, PYZ5S5, MS, GXS5, GYS5, GZ5S,
5 PX5P7, PXYP7, PXZP7, PYYP7, PYZP7, MP, GXP7, GYP7, GZP7,
6 PX5R9, PXYR9, PXZR9, PYYR9, PYZR9, MR, GXR9, GYP9, GZR9,
7 PX5A11, PXYA11, PXZA11, PYYA11, PYZA11, MA, GX11, GY11, GA11,
8 PX5E13, PXYE13, PXZE13, PYYE13, PYZE13, PZZE13, ME, GXE13, GYE13, GZE13,
31 FORMAT (1H1, 5X, -TO-, 8X, -DT-, 8X, -RF-, 7X, -NRUN-/
1 2X, 3E10.4, 3X, 13//
8X, -M-, 8X, -GX-, 8X, -GY-, 8X, -GZ-/ 2X, 3E10.4, F7.4, 2F10.4, 3X, 3E10.4/
3 2X, 6E10.4, F7.4, 2F10.4, 3X, 3E10.4/
3 2X, 6E10.4, F7.4, 2F10.4, 3X, 3E10.4/
3 2X, 6E10.4, F7.4, 2F10.4, 3X, 3E10.4/
6 2X, -PLATFORM--2X, 10E10.4/
7 2X, -ROTOR --, 2X, 10E10.4/
8 2X, -YAW --, 2X, 10E10.4/
9 2X, -PITCH --, 2X, 10E10.4/) PRINT 32,
1 KDYBS4, FIYOS4, KVYBS4, KCYB4,
2 KDXSP6, FIXOP6, KVXSP6, KCXSP6,
3 KVZPR8, KCPZ8,
4 KDYBA4, FIYOA4, KVYBA4, KCYBA4,
C CONVERT ANGLES FROM DEGREES TO RADIANS

FIY4 = FIY4 * RAD
FIX6 = FIX6 * RAD
FIZ8 = FIZ8 * RAD
FIYOS4 = FIYOS4 * RAD
FIYOA4 = FIYOA4 * RAD
FIXOE4 = FIXOE4 * RAD

C CONVERT ANGULAR VELOCITIES FROM DEG/SEC TO RAD/SEC

WXBl = WXBl * RAD
WYBl = WYBl * RAD
WZBl = WZBl * RAD
WXBl = WXBl * RAD
WYB3 = WYB3 * RAD
WZB3 = WZB3 * RAD
WXR1 = WXR1 * RAD
WYR1=WYR1*RAD
WZR1=WZR1*RAD
WXR7=WXR7*RAD
WYR7=WYR7*RAD
WZR7=WZR7*RAD
WZOR8=WZOR8*RAD

C CONVERT SPRING CONSTANTS FROM (N-M)/DEG TO (N-M)/RAD

KDYBS4=KDYBS4*DEG
KDXSP6=KDXSP6*DEG
KDYBA4=KDYBA4*DEG
KDXBE4=KDXBE4*DEG

C CONVERT VISCOUS CONSTANTS FROM (N-M)/(DEG/SEC) TO (N-M)/(RAD/SEC)

C CONVERT GYRO SPIN MOTOR CONSTANT FROM (N-M)/(DEG/SEC) TO (N-M)/(RAD/SEC)

KVYBS4=KVYBS4*DEG
KVXSP6=KVXSP6*DEG
KVZPR8=KVZPR8*DEG
KVYBA4=KVYBA4*DEG
KVXBE4=KVXBE4*DEG
KMZPR8=KMZPR8*DEG

C COMPUTE L17

SY4=SIN(FIY4)
CY4=COS(FIY4)
SX6=SIN(FIX6)
CX6=COS(FIX6)
SZ8=SIN(FIZ8)
CZ8=COS(FIZ8)
L1711=L1311*CY4-L1313*SY4
L1712=L1311*SY4*SX6+L1312*CX6+L1313*CY4*SX6
L1713=L1311*SY4*CX6-L1312*SX6+L1313*CY4*CX6
L1721 = L1321 * CY4 - L1323 * SY4  
L1722 = L1321 * SY4 * SX6 + L1322 * CX6 + L1323 * CY4 * SX6  
L1723 = L1321 * SY4 * CX6 - L1322 * SX6 + L1323 * CY4 * CX6  
L1731 = L1331 * CY4 - L1333 * SY4  
L1732 = L1331 * SY4 * SX6 + L1332 * CX6 + L1333 * CY4 * SX6  
L1733 = L1331 * SY4 * CX6 - L1332 * SX6 + L1333 * CY4 * CX6  

C CONVERT INITIAL CONDITIONS TO XOB1, YOB1, WOB1, WR1

XOB1 = XOB1 + L1311 * XOB3 + L1312 * YOB3 + L1313 * ZOB3  
YOB1 = YOB1 + L1321 * XOB3 + L1322 * YOB3 + L1323 * ZOB3  
ZOB1 = ZOB1 + L1331 * XOB3 + L1332 * YOB3 + L1333 * ZOB3  

VXOB1 = VXOB1 + L1311 * VXOB3 + L1312 * VYOB3 + L1313 * VZOB3  
VYOB1 = VYOB1 + L1321 * VXOB3 + L1322 * VYOB3 + L1323 * VZOB3  
VZOB1 = VZOB1 + L1331 * VXOB3 + L1332 * VYOB3 + L1333 * VZOB3  

WXOB1 = WXOB1 + L1311 * WXOB3 + L1312 * WYOB3 + L1313 * WZOB3  
WYOB1 = WYOB1 + L1321 * WXOB3 + L1322 * WYOB3 + L1323 * WZOB3  
WZOB1 = WZOB1 + L1331 * WXOB3 + L1332 * WYOB3 + L1333 * WZOB3  

WXRI = WXRI + L1711 * WXRI + L1712 * WYR7 + L1713 * WZR7  
WYRI = WYRI + L1721 * WXRI + L1722 * WYR7 + L1723 * WZR7  
WZRI = WZRI + L1731 * WXRI + L1732 * WYR7 + L1733 * WZR7  

C ITERATION ENTRY POINT

340 DO 890 IA=1, ITER  
T = TO + (IA-1) * DT  

C COMPUTE TRIG VALUES OF AIRFRAME AND GIMBAL ANGLES

C SXB = SIN(FIXB)  
CXB = COS(FIXB)  
SYB = SIN(FLYB)
\[
\begin{align*}
\text{CYB} &= \cos(\text{FIYB}) \\
\text{SZB} &= \sin(\text{FIZB}) \\
\text{CZB} &= \cos(\text{FIZB}) \\
\text{SY4} &= \sin(\text{FIY4}) \\
\text{CY4} &= \cos(\text{FIY4}) \\
\text{SX6} &= \sin(\text{FIX6}) \\
\text{CX6} &= \cos(\text{FIX6}) \\
\text{TX6} &= \text{SX6} / \text{CX6} \\
\text{SZ8} &= \sin(\text{FIZ8}) \\
\text{CZ8} &= \cos(\text{FIZ8})
\end{align*}
\]

\[
\begin{align*}
\text{C} &\quad \text{COMPUTE L13} \\
\text{L1311} &= \text{CYB} \cdot \text{CZB} + \text{SX6} \cdot \text{SYB} \cdot \text{SZB} \\
\text{L1312} &= -\text{CYB} \cdot \text{SZB} + \text{SX6} \cdot \text{SYB} \cdot \text{CZB} \\
\text{L1313} &= \text{CX6} \cdot \text{SYB} \\
\text{L1321} &= \text{CX6} \cdot \text{SZB} \\
\text{L1322} &= \text{CX6} \cdot \text{CZB} \\
\text{L1323} &= -\text{SX6} \\
\text{L1331} &= -\text{SY4} \cdot \text{CZB} + \text{SX6} \cdot \text{CYB} \cdot \text{SZB} \\
\text{L1332} &= \text{SY4} \cdot \text{SZB} + \text{SX6} \cdot \text{CYB} \cdot \text{CZB} \\
\text{L1333} &= \text{CX6} \cdot \text{CYB}
\end{align*}
\]

\[
\begin{align*}
\text{C} &\quad \text{COMPUTE L35, L37, L39} \\
\text{L3511} &= \text{CY4} \\
\text{L3512} &= 0. \\
\text{L3513} &= \text{SY4} \\
\text{L3521} &= 0. \\
\text{L3522} &= 1. \\
\text{L3523} &= 0. \\
\text{L3531} &= -\text{SY4} \\
\text{L3532} &= 0. \\
\text{L3533} &= \text{CY4}
\end{align*}
\]
L3711 = CY4
L3712 = SY4 * SX6
L3713 = SY4 * CX6
L3721 = 0.
L3722 = CX6
L3723 = -SX6
L3731 = -SY4
L3732 = CY4 * SX6
L3733 = CY4 * CX6

L3911 = CY4 * CZ8 + SY4 * SX6 * SZ8
L3912 = -CY4 * SZ8 + SY4 * SX6 * CZ8
L3913 = SY4 * CX6
L3921 = CX6 * SZ8
L3922 = CX6 * CZ8
L3923 = -SX6
L3931 = -SY4 * CZ8 + CY4 * SX6 * SZ8
L3932 = SY4 * SZ8 + CY4 * SX6 * CZ8
L3933 = CY4 * CX6

C COMPUTE FIYA4, FIXE4, AND TRIG VALUES

RXAP4 = L3711 * RXAP8 + L3712 * RYAP8 + L3713 * RZAP8
RYAP4 = L3721 * RXAP8 + L3722 * RYAP8 + L3723 * RZAP8
RZAP4 = L3731 * RXAP8 + L3732 * RYAP8 + L3733 * RZAP8
SA4 = -RZAP4 / RAP8
FIYA4 = ASIN (SA4)
CA4 = COS (FIYA4)
FXA = RXAP4 / (RAP8 * CA4)
FYA = RYAP4 / (RAP8 * CA4)
FZA = RZAP4 / (RAP8 * CA4)

RXEP4 = L3711 * RXEP8 + L3712 * RYEP8 + L3713 * RZEP8
RYEP4 = L3721 * RXEP8 + L3722 * RYEP8 + L3723 * RZEP8
RZEP4 = L3731 * RXEP8 + L3732 * RYEP8 + L3733 * RZEP8
SE4 = RZEP4 / REP8
FIXE4=ASIN(SE4)
CE4=COS(FIXE4)
FXE=RXEP4/(REP8*CE4)
FYE=RYEP4/(REP8*CE4)
FZE=RZEP4/(REP8*CE4)

C COMPUTE L311, L313

L31111=CA4
L31112=0.
L31113=SA4
L31121=0.
L31122=1.
L31123=0.
L31131=-SA4
L31132=0.
L31133=CA4
L31311=1.
L31312=0.
L31313=0.
L31321=0.
L31322=CE4
L31323=-SE4
L31331=0.
L31332=SE4
L31333=CE4

C COMPUTE L15, L17, L19, L111, L113

C L1511=L1311*CY4-L1313*SY4
C L1512=L1312
C L1513=L1311*SY4+L1313*CY4
C L1521=L1321*CY4-L1323*SY4
C L1522=L1322
C L1523=L1321*SY4+L1323*CY4
L1531 = L1331 * CY4 - L1333 * SY4
L1532 = L1332
L1533 = L1331 * SY4 + L1333 * CY4

L1711 = L1311 * CY4 - L1313 * SY4
L1712 = L1311 * SY4 * SX6 + L1312 * CX6 + L1313 * CY4 * SX6
L1713 = L1311 * SY4 * CX6 - L1312 * SX6 + L1313 * CY4 * CX6
L1721 = L1321 * CY4 - L1323 * SY4
L1722 = L1321 * SY4 * SX6 + L1322 * CX6 + L1323 * CY4 * SX6
L1723 = L1321 * SY4 * CX6 - L1322 * SX6 + L1323 * CY4 * CX6
L1731 = L1331 * CY4 - L1333 * SY4
L1732 = L1331 * SY4 * SX6 + L1332 * CX6 + L1333 * CY4 * SX6
L1733 = L1331 * SY4 * CX6 - L1332 * SX6 + L1333 * CY4 * CX6

L1911 = L1311 * L3911 + L1312 * L3921 + L1313 * L3931
L1912 = L1311 * L3912 + L1312 * L3922 + L1313 * L3932
L1913 = L1311 * L3913 + L1312 * L3923 + L1313 * L3933
L1921 = L1321 * L3911 + L1322 * L3921 + L1323 * L3931
L1922 = L1321 * L3912 + L1322 * L3922 + L1323 * L3932
L1923 = L1321 * L3913 + L1322 * L3923 + L1323 * L3933
L1931 = L1331 * L3911 + L1332 * L3921 + L1333 * L3931
L1932 = L1331 * L3912 + L1332 * L3922 + L1333 * L3932
L1933 = L1331 * L3913 + L1332 * L3923 + L1333 * L3933

C
L11111 = L1311 * CA4 - L1313 * SA4
C
L11112 = L1312
C
L11113 = L1311 * SA4 + L1313 * CA4
C
L11121 = L1321 * CA4 - L1323 * SA4
C
L11122 = L1322
C
L11123 = L1321 * SA4 + L1323 * CA4
C
L11131 = L1331 * CA4 - L1333 * SA4
C
L11132 = L1332
C
L11133 = L1331 * SA4 + L1333 * CA4
C
L11311 = L1311
C L11312=L1312*CE4+L1313*SE4
C L11313=-L1312*SE4+L1313*CE4
C L11321=L1321
C L11322=L1322*CE4+L1323*SE4
C L11323=-L1322*SE4+L1323*CE4
C L11331=L1331
C L11332=L1332*CE4+L1333*SE4
C L11333=-L1332*SE4+L1333*CE4

C COMPUTE LOCATION OF CENTER OF MASS FOR EACH COMPONENT

GXS3=L3511*GXS5+L3512*GYS5+L3513*GZS5
GYS3=L3521*GXS5+L3522*GYS5+L3523*GZS5
GZS3=L3531*GXS5+L3532*GYS5+L3533*GZS5

GXP3=L3711*GXP7+L3712*GYP7+L3713*GZP7
GYP3=L3721*GXP7+L3722*GYP7+L3723*GZP7
GZP3=L3731*GXP7+L3732*GYP7+L3733*GZP7

GXR3=L3911*GXR9+L3912*GYR9+L3913*GZR9
GYR3=L3921*GXR9+L3922*GYR9+L3923*GZR9
GZR3=L3931*GXR9+L3932*GYR9+L3933*GZR9

GXA3=L31111*GXA11+L31112*GYA11+L31113*GZA11
GYA3=L31121*GXA11+L31122*GYA11+L31123*GZA11
GZA3=L31131*GXA11+L31132*GYA11+L31133*GZA11

GXE3=L31311*GXE13+L31312*GYE13+L31313*GZE13
GYE3=L31321*GXE13+L31322*GYE13+L31323*GZE13
GZE3=L31331*GXE13+L31332*GYE13+L31333*GZE13

C COMPUTE PRODUCTS AND MOMENTS OF INERTIA FOR EACH COMPONENT

PXXS3=L3511*(PXXS5*L3511+PXY55*L3512+PXZS5*L3513)
  +L3512*(PXY55*L3511+PYY55*L3512+PYZ55*L3513)
\begin{align*}
2 + & L_{3513} (P_{XZS5} L_{3511} + P_{YZS5} L_{3512} + P_{ZZS5} L_{3513}) \\
P_{XYS3} = & L_{3511} (P_{XXS5} L_{3521} + P_{XYS5} L_{3522} + P_{XZS5} L_{3523}) \\
& + L_{3512} (P_{XYS5} L_{3521} + P_{YYS5} L_{3522} + P_{YZS5} L_{3523}) \\
& + L_{3513} (P_{XZS5} L_{3521} + P_{YZS5} L_{3522} + P_{ZZS5} L_{3523}) \\
P_{ZS3} = & L_{3511} (P_{XXS5} L_{3531} + P_{XYS5} L_{3532} + P_{XZS5} L_{3533}) \\
& + L_{3512} (P_{XYS5} L_{3531} + P_{YYS5} L_{3532} + P_{YZS5} L_{3533}) \\
& + L_{3513} (P_{XZS5} L_{3531} + P_{YZS5} L_{3532} + P_{ZZS5} L_{3533}) \\
P_{YY3} = & L_{3521} (P_{XXS5} L_{3521} + P_{XYS5} L_{3522} + P_{XZS5} L_{3523}) \\
& + L_{3522} (P_{XYS5} L_{3521} + P_{YYS5} L_{3522} + P_{YZS5} L_{3523}) \\
& + L_{3523} (P_{XZS5} L_{3521} + P_{YZS5} L_{3522} + P_{ZZS5} L_{3523}) \\
P_{ZZS3} = & L_{3521} (P_{XXS5} L_{3531} + P_{XYS5} L_{3532} + P_{XZS5} L_{3533}) \\
& + L_{3522} (P_{XYS5} L_{3531} + P_{YYS5} L_{3532} + P_{YZS5} L_{3533}) \\
& + L_{3523} (P_{XZS5} L_{3531} + P_{YZS5} L_{3532} + P_{ZZS5} L_{3533}) \\
\end{align*}
PZZP3 = L3731*(PXXP7*L3731+PYYP7*L3732+PXYP7*L3733)  
1 + L3732*(PYYP7*L3731+PYYP7*L3732+PYYP7*L3733)  
2 + L3733*(PXYP7*L3731+PYYP7*L3732+PZZP7*L3733)  
PIXXP3 = PYYP3 + PZZP3  
PIYYP3 = PXXP3 + PZZP3  
PIZZP3 = PXXP3 + PYYP3

PXXR3 = L3911*(PXXR9*L3911+PXYR9*L3912+PXZR9*L3913)  
1 + L3912*(PXYR9*L3911+PYR9*L3912+PZZR9*L3913)  
2 + L3913*(PXZR9*L3911+PYR9*L3912+PZZR9*L3913)  
PXYR3 = L3911*(PXXR9*L3921+PXYR9*L3922+PXZR9*L3923)  
1 + L3922*(PXYR9*L3921+PYR9*L3922+PZZR9*L3923)  
2 + L3923*(PXZR9*L3921+PYR9*L3922+PZZR9*L3923)  
PXZ3R = L3911*(PXXR9*L3931+PXYR9*L3932+PXZR9*L3933)  
1 + L3932*(PXYR9*L3931+PYR9*L3932+PZZR9*L3933)  
2 + L3933*(PXZR9*L3931+PYR9*L3932+PZZR9*L3933)  
PZYR3 = L3921*(PXXR9*L3921+PXYR9*L3922+PXZR9*L3923)  
1 + L3922*(PXYR9*L3921+PYR9*L3922+PZZR9*L3923)  
2 + L3923*(PXZR9*L3921+PYR9*L3922+PZZR9*L3923)  
PZZR3 = L3931*(PXXR9*L3931+PXYR9*L3932+PXZR9*L3933)  
1 + L3932*(PXYR9*L3931+PYR9*L3932+PZZR9*L3933)  
2 + L3933*(PXZR9*L3931+PYR9*L3932+PZZR9*L3933)  
PIXXR3 = PYYR3 + PZZR3  
PIYR3 = PXXR3 + PZZR3  
PIZ3R = PXXR3 + PYYR3

PXXA3 = L31111*(PXAA11*L31111+PXAA11*L31112+PXAA11*L31113)  
1 + L31112*(PXAA11*L31111+PYA11*L31112+PYA11*L31113)  
2 + L31113*(PXAA11*L31111+PYA11*L31112+PZA11*L31113)  
PXYA3 = L31111*(PXAA11*L31121+PYA11*L31122+PXAA11*L31123)  
1 + L31122*(PXAA11*L31121+PYA11*L31122+PZA11*L31123)  
2 + L31123*(PXAA11*L31121+PYA11*L31122+PZA11*L31123)
\[
\begin{align*}
PXZA_3 &= L_{31111} (PXXA_{11} L_{31131} + PXYA_{11} L_{31132} + PXZ_{11} L_{31133}) \\
&+ L_{31112} (PXYA_{11} L_{31131} + PYYA_{11} L_{31132} + PYZ_{11} L_{31133}) \\
&+ L_{31113} (PXZA_{11} L_{31131} + PYZA_{11} L_{31132} + PZZA_{11} L_{31133}) \\
PYYA_3 &= L_{31121} (PXXA_{11} L_{31121} + PXYA_{11} L_{31122} + PXZ_{11} L_{31123}) \\
&+ L_{31122} (PXYA_{11} L_{31121} + PYYA_{11} L_{31122} + PYZ_{11} L_{31123}) \\
&+ L_{31123} (PXZA_{11} L_{31121} + PYZA_{11} L_{31122} + PZZA_{11} L_{31123}) \\
PYZA_3 &= L_{31131} (PXXA_{11} L_{31131} + PXYA_{11} L_{31132} + PXZ_{11} L_{31133}) \\
&+ L_{31132} (PXYA_{11} L_{31131} + PYYA_{11} L_{31132} + PYZ_{11} L_{31133}) \\
&+ L_{31133} (PXZA_{11} L_{31131} + PYZA_{11} L_{31132} + PZZA_{11} L_{31133}) \\
PYYA_3 &= PXA_{3} + PZZA_{3} \\
PIYYA_3 &= PXA_{3} + PZZA_{3} \\
PZZA_3 &= PXA_{3} + PYA_{3} \\
\end{align*}
\]

\[
\begin{align*}
PXZ_{3} &= L_{31311} (PXXE_{13} L_{31311} + PXYE_{13} L_{31312} + PXZE_{13} L_{31313}) \\
&+ L_{31312} (PXYE_{13} L_{31311} + PYYE_{13} L_{31312} + PYZE_{13} L_{31313}) \\
&+ L_{31313} (PXZE_{13} L_{31311} + PYZE_{13} L_{31312} + PZZE_{13} L_{31313}) \\
PYYE_{3} &= L_{31321} (PXXE_{13} L_{31321} + PXYE_{13} L_{31322} + PXZE_{13} L_{31323}) \\
&+ L_{31322} (PXYE_{13} L_{31321} + PYYE_{13} L_{31322} + PYZE_{13} L_{31323}) \\
&+ L_{31323} (PXZE_{13} L_{31321} + PYZE_{13} L_{31322} + PZZE_{13} L_{31323}) \\
PYZE_{3} &= L_{31331} (PXXE_{13} L_{31331} + PXYE_{13} L_{31332} + PXZE_{13} L_{31333}) \\
&+ L_{31332} (PXYE_{13} L_{31331} + PYYE_{13} L_{31332} + PYZE_{13} L_{31333}) \\
&+ L_{31333} (PXZE_{13} L_{31331} + PYZE_{13} L_{31332} + PZZE_{13} L_{31333}) \\
PZZE_{3} &= L_{31331} (PXXE_{13} L_{31331} + PXYE_{13} L_{31332} + PXZE_{13} L_{31333}) \\
&+ L_{31332} (PXYE_{13} L_{31331} + PYYE_{13} L_{31332} + PYZE_{13} L_{31333}) \\
&+ L_{31333} (PXZE_{13} L_{31331} + PYZE_{13} L_{31332} + PZZE_{13} L_{31333}) \\
\end{align*}
\]
PIXXE3=PYYE3+PZZE3
PIYYE3=PXEX3+PZZE3
PIZZE3=PXEX3+PYY3

C  COMPUTE ANGULAR VELOCITIES OF GIMBAL COMPONENTS

\[
\begin{align*}
WXB3 &= L_{1311} * WXBl + L_{1321} * WYBl + L_{1331} * WZB1 \\
WYB3 &= L_{1312} * WXBl + L_{1322} * WYBl + L_{1332} * WZB1 \\
WZB3 &= L_{1313} * WXBl + L_{1323} * WYBl + L_{1333} * WZB1 \\
WXR7 &= L_{1711} * WXR1 + L_{1721} * WYR1 + L_{1731} * WZR1 \\
WYR7 &= L_{1712} * WXR1 + L_{1722} * WYR1 + L_{1732} * WZR1 \\
WZR7 &= L_{1713} * WXR1 + L_{1723} * WYR1 + L_{1733} * WZR1 \\
WXS3 &= WX3 \\
WYS3 &= (WYR7 - SX6 * (SY4 * WX3 + CY4 * WZ3)) / CX6 \\
WZS3 &= WZB3 \\
WXS5 &= CY4 * WX3 - SY4 * WZ3 \\
WYS5 &= WYS3 \\
WZS5 &= SY4 * WX3 + CY4 * WZ3 \\
WXSI &= L_{1311} * WX3 + L_{1312} * WY3 + L_{1313} * WZ3 \\
WYSI &= L_{1321} * WX3 + L_{1322} * WY3 + L_{1323} * WZ3 \\
WZSI &= L_{1331} * WX3 + L_{1332} * WY3 + L_{1333} * WZ3 \\
WXP7 &= WXR7 \\
WYP7 &= WYR7 \\
WZP7 &= ((SY4 * WX3 + CY4 * WZ3) - SX6 * WYR7) / CX6 \\
WXP5 &= WXR7 \\
WYP5 &= WYS3 \\
WZP5 &= WZS3 \\
WX3 &= + CY4 * WX5 + SY4 * WZ5 \\
WYP3 &= WYP5 \\
WZP3 &= - SY4 * WX5 + CY4 * WZ5 \\
WXPI &= L_{1711} * WX7 + L_{1712} * WY7 + L_{1713} * WZ7 \\
WYP1 &= L_{1721} * WX7 + L_{1722} * WY7 + L_{1723} * WZ7 \\
WZPI &= L_{1731} * WX7 + L_{1732} * WY7 + L_{1733} * WZ7
\end{align*}
\]
WXR3=L1311*WXR1+L1321*WYR1+L1331*WZR1
WYR3=L1312*WXR1+L1322*WYR1+L1332*WZR1
WZR3=L1313*WXR1+L1323*WYR1+L1333*WZR1

C COMPUTE TIME DERIVATIVE OF GIMBAL AND PUSH-ROD ANGLES

DFIY4=WYS3-WYB3
DFIX6=WXP5-WXS5
DFIZ8=WZR7-WZP7
DFIYA4=-(WXP3-WXB3)*FYA+(WYP3-WYB3)*FXA
DFIXE4=(WXP3-WXB3)*FYE-(WYP3-WYB3)*FXE

C COMPUTE ANGULAR VELOCITIES OF PUSH-ROD TORQUE MOTOR ROTORS

WXA3=WXB3
WYA3=WYB3+DFIYA4
WZA3=WZB3

C WXA1=L1311*WXA3+L1312*WYA3+L1313*WZA3
C WYA1=L1321*WXA3+L1322*WYA3+L1323*WZA3
C WZA1=L1331*WXA3+L1332*WYA3+L1333*WZA3

WXE3=WXB3+DFIXE4
WYE3=WYB3
WZE3=WZB3

C WXE1=L1311*WXE3+L1312*WYE3+L1313*WZE3
C WYE1=L1321*WXE3+L1322*WYE3+L1323*WZE3
C WZE1=L1331*WXE3+L1332*WYE3+L1333*WZE3

C - D COMPUTE AIRFRAME FORCING FUNCTIONS

C UNIT OF ANGULAR ACCELERATION IS (DEG/SEC)/SEC

400 AXOB1=0.
AYOB1=0.
AZOB1=0.
AXOB3=0.

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AYOB3=0.
AZOB3=0.
DWXBl=0.
DWYBl=0.
DWZBl=0.
DWZB3=0.
DWYB3=0.
DWZB3=0.

C CONVERT FORCING FUNCTIONS TO AOBl, DWBl

AXOB1=AXOB1+L1311*AXOBl+L1312*AYOB3+L1313*AXOB3
AYOB1=AYOB1+L1312*AXOBl+L1322*AYOB3+L1323*AZOB3
AXOB1=AZOB1+L1331*AXOBl+L1332*AYOB3+L1333*AZOB3

DWXB1=DWXB1+L1311*DWXBl+L1312*DWYBl+L1313*DWZB3
DWYB1=DWYB1+L1321*DWXBl+L1322*DWYBl+L1323*DWZB3
DWZB1=DWZB1+L1331*DWXBl+L1332*DWYBl+L1333*DWZB3

C CONVERT ANGULAR ACCELERATION TO (RAD/SEC)/SEC

DWXB1=DWXB1*RAD
DWYB1=DWYB1*RAD
DWZB1=DWZB1*RAD

C COMPUTE AOBl, DWBl

AXOB3=L1311*AXOB1+L1321*AYOB1+L1331*AZOB1
AYOB3=L1312*AXOB1+L1322*AYOB1+L1332*AZOB1
AZOB3=L1313*AXOB1+L1323*AYOB1+L1333*AZOB1

DWXB3=L1311*DWXB1+L1321*DWYB1+L1331*DWZB1
DWYB3=L1312*DWXB1+L1322*DWYB1+L1332*DWZB1
DWZB3=L1313*DWXB1+L1323*DWYB1+L1333*DWZB1

C - E COMPUTE TRACKER FORCING FUNCTIONS
500 TFYBS4=0.
    TFXSP6=0.
    IF(T.LT.DT2) TFXSP6=1./DT
    TFYBA4=0.
    TFXBE4=0.

C - F    COMPUTE DESIRED DATA

C    COMPUTE PSL, PPL, PRL, PAL, PEL

600 PSL11=+PIXX3
    PSL12=- PXYS3
    PSL13=- PXZS3
    PSL21=- PXYS3
    PSL22=+PIYYS3
    PSL23=- PYZS3
    PSL31=- PXZS3
    PSL32=- PYZS3
    PSL33=+PIZZS3

    PPL11= PIXXP3*L3511- PXYP3*L3521- PXZP3*L3531
    PPL12= PIXXP3*L3512- PXYP3*L3522- PXZP3*L3532
    PPL13= PIXXP3*L3513- PXYP3*L3523- PXZP3*L3533
    PPL21= PXYP3*L3511+PIYYP3*L3521- PYZP3*L3531
    PPL22= PXYP3*L3512+PIYYP3*L3522- PYZP3*L3532
    PPL23= PXYP3*L3513+PIYYP3*L3523- PYZP3*L3533
    PPL31= PXZP3*L3511- PYZP3*L3521+PIZZP3*L3531
    PPL32= PXZP3*L3512- PYZP3*L3522+PIZZP3*L3532
    PPL33= PXZP3*L3513- PYZP3*L3523+PIZZP3*L3533

    PRL11= PIXXR3*L3711- PXYR3*L3721- PXZR3*L3731
    PRL12= PIXXR3*L3712- PXYR3*L3722- PXZR3*L3732
    PRL13= PIXXR3*L3713- PXYR3*L3723- PXZR3*L3733
    PRL21= PXYR3*L3711+PIYR3*L3721- PYZR3*L3731
    PRL22= PXYR3*L3712+PIYR3*L3722- PYZR3*L3732
C COMPUTE GYROSCOPIC TORQUES

\[
\begin{align*}
TWSA &= (WYS3^2 - WZS3^2)PYZS3 - WYS3WZS3(PYYS3 - PZZS3) \\
&\quad + WXS3(WYS3 - WZS3)(PXZS3 - WX3WZS3 + PXY3) \\
\text{TWSB} &= (WZS3^2 - WX3^2)PXZS3 + WXS3WX3(PXXS3 - PZZS3) \\
&\quad - WX3WYS3PYZS3 + WYS3WX3PXYS3 \\
\text{TWSC} &= (WX3^2 - WYS3^2)PXYS3 - WX3WYS3PWYS3 \\
&\quad + WX3WZS3PYZS3 - WYS3WZS3PXYS3 \\
\text{TPWA} &= (WYP3^2 - WZP3^2)PYZP3 + WYP3WZP3(PYYP3 - PZZP3) \\
&\quad - WYP3WZP3PYZP3 + WZP3WYP3PYP3 \\
\end{align*}
\]
1 +WXP3*WYP3*PXZP3-WXP3*WZP3*PYZP3
TWPB=(WZP3*WZP3-WXP3*WXP3)*PXZP3+WXP3*WZP3*(PXXP3-PZZP3)
1 -WXP3*WYP3*PYZP3+WYP3*WZP3*PXYP3
TWPC=(WXP3*WXP3-WYP3*WYP3)*PXYP3-WXP3*WYP3*(PXXP3-PYYP3)
1 +WXP3*WZP3*PYZP3-WYP3*WZP3*PXZP3

TWRA=(WYR3*WYR3-WZR3*WZR3)*PYZR3-WYR3*WZR3*(PYYR3-PZZR3)
1 +WXR3*WYR3*PXZR3-WXR3*WZR3*PXYR3
TWRB=(WZR3*WZR3-WXR3*WXR3)*PXZR3+WXR3*WZR3*(PXXR3-PZZR3)
1 -WXR3*WYR3*PYZR3+WYR3*WZR3*PXYR3
TWRC=(WXR3*WXR3-WYR3*WYR3)*PXYR3-WXR3*WYR3*(PXXR3-PYYR3)
1 +WXR3*WZR3*PYZR3-WYR3*WZR3*PXZR3

TWAA=(WYA3*WYA3-WZA3*WZA3)*PYZA3-WYA3*WZA3*(PYYA3-PZZA3)
1 +WXA3*WYA3*PXZA3-WXA3*WZA3*PXYA3
TWAB=(WZA3*WZA3-WXA3*WXA3)*PXZA3+WXA3*WZA3*(PXXA3-PZZA3)
1 -WXA3*WYA3*PYZA3+WYA3*WZA3*PXYA3
TWAC=(WXA3*WXA3-WYA3*WYA3)*PXYA3-WXA3*WYA3*(PXXA3-PYYA3)
1 +WXA3*WZA3*PYZA3-WYA3*WZA3*PXZA3

TWEA=(WYE3*WYE3-WZE3*WZE3)*PYZE3-WYE3*WZE3*(PYYE3-PZZE3)
1 +WXE3*WYE3*PXZE3-WXE3*WZE3*PXYE3
TWEB=(WZE3*WZE3-WXE3*WXE3)*PXZE3+WXE3*WZE3*(PXXE3-PZZE3)
1 -WXE3*WYE3*PYZE3+WE3*WZE3*PXYE3
TWEC=(WXE3*WXE3-WYE3*WYE3)*PXYE3-WXE3*WYE3*(PXXE3-PYYE3)
1 +WXE3*WZE3*PYZE3-WYE3*WZE3*PXZE3

C COMPUTE KNOWN PIVOT TORQUES

TYBS3=TFYBS4
1 -((KDYBS4*(FIY4-FIYOS4)+KVYBS4*DFIY4+KCYBS4*DFIY4/(ABS(DFIY4)))
TXSP6=TFXSP6
1 -(KDXSP6*(FIX6-FIX0P6)+KVXSP6*DFIX6+KCXSP6*DFIX6/(ABS(DFIX6)))
WZR8=DFIZ8
TZPR7=-(KMZPR8*(WZR8-WZOR8)+KVZPR8*WZR8+K CZPR8*WZR8/(ABS(WZR8)))
TYBA3=TYBA4-KDYBA4*(FIYA4-FIYOA4)
    -KVYBA4*DFIYA4-KCYBA4*DFIYA4/(ABS(DFIYA4))
TXBE3=TXBE4-KDXBE4*(FIXE4-FIXE4)
    -KVXBE4*DFIXE4-KCXBE4*DFIXE4/(ABS(DFIXE4))

C COMPUTE TORQUES DUE TO LINEAR ACCELERATION OF THE AIRFRAME

TMGASA=MS*(GZS3*AYOB3-GYS3*AZOB3)
TMGASB=MS*(GXS3*AZOB3-GZS3*AXOB3)
TMGASC=MS*(GYS3*AXOB3-GXS3*AYOB3)

TMGAPA=MP*(GZP3*AYOB3-GYP3*AZOB3)
TMGAPB=MP*(GXP3*AZOB3-GZP3*AXOB3)
TMGAPC=MP*(GYP3*AXOB3-GXP3*AYOB3)

TMGARA=MR*(GZR3*AYOB3-GYR3*AZOB3)
TMGARB=MR*(GXR3*AZOB3-GZR3*AXOB3)
TMGARC=MR*(GYR3*AXOB3-GXR3*AYOB3)

TMGAAA=MA*(GZA3*AYOB3-GYA3*AZOB3)
TMGAAB=MA*(GXA3*AZOB3-GZA3*AXOB3)
TMGAAC=MA*(GYA3*AXOB3-GXA3*AYOB3)

TMGAEA=ME*(GZE3*AYOB3-GYE3*AZOB3)
TMGAEB=ME*(GXE3*AZOB3-GZE3*AXOB3)
TMGAE=ME*(GYE3*AXOB3-GXE3*AYOB3)

C COMPUTE TORQUES DUE TO ANGULAR ACCELERATION OF THE AIRFRAME

XA=(SY4*DWXB3+CY4*DWZB3)
XB=XA*SX6/CX6
TDWBSA=-PSL11*DWXB3+PSL12*XB-PSL13*DWZB3
TDWBSB=-PSL12*DWXB3+PSL22*XB-PSL23*DWZB3
TDWBS=-PSL13*DWXB3+PSL32*XB-PSL33*DWZB3
XA, XB DEFINED BY SPIDER SET
TDWBPA=PPL12*XB-PPL13*XA
TDWPB=PPL22*XB-PPL23*XA
TDWPC=PPL32*XB-PPL33*XA

THERE ARE NO TDWR TERMS - THE ROTOR IS ISOLATED FROM THE BASE MOTION EXCEPT FOR GIMBAL MASS TORQUES

\[ XB = +((SY4*(FXA*SX6/CX6+SY4*FYA)-FYA)*DWXB3) \]
\[ + ((FXA-1.)*DWB3) \]
\[ + ((CY4*(FXA*SX6/CX6+SY4*FYA))*DWZB3) \]

TDWBAA=-PAL11*DWXB3+PAL12*XB-PAL13*DWZB3
TDWBAB=-PAL21*DWXB3+PAL22*XB-PAL23*DWZB3
TDWBAC=-PAL31*DWXB3+PAL32*XB-PAL33*DWZB3

\[ XA = +(((FYE-1.)-SY4*(SY4*FYE+FXE*SX6/CX6))*DWXB3) \]
\[ - FXE*DWYB3 \]
\[ - CY4*(FYE*SY4+FXE*SX6/CX6)*DWZB3 \]

TDWBEA=PEL11*XA-PEL12*DWYB3-PEL13*DWZB3
TDWBEB=PEL21*XA-PEL22*DWYB3-PEL23*DWZB3
TDWVEC=PEL31*XA-PEL32*DWYB3-PEL33*DWZB3

COMPUTE CONSTRAINT TORQUES - THESE TORQUES DO NOT EXIST PHYSICALLY
THEY ARE MATHEMATICAL QUANTITIES CREATED BY THE EXPRESSIONS FOR DWS3, DW5, DWA3, AND DWE3 IN TERMS OF DWB3 AND DWR7

\[ XA = +WZB3*DFIY4 \]
\[ XB = -(SX6*(-WXS5*DFIY4+WYS5*DFIX6)+WXP5*DFIZ8)/CX6 \]
\[ XA = +WZS5*DFIY6 \]

XWWSA=PSL11*XA+PSL12*XB+PSL13*XC
XWWSB=PSL21*XA+PSL22*XB+PSL23*XC
XWWSC=PSL31*XA+PSL32*XB+PSL33*XC

\[ XA = +WYP7*DFIZ8 \]
XB = -(SX6*(-WXS5*DFIY4+WYS5*DFIX6)+WXP5*DFIZ8)/CX6
XC = -WXS5*DFIY4+WYS5*DFIX6
TWWPA = PPL11*XA + PPL12*XB + PPL13*XC
TWWPB = PPL21*XA + PPL22*XB + PPL23*XC
TWWPC = PPL31*XA + PPL32*XB + PPL33*XC

C THERE ARE NO TWWR TERMS

XA = +WZB3*DFIYA4
XB = +(FXA*SX6/CX6+FYA*SY4)*(WXS5*DFIY4-WYS5*DFIX6)
1  -(FXA*WXP7/CX6+FYA*CY4*WYP7)*DFIZ8
2  +(WXP3*WZP3+WXB3*WZB3-2.*WZP3*WXB3)*FXA
3  +(WYP3*WZP3+WB3*WZB3-2.*WZP3*WB3)*FYA
4  -((WXP3-WXB3)**2+(WYP3-WYB3)**2)*FZA
5  -DFIYA4*DFIYA4*SA4/CA4
XC = WB3*DFIYA4
TWWAA = PAL11*XA + PAL12*XB + PAL13*XC
TWWAB = PAL21*XA + PAL22*XB + PAL23*XC
TWWAC = PAL31*XA + PAL32*XB + PAL33*XC

XA = -(FXE*TX6+FYE*SY4)*(WXS5*DFIY4-WYS5*DFIX6)
1  +(FXE*WXP7/CX6+FYE*CY4*WYP7)*DFIZ8
2  -(WXP3*WZP3+WXB3*WZB3-2.*WZP3*WXB3)*FXE
3  -(WYP3*WZP3+WB3*WZB3-2.*WZP3*WB3)*FYE
4  +( (WXP3-WXB3)**2 + (WYP3-WYB3)**2 )*FZE
5  -DFIXE4*DFIXE4*SE4/CE4
XB = -WZB3*DFIXE4
XC = +WYB3*DFIXE4
TWWEA = PEL11*XA + PEL12*XB + PEL13*XC
TWWEB = PEL21*XA + PEL22*XB + PEL23*XC
TWWEC = PEL31*XA + PEL32*XB + PEL33*XC

C COMPUTE RESULTANT OF ALL TORQUES

TFA = TDWBSA + TWWSA + TMGASA + TWSA  - L3511*TXSP5
TFB = TDWBSB + TWWSB + TMGASB + TWSB  + TYP3-L3521*TXSP5
TFC = TDBSC + TWSC + TMGASC + TWSC - L3531 * TXSP5

TFD = TDBPA + TWWPA + TMGAPA + TWPA + L3511 * TXSP5 - L3713 * TZPR7

TFE = TDBPB + TWWPB + TMGAPB + TWPB + L3521 * TXSP5 - L3723 * TZPR7

TFF = TDBPC + TWWPC + TMGAPC + TWPC + L3531 * TXSP5 - L3733 * TZPR7

TFG =

TFH =

TFI =

TMGARA + TWRA + L3713 * TZPR7

TMGARB + TWRB + L3723 * TZPR7

TMGARC + TWRC + L3733 * TZPR7

TFJ = TDBAA + TWWAA + TMGAAA + TWAA + TYBA3

TFK = TDBAB + TWWAB + TMGAAB + TWAB

TFL = TDBAC + TWWAC + TMGAAC + TWAC

TFM = TDBEA + TWEA + TMGAEA + TWEA + TXBE3

TFN = TDBEB + TWEB + TMGAEB + TWEB

TFO = TDBEC + TWEB + TMGAEC + TWEB

C COMPUTE MATRIX A

C DWXR7 COEFFICIENTS

A1WX = PPL11 + PRL11 + CY4 * (PAL22 * FYA * FYA + PEL11 * FYE * FYE)
A2WX = PPL21 + PRL21 - CY4 * (PAL22 * FXA * FYA + PEL11 * FXE * FYE)
A3WX = PPL31 + PRL31
A4WX = PRL31 + (PRL11 * SY4 - PRL21 * SX6 / CX6) / CY4
A5WX = 0.

C DWYR7 COEFFICIENTS

A1WY = (PSL12 + PPL12 - PAL22 * FXA * FYA - PEL11 * FXE * FYE) / CX6 + PRL12
A2WY = (PSL22 + PPL22 + PAL22 * FXA * FYA + PEL11 * FXE * FXE) / CX6 + PRL22
A3WY = (PSL32 + PPL32) / CX6 + PRL32
A4WY = PRL32 + (PRL12 * SY4 - PRL22 * SX6 / CX6) / CY4
A5WY = (PSL12 - PSL32 * SY4 / CY4) / CX6

C DWZR7 COEFFICIENTS
A1WZ=PRL13
A2WZ=PRL23
A3WZ=PRL33
A4WZ=PRL33+(PRL13*SY4-PRL23*SX6/CX6)/CY4
A5WZ= 0.

C TXBS3 COEFFICIENTS
A1TX=-1.
A2TX= 0.
A3TX= 0.
A4TX= 0.
A5TX=-1.

C TZBS3 COEFFICIENTS
A1TZ= 0.
A2TZ= 0.
A3TZ=-1.
A4TZ= 0.
A5TZ=SY4/CY4

C MATRIX A FORCING FUNCTIONS
TA1=TFA+TFD+TFG-FYA*TFK+FYE*TFM
TA2=TFB+TFE+TFH+FXA*TFK-FXE*TFM
TA3=TFC+TFF+TFI
TA4=TFI+(TFG*SY4-TFH*SX6/CX6)/CY4
TA5=TFA-TFC*SY4/CY4

C COMPUTE MATRIX B
B1WX=A1WX-A3WX*SY4/CY4
B2WX=A2WX
B3WX=A4WX
B1WY=A1WY-A3WY*SY4/CY4-A5WY
B2WY=A2WY
B3WY=A4WY
B1WZ=A1WZ-A3WZ*SY4/CY4
B2WZ=A2WZ
B3WZ=A4WZ
TB1=TA1-TA3*SY4/CY4-TA5
TB2=TA2
TB3=TA4

C COMPUTE DWXR, DWYR, DWZR

DETB=B1WX*(B2WY*B3WZ-B3WY*B2WZ) -B2WX*(B1WY*B3WZ-B3WY*B1WZ)
 1 +B3WX*(B1WY*B2WZ-B2WY*B1WZ) )/DETB
DWXR7=(TB1*(B2WY*B3WZ-B3WY*B2WZ) -TB2*(B1WY*B3WZ-B3WY*B1WZ)
 1 +TB3*(B1WY*B2WZ-B2WY*B1WZ) )/DETB
DWYR7=(TB1*(B3WX*B2WZ-B2WX*B3WZ) -TB2*(B3WX*B1WZ-B1WX*B3WZ)
 1 +TB3*(B2WX*B1WZ-B1WX*B2WZ) )/DETB
DWZR7=(TB1*(B2WX*B3WY-B3WX*B2WY) -TB2*(B1WX*B3WY-B3WX*B1WY)
 1 +TB3*(B1WX*B2WY-B2WX*B1WY) )/DETB

DWXR1=L1711*DWXR7+L1712*DWYR7+L1713*DWZR7
DWYR1=L1721*DWXR7+L1722*DWYR7+L1723*DWZR7
DWZR1=L1731*DWXR7+L1732*DWYR7+L1733*DWZR7

DWXR3=L3711*DWXR7+L3712*DWYR7+L3713*DWZR7
DWYR3=L3721*DWXR7+L3722*DWYR7+L3723*DWZR7
DWZR3=L3731*DWXR7+L3732*DWYR7+L3733*DWZR7

C COMPUTE ADDITIONAL DATA AS DESIRED

TXBS3=(TA1-A1WX*DWXR7-A1WY*DWYR7-A1WZ*DWZR7)/A1TX
TZBS3=(TA3-A3WX*DWXR7-A3WY*DWYR7-A3WZ*DWZR7)/A3TZ

C - G COMPARE BOUNDARY VALUES AGAINST EXISTING VALUES
C CHECK FOR PUNCH OUTPUT

700 IF(FLAGD.GT.1) GO TO 710
   FLAGD=NPUNCH+1
   NCARD=NCARD+1
   PUNCH  70,T,L1913,L1921,L1922,L1923,L1933,NCARD,NRUN
   70 FORMAT(6E11.4,6X,214)

C CHECK FOR PRINT OUTPUT

710 IF((T+0.5*DT).LT.TOPRNT) FLAGB=2
   IF(FLAGB.GT.1) GO TO 750
   FLAGB=NPRINT+1

C GENERATE PRINT OUT

C COMPUTE DESIRED READOUT PARAMETERS
C CONVERT ANGLES TO DEGREES
C CONVERT ANGULAR VELOCITIES TO DEG/SEC

   FIXBD=FIXB*DEG
   FIYBD=FIYB*DEG
   FIZBD=FIZB*DEG
   WXB1D=WXB1*DEG
   WYB1D=WYB1*DEG
   WZB1D=WZB1*DEG
   DWXB1D=DWXB1*DEG
   DWYB1D=DWYB1*DEG
   DWZB1D=DWZB1*DEG
   WXB3D=WXB3*DEG
   WYB3D=WYB3*DEG
   WZB3D=WZB3*DEG
   DWXB3D=DWXB3*DEG
   DWYB3D=DWYB3*DEG
   DWZB3D=DWZB3*DEG
IF(FLAGC.GT.1) GO TO 720
FLAGC=7
PRINT 71
PRINT 72

71 FORMAT(1H1,5X,-TIME-,7X,-FIX6-,7X,-FIY4-,7X,-FIZ8-,6X,-L1311-,  
1 6X,-L1312-,6X,-L1313-,6X,-L1911-,6X,-L1912-,6X,-L1913-/  
2 17X,-WXR7-,7X,-WYR7-,7X,-WZR7-,6X,-L1321-,  
2 6X,-L1322-,6X,-L1323-,6X,-L1921-,6X,-L1922-,6X,-L1923-/  
3 16X,-DWXR7-,6X,-DWYR7-,6X,-DWZR7-,6X,-L1331-,  
3 6X,-L1332-,6X,-L1333-,6X,-L1931-,6X,-L1932-,6X,-L1933-/  
4 17X,-WX1-,7X,-WF2-,7X,-WZR1-,7X,-A1WX-,7X,-A1WY-,  
4 7X,-A1WZ-,7X,-A1WX-,7X,-A1TX-,7X,-A1TZ-,7X,-TA1-/  
5 16X,-DWXR1-,6X,-DWYR1-,6X,-DWZR1-,7X,-A2WX-,7X,-A2WY-,  
5 7X,-A2WZ-,7X,-A2TX-,7X,-A2TZ-,7X,-TA2-)

72 FORMAT(50X,-A3WX-,7X,-A3WY-,  
1 7X,-A3WZ-,7X,-A3TX-,7X,-A3TZ-,7X,-TA3-/  
2 17X,-FX1-,7X,-FY1-,7X,-FZ1-,7X,-A4WX-,7X,-A4WY-,  
2 7X,-A4WX-,7X,-A4TX-,7X,-A4TZ-,7X,-TA4-/  
3 17X,-WX1-,7X,-WY1-,7X,-WZ1-,7X,-A5WX-,7X,-A5WY-,  

3 7X,-A5WZ-,7X,-A5TX-,7X,-A5TZ-,7X,-TA5-/
4 16X,-DWXB1-,6X,-DWYB1-,6X,-DWZB1-/
5 50X,-XOB1-,7X,-YOB1-,7X,-ZOB1-,6X,-TXBS3-,17X,-TZBS3-/
6 17X,-WXB3-,7X,-WYB3-,7X,-WZB3-,6X,-VXOB1-,6X,-VYOB1-,
7 6X,-VZOB1-,6X,-TFXSP6-,5X,-TFYBS4-/
8 16X,-DWXB3-,6X,-DWYB3-,6X,-DWZB3-,6X,-AXOB1-,6X,-AYOB1-,
9 6X,-AZOB1-,6X,-TFXBE4-,5X,-TFYBA4-/

720 FLAGC=FLAGC-1

73 FORMAT(E12.4,3E11.4,F9.4,5F11.4/,12X,3E11.4,F9.4,5F11.4/)
1 12X,3E11.4,F9.4,5F11.4/
2 12X,3E11.4,F9.4,5F11.4//
3 12X,9E11.4/
4 12X,9E11.4/
5 45X,6E11.4/
6 12X,9E11.4/
7 12X,9E11.4/
8 12X,3E11.4/ 45X,4E11.4,11X,E11.4/
9 12X,8E11.4/
10 12X,8E11.4//

750 IF(T.GE.TF) GO TO 900
FLAGB=FLAGB-1
FLAGD=FLAGD-1

C - H  PREDICT AIRFRAME STATE VECTOR FOR (T+DT)

\[
\begin{align*}
XOB1 &= XOB1 + VXOB1 \times DT + AXOB1 \times DT^2 \\
YOB1 &= YOB1 + VYOB1 \times DT + AYOB1 \times DT^2 \\
ZOB1 &= ZOB1 + VZOB1 \times DT + AZOB1 \times DT^2 \\
VXOB1 &= VXOB1 + AXOB1 \times DT \\
VYOB1 &= VYOB1 + AYOB1 \times DT \\
VZOB1 &= VZOB1 + AZOB1 \times DT
\end{align*}
\]

\[
\begin{align*}
WL1311 &= -(WYB1 \times WYB1 + WZB1 \times WZB1) \times DT^2 \\
WL1312 &= -WZB1 \times DT + (WXB1 \times WYB1 - DWZB1) \times DT^2 \\
WL1313 &= WYB1 \times DT + (WXB1 \times WZB1 + DWYB1) \times DT^2 \\
WL1321 &= WZB1 \times DT + (WXB1 \times WYB1 + DWZB1) \times DT^2 \\
WL1322 &= -(WXB1 \times WX1 + WZB1 \times WZB1) \times DT^2 \\
WL1323 &= -WXB1 \times DT + (WYB1 \times WZB1 - DWXB1) \times DT^2 \\
WL1331 &= -WYB1 \times DT + (WXB1 \times WZB1 - DWYB1) \times DT^2 \\
WL1332 &= WXB1 \times DT + (WYB1 \times WZB1 + DWXB1) \times DT^2 \\
WL1333 &= -(WXB1 \times WX1 + WYB1 \times WYB1) \times DT^2
\end{align*}
\]

\[
\begin{align*}
C &= DL1311 = WL1311 \times L1311 + WL1312 \times L1321 + WL1313 \times L1331 \\
C &= DL1312 = WL1311 \times L1312 + WL1312 \times L1322 + WL1313 \times L1332 \\
C &= DL1313 = WL1311 \times L1313 + WL1312 \times L1323 + WL1313 \times L1333 \\
C &= DL1321 = WL1321 \times L1311 + WL1322 \times L1321 + WL1323 \times L1331 \\
C &= DL1322 = WL1321 \times L1312 + WL1322 \times L1322 + WL1323 \times L1332 \\
C &= DL1323 = WL1321 \times L1313 + WL1322 \times L1323 + WL1323 \times L1333 \\
C &= DL1331 = WL1331 \times L1311 + WL1332 \times L1321 + WL1333 \times L1331 \\
C &= DL1332 = WL1331 \times L1312 + WL1332 \times L1322 + WL1333 \times L1332 \\
C &= DL1333 = WL1331 \times L1313 + WL1332 \times L1323 + WL1333 \times L1333 \\
C &= L1311 = L1311 + DL1311
\end{align*}
\]
C L1312=L1312+DL1312
L1313=L1313+DL1313
L1321=L1321+DL1321
L1322=L1322+DL1322
L1323=L1323+DL1323
C L1331=L1331+DL1331
C L1332=L1332+DL1332
L1333=L1333+DL1333

C COMPUTE AIRFRAME ATTITUDE ANGLES
C THE ANGLES FIYB, FIXB ARE RESTRICTED TO QUADRANTS 1 AND 4

IF(L1333.LE.0.) GO TO 900
IF(L1323.GE.+1.) GO TO 900
IF(L1323.LE.-1.) GO TO 900
SXB=-L1323
FIXB=ASIN(SXB)
FIYB=ATAN2(L1331,L1333)
FIZB=ATAN2(L1321,L1322)

WXBl=WXBl+DWXBl*DT
WYBl=WYBl+DWYBl*DT
WZBl=WZBl+DWZBl*DT

C - I COMPUTE TRACKER STATE VECTOR FOR (T+DT)

850 WXRB4=WXR3-WXB3
WYRB4=WXRB4=WXR3-WYB3
WZRB4=WXR3-WZB3

DWXRB4=DWXR3-DWXB3+DZB3*WYRB4-WYB3*WZRB4
DWYRB4=DWYR3-DWYB3-WZB3*WXRB4+WXB3*WZRB4
DWZRB4=DWZR3-DWZB3+WYB3*WXRB4-WXB3*WYRB4
WL3911 = -(WYRB4*WYRB4+WZRB4*WZRB4)*DT22
WL3912 = -WZRB4*DT+(WXRB4*WYRB4-DWZRB4)*DT22
WL3913 = WYRB4*DT+(WXRB4*WZRB4+DWYRB4)*DT22
WL3921 = WZRB4*DT+(WXRB4*WYRB4+DZWRB4)*DT22
WL3922 = -(WXRB4*WXRB4+WZRB4*WZRB4)*DT22
WL3923 = -WXRB4*DT+(WYRB4*WZRB4-DWXRB4)*DT22
WL3931 = -WYRB4*DT+(WXRB4*WZRB4-DWYRB4)*DT22
WL3932 = WXRB4*DT+(WYRB4*WZRB4+DZWRB4)*DT22
WL3933 = -(WXRB4*WXRB4+WYRB4*WYRB4)*DT22

DL3911 = WL3911*L3911+WL3912*L3921+WL3913*L3931
DL3912 = WL3911*L3912+WL3912*L3922+WL3913*L3932
DL3913 = WL3911*L3913+WL3912*L3923+WL3913*L3933
DL3921 = WL3921*L3911+WL3922*L3921+WL3923*L3931
DL3922 = WL3921*L3912+WL3922*L3922+WL3923*L3932
DL3923 = WL3921*L3913+WL3922*L3923+WL3923*L3933
DL3931 = WL3931*L3911+WL3932*L3921+WL3933*L3931
DL3932 = WL3931*L3912+WL3932*L3922+WL3933*L3932
DL3933 = WL3931*L3913+WL3932*L3923+WL3933*L3933

L3911 = L3911+DL3911
L3912 = L3912+DL3912
L3913 = L3913+DL3913
L3921 = L3921+DL3921
L3922 = L3922+DL3922
L3923 = L3923+DL3923
L3931 = L3931+DL3931
L3932 = L3932+DL3932
L3933 = L3933+DL3933

C COMPUTE TRACKER GIMBAL ANGLES
C THE ANGLES FIY4, FIX6 ARE RESTRICTED TO QUADRANTS 1 AND 4

IF(L3933.LE. 0.) GO TO 900
IF(L3923.GE.+1.) GO TO 900
IF(L3923.LE.-1.) GO TO 900
SX6=-L3923
FIY4=ATAN2(L3913,L3933)
FIZ8=ATAN2(L3921,L3922)

WXR1=WXR1+DWXR1*DT
WYR1=WYR1+DWYR1*DT
WZR1=WZR1+DWZR1*DT

890 CONTINUE

900 PRINT 91
91 FORMAT(1H1)

STOP
END
XII Bibliography


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XIII VITA

Floyd Stanley Hall was born on December 13, 1941, in Nevada, Missouri. He completed his primary and secondary education in Kansas City, Missouri, in June of 1959. He received a Bachelor of Science degree in Electrical Engineering from the University of Missouri School of Mines and Metallurgy, in Rolla, Missouri in July 1962.

He completed the academic requirements of the Master of Science degree in Electrical Engineering at the University of Missouri in Rolla in May 1963. He left campus for employment by the Naval Ordnance Test Station, China Lake, California returning in December 1965 to submit the required thesis. He received the Master of Science degree in Electrical Engineering in May 1966.

Although formally entering the graduate school of the University of Missouri at Rolla in September 1966, he did not begin active participation until June 1967 when he returned to campus on an Naval Weapons Center Fellowship. He is presently employed as Head, Infrared Weapons Systems Branch, Infrared Systems Division of the Naval Weapons Center, China Lake, California.