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Dynamic behavior of eccentrically stiffened plates

Charles Stuart Ferrell

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DYNAMIC BEHAVIOR OF ECCENTRICALLY STIFFENED PLATES

BY

CHARLES STUART FERRELL

A THESIS

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ABSTRACT

DYNAMIC BEHAVIOR OF ECCENTRICALLY STIFFENED PLATES

By Charles Stuart Ferrell

The dynamic behavior of eccentrically stiffened rectangular plates is studied by the finite element method. The addition of stiffeners to a plate structure is a common method of increasing the rigidity of the structure with a minimum of weight increase. However, this stiffened plate configuration does not easily lend itself to analysis. This study is an approach to finding a rational method by which the dynamic behavior of such a stiffened plate structure may be determined.

The governing differential equations for the finite element model are derived and the stiffness and mass matrices for the plate and stiffener elements are presented in terms of the element properties. A comparison of the natural frequencies and mode shapes of a series of finite element models is made with those obtained from tests conducted on square, aluminum, stiffened plates. Also included is the computer program from which the frequencies of the finite element model were determined, along with the input and output data corresponding to one of the test problems.

Some difficulties were encountered in matching the finite element model boundary conditions with the test plate supports. Where boundary conditions were matched successfully, the finite element model and test plate values of the lower natural frequencies were in close agreement.
Other general trends of stiffened plate behavior which became evident from the test plate data were also predicted by the finite element computer program. Results were also obtained which compare favorably with those reported elsewhere in the literature.
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DYNAMIC BEHAVIOR OF ECCENTRICALLY STIFFENED PLATES

By Charles Stuart Ferrell

INTRODUCTION

Stiffened plate construction is widely used in modern engineering structures. Examples may be found in highway and railroad bridges, flooring systems, ship hulls and decks, and various aircraft components. A problem arises, however, in analysing the behavior of such a structure because classical theories are not applicable when the plate contains an abrupt change in cross section.

The most common type of approach for determining the behavior of stiffened plates is by the adoption of orthotropic plate theory. Huber (14) presented a method to describe the static behavior of plates which are stiffened symmetrically with respect to the middle surface of the plate. In this method, the orthogonally stiffened plate is replaced by an equivalent orthotropic plate of constant thickness. Hoppmann and Huffington (9,10,15) have done analytical and experimental work in determining the stiffness properties for both symmetrically and eccentrically stiffened plates which may be used in Huber's equation.

1Graduate Student, Dept. of Civ. Engrg., Univ. of Missouri-Rolla, Rolla, Mo.

2Numerals in parentheses refer to corresponding items in Appendix I.

References.
A Huber type of solution for free vibration of orthogonally stiffened, rectangular, simply supported plates has been formulated by Hoppmann, Huffington and Magness (12,13).

In 1947, Pflüger (21) determined a more exact orthotropic plate theory by deriving a set of differential equations which include the effects of stiffener eccentricity and which are not dependent upon fictitious anisotropic properties. This theory was applied by Clifton, Chang and Au (3) for determining the static behavior of orthotropic plate bridges. Feezer (6) later extended the use of the Pflüger type of solution to the dynamic behavior of eccentrically stiffened orthotropic plates. The main drawback to an orthotropic plate theory approach to stiffened plates is the stipulation that the stiffener spacing be regular and that it must be small in comparison to the span length.

The dynamic behavior of stiffened plates has been investigated by other methods as well. Kirk (16,17) has approached the problem by applying the Rayleigh method to rectangular plates with stiffeners parallel to one edge. An analysis of eccentrically stiffened plates by the simultaneous solution of plate bending equations, plane stress equations, beam equations, and displacement compatibility equations was advanced by Long (18) for a rectangular plate having two simply supported edges normal to the stiffener direction.

The finite element method is a rather recent development for the idealization of an elastic continua by a system with a finite number of degrees of freedom. Although not restricted to these cases, it is ideally suited for handling structures with difficult boundary conditions and elastic discontinuities as well as geometric discontinuities such
as the abrupt change in cross section which occurs in a stiffened plate. Books by Zienkiewicz (26) and Przemieniecki (22) provide an excellent insight into the development and application of the finite element method. The finite element method has been used by Gustafson and Wright (8) in the static analysis of skewed composite girder bridges and by Damle (5) in the dynamic analysis of stiffened plates.

This paper is a presentation of a particular finite element formulation for the solution of the dynamic behavior of a stiffened plate. A comparison of this solution with corresponding test data for various stiffener sizes and arrangements is also discussed.
The structure considered in this study is a thin rectangular plate of elastic, isotropic material reinforced in one direction by monolithic stiffeners. In this particular case, the stiffeners are rectangular in cross section, equally spaced, and of the same material as the plate. However, for the general finite element solution of the behavior of stiffened plates, none of these limitations are necessary.

The following assumptions as to the behavior of the plate are based on the classical plate theory for thin plates:

1. The deflection of the plate in the z-direction, \( w \), is small in comparison to the plate thickness.

2. The normal stresses in the z-direction can be disregarded, i.e., \( \sigma_z = 0 \).

3. Normals to the reference surface before deformation remain straight and normal after deformation. This is equivalent to the assumption that \( \varepsilon_{xz} = \varepsilon_{yz} = 0 \) (23). It has been shown (20) that in general for the lower frequencies (\( \lambda/t > 5 \), where \( \lambda \) is the wave length and \( t \) is the plate thickness), the shearing strains \( \varepsilon_{xz} \) and \( \varepsilon_{yz} \) have little effect on the plate behavior.

4. The lateral deflection of the middle surface of the plate, \( w \), is geometrically a function of \( x \) and \( y \) only.

Similarly, the stiffeners are assumed to behave as classical shallow beams. The shearing strain has also been shown to have little effect upon the lower frequencies of simply supported beams (24).

Denoting \( u \) and \( v \) as the horizontal displacements of the middle
surface of the plate in the x- and y-directions, respectively, then the horizontal displacements $U$ and $V$ at a distance $z$ from the middle surface are

$$ U = u - z \left( \frac{\partial w}{\partial x} \right) \quad (1a) $$

$$ V = v - z \left( \frac{\partial w}{\partial y} \right) \quad (1b) $$

Differentiation of Eqs. 1 yield strain components in the plate of

$$ \varepsilon_x = \frac{\partial U}{\partial x} = \frac{\partial u}{\partial x} - z \frac{2}{\partial x^2} \left( \frac{\partial w}{\partial x} \right) \quad (2a) $$

$$ \varepsilon_y = \frac{\partial V}{\partial y} = \frac{\partial v}{\partial y} - z \frac{2}{\partial y^2} \left( \frac{\partial w}{\partial y} \right) \quad (2b) $$

$$ \varepsilon_{xy} = \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z \frac{2}{\partial x \partial y} \left( \frac{\partial w}{\partial x} \right) \quad (2c) $$

The strains in the plate are related to the stresses according to Hooke's Law, namely:

$$ \varepsilon_x = \frac{1}{E}(\sigma_x - \nu \sigma_y) \quad (3a) $$

$$ \varepsilon_y = \frac{1}{E}(-\nu \sigma_x + \sigma_y) \quad (3b) $$

$$ \varepsilon_{xy} = \frac{2(1+\nu)}{E}(\sigma_{xy}) \quad (3c) $$

where $E$ is the modulus of elasticity and $\nu$ is Poisson's ratio for the plate element material.
Equations 2 and 3 may be combined to yield stress-displacement relations for the plate of

\[
\begin{align*}
\sigma_x &= \frac{E}{(1-\nu^2)} \left[ (\frac{\partial^2 u}{\partial x^2} - z \frac{\partial^2 w}{\partial x^2}) + \nu \left( \frac{\partial^2 v}{\partial y^2} - z \frac{\partial^2 w}{\partial y^2} \right) \right] \\
\sigma_y &= \frac{E}{(1-\nu^2)} \left[ \nu (\frac{\partial^2 u}{\partial x^2} - z \frac{\partial^2 w}{\partial x^2}) + (\frac{\partial^2 v}{\partial y^2} - z \frac{\partial^2 w}{\partial y^2}) \right] \\
\sigma_{xy} &= \frac{E}{2(1+\nu)} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z \frac{\partial^2 w}{\partial x \partial y} \right)
\end{align*}
\] (4a) (4b) (4c)

One of the results in using the finite element method for analysing a continuous system is that the continuous system is replaced with one which has a finite number of degrees of freedom. The equations of motion for the free vibration of an undamped multi-degree of freedom system are given in matrix form by

\[
\ddot{X} + KX = 0
\] (5)

in which M is the mass matrix of the system, K is the stiffness matrix of the system, X is the deflection vector of the system and the notation \( \dot{X} \) refers to the second derivative of X with respect to time. Assuming a harmonic solution of the form \( X = \{a\} \exp(j\omega t) \), where \( \{a\} \) is a vector of vibration amplitudes, \( j = (-1)^{1/2} \), \( \omega \) is the circular frequency of the system, and \( t \) is the time variable, Eq. 5 reduces to

\[
(-\omega^2 M + K) \{a\} = 0
\] (6)

In order that Eq. 6 have a non-trivial solution, the coefficient matrix of \( \{a\} \) must be singular. This will be true only if the determinant
\[-\omega^2 M + K \] = 0 \quad \quad (7)

Solving for values of \( \omega^2 \) which satisfy Eq. 7 will yield the natural circular frequencies of the system. The mode shape corresponding to each natural frequency may then be found by substituting the value of \( \omega \) for which the mode shape is desired into Eq. 6 and solving for the modal vector \( \{a\} \).
THE FINITE ELEMENT MODEL

Equation 5 describes the dynamic behavior only of a system which has a finite number of degrees of freedom. A continuous system, such as the stiffened plate, does not fall into this category. However, by the use of the finite element method, the continuous plate system is divided into a finite number of discrete plate and stiffener elements as shown in Fig. 1. The elements are considered to be connected by the nodal points which lie in the middle surface of the plate, where continuity of displacements and equilibrium of forces are established. There are five degrees of freedom at each node: $u, v, w,$ and $\theta_x$ and $\theta_y$, the displacements in the $x$-, $y$-, and $z$-directions and the slopes of the structure in the $x$- and $y$-directions respectively. Nodal deflections and the corresponding nodal forces for a plate and a stiffener element, along with the element dimensions, are shown in Fig. 2. For the sake of simplicity, deflections and forces are shown at one node only. Naturally, the same types of deflections and forces apply for each node of the element. Thus, for a plate which is divided into $m$ elements in the $x$-direction and $n$ elements in the $y$-direction there are $5(m+1)(n+1)$ degrees of freedom in the unsupported plate. Displacement functions which are assumed to represent the deformation behavior of the elements are chosen for $u, v, w$, and, for the stiffener element, $\theta_y$. For the plate element, $\theta_y = \frac{\partial w}{\partial y}$, while for both the plate and stiffener elements, $\theta_x = \frac{\partial w}{\partial x}$. The displacement function establishes a relationship between the element deflections and the displacement of the nodes which define the elements. The element deflections may then be found in terms of the nodal displacements. The continuity of the elements at the
Figure 1. Finite Element Model of Continuous Plate Structure
a) PLATE ELEMENT

b) STIFFENER ELEMENT

Figure 2. Plate and Stiffener Elements
nodes then allows the deformations of the entire stiffened plate structure to be described by the nodal displacements. Hence, the original continuous structure has been replaced by a model with a finite number of degrees of freedom. The mass matrix, $M$, and the stiffness matrix, $K$, for the complete structure may be constructed from the individual element mass and stiffness matrices. The equation of motion of the finite element model may now be written in the form of Eq. 5, and the natural frequencies of the system found by use of Eq. 7.

Depending upon the degree to which the behavior of the finite element model approximates the original continuum, the lower natural frequencies of the stiffened plate structure will be equal to those found for the model. Theoretically, a finer mesh of elements results in an assemblage that more closely represents the continuum. A discussion of the requirements for convergence of the behavior of the finite element model to that of the continuum may be found in several references (1,4,19,22,26).

The functions chosen to represent the displacement vector of the reference surface of the plate element are those which have been used by several authors (4,5,8,26). The functions are:

$$w = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 + a_7x^3 + a_8x^2y +$$
$$a_9xy^2 + a_{10}y^3 + a_{11}x^3y + a_{12}xy^3 \quad (8a1)$$

$$u = \beta_1 + \beta_2x + \beta_3xy + \beta_4y \quad (8b1)$$

$$v = \gamma_1 + \gamma_2x + \gamma_3xy + \gamma_4y \quad (8c1)$$
or

\[ w = X \alpha \]  
(8a2)

\[ u = Y \beta \]  
(8b2)

\[ v = Y \gamma \]  
(8c2)

The displacement vector, \( \mathbf{\delta} \), of a point on the reference surface may now be defined in matrix terms by

\[
\mathbf{\delta} = \begin{bmatrix} w \\ \theta_x \\ \theta_y \\ u \\ v \end{bmatrix} = \begin{bmatrix} X & 0 & 0 \\ \frac{\partial X}{\partial y} & 0 & 0 \\ \frac{\partial X}{\partial x} & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & Y \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = B \alpha^* 
\]  
(9a)

By substituting the \( x \) and \( y \) coordinates of the nodal points \( i, j, k, \) and \( l \) of the element, the nodal displacement vector, \( \mathbf{\delta}^e \), becomes

\[
\mathbf{\delta}^e = \begin{bmatrix} \delta_i \\ \delta_j \\ \delta_k \\ \delta_l \end{bmatrix} = \begin{bmatrix} B_i \\ B_j \\ B_k \\ B_l \end{bmatrix} \alpha^* = A \alpha^* 
\]  
(10a)
or

\[ \alpha^* = A^{-1}\delta^e \]  

(10b)

Equation (9a) now becomes

\[ \delta = BA^{-1}\delta^e \]  

(9b)

Thus the displacement vector, \( \delta \), for a given point on the reference surface may be found as a function of the coordinates of the point, \( B \), the coordinates of the nodal points, \( A \), and the nodal deflections of the element, \( \delta^e \).

Equations 2 may be rewritten in matrix form as

\[ \varepsilon = [\varepsilon_x, \varepsilon_y, \varepsilon_{xy}]^T = CA^{-1}\delta^e \]  

(11)

where the matrix \( C \) relates the state of strain to the state of deformation. Equations 4 now become

\[ \sigma = [\sigma_x, \sigma_y, \sigma_{xy}]^T = DCA^{-1}\delta^e \]  

(12)

where \( D \) is the matrix of constants of elasticity relating stress to strain.

By applying a virtual deflection vector at the element nodes \( \delta^e \), the work done by the nodal force vector, \( F^e \), composed of the forces corresponding to the nodal deflections, can be equated to the internal strain energy of the plate element due to those nodal deflections. This yields
The stiffness matrix for the plate element, then is

\[ K = (A^{-1})^T \int_V C^T D C dV A^{-1} \]

The mass matrix for the plate element is found by the method of consistent masses. This method consists of "lumping" equivalent masses at the nodal points, such that the work done by the d'Alembert nodal forces as a virtual deflection vector is applied at the nodes is equal to the work done by the actual distributed d'Alembert force on the plate element due to that virtual nodal deflection vector. Denoting the plate element mass matrix as \( M \), this yields

\[ (\delta^T e)^T M \delta e = \rho (\delta^T e)^T (A^{-1})^T \int_V B_m^T B_m dV A^{-1} \delta e \]

where \( \rho \) is the mass per unit volume of the element material and \( B_m \) is the matrix relating the transverse and in-plane displacements of the mass centroidal surface, \( \delta_m \), to the nodal deflections by

\[ \delta_m = [w, u, v]^T = B_m A^{-1} \delta e \]
The mass matrix for the plate element is then defined by

\[ M = \rho (A^{-1})^T \int V B^T R_m dV A^{-1} \]  \hspace{1cm} (18)

The mass matrix for the plate element will not include the effects of rotatory inertia. Rotatory inertia has been shown to have even less effect upon the lower natural frequencies of beams and plates than does the shearing strain (20,24).

The derivation for the stiffness and mass matrices for the stiffener element is quite similar to that for the plate element. Considering a stiffener which lies along the plate element boundary \( x = 0 \), continuity between the plate and stiffener elements dictates that

\[ \bar{w} = \alpha_1 + \alpha_3 y + \alpha_6 y^2 + \alpha_{10} y^3 \]  \hspace{1cm} (19a)

\[ \bar{u} = \beta_1 + \beta_4 y \]  \hspace{1cm} (19b)

\[ \bar{v} = \gamma_1 + \gamma_4 y \]  \hspace{1cm} (19c)

where the bar over a term indicates a value pertaining to the stiffener element.

As \( \bar{\theta}_y \) is now the rotation of the stiffener about the \( y \)-axis instead of the slope in the \( x \)-direction, it is necessary to also assume a displacement vector

\[ \bar{\theta}_y = \zeta_1 + \zeta_2 y \]  \hspace{1cm} (19d)
Considering the stiffener to exhibit strain only in the longitudinal direction due to the bending and in-plane forces, the strain vector due to these forces becomes

\[ \bar{\varepsilon} = \varepsilon_y = C_1A^{-1}\delta e \]  

(20a)

The flexural and axial stiffness matrix for the stiffener element may now be found by substituting the appropriate values into Eq. 15. The torsional strain of the stiffener element is equal to the angle of twist, \( \bar{\phi}_y \), where

\[ \bar{\phi}_y = \frac{\partial \delta y}{\partial y} = C_2A^{-1}\delta e \]  

(20b)

and the torque at a section is

\[ \bar{T}_y = GJ\bar{\phi}_y = GJC_2A^{-1}\delta e \]  

(21)

where \( G \) is the shearing modulus of elasticity and \( J \) is the St. Venant's modified polar moment of inertia. The torsional stiffness matrix can now be found by again applying a virtual deflection at the node points and equating the virtual work terms. The two stiffness matrices are combined to obtain the total stiffness matrix for the stiffener element.

The stiffener mass matrix can be found using Eq. 18. The only modification is in \( \bar{\delta}_m \). The mass of the stiffener element must be considered to be distributed throughout the stiffener instead of being considered to lie along the centroid as was the case of the plate element. This distribution of mass must be reflected in the determination of \( \bar{\delta}_m \). The mass matrix for the stiffener element does not
completely account for rotatory inertia. It does account for inertia forces due to the mass centroid of the stiffener not lying in the reference surface. A more detailed description of the derivation of the finite element matrices, along with a discussion of the validity of the method, may be found in reference (7). Also included in reference (7) is a step-by-step evaluation of those matrices needed for the dynamic analysis of stiffened plates.

Once the stiffness and mass matrices for the individual plate and stiffener elements have been determined, they must be combined to obtain the stiffness and mass matrix for the stiffened plate structure. Each element, \((k_s)_{ij}\), of the stiffness matrix of the structure is equal to the nodal force \(i\) resulting from a unit nodal displacement \(j\). Similarly, the elements of the mass matrix of the structure are equal to the nodal forces caused by unit accelerations of the nodes. After assembling the stiffness and mass matrices of the structure, the natural frequencies of the finite element model may be found from Eq. 7.

The principal objective of this work was the derivation of a method of determining the dynamic behavior of stiffened plates. A computer program was written which can construct the stiffened plate mass and stiffness matrices from input data concerning the mechanical and physical properties and specific boundary conditions of the structure. The eigenvalue problem defined by Eqs. 6 and 7 can be solved using the standard IBM library subroutine NROOT, yielding the natural frequencies and mode shapes of the structure.
EXPERIMENTAL WORK

In order to obtain an indication of the accuracy and acceptability of the finite element solution, a series of tests were made on a variety of square stiffened plate models. It was proposed that three different stiffener dimensions be studied for plates with both simply supported and clamped edge conditions, yielding 6 basic plate models. Initially, there were 13 equally spaced stiffeners on each plate. After each of these plates was tested, every other stiffener was removed such that each plate then had 7 equally spaced stiffeners. This procedure was continued until test information had been gathered on each plate when stiffened with 13, 7, and 3 equally spaced stiffeners; one central stiffener; and on the unstiffened plate. Thus, 30 separate plate configurations were tested. The plates were machined from 7075-T6 aluminum. The physical properties and plate dimensions of the 6 basic plates are given in Table 1.

In manufacturing the test plates, a one-inch-wide, uniform thickness border was machined around the perimeter of each plate. This border was placed between two steel clamping frames and the whole assembly was bolted to an aluminum support head which was in turn bolted to the head of an MB Electronics shaker table. Figure 3 shows a diagram of the support arrangement, while Fig. 4 gives a picture of the overall test set up. An approximation of the simply supported edge condition was to have been made by machining a groove in the face of the plate such that the center of the groove was in line with the inner edges of the clamping frames. The depth of the groove was to be between 70% to 80% of the thickness of the plate. As the depth of the
TABLE 1.-PROPERTIES OF TEST PLATES

Side dimension = 11 inches
Plate Thickness = 0.0625 inches
Modulus of Elasticity = 10,400,000 pounds per square inch
Poisson's Ratio = 0.33
Mass Density = 0.000262 slugs per cubic foot

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<th>Plate Number</th>
<th>Stiffener Width, in Inches</th>
<th>Stiffener Depth, in Inches</th>
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Figure 3. Clamping Arrangement of Test Plates
Figure 4. Test Plate Mounted on Shaker Table
groove approaches 75% of the plate thickness, the rotational resistance at the edge of the plate becomes negligible (11). The plate then behaves as a simply supported plate, the size of which is equal to the distance from center line to center line of the grooves. A subsequent test of one of these theoretically simply supported plates in the unstiffened condition revealed that the static deflection of the plate was closer to the value expected from a clamped plate rather than that expected from a simply supported one. Thus, these plates are referred to as "partially clamped." In approximating the clamped edge condition, the same type of support arrangement was used, except that no groove was present. The border was merely clamped between the two steel frames.

The test objectives were to determine the natural frequencies and mode shapes of the various test plates along with checking the assumption of no stress in the x-direction of the stiffeners. The plates were excited by sinusoidal movement of the shaker table head. The shaker head was excited through a frequency range of 0-2000 Hz. Up to about 45 Hz. the shaker head operated at a constant value of maximum displacement while from 45-2000 Hz. it operated at a constant maximum acceleration. Thus the forcing function of the test plates consisted of a sinusoidal displacement of the plate supports over the range of 0-45 Hz. and a sinusoidal acceleration of the supports from 45-2000 Hz. Due to the symmetrical nature of the plate excitation, only information concerning those natural frequencies whose mode shape was symmetrical in both directions could be obtained.

Foil strain gages were mounted at the middle of the central
stiffener, with one gage oriented in the direction of the stiffener and another gage oriented transverse to the stiffener direction. The gages were then connected to a cathode ray tube oscilloscope, from which the double amplitude of the strain in the plate at the location of the gages could be taken. The natural frequencies were obtained by observing the frequency of the shaker head at which peak values of the strains occurred. It was not possible to obtain exact mode shapes in that the deflections of the plate could not be measured without disturbing the motion of the plate. However, the location of the nodal lines could be found by sprinkling a small amount of sand onto the plate surface and noting the buildup of sand along the nodal lines during resonance. Figure 5 illustrates a typical nodal pattern exhibited by this sand distribution.
Figure 5. Typical Nodal Pattern as Designated by Build Up of Sand
RESULTS AND CONCLUSIONS

The purpose of the experimental work was to have some basis for determining the ability of the proposed finite element method to accurately predict the resonance frequencies of stiffened plates and to verify the assumptions made as to the state of strain in the stiffener elements. Due to the symmetry of the problem, the finite element model could be analyzed by considering only one-quarter of the plate. The initial attachment of 13 stiffeners to the entire plate dictated that the quarter plate be composed of 7 elements in the x-direction, perpendicular to the stiffener axes. The size of core storage available on the IBM 360 system used for solving the eigenvalue program, along with time requirements, limited the number of elements to be used in the y-direction of the quarter plate to 2. A comparison of the predicted natural frequencies and the results of the plate testing is shown in Tables 2 through 7. These tables show the natural frequencies as found from the test procedure and as found from the finite element computer program for each of the 5 stiffener configurations studied. While Tables 2, 3, and 4 are labeled as pertaining to partially clamped plates, the finite element results are those obtained from the finite element model with simply supported boundary conditions. Tables 8 and 9 show a comparison of the test frequencies, finite element frequencies, and frequencies found from the solution to the governing differential equation of dynamic plate behavior for the unstiffened plates. Results are shown for mode shapes of $m=1, n=1$ up to $m=3, n=3$, where $m =$ number of half-waves in the $x$-direction and $n =$ number of half-waves in the $y$-direction, which is the direction of orientation of the stiffeners.
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<th>Mode Shape (2)</th>
<th>Frequency, in Hertz (3)</th>
<th>Finite Element Method (4)</th>
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### TABLE 3.-NATURAL FREQUENCIES, PLATE 2 (PARTIALLY CLAMPED)

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*Mode shape not apparent during testing.*
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<td>700</td>
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*Mode shape not apparent during testing.*
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<th>Frequency, in Hertz</th>
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<td>1,3</td>
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</tr>
<tr>
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<td>315</td>
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<tr>
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<td>3,1</td>
<td>525</td>
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<tr>
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<td>1,3</td>
<td>695</td>
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*aMode shape not apparent during testing.*
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<tr>
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</thead>
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<td>1,3</td>
<td>3,3</td>
</tr>
<tr>
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<td>(3)</td>
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<td>Plate 1</td>
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<td>515</td>
<td>835</td>
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<tr>
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<td>a</td>
<td>490</td>
<td>785</td>
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<td>Plate 3</td>
<td>245</td>
<td>490</td>
<td>525</td>
<td>875</td>
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<td>Finite Element</td>
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<td>473</td>
<td>494</td>
<td>780</td>
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<tr>
<td>Differential</td>
<td>99</td>
<td>494</td>
<td>494</td>
<td>890</td>
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<tr>
<td>Equation-Ref.</td>
<td>(23)</td>
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</table>

*aMode shape not apparent during testing.*
### TABLE 9.-NATURAL FREQUENCIES, IN HERTZ, OF UNSTIFFENED PLATES-CLAMPED

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<tr>
<td></td>
<td>(1)</td>
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<tr>
<td>Plate 4</td>
<td>295</td>
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<td>Plate 5</td>
<td>215</td>
</tr>
<tr>
<td>Plate 6</td>
<td>255</td>
</tr>
<tr>
<td>Finite Element</td>
<td>175</td>
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<tr>
<td>Differential Equation-Ref. (25)</td>
<td>180</td>
</tr>
</tbody>
</table>

*Frequency for 3,3 mode shape not included.*
Frequencies for mode shapes other than those shown were obtained from both the tests and the computer program, but due to the 2000 Hz. frequency limitation of the shaker table, the results shown here represent the most complete set of data available.

A brief explanation of the difficulties encountered in the machining of the test plates will help in discussing the data shown in Tables 2 through 9. In light of the 2000 Hz. limitation of the shaker head frequency and the expected frequencies of the test plates, it was necessary to choose a very small plate thickness (0.0625 inches). This small plate thickness in itself tended to magnify the percentage error of small differences of plate thickness and groove depth. In addition, cutting out the material between stiffeners relieved some of the stresses induced into the original aluminum plate during the rolling process. The effect of this stress relief was to cause warping of the plates thus increasing the difficulty of maintaining close tolerances on most of the plate dimensions. As a result the grooves in the "simply supported" plates did not produce the desired effect. Although it was not possible to measure the final depth of the grooves, at some points it appeared to be barely 50% of the plate thickness. The plate thickness itself was also affected by these factors. The average thicknesses of plates 1, 2, 3, and 6 were in the neighborhood of 10% smaller than the specified thickness of 0.0625 inches, while plates 4 and 5 appeared to have the required thickness. In addition, when the stiffeners were taken out, either too much material would be removed or some would be left in place, resulting in a rather rough surface on the machined side of the plates. The effect of these factors upon the plate behavior are included in the discussion of the test results.
From Tables 5, 6, and 7 it appears that the finite element method predicts fundamental frequencies for a clamped plate which are much too low. However, examination of Table 9 reveals that the fundamental test frequencies are too high while the fundamental finite element frequency is in rather close agreement with the differential equation solution. For the remaining mode shapes, the test plate frequencies, finite element frequencies and differential equation frequencies are approximately equal for their respective shapes except for those of plate 6. The warping of the plates during the machining process caused them to become very shallow shells instead of flat plates. The natural frequencies of a shell are higher than those of a similar flat plate, the increase depending upon the radius of curvature and the mode shape. It has been shown (2) that for a simply supported shallow shell with a large radius of curvature, the influence of curvature is much greater in the fundamental mode than in the higher modes, and it is assumed that this phenomenon is true in the case of clamped shells as well. Thus, it appears that the warping of the plates is responsible for the higher values of fundamental test frequencies, but that the warping was not great enough to have much effect upon the larger natural frequencies. The relatively low value of the test frequencies of plate 6 may be explained by the fact that the actual plate thickness was about 10% smaller than that of the assumed thickness.

The machining difficulties had somewhat offsetting effects upon the partially clamped plates. The insufficient depth of the edge grooves tended to increase the natural frequencies, while the smaller plate thicknesses tended to decrease them. Table 8 shows that the warping of the plates has caused an additional increase in the funda-
mental frequencies and also shows that the finite element frequencies coincide with those of the differential equation solution for an unstiffened plate. While Tables 2, 3, and 4 show that for the higher mode shapes, the test frequencies of the partially clamped plates are about the same as the corresponding finite element frequencies for simply supported plates, no definite conclusions can be made from these results due to the previously mentioned discrepancies in groove depths and plate thicknesses. Thus, for purposes of a quantitative analysis, only the results concerned with plates 4 and 5 may be examined with any degree of confidence.

Excluding the fundamental modes, there are 29 frequency comparisons given in Tables 5 and 6. Of these 29 comparisons there are 9 cases in which the percentage difference between the test frequencies and the finite element frequencies is greater than 5%, and only 3 cases in which it is greater than 10%. These cases are shown in Table 10, where the difference in frequencies is found as a percentage of the respective test frequency. Of these 9 cases, 3 of them occurred in plates with 13 stiffeners, 3 occurred in plates with 7 stiffeners, 2 occurred in plates with 3 stiffeners, and only once in either the unstiffened plates or plates with one stiffener. Hence Table 10 indicates that as the number of stiffeners increases, the finite element model usually becomes increasingly more flexible than the actual plate structure. This increased flexibility is probably due to the lack of continuity between the plate and stiffener elements. The assumed deflection equations of the plate elements yield a function which is cubic with respect to y for the slope in the x-direction, $\theta_y = \frac{\partial w}{\partial x}$, along the element boundaries to which stiffeners are attached. However, the deflection equation for the rotation of the
TABLE 10.-CASES IN WHICH TEST AND FINITE ELEMENT FREQUENCIES DIFFERED BY MORE THAN FIVE PERCENT, PLATES 4 AND 5

<table>
<thead>
<tr>
<th>Plate Number</th>
<th>Number of Stiffeners</th>
<th>Mode Shape</th>
<th>Percentage Difference</th>
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</thead>
<tbody>
<tr>
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<td>3,1</td>
<td>+6.1</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>1,3</td>
<td>-5.1</td>
</tr>
<tr>
<td>4</td>
<td>13</td>
<td>3,1</td>
<td>+10.1</td>
</tr>
<tr>
<td>4</td>
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</tr>
<tr>
<td>5</td>
<td>7</td>
<td>3,1</td>
<td>+11.6</td>
</tr>
<tr>
<td>5</td>
<td>13</td>
<td>3,1</td>
<td>+15.6</td>
</tr>
</tbody>
</table>
stiffener elements about the y-axis, \( \bar{\alpha}_y \), is linear with respect to \( y \). Thus, in general, the rotation of the plate and stiffener elements about the y-axis will be equal only at the nodal points, thereby increasing the flexibility of the finite element model. The effects of this lack of continuity would increase with the addition of more stiffeners, producing the results found in Table 10.

Also shown in Tables 4 and 5 is the fact that, excluding the fundamental modes, in all but 4 cases the test plate frequencies were greater than the finite element frequencies when the plates were vibrating in the 3,1 or 3,3 modes, while in all cases the plate frequencies were lower than the finite element frequencies when the plates were vibrating in the 1,3 mode. This indicates that the finite element model exhibits relatively less stiffness in the x-direction or more stiffness in the y-direction than the respective test plates. As there are over twice as many nodal lines in the x-direction than in the y-direction, the discontinuity present in normal slope at the element boundaries (7), along with the lack of continuity between the plate and stiffener elements, would thus result in the increased flexibility in the x-direction. The use of rectangular elements to represent a square plate can also account for the non-isotropic behavior of the unstiffened plates as shown in the fact that the natural frequency of the simply supported finite element model for the 3,1 mode shape is not equal to that for the 1,3 mode shape. The results of an additional finite element analysis of the unstiffened plates are included in Table 11. In this analysis, an isotropic element arrangement of 4 elements in both the x- and y-directions was used. Table 11 shows that this finite element model is isotropic in that the natural frequencies of the 3,1 and 1,3 mode shapes of the simply
TABLE 11.-FINITE ELEMENT NATURAL FREQUENCIES, IN HERTZ, ISOTROPIC AND ANISOTROPIC ELEMENT ARRANGEMENT

<table>
<thead>
<tr>
<th>Support Condition</th>
<th>Mode Shape</th>
<th>Finite Element Method</th>
<th>Differential Equation References (22) and (24)</th>
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<td>4 elements, x-direction</td>
<td>7 elements, x-direction</td>
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<tr>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
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<td>98</td>
<td>97</td>
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<td>473</td>
</tr>
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<td>1,3</td>
<td>488</td>
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<td></td>
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<td>836</td>
<td>780</td>
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<td>175</td>
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<td>1,022</td>
<td>1,304</td>
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</table>

* Frequency for 3,3 mode shape, clamped supports, not included in reference (24)
supported plate are now equal. Table 11 also shows that the predicted frequencies for the isotropic element arrangement are in closer agreement with the theoretical frequencies than those predicted using the anisotropic element arrangement. This seems to indicate that the shape of the plate elements should be as close as possible to the shape of the actual plate.

Although the frequency comparisons for the remaining plates 1, 2, 3, and 6 may be used only in a qualitative manner, Tables 2, 3, 4, and 7 show that these plates do follow the same general trends as plates 4 and 5. The finite element model becomes relatively more flexible with increasing numbers of stiffeners, and the finite element models is comparatively stiffer in the 1,3 mode than in the 3,1 and 3,3 modes, reflecting the greater flexibility in the x-direction.

A comparison of one of the test mode shapes with the corresponding mode shape obtained from the computer program is shown in Figs. 6 and 7. Figure 6 is a photograph of the nodal pattern of plate 5 with 3 stiffeners when vibrating in the 3,1 mode shape at a frequency of 650 Hz. The light lines on the surface of the test plate represent the element boundaries of the finite element model. Figure 7 shows a line drawing of the corresponding mode shape at the center lines of the finite element model, whose natural frequency was 596 Hz. It was not possible to measure the deflection amplitudes of the test plates, and thus it is impossible to compare the mode shapes directly. However, Fig. 7 shows that the nodal lines of the finite element model intersect the x-direction center line at about the same points as do those of the test plate, and that the mode shape given by the finite element model is a reasonable one.
Figure 6. Test Plate 5, With 3 Stiffeners, Vibrating at 650 Hertz
Figure 7. Mode Shape of Finite Element Plate 5, With 3 Stiffeners, $\omega = 596$ Hertz
Ratios of the strain in the x-direction to the strain in the y-direction at the mid-point of the surface of the central stiffeners of each of the test plates is shown in Table 12. The assumption that the stiffener element will exhibit strain due to bending and in-plane forces in the longitudinal direction only is actually an assumption of stress in the longitudinal direction only, for Poisson's effect would normally cause strains in directions normal to the longitudinal axis. Thus, if the assumption that longitudinal stress only will be present in the stiffener is valid, the ratio of $\varepsilon_x$ to $\varepsilon_y$ should equal Poisson's ratio of 0.33 for the aluminum plates. Examination of Table 12 reveals that, except for plate 4, over 95% of the ratios fall between the limits of 0.25 and 0.40. This seems to be an acceptable enough range to validate the assumption when the following two points are considered. One is that the gages were not "stacked", that is, they did not measure strains at exactly the same point on the stiffener. The other is the fact that the test equipment was such that each strain reading had to be taken separately, and the test procedure was such that all of the strain readings in one direction were taken as the plate was excited through the frequency test range and then the remaining gage was connected to the oscilloscope and strain readings in the other direction were taken. This obviously introduced the possibility that strain readings corresponding to a particular mode shape could have been taken at slightly different frequencies.

One interesting point about the behavior of stiffened plates is also illustrated in Tables 2 through 7. In all but 4 cases, the frequency predicted by the finite element program for the 3,1 mode shape of each plate decreases with the addition of more stiffeners.
## TABLE 12.-RATIO OF STIFFENER STRAIN IN THE X-DIRECTION TO STIFFENER STRAIN IN THE Y-DIRECTION

<table>
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<th>Number of Stiffeners</th>
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<th>Plate 2</th>
<th>Plate 3</th>
<th>Plate 4</th>
<th>Plate 5</th>
<th>Plate 6</th>
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<td>(3)</td>
<td>(4)</td>
<td>(5)</td>
<td>(6)</td>
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<td>(8)</td>
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<tr>
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<td>0.34</td>
<td>0.26</td>
<td>0.32</td>
<td>0.30</td>
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<td>d</td>
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<tr>
<td>7</td>
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<td>0.32</td>
<td>0.31</td>
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</tr>
<tr>
<td></td>
<td>3,1</td>
<td>c</td>
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<td>0.95</td>
<td>0.33</td>
<td>0.30</td>
<td></td>
</tr>
<tr>
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<td>1,3</td>
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<td>0.29</td>
<td>0.33</td>
<td>0.27</td>
<td>0.30</td>
<td>0.31</td>
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</tr>
<tr>
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<td>0.25</td>
<td>0.30</td>
<td>0.20</td>
<td>0.30</td>
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<td>a</td>
<td>0.32</td>
<td>0.33</td>
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<td>0.29</td>
<td>b</td>
<td>0.32</td>
<td>0.29</td>
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<td>d</td>
<td>0.32</td>
<td>0.25</td>
<td>0.31</td>
<td>0.31</td>
<td>0.29</td>
<td></td>
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<tr>
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<td>3,3</td>
<td>0.24</td>
<td>0.35</td>
<td>d</td>
<td>0.10</td>
<td>d</td>
<td>d</td>
<td></td>
</tr>
</tbody>
</table>

\( a \) Value of \( \epsilon_x \) indeterminable due to non-harmonic wave form.

\( b \) Value of \( \epsilon_y \) indeterminable due to non-harmonic wave form.

\( c \) Value of \( \epsilon_x \) and \( \epsilon_y \) indeterminable due to non-harmonic wave form.

\( d \) Mode shape not apparent during testing.
In one of the exceptions the frequency remained unchanged while in the other 3 cases the frequency increased a relatively small amount. This same pattern is not so apparent in the results of the test plate frequencies. However, many of the test plate frequencies for the 3,1 mode did decrease with an increasing number of stiffeners. In cases where the test frequencies increased, this increase was not nearly as great as the increase which occurred in the other mode shapes. Kirk (16) has indicated that the addition of stiffeners to a plate will increase the natural frequency of a particular mode only if the stiffness and mass of the stiffener are such that their addition increases the ratio of the maximum strain energy of the plate to the maximum kinetic energy of the plate when the plate is vibrating in that mode. Thus, while it might be assumed that the addition of stiffeners to a plate structure would have the effect of increasing the natural frequency of the system, only a rational analysis of the problem can predict the effect of the added stiffeners upon the dynamic behavior of the system.

The lack of success in this study in attempting to obtain a simply supported edge condition by machining a groove along the outer edge of the plates and then clamping the boundaries should not be taken as condemnation of the method. The method has been used successfully in a previous study (12) and was also proven worthwhile in the early stages of this study. During the preliminary organization of the testing equipment and procedures a one-quarter inch thick aluminum plate was grooved in this manner and tested on the shaker table. There were no stiffeners on this plate, allowing it to be clamped more securely during the machining of the groove. The plate was also not warped and thus the groove depth could be controlled better than those of the test plates. The
first three natural frequencies of this preliminary plate were found to be within about 10% of the theoretical solution.
SUMMARY

This study examines the use of the finite element method in determining the dynamic behavior of eccentrically stiffened rectangular plates. The finite element method is particularly well suited to the study of continuous structures with irregular characteristics. In this case the irregularity is the abrupt change in cross section encountered in a stiffened plate. Previous work has shown the finite element method to be a useful tool in studying the vibration characteristics of the component parts of a stiffened plate - the unstiffened thin plate and the elastic beam. Here the method is expanded to combine the two types of elements into a finite element model of the stiffened plate structure.

The finite element method represents a more accurate approach to the behavior of stiffened plates than does the orthotropic plate theory. Orthotropic plate analyses demand that the stiffeners be closely spaced, while this limitation is not necessary in the finite element method. However, the presence of many stiffeners necessitates the use of a large number of elements, which may result in the need of an unacceptable amount of computer time in solving the finite element eigenvalue problem.

A comparison of the frequencies of finite element models with those obtained from the test plates show that the finite element model can be a useful tool in predicting lower resonance frequencies of stiffened plates. Of the valid frequency comparisons made, in almost 90% of the cases the finite element method predicted natural frequencies which were within 10% of the test plate frequencies. Direct frequency com-
parisons could not be made for 4 of the test plates because of problems related to the machining of the plates. However, a quantitative comparison for all the test plates revealed the same general trend. Due to element discontinuities inherent in the method and to the element pattern, the finite element models exhibited relatively more flexibility in the direction perpendicular to the stiffeners than in the stiffener direction and this flexibility increased with added stiffeners. This anisotropic behavior was not present in finite element models of unstiffened plates with a symmetric element arrangement. The test results also disclosed that the assumption of a uniaxial state of stress, with superposed torsion, in the stiffener element is a valid one.

Although they have not been included in this study, the finite element computer program developed during this work has produced results which compare favorably with those reported elsewhere in the literature. Predicted frequencies were within 20% of the test frequencies for a rectangular, simply supported stiffened plate as found by Hoppmann, Huffington, and Magness (12). In this case, the predicted frequencies were all lower than the test frequencies. This condition may be due to the fact that the simply supported boundary condition for the test plate was approximated using the same grooving technique as discussed herein. The finite element program also produced results which compared favorably with those obtained by Long (18) for a simply supported plate with a single stiffener and with the orthotropic plate theory frequencies of a single-span highway bridge as studied by Feezer (6).
ACKNOWLEDGMENTS

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APPENDIX I.-REFERENCES


APPENDIX II.-NOTATION

The following symbols are used in this paper:

$A = \text{matrix relating plate element nodal deflection to generalized plate element coordinates;}$

$\bar{A} = \text{matrix relating stiffener element nodal deflections to generalized stiffener element coordinates;}$

$B = \text{matrix relating displacement vector of plate element reference surface to generalized plate element coordinates;}$

$B_m = \text{matrix relating displacement vector of plate element mass surface to generalized plate element coordinates;}$

$C = \text{matrix relating plate element strain vector to generalized plate element coordinates;}$

$\bar{C}_1 = \text{matrix relating stiffener element longitudinal strain vector to generalized stiffener element coordinates;}$

$\bar{C}_2 = \text{matrix relating stiffener element torsional strain vector to generalized stiffener element coordinates;}$

$D = \text{matrix of elastic constants relating plate element stress and strain vectors;}$

$E = \text{modulus of elasticity of plate element material;}$

$F_e = \text{vector of plate element nodal forces;}$

$G = \text{shearing modulus of elasticity of stiffener element material;}$

$J = \text{St. Venant's modified polar moment of inertia of stiffener element;}$

$K = \text{plate element stiffness matrix;}$

$M = \text{plate element mass matrix;}$

$\bar{T}_y = \text{torque about y-axis of stiffener element;}$

$U, V = \text{displacements of plate element in x- and y-directions at any distance, z, from reference surface;}$

$u, v, w = \text{components of displacement vector of plate element reference surface in x-, y-, and z-directions;}$
\( \bar{u}, \bar{v}, \bar{w} = \) components of displacement vector of stiffener element reference axis in the x-, y-, and z-directions;

\( X = \) matrix relating lateral deflection of plate element reference surface to generalized bending coordinates;

\( x, y, z = \) orthogonal coordinates of plate element reference surface and stiffener element reference axis;

\( Y = \) matrix relating in-plane deflections of plate element reference surface to generalized in-plane coordinates;

\( \alpha = \) matrix of generalized coordinates for bending displacements;

\( \alpha^* = \) matrix of generalized plate element coordinates;

\( \beta, \gamma = \) matrices of generalized coordinates for in-plane deflections;

\( \delta = \) displacement vector of plate element reference surface;

\( \delta_m = \) displacement vector of plate element mass surface;

\( \delta_m^e = \) displacement vector of stiffener element mass;

\( \delta^e = \) plate element nodal displacement vector;

\( \ddot{\delta}^e = \) plate element nodal acceleration vector;

\( \delta'^e = \) plate element virtual nodal displacement vector;

\( \delta^e = \) stiffener element nodal displacement vector;

\( \varepsilon = \) plate element strain vector;

\( \varepsilon^* = \) stiffener element strain vector;

\( \varepsilon_x, \varepsilon_y = \) components of plate element strain vector in x- and y-directions;

\( \varepsilon_y = \) component of stiffener element strain vector in y-direction;

\( \varepsilon_{xy} = \) shearing component of plate element strain vector;

\( \zeta = \) matrix of generalized stiffener coordinates for twisting displacements;

\( \theta_x, \theta_y = \) slope of plate element plate element in y- and x-directions;

\( \beta_x = \) slope of stiffener element in y-direction;

\( \beta_y = \) rotation of stiffener element about y-axis;
ν = Poisson's ratio;
ρ = mass density;
σ = plate element stress vector;
σ_x, σ_y = components of plate element stress vector in x- and y-directions;
σ_{xy} = shearing component of plate element stress vector;
ϕ_y = angle of twist of stiffener element about y-axis; and
ω = circular natural frequency.
VITA

Charles Stuart Ferrell was born on May 29, 1942, in Provo, Utah. He received his secondary education in Longview, Texas. He received a Bachelor of Science degree in Civil Engineering from the University of Missouri-Rolla, Rolla, Missouri, in May 1964 and received a Master of Science degree in Civil Engineering from Rice University, Houston, Texas, in May 1966.

He has been enrolled in the Graduate School of the University of Missouri-Rolla since September 1966 and held an NDEA Title IV Fellowship from September 1966 to August 1969. In June 1968 he married the former Twyla Griffie of Kansas City, Kansas. He presently lives in Murphysboro, Illinois and is an instructor in the School of Engineering and Technology at Southern Illinois University, Carbondale, Illinois.
APPENDIX III.-DESCRIPTION OF THE FINITE ELEMENT METHOD USED IN THE SOLUTION OF THE PROBLEM

INTRODUCTION

Many engineering problems involve the behavior of a continuous system, i.e., one which has an infinite number of degrees of freedom. The solution of these problems usually involves the determination and solution of a set of differential equations. Sometimes problems arise in merely determining the differential equations and boundary conditions which describe the behavior of the system, and in other cases it may be the solution of these equations which presents problems. Consider the lateral vibration of a uniform rectangular plate. Using the classical assumptions of a perfectly elastic, homogeneous, isotropic material and neglecting the effects of shear and rotatory inertia, it is a rather simple matter to develop the governing differential equation of the plate if the lateral deflection is assumed to be small in relation to the plate thickness \((5A)\)^1. A fairly straightforward approach will yield a solution for the natural frequencies of the plate for the case of the plate being simply supported along all the edges, but if the natural frequencies for a plate which is clamped all around are required, a more sophisticated approach, such as the Ritz method \((6A)\), is necessary.

^1Numerals in parentheses refer to corresponding items in references at the end of this appendix.
When an analytic solution to a differential equation and boundary value problem cannot be found satisfactorily it becomes necessary to use a numerical approach. Numerical methods for the solution of differential equations can generally be broken down into two types (3A). The first type uses a mathematical approximation, such as the finite-difference technique, to obtain a solution to the differential equation. The second type replaces the original problem with an approximate mathematical model and then the behavior of the model is found. Basically then, the two approaches are either to find an exact set of equations which describe the behavior of the system and solve them by a numerical procedure, or to make a numerical approximation of the system and find an exact solution for the behavior of this approximate system. The validity of the "exact set of equations" and the "exact solution" may of course be limited by the assumptions which have been made in their derivation and solution. One result of using either of these types of numerical procedures is usually to reduce the problem of dealing with an infinite number of degrees of freedom to one of dealing with a finite number of degrees of freedom. The finite element method of structural analysis falls into the second category of numerical solutions described above.
FINITE ELEMENT METHOD

The basic idea of replacing a continuous structural system with a system of interconnected structural elements is not a new one. Various methods have been derived for handling structures which may be approximated by a system of structural elements, each of which is subjected to a uniaxial state of stress. In the stiffness or displacement method the differential equations of continuum mechanics (elasticity) are transformed into a set of algebraic (matrix) operations (3A). The differential equations representing the elasticity for each structural element can be solved initially in terms of the element boundary values, yielding element boundary force-displacement relationships. Force-equilibrium and displacement-compatibility relations for connected boundaries then yield a set of algebraic equations which may be solved using matrix algebra to find the unknown boundary displacements. The finite element method enables a similar type of solution to be obtained for an elastic continuum in which the continuum may be approximated by a system of elements which may be subjected to a general state of stress.

The basic steps in using finite elements for analysing a continuum are as follows (3A,8A):

1. The continuum is separated by imaginary lines or surfaces into a finite number of elements.

2. The elements are assumed to be continuously attached at a discrete number of nodal points along the element boundaries. The displacements of these nodal points now become the basic unknown parameters of the problem.
3. Assume that the element displacement vector $u = [u_x, u_y, u_z]^T$ is expressible in terms of the nodal displacements $U = [U_1, U_2, \ldots, U_n]^T$ by the matrix equation

$$u = aU \tag{A.1}$$

The element strains may be found by differentiation of Eq. A.1 leading to the matrix equation

$$\varepsilon = bU \tag{A.2}$$

From Hooke's Law, the stress may be found and may be described in terms of the nodal displacements by

$$\sigma = dBU \tag{A.3}$$

4. By using the principle of virtual work the external virtual work may be equated to the internal virtual work and the behavior of the continuum may be expressed in matrix form by

$$M\ddot{U} + KU = P \tag{A.4}$$

where

$$M = \int_V (ba^Ta)dV = \text{mass matrix}$$

$$K = \int_V (b^Tdb)dV = \text{stiffness matrix}$$

$$P = \text{element nodal forces, which include the applied nodal}$$
forces and the equivalent nodal forces due to thermal forces, body forces, and surface forces.

The accuracy which may be obtained by the finite element method depends directly on the extent to which the assumed displacement patterns are able to represent the actual deformation of the continuum; thus, the most critical factor in finite element analysis is the selection of the element displacement function (2A).

Basic guides to the selection of suitable element displacement functions have been summarized as follows (1A,2A):

1. The displacement function is such that self-straining due to rigid body motion of the element is not permitted.
2. The displacement function within each element is such that it can express constant strain conditions.
3. Conditions of compatibility should be satisfied at the boundaries between elements as well as within each element.

If all of these conditions are met, the finite element idealization will provide a lower bound to the strain energy of the system, and the results will converge toward the true state of deformation as the mesh size is reduced (2A).

In some cases it may be difficult to satisfy a part of the third criteria—that of compatible displacements along the element boundaries. Such non-conformity at the boundaries will cause infinite strains at those boundaries, thus the true strain energy is not obtained by restricting the energy contribution to the elements themselves. However, if, in the limit, as the size of the element subdivisions decreases continuity is restored, the preceding finite element process must tend to the correct solution. If the constant
strain criteria is satisfied by the displacement functions, then as the elements decrease indefinitely in size, the continuity at the nodes will require a constant strain state within the element and this constant strain state will automatically insure compatibility of deformations across the element boundaries (1A).
DESCRIPTION OF THE PROBLEM

The continuum analysed in this paper is a thin rectangular plate of elastic isotropic material reinforced by monolithic equidistant stiffeners in one direction and will be considered to be composed of a system of identical plate elements and identical beam elements, as shown in Fig. 1. The stipulation that the plate elements all be alike and that the beam elements all be alike is not a necessary criteria for the finite element method, but has been used here because of the simplicity which it introduces into the computer programming.

If the deflections of the plate are assumed to be small in comparison with the plate thickness, classical plate theory allows the state of deformation of the plate to be described entirely by one quantity, \( w \), the lateral displacement of the middle plane of the plate. However, while it was previously stated that the satisfaction of the conditions of compatibility along the element boundaries is not an absolute necessity, it is necessary that this compatibility exist at the node points where the elements are considered to be connected. This dictates that not only the lateral plate displacement, \( w \), but also its derivatives with respect to \( x \) and \( y \) be compatible at the nodes. Thus, at each node three conditions of continuity must be imposed for a plate element in bending.

Two additional displacements must be considered in the case of a stiffened plate. The neutral axis of bending in the direction of the stiffeners will not, in general, be at the middle surface of the plate for a stiffened plate. For closely spaced stiffeners, the neutral axis would most likely be located at a fairly constant depth and would
lie near the neutral axis of a corresponding T-section whose web is composed of one-half of the length of plate between stiffeners. For very widely spaced stiffeners the neutral axis might actually lie at the middle surface of the plate at a section located mid-way between stiffeners, but would again lie below the middle surface at a section of the plate closer to a stiffener. This shifting of the neutral axis will be considered to be the result of a strain which is constant with depth in combination with the strain associated with bending about the neutral axis of the plate. This constant strain will result in a displacement of the nodes in the y-direction, parallel to the stiffeners. In addition, there will be strain in the perpendicular x-direction due to Poisson's effect. There will be additional strains in the x-direction, perpendicular to the stiffeners, due to the stress caused by the inertia force of the rotating stiffener. Thus, it is also necessary to consider nodal displacements in the x-direction. Note that these in-plane strains in the x- and y-directions are not a result of a straining of the middle surface of the plate due to bending, since only small deflections are to be considered.

Thus, at each node of the finite element model, there are five nodal displacements or degrees of freedom (displacements in the x-, y-, and z-directions and slopes in the x- and y-directions) to be considered. Corresponding to those displacements will be five nodal forces (horizontal forces in the x- and y-directions, a vertical force, and moments about the y- and x-axes) at each node.
PLATE ELEMENT STIFFNESS AND MASS MATRICES

A plate element (with nodal displacements and nodal forces shown for one node only) is shown in Fig. 2.a. Denoting the displacement of node \( i \) as

\[
\delta_i = \begin{bmatrix} w_i \\ \theta_xi \\ \theta_yi \\ u_i \\ v_i \end{bmatrix}
\] (A.5)

the element nodal displacement vector, \( \delta^e \), becomes

\[
\delta^e = \begin{bmatrix} \delta_i \\ \delta_j \\ \delta_k \\ \delta_l \end{bmatrix}
\] (A.6)

The simplest expression for the displacement function of a rectangular element in bending is one which has been credited to many sources. The earliest paper referred to seems to be a paper by Adini and Clough referred to by Clough and Toucher (2A). The function has also been used by Zienkiewicz (1A,7A). This function is a twelve term, fourth order polynomial of the form:
\[
  w(x,y) = \alpha_1 + \alpha_2x + \alpha_3y + \alpha_4x^2 + \alpha_5xy + \alpha_6y^2 + \alpha_7x^3 + \alpha_8x^2y + \alpha_9xy^2 + \alpha_{10}y^3 + \alpha_{11}x^3y + \alpha_{12}xy^3
  \]  
(A.7a)

or, in matrix form

\[
  w(x,y) = [1,x,y,x^2,...,x^3y,xy^3] \begin{pmatrix}
  \alpha_1 \\
  \alpha_2 \\
  \alpha_3 \\
  \vdots \\
  \alpha_9 \\
  \alpha_{10} \\
  \alpha_{11} \\
  \alpha_{12}
  \end{pmatrix} = X_\alpha
  \]

(A.7b)

Then

\[
  \theta_x(x,y) = \frac{\partial w}{\partial y} = \frac{\partial X}{\partial y} \alpha
  \]  
(A.8)

and

\[
  \theta_y(x,y) = \frac{\partial w}{\partial x} = \frac{\partial X}{\partial x} \alpha
  \]  
(A.9)

Along any line where \( x = \) constant or \( y = \) constant, \( w(x,y) \) will vary as a cubic, which is uniquely defined by four conditions. Since the element boundaries are composed of such lines, the boundary deflections along two adjacent elements will be identical since they will
both be cubic and will be defined by the same four conditions; the lateral displacements and the slopes at the two ends (node points) of the boundaries. Thus the conditions of compatibility along the boundary between elements has been met. However, the compatibility condition will not exist across element boundaries. In the direction normal to an element boundary, the deflections of adjacent elements will have only two common conditions, the normal slopes at the node points. Since these two conditions are not enough to guarantee uniqueness, a discontinuity of the normal slope will generally occur across an element boundary. However, it will be shown later that this displacement function will satisfy the necessary criteria for convergence toward the true state of deformation for decreasing mesh size.

Concerning the in-plane deflections, if the displacement function is chosen such that it varies linearly along the element boundaries (where \( x = a \) constant and \( y = a \) constant), then continuity of the two ends of the boundaries will insure continuity of deflections all along the boundaries. Displacement functions of the form:

\[
\begin{align*}
\mathbf{u}(x,y) &= \beta_1 + \beta_2 x + \beta_3 xy + \beta_4 y = \begin{bmatrix} 1, x, xy, y \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} = \gamma \beta \\
\end{align*}
\]

and

\[
\mathbf{v}(x,y) = \gamma_1 + \gamma_2 x + \gamma_3 xy + \gamma_4 y = \gamma y
\]
will satisfy this criteria.

Rearranging the components of $\delta_i$, $\delta_j$, $\delta_k$, and $\delta_l$ in Eq. A.6 yields the nodal displacement vector in terms of the assumed displacement functions as

$$
\delta^e = \begin{bmatrix}
X_i \\
X_j \\
X_k \\
X_l \\
\frac{\partial X_i}{\partial y} \\
\frac{\partial X_j}{\partial y} \\
\frac{\partial X_k}{\partial y} \\
\frac{\partial X_l}{\partial y} \\
\frac{\partial X_i}{\partial x} \\
\frac{\partial X_j}{\partial x} \\
\frac{\partial X_k}{\partial x} \\
\frac{\partial X_l}{\partial x}
\end{bmatrix} \begin{bmatrix}
\alpha \\
\beta \\
\gamma
\end{bmatrix} = \begin{bmatrix}
\delta^e_b \\
\delta^e_p
\end{bmatrix}
$$
Then

\[
\begin{bmatrix}
0 & 0 & \gamma_1 \\
0 & 0 & \gamma_2 \\
0 & 0 & \gamma_3 \\
0 & 0 & \gamma_4
\end{bmatrix}
\]

or

\[
\delta^e = \mathbf{A}_{\alpha^*} = \begin{bmatrix} A_b & 0 \\ 0 & A_p \end{bmatrix} \alpha^* 
\]

(A.12)

Then

\[
\alpha^* = A^{-1} \delta^e = \begin{bmatrix} A^{-1}_b & 0 \\ 0 & A^{-1}_p \end{bmatrix} \delta^e
\]

(A.13)

where the subscripts of the submatrices apply to the bending deflections (b) and to the in-plane deflections (p). The matrices \(A_b, A_p, A_b^{-1}, \) and \(A_p^{-1}\) are given in terms of the plate dimensions in Tables A.1 and A.2.

Now the displacement of any point on the middle surface of the plate, \(\delta(x,y)\), can be found in terms of the nodal displacements, \(\delta^e\), by the equation
TABLE A.1a.-SUBMATRIX $A_b$ IN TERMS OF PLATE DIMENSIONS $a$ AND $b$.

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & b & 0 & 0 & b^2 & 0 & 0 & 0 & b^3 & 0 & 0 \\
1 & a & b & a^2 & ab & b^2 & a^3 & a^2b & ab^2 & b^3 & a^3b & ab^3 \\
1 & a & 0 & a^2 & 0 & 0 & a^3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 2b & 0 & 0 & 0 & 3b^2 & 0 & 0 \\
0 & 0 & 1 & 0 & a & 2b & 0 & a^2 & 2ab & 3b^2 & a^3 & 3ab^2 \\
0 & 0 & 1 & 0 & a & 0 & 0 & a^2 & 0 & 0 & a^3 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & b & 0 & 0 & 0 & b^2 & 0 & 0 & b^3 \\
0 & 1 & 0 & 2a & b & 0 & 3a^2 & 2ab & b^2 & 0 & 3a^2b & b^3 \\
0 & 1 & 0 & 2a & 0 & 0 & 3a^2 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
TABLE A.1b.-SUBMATRIX $A_p$ IN TERMS OF PLATE DIMENSIONS $a$ AND $b$.

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & b & 0 & 0 & 0 & 0 \\
1 & a & ab & b & 0 & 0 & 0 & 0 \\
1 & a & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & b \\
0 & 0 & 0 & 0 & 1 & a & ab & b \\
0 & 0 & 0 & 0 & 1 & a & 0 & 0
\end{bmatrix}
\]
TABLE A.2a.-SUBMATRIX $A_b^{-1}$ IN TERMS OF PLATE DIMENSIONS $a$ AND $b$.

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-3}{a^2} & 0 & 0 & \frac{3}{a^2} & 0 & 0 & 0 & 0 & \frac{-2}{a} & 0 & 0 & \frac{-1}{a} & 0 & 0 \\
\frac{-1}{ab} & \frac{1}{ab} & \frac{-1}{ab} & \frac{1}{ab} & \frac{-1}{a} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-3}{b^2} & \frac{3}{b^2} & 0 & 0 & \frac{-2}{b} & \frac{-1}{b} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{2}{a^3} & 0 & 0 & \frac{-2}{a^3} & 0 & 0 & 0 & 0 & \frac{1}{a^2} & 0 & 0 & \frac{1}{a^2} & 0 & 0 \\
\frac{3}{a^2b} & \frac{-3}{a^2b} & \frac{3}{a^2b} & \frac{-3}{a^2b} & 0 & 0 & 0 & 0 & \frac{2}{ab} & \frac{-2}{ab} & \frac{-1}{ab} & \frac{1}{ab} & 0 & 0 \\
\frac{3}{ab^2} & \frac{-3}{ab^2} & \frac{3}{ab^2} & \frac{-3}{ab^2} & \frac{2}{ab} & \frac{1}{ab} & \frac{-1}{ab} & \frac{-2}{ab} & 0 & 0 & 0 & 0 & 0 \\
\frac{2}{b^3} & \frac{-2}{b^3} & 0 & 0 & \frac{1}{b^2} & \frac{1}{b^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-2}{a^3b} & \frac{2}{a^3b} & \frac{-2}{a^3b} & \frac{2}{a^3b} & 0 & 0 & 0 & 0 & \frac{-1}{a^2b} & \frac{1}{a^2b} & \frac{1}{a^2b} & \frac{-1}{a^2b} & 0 & 0 \\
\frac{-2}{a^3b} & \frac{2}{ab^3} & \frac{-2}{ab^3} & \frac{2}{ab^3} & \frac{-1}{ab^2} & \frac{-1}{ab^2} & \frac{1}{ab^2} & \frac{1}{ab^2} & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
TABLE A.2b.-SUBMATRIX $A_p^{-1}$ IN TERMS OF PLATE DIMENSIONS $a$ AND $b$.

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{a} & 0 & 0 & \frac{1}{a} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{ab} & -\frac{1}{ab} & \frac{1}{ab} & -\frac{1}{ab} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{b} & \frac{1}{b} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{a} & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{ab} & -\frac{1}{ab} & \frac{1}{ab} & -\frac{1}{ab} & \frac{1}{ab} & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{b} & \frac{1}{b} & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
\[ \delta(x,y) = \begin{bmatrix} w(x,y) \\ e_x(x,y) \\ e_y(x,y) \\ u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} x & 0 & 0 \\ \frac{\partial x}{\partial y} & 0 & 0 \\ \frac{\partial x}{\partial x} & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & y \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = BA^e = B \delta e = a \delta e \]

which is of the form of Eq. A.1

Denoting \( u(x,y,0) \) as \( u \) and \( v(x,y,0) \) as \( v \), the horizontal displacements at any point in the plate are

\[ u(x,y,z) = u - z \frac{\partial w}{\partial x} = u - z e_y \quad \text{(A.15a)} \]

\[ v(x,y,z) = v - z \frac{\partial w}{\partial y} = v - z e_x \quad \text{(A.15b)} \]

The strain vector is then defined as

\[ \varepsilon(x,y,z) = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u(x,y,z)}{\partial x} \\ \frac{\partial v(x,y,z)}{\partial y} \end{bmatrix} \]

\[ \varepsilon_{xy} = \frac{\partial u(x,y,z)}{\partial y} + \frac{\partial v(x,y,z)}{\partial x} \quad \text{(A.16a)} \]

In terms of the displacement functions, Eq. A.16a becomes
\[
\varepsilon(x,y,z) = \begin{bmatrix}
-z\frac{\partial^2 w}{\partial x^2} + \frac{\partial u}{\partial x} \\
-z\frac{\partial^2 w}{\partial y^2} + \frac{\partial v}{\partial y} \\
-2z\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}
\end{bmatrix} = \begin{bmatrix}
-z\frac{\partial^2 x}{\partial x^2} \\
-z\frac{\partial^2 x}{\partial y^2} \\
-2z\frac{\partial^2 x}{\partial x \partial y}
\end{bmatrix}
\]

\[
\alpha^* = C\alpha = \begin{bmatrix}
\alpha_x \\
\alpha_y \\
\alpha_z
\end{bmatrix}
\]

which is of the form of Eq. A.2. The submatrices \(C_b\) and \(C_p\), where \(C = [C_b C_p]\), are given in terms of the point coordinates in Table A.3.

The stress-strain relationships for an elastic, isotropic material are

\[
\sigma_x = \frac{E}{(1-\nu^2)} \varepsilon_x + \frac{\nu E}{(1-\nu^2)} \varepsilon_y
\]  
(A.17a)

\[
\sigma_y = \frac{\nu E}{(1-\nu^2)} \varepsilon_x + \frac{E}{(1-\nu^2)} \varepsilon_y
\]  
(A.17b)

\[
\sigma_{xy} = \frac{E}{2(1+\nu)} \varepsilon_{xy} = \frac{(1-\nu)E}{2(1-\nu^2)} \varepsilon_{xy}
\]  
(A.17c)

The strain vector for the plate element then becomes

\[
\sigma(x,y,z) = \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z
\end{bmatrix} = \begin{bmatrix}
E \\
(1-\nu^2) \\
\nu
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \frac{1-\nu}{2}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_{zy}
\end{bmatrix} = d\varepsilon = d\delta^e
\]  
(A.18)

which is of the form of Eq. A.3.
**TABLE A.3.-SUBMATRICES $C_b$ AND $C_p$ IN TERMS OF POINT COORDINATES $x$, $y$, AND $z$.**

a.-SUBMATRIX $C_b$

\[
\begin{bmatrix}
0 & 0 & 0 & -2z & 0 & 0 & -6xz & -2yz & 0 & 0 & -6xyz & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2z & 0 & 0 & -2xz & -6yz & 0 \\
0 & 0 & 0 & 0 & -2z & 0 & 0 & -4xz & -4yz & 0 & -6x^2z & -6y^2z \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

b.-SUBMATRIX $C_p$

\[
\begin{bmatrix}
0 & 1 & y & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & x & 1 \\
0 & 0 & x & 1 & 0 & 1 & y & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
The element nodal force vector is defined as

\[ F_e = \begin{pmatrix} V_i \\ V_j \\ V_k \\ V_l \\ T_{xi} \\ T_{xj} \\ T_{xk} \\ T_{xl} \\ T_{yi} \\ T_{yj} \\ T_{yk} \\ T_{yl} \\ S_{xi} \\ S_{xj} \\ S_{xk} \\ S_{xl} \\ S_{yi} \\ S_{yj} \\ S_{yk} \\ S_{yl} \end{pmatrix} \]

(A.19)

to correspond to the element nodal displacement vector.

Applying virtual nodal displacements \( \delta^e \) at the element nodes, the external work done by the element forces is
The internal strain energy per differential volume of the plate element due to the stresses moving through the "virtual" strains is

\[ W_e = (\delta^e)^T \sigma e \]  \hspace{1cm} (A.20)

Equating the external virtual work to the total internal virtual strain energy yields

\[ dW_i = (\epsilon_i)^T \sigma dV = (b \delta^e)^T T b \delta e dV = (\delta^e)^T T b T b \delta e dV \]  \hspace{1cm} (A.21)

Equating the external virtual work to the total internal virtual strain energy yields

\[ (\delta^e)^T T b T b \delta e dV = (\delta^e)^T \left[ \int_V (b T b) dV \right] \delta e \]  \hspace{1cm} (A.22)

since \( \delta^e \) and \( \delta e \) are not functions of the volume. Thus

\[ \delta^e \left[ F^e - \left( \int_V (b T b) dV \right) \delta e \right] = 0 \]  \hspace{1cm} (A.23)

Since Eq. A.23 must hold true for all possible values of virtual displacement, \( \delta^e \), then

\[ \left[ F^e - \left( \int_V (b T b) dV \right) \delta e \right] = 0 \]

or

\[ F^e = \left( \int_V (b T b) dV \right) \delta e \]

or

\[ F^e = K \delta e \]  \hspace{1cm} (A.24)
in which

\[ K = \int_V (b^T db) dV \]  \hfill (A.25a)

where \( K \) is the stiffness matrix as defined in Eq. A.4.

Rewriting Eq. A.25a using the relationships given in Eq. A.16a yields

\[ K = (A^{-1})^T (\int_V (C^T dC) dV) A^{-1} \]  \hfill (A.25b)

As shown in Eqs. A.13 and A.16, the matrices \( A^{-1} \) and \( C \) can be partitioned into submatrices which apply to bending deflections \( (b) \) and to in-plane deflections \( (p) \). Partitioning \( C \) in this manner and expanding the integral over the volume, an intermediate stiffness matrix, \( k \), may be defined as

\[
\begin{bmatrix}
  k_b & k_{bp} \\
  k_{pb} & k_p
\end{bmatrix} = \int_V (C^T dC) dV = \int_a^b \int_{-t/2}^{t/2} [C_b C_p]^T d[C_b C_p] dz dy dx
\]

\[ k_{bp} = \int_V (C^T dC) dV = \int_a^b \int_{-t/2}^{t/2} [C_b C_p]^T [C_b C_p] dz dy dx \]  \hfill (A.26)

A step-by-step evaluation of \( k_b \) and \( k_p \) is given in Tables A.4 through A.6. An examination of these tables shows that each element, \( k_{ij} \), of \( k_{bp} \) and \( k_{pb} \) is zero, since

\[
k_{ij} = F_{ij}(x,y)\int_{-t/2}^{t/2} zdz = 0 \]  \hfill (A.27)

Hence, the matrices \( k_{bp} \) and \( k_{pb} \) are null.
TABLE A.4.-PRODUCT MATRICES $dC_b$ AND $dC_p$ IN TERMS OF POINT COORDINATES $x$, $y$, AND $z$.

a.-PRODUCT MATRIX $dC_b$

$$\begin{bmatrix}
0 & 0 & 0 & -2z & 0 & -2vz & -6xz & -2yz & -2vxz & -6vyz & -6xyz & -6vxyz \\
0 & 0 & 0 & -2vz & 0 & -2z & -6vzx & -2vyz & -2xz & -6yz & -6vxyz & -6xyz \\
0 & 0 & 0 & \frac{-2(1-v)z}{2} & 0 & 0 & \frac{-4(1-v)xz}{2} & \frac{-4(1-v)yz}{2} & 0 & \frac{-6(1-v)x^3z}{2} & \frac{-6(1-v)y^3z}{2}
\end{bmatrix}$$

b.-PRODUCT MATRIX $dC_p$

$$\frac{E}{(1-v^2)}\begin{bmatrix}
0 & 1 & y & 0 & 0 & 0 & \nu x & \nu \\
0 & \nu & \nu y & 0 & 0 & 0 & x & 1 \\
0 & 0 & \frac{(1-v)x}{2} & \frac{(1-v)}{2} & 0 & \frac{(1-v)y}{2} & \frac{(1-v)z}{2} & 0
\end{bmatrix}$$
TABLE A.5a.-PRODUCT MATRIX $C_b^T d C_b$ IN TERMS OF POINT COORDINATES $x$, $y$, AND $z$.

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 4 \\
0 & 0 & 0 & 0 & 4D_0 \\
0 & 0 & 0 & 4\nu & 0 & 4 \\
0 & 0 & 0 & 12x & 0 & 12\nu x & 36x^2 \\
4y & 8x^2 & 4\nu y & 12xy & 4y^2 & +16x^2 D_0 \\
4\nu x & 8yD_0 & 4x & 12x^2 & 4\nu xy & 4x^2 & +16xyD_0 & +16y^2 D_0 \\
0 & 0 & 0 & 12\nu y & 0 & 12y & 36\nu xy & 12\nu y^2 & 12xy & 36y^2 \\
0 & 0 & 0 & 12xy & 12x^2 D_0 & 12\nu xy & 36x^2 y & 12x^2 y & 36x^2 y^2 & +24x^3 D_0 & +24x^2 y D_0 & 36\nu xy^2 & +36x^4 D_0 \\
0 & 0 & 0 & 12\nu xy & 12y^2 D_0 & 12xy & 36\nu x^2 y & 12x^2 y & 36x^2 y & 12x^2 y^2 & +24xy^2 D_0 & +24y^3 D_0 & 36\nu x^2 y^2 & +36x^2 y^2 D_0 & +36y^4 D_0 \\
0 & 0 & 0 & 12\nu xy & 12y^2 D_0 & 12xy & 36\nu x^2 y & 12x^2 y & 36x^2 y & 12x^2 y^2 & +24xy^2 D_0 & +24y^3 D_0 & 36\nu x^2 y^2 & +36x^2 y^2 D_0 & +36y^4 D_0 \\
\end{array}
\]

\[D = \frac{E}{(1-\nu^2)}\]

\[D_0 = \frac{1-\nu}{2}\]
TABLE A.5b.-PRODUCT MATRIX $C^T_dC_p$ IN TERMS OF POINT COORDINATES $x$, $y$, and $z$.

$$
egin{bmatrix}
0 \\
0 & 1 \\
0 & y & y^2 + x^2D_o \\
0 & 0 & xD_o & D_o \\
0 & 0 & 0 & 0 & 0 & D_o \\
0 & 0 & xD_o & D_o & 0 & D_o \\
0 & vx & xyv + xyD_o & yD_o & 0 & yD_o & x^2 + y^2D_o \\
0 & v & vy & 0 & 0 & 0 & x & 1
\end{bmatrix}
$$

$$
D = \frac{E}{(1-\nu^2)}
$$

$$
D_o = \frac{1-\nu}{2}
$$
TABLE A.6a.-INTERMEDIATE STIFFNESS SUBMATRIX $k_b$ IN TERMS OF PLATE DIMENSIONS $a$, $b$, AND $t$. 

\[
\begin{bmatrix}
0 & 0 & 0 & 4 & \text{SYMMETRIC} \\
0 & 0 & 0 & 4D_0 & 0 \\
0 & 0 & 0 & 4v & 0 & 4 \\
0 & 0 & 0 & 6a & 0 & 6va & 12a^2 \\
0 & 0 & 0 & 2b & 4aD_0 & 2vb & 3ab & 4b^2/3 + 16a^2D_0/3 \\
0 & 0 & 0 & 2va & 4bD_0 & 2a & 4va^2 & vab & 4a^2/3 + 4abD_0 + 16b^2D_0/3 \\
0 & 0 & 0 & 6vb & 0 & 6b & 9vab & 4vb^2 & 3ab & 12b^2 \\
0 & 0 & 0 & 3ab & 4a^2D_0 & 3vab & 6a^2b & 2ab & 2va^2b & 4a^2b^2 + 6a^3D_0 + 4a^2bD_0 + 36a^4D_0/5 \\
0 & 0 & 0 & 3vab & 4b^2D_0 & 3ab & 6va^2b & 2vab^2 & 2a^2b & 4va^2b^2 + 4a^2b^2D_0 + 36b^4D_0/5 \\
\end{bmatrix}
\]

$D_1 = \frac{Eabt^3}{12(1-v^2)}$

$D_0 = \frac{1-v}{2}$
TABLE A.6b.-INTERMEDIATE STIFFNESS SUBMATRIX $k_0$ IN TERMS OF PLATE DIMENSIONS $a$, $b$, AND $t$.

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & aD_0 & 0 & 0 & 0 \\
0 & b/2 & b^2 + a^2D_0 & 0 & 0 & D_0 & 0 & 0 \\
0 & 0 & aD_0/2 & D_0 & 0 & D_0 & 0 & 0 \\
0 & v^a/2 & vab + abD_0 & bD_0/2 & 0 & bD_0/2 & a^2 + b^2D_0 & 0 \\
0 & v & vb/2 & 0 & 0 & 0 & a/2 & 1 \\
\end{array}
\]

$D_2 = \frac{Eabt}{(1-v^2)}$

$D_0 = \frac{1-v}{2}$
Combining Eq. A.26 with Eq. A.25b yields

\[
K = \begin{bmatrix} K_b & 0 \\ 0 & K_p \end{bmatrix} = \begin{bmatrix} [A_b^{-1}]^T k_b A_b^{-1} & 0 \\ 0 & [A_p^{-1}]^T k_p A_p^{-1} \end{bmatrix}
\]

(A.25c)

A step-by-step evaluation of \( K_b \) and \( K_p \) is given in Tables A.7 and A.8.

In the case of a vibrating system, the inertia loading may be considered as a static loading by use of d'Alembert's principle.

Since only forces at the nodes are being considered, a method of choosing nodal forces which will closely approximate the effect of the distributed inertia forces must be found. One way of approximating this distributed force is simply to assume that one-quarter of the mass of the element is concentrated at the element nodes. A more recently developed method, commonly called the method of consistent masses, consists of "lumping" masses at the nodes in a manner which will result in more realistic inertia forces at the nodes.

These inertia forces will be equal to the product of the nodal "masses" and their respective nodal accelerations. The masses are chosen such that the work done by the nodal inertia forces moving through the nodal deflections is equal to the work done by the distributed inertia force of the plate moving through the actual plate deflection corresponding to those nodal deflections.

Assuming deflection functions of the form:

\[
w(x, y, \tau) = w(x, y) \cdot T(\tau) = T \cdot x \alpha
\]

(A.28a)
TABLE A.7a.-PRODUCT MATRIX, $k_{bA^{-1}}$, IN TERMS OF PLATE DIMENSIONS $a$, $b$, AND $t$. 

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4D_0 & -4D_0 & 4D_0 & -4D_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -a & a & a & -a & -vb & -vb & vb & vb & vb \\
3b & 3b & -3b & -3b & -va^2 & va^2 & 2va^2 & -2va^2 & 0 & 0 & 3ab & 3ab \\
va & -va & -va & va & 0 & vab & vab & 0 & -b^2/3 & -2b^2/3 & 2b^2/3 & b^2/3 \\
+4aD_0 & -4aD_0 & +4aD_0 & -4aD_0 & 0 & vab & vab & 0 & +2a^2D_0/3 & -2a^2D_0/3 & +2a^2D_0/3 & -2a^2D_0/3
\end{bmatrix}
\]
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\[
D_1 = \frac{Eab^2}{12(1-v^2)}; \quad D_0 = \frac{1-v}{2}
\]
TABLE A.7b.-PLATE ELEMENT STIFFNESS SUBMATRIX $K_b$ IN TERMS OF PLATE DIMENSIONS $a$, $b$, AND $t$.

$$
\begin{array}{ccc}
\frac{4b+4a+2\nu}{a^3 \: b^3 \: ab} & \frac{-2b+4a+2\nu}{a^3 \: b^3 \: ab} & \frac{2b-4a-2\nu}{a^3 \: b^3 \: ab} \\
+28D_0 & -28D_0 & \frac{D_1}{ab} \\
\frac{-28D_0}{5ab} & \frac{-28D_0}{5ab} & \frac{28D_0}{5ab} \\
\frac{2b-4a-2\nu}{a^3 \: b^3 \: ab} & \frac{4b+4a+2\nu}{a^3 \: b^3 \: ab} & \frac{-2b-2a+2\nu}{a^3 \: b^3 \: ab} \\
\frac{-28D_0}{5ab} & \frac{-28D_0}{5ab} & \frac{28D_0}{5ab} \\
\frac{-4b+2a+2\nu}{a^3 \: b^3 \: ab} & \frac{4b-4a-2\nu}{a^3 \: b^3 \: ab} & \frac{-2b-2a+2\nu}{a^3 \: b^3 \: ab} \\
\frac{-28D_0}{5ab} & \frac{28D_0}{5ab} & \frac{-28D_0}{5ab} \\
\end{array}
$$

$D_0 = \frac{1-\nu}{2}$

$D_1 = \frac{Eabt^3}{12(1-\nu^2)}$

Continued
TABLE A.7b.-Continued

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$D_1$

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$\frac{D_1}{ab}$

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SYMMETRIC
TABLE A.8a.-PRODUCT MATRIX \((A_p^{-1})^T k_p\) IN TERMS OF PLATE DIMENSIONS a, b, AND t.

\[
\begin{bmatrix}
0 & -b & -b^2 \cdot a^2 D_0 & -a D_0 & -a D_0 & -\frac{ab - 2abD_0}{2} & -\frac{b}{2} \\
0 & -b & -b^2 + a^2 D_0 & a D_0 & 0 & \frac{a D_0}{2} & -\frac{ab + 2abD_0}{2} & -\frac{b}{2} \\
0 & b & b^2 - a^2 D_0 & a D_0 & 0 & \frac{a D_0}{2} & \frac{ab + 2abD_0}{2} & \frac{b}{2} \\
0 & b & b^2 - a^2 D_0 & -a D_0 & 0 & -\frac{a D_0}{2} & \frac{ab - 2abD_0}{2} & \frac{b}{2} \\
0 & -a & -ab - abD_0 & -b D_0 & 0 & -\frac{b D_0}{2} & -\frac{a^2 - 2b^2 D_0}{2} & -\frac{a}{2} \\
0 & a & ab + abD_0 & b D_0 & 0 & -\frac{b D_0}{2} & \frac{a^2 - 2b^2 D_0}{2} & \frac{a}{2} \\
0 & a & ab + abD_0 & b D_0 & 0 & -\frac{b D_0}{2} & \frac{a^2 - 2b^2 D_0}{2} & \frac{a}{2} \\
0 & -a & -ab - abD_0 & -b D_0 & 0 & -\frac{b D_0}{2} & -\frac{a^2 - 2b^2 D_0}{2} & -\frac{a}{2} \\
\end{bmatrix}
\]

\[
D_2 = \frac{Eabt}{(1-\nu^2)} \quad \text{and} \quad D_0 = \frac{1-\nu}{2}
\]
TABLE A.8b.-PLATE ELEMENT STIFFNESS SUBMATRIX \( K_p \) IN TERMS OF PLATE DIMENSIONS \( a, b, \) AND \( t.\)

\[
\begin{bmatrix}
\frac{4b+4aD_0}{ab} & 0 \\
\frac{2b-4aD_0}{ab} & \frac{4b+4aD_0}{ab} \\
\frac{-4b+2aD_0}{ab} & \frac{-2b-2aD_0}{ab} & \frac{2b-4aD_0}{ab} & \frac{4b+4aD_0}{ab} \\
\frac{3(1+v)}{2} & -\frac{3(1-3v)}{2} & -\frac{3(1+v)}{2} & \frac{3(1-3v)}{2} & \frac{4a+4bD_0}{b} & \frac{4a+4bD_0}{a} \\
\frac{3(1-3v)}{2} & -\frac{3(1+v)}{2} & -\frac{3(1-3v)}{2} & \frac{3(1+v)}{2} & -\frac{4a+2bD_0}{b} & \frac{4a+4bD_0}{b} & \frac{4a+4bD_0}{a} \\
\frac{-3(1+v)}{2} & \frac{3(1-3v)}{2} & \frac{3(1+v)}{2} & -\frac{3(1-3v)}{2} & -\frac{2a-2bD_0}{b} & \frac{2a-4bD_0}{b} & \frac{4a+4bD_0}{b} & \frac{4a+4bD_0}{b} \\
\frac{-3(1-3v)}{2} & \frac{3(1+v)}{2} & \frac{3(1-3v)}{2} & -\frac{3(1+v)}{2} & \frac{2a-4bD_0}{b} & \frac{-2a-2bD_0}{b} & \frac{-4a+2bD_0}{b} & \frac{4a+4bD_0}{a} \\
\end{bmatrix}
\]

\( D_2 = \frac{Eabt}{(1-v^2)} \)

\( D_0 = \frac{1-v}{2} \)
\[ \theta_x(x, y, \tau) = \theta_x(x, y) \cdot T(\tau) = T \cdot \frac{\partial x}{\partial y} \]  
(A.28b)

\[ \theta_y(x, y, \tau) = \theta_y(x, y) \cdot T(\tau) = T \cdot \frac{\partial x}{\partial x} \]  
(A.28c)

\[ u(x, y, \tau) = u(x, y) \cdot T(\tau) = T \cdot \gamma \]  
(A.28d)

\[ v(x, y, \tau) = v(x, y) \cdot T(\tau) = T \cdot \gamma \]  
(A.28e)

where \( \tau \) is the time coordinate, yields a matrix equation of the same form as Eq. A.12,

\[ \delta^e = T \cdot A^e \]  
(A.29a)

or

\[ A^e = \frac{1}{T} A^{-1} \delta^e \]  
(A.29b)

then

\[ \ddot{\delta}^e = \ddot{T} \cdot A^e \]  
(A.29c)

or

\[ A^e = \frac{1}{T} A^{-1} \delta^e \]  
(A.29d)

where \( \ddot{\cdot} \) denotes \( \frac{\partial^2}{\partial t^2} \).

The nodal inertia forces, \( F_i \), will be equal to the nodal mass matrix, \( M \), times the negative of the nodal accelerations, \( \ddot{\delta}^e \), where \( F_i \) has the same form as Eq. A.19. Then
Neglecting rotatory inertia, the distributed inertia forces of the plate will be due only to the transverse and in-plane motions. The deflection vector of the middle (reference) surface of the plate, $\delta_m$, will be

$$\delta_m(x,y,\tau) = T \cdot \delta_m(x,y)$$  \hspace{1cm} (A.31a)

where

$$\delta_m(x,y) = \begin{bmatrix} w(x,y) \\ u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & Y \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} B_{mb} & 0 \\ 0 & B_{mp} \end{bmatrix} \alpha^* = B_m \alpha^*$$  \hspace{1cm} (A.32)

The submatrices $B_{mb}$ and $B_{mp}$ are given in Table A.9.

Then

$$\delta_m(x,y,\tau) = T \cdot B_m \alpha^*$$  \hspace{1cm} (A.31b)

and

$$\ddot{\delta}(x,y,\tau) = \ddot{T} \cdot B_m \alpha^* = \ddot{T} \cdot B_m \frac{1}{T} \cdot A^{-1} \ddot{\delta}e = B_m A^{-1} \ddot{\delta}e$$  \hspace{1cm} (A.33)

If the mass per unit volume of the plate is $\rho$, then the inertia force per unit volume will be

$$dI_f = -\rho \ddot{\delta}dV = -\rho B_m A^{-1} \ddot{\delta}e dV$$  \hspace{1cm} (A.34)
TABLE A.9-SUBMATRICES $B_{mb}$ AND $B_{mp}$ IN TERMS OF POINT COORDINATES $x$, $y$, AND $z$.

a.-SUBMATRIX $B_{mb}$

\[
\begin{bmatrix}
1 & x & y & x^2 & xy & y^2 & x^3 & x^2y & xy^2 & y^3 & x^3y & xy^3
\end{bmatrix}
\]

b.-SUBMATRIX $B_{mp}$

\[
\begin{bmatrix}
1 & x & xy & y & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 1 & x & xy & y
\end{bmatrix}
\]
Again applying a virtual displacement $\delta^i e$, at the element nodes, the virtual work done by the nodal inertia forces is

$$W_v = (\delta^i e)^T F_I$$  \hfill (A.35)

The virtual plate deflection, $\delta'$ due to the virtual nodal displacements will become

$$\delta' = T \cdot B_m \alpha^* = T \cdot B_m (A^{-1} \delta^i e) = B_m A^{-1} \delta^i e$$ \hfill (A.36)

and the work done by a differential volume of the plate element will be

$$dW_4 = -\rho (\delta')^T B_m A^{-1} \delta^i e dV$$ \hfill (A.37)

Equating the virtual work done by the nodal forces to the total virtual work done by the distributed inertia forces yields,

$$(\delta^i e)^T F_I = -\int_V (\rho (\delta')^T B_m A^{-1} \delta^i e) dV$$

or

$$-(\delta^i e)^T M \ddot{\delta}^i e = -\rho \int_V (\delta^i e)^T (A^{-1})^T B_m T B_m A^{-1} \ddot{\delta}^i e dV$$

or

$$(\delta^i e)^T [M \ddot{\delta}^i e - \rho (A^{-1}) \int_V (B_m^T B_m) dV A^{-1} \ddot{\delta}^i e] = 0$$ \hfill (A.38)
Since this relationship also must hold true for any value of virtual displacements

\[ M \ddot{\mathbf{e}} - \rho (A^{-1})^T \int_{V} (B_m^T B_m) dV A^{-1} \mathbf{e} = 0 \]

or

\[ \left[ M - \rho (A^{-1})^T \int_{V} (B_m^T B_m) dV A^{-1} \right] \ddot{\mathbf{e}} = 0 \] (A.39)

For the non-trivial solution of \( \ddot{\mathbf{e}} \neq 0 \), then

\[ M = \rho (A^{-1})^T \int_{V} \int_{0}^{b} \int_{0}^{t/2} (B_m^T B_m) dz dy dz A^{-1} \] (A.40)

As in the case of the stiffness matrix, the matrices \( A^{-1} \) and \( B_m \) can be partitioned into the bending and in-plane submatrices and an intermediate mass matrix, \( m \), may be defined as

\[ m = \begin{bmatrix} m_b & 0 \\ 0 & m_p \end{bmatrix} = \int_{V} (B_m^T B_m) dV = \begin{bmatrix} \int_{V} (B_{mb}^T B_{mb}) dV & 0 \\ 0 & \int_{V} (B_{mp}^T B_{mp}) dV \end{bmatrix} \] (A.41)

An evaluation of \( m_b \) and \( m_p \) is given in Tables A.10 and A.11. Then,

\[ M = \begin{bmatrix} M_b & 0 \\ 0 & M_p \end{bmatrix} = \rho \begin{bmatrix} (A_b^{-1})^T m_b A_b^{-1} & 0 \\ 0 & (A_p^{-1})^T m_p A_p^{-1} \end{bmatrix} \] (A.42)

A step-by-step evaluation of \( M_b \) and \( M_p \) is given in Tables A.12 and A.13.
TABLE A.10a.-PRODUCT MATRIX $b_{mb}^T b_{mb}$ IN TERMS OF POINT COORDINATES $x, y, \text{AND} z$.

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<th>(y)</th>
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<th>(x^8)</th>
<th>(x^9)</th>
<th>(x^{10})</th>
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SYMMETRIC

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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{array}\]
TABLE A.10b. - PRODUCT MATRIX $B_{mp}^T B_{mp}$ IN TERMS OF POINT COORDINATES $x$, $y$, AND $z$.

\[
\begin{bmatrix}
1 \\
\times & x^2 \\
xy & x^2y & x^2y^2 \\
y & xy & xy^2 & y^2 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & x & x^2 \\
0 & 0 & 0 & 0 & xy & x^2y & x^2y^2 \\
0 & 0 & 0 & 0 & y & xy & xy^2 & y^2
\end{bmatrix}
\]

SYMMETRIC
### TABLE A.11a. - INTERMEDIATE MASS SUBMATRIX $m_b$ IN TERMS OF PLATE DIMENSIONS $a$, $b$, AND $t$.

$$
\begin{array}{cccccccc}
\frac{a}{2} & \frac{a^2}{3} \\
b & \frac{ab}{4} & \frac{b^2}{3} \\
\frac{a^2}{3} & \frac{a^3}{4} & \frac{a^2b}{6} & \frac{a^4}{5} \\
\frac{ab}{4} & \frac{a^2b}{6} & \frac{ab^2}{8} & \frac{a^2b^2}{9} \\
\frac{b^2}{3} & \frac{ab^2}{6} & \frac{b^3}{4} & \frac{a^2b^2}{9} & \frac{ab^3}{8} & \frac{b^4}{5} \\
\frac{a^3}{4} & \frac{a^4}{5} & \frac{a^3b}{8} & \frac{a^5}{6} & \frac{a^4b}{10} & \frac{a^3b^2}{12} & \frac{a^6}{7} \\
\frac{a^2b}{6} & \frac{a^3b}{8} & \frac{a^2b^2}{9} & \frac{a^4b}{10} & \frac{a^3b^2}{12} & \frac{a^5b}{12} & \frac{a^4b^2}{15} \\
\frac{ab^2}{6} & \frac{a^2b^2}{9} & \frac{ab^3}{8} & \frac{a^2b^3}{10} & \frac{a^4b}{12} & \frac{a^3b^3}{15} & \frac{a^2b^4}{16} \\
\frac{b^3}{4} & \frac{ab^3}{8} & \frac{b^4}{5} & \frac{a^2b^3}{12} & \frac{ab^4}{10} & \frac{b^5}{6} & \frac{a^3b^3}{16} & \frac{a^2b^4}{15} & \frac{ab^5}{12} & \frac{b^6}{7} \\
\frac{a^3b}{8} & \frac{a^4b}{10} & \frac{a^3b^2}{12} & \frac{a^5b}{12} & \frac{a^3b^2}{15} & \frac{a^6b}{14} & \frac{a^5b^2}{18} & \frac{a^4b^3}{20} & \frac{a^3b^4}{20} & \frac{a^6b^2}{21} \\
\frac{ab^3}{8} & \frac{a^2b^3}{12} & \frac{ab^4}{10} & \frac{a^3b^3}{16} & \frac{a^2b^4}{15} & \frac{ab^5}{12} & \frac{a^4b^3}{20} & \frac{a^3b^4}{20} & \frac{a^2b^5}{18} & \frac{ab^6}{14} & \frac{a^4b^4}{25} & \frac{a^2b^6}{21}
\end{array}
$$
TABLE A.11b.-INTERMEDIATE MASS SUBMATRIX $m_0$

IN TERMS OF PLATE DIMENSIONS $a$, $b$, AND $t$.

\[
\begin{bmatrix}
1 \\
\frac{a}{2} & \frac{a^2}{3} \\
\frac{ab}{4} & \frac{a^2b}{6} & \frac{a^2b^2}{9} \\
\frac{b}{2} & \frac{ab}{4} & \frac{ab^2}{6} & \frac{b^2}{3} \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & \frac{a}{2} & \frac{a^2}{3} \\
0 & 0 & 0 & 0 & \frac{ab}{4} & \frac{a^2b}{6} & \frac{a^2b^2}{9} \\
0 & 0 & 0 & 0 & \frac{b}{2} & \frac{ab}{4} & \frac{ab^2}{6} & \frac{b^2}{3}
\end{bmatrix}
\]

SYMMETRIC
<table>
<thead>
<tr>
<th></th>
<th>( b^2 )</th>
<th>( ab )</th>
<th>( a^2 )</th>
<th>( \frac{3a}{40} )</th>
<th>( 3b )</th>
<th>( \frac{3a}{40} )</th>
<th>( \frac{1}{4} )</th>
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<tr>
<td>1</td>
<td>( \frac{b^2}{30} )</td>
<td>( \frac{19ab}{150} )</td>
<td>( \frac{a^2}{30} )</td>
<td>( \frac{7b}{40} )</td>
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<td>( \frac{7b}{40} )</td>
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</tr>
<tr>
<td>2</td>
<td>( \frac{2b}{15} )</td>
<td>( \frac{9ab}{150} )</td>
<td>( \frac{a^2}{15} )</td>
<td>( \frac{3b}{40} )</td>
<td>( \frac{5ab}{360} )</td>
<td>( \frac{ab}{24} )</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{b^2}{30} )</td>
<td>( \frac{ab}{180} )</td>
<td>( \frac{a^2}{15} )</td>
<td>( \frac{3b}{40} )</td>
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<td>( \frac{ab}{36} )</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{b^2}{120} )</td>
<td>( \frac{-ab}{180} )</td>
<td>( \frac{a^2}{144} )</td>
<td>( \frac{2b}{40} )</td>
<td>( \frac{ab}{48} )</td>
<td>( \frac{ab}{48} )</td>
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</tr>
<tr>
<td>5</td>
<td>( \frac{-b^3}{60} )</td>
<td>( \frac{-ab}{120} )</td>
<td>( \frac{-a^2}{90} )</td>
<td>( \frac{2b}{40} )</td>
<td>( \frac{ab}{90} )</td>
<td>( \frac{ab}{90} )</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>6</td>
<td>( \frac{-b^3}{120} )</td>
<td>( \frac{-ab}{180} )</td>
<td>( \frac{-a^2}{180} )</td>
<td>( \frac{2b}{40} )</td>
<td>( \frac{ab}{180} )</td>
<td>( \frac{ab}{180} )</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>7</td>
<td>( \frac{ab}{48} )</td>
<td>( \frac{2b}{90} )</td>
<td>( \frac{a^2}{36} )</td>
<td>( \frac{-ab}{45} )</td>
<td>( \frac{-a^2}{45} )</td>
<td>( \frac{-a^2}{45} )</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>8</td>
<td>( \frac{-ab}{72} )</td>
<td>( \frac{-ab}{120} )</td>
<td>( \frac{-a^2}{72} )</td>
<td>( \frac{-ab}{40} )</td>
<td>( \frac{-a^2}{72} )</td>
<td>( \frac{-a^2}{120} )</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>9</td>
<td>( \frac{-ab}{120} )</td>
<td>( \frac{-ab}{180} )</td>
<td>( \frac{-a^2}{180} )</td>
<td>( \frac{-ab}{40} )</td>
<td>( \frac{-a^2}{180} )</td>
<td>( \frac{-a^2}{180} )</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>10</td>
<td>( \frac{-ab}{120} )</td>
<td>( \frac{-ab}{180} )</td>
<td>( \frac{-a^2}{180} )</td>
<td>( \frac{-ab}{40} )</td>
<td>( \frac{-a^2}{180} )</td>
<td>( \frac{-a^2}{180} )</td>
<td>( \frac{1}{4} )</td>
</tr>
</tbody>
</table>

**TABLE A.12a.** PRODUCT MATRIX \( M_{ab} \) IN TERMS OF PLATE DIMENSIONS \( a, b, \) AND \( t \).
<table>
<thead>
<tr>
<th></th>
<th>3</th>
<th>3</th>
<th>3</th>
<th>3</th>
<th>b</th>
<th>b</th>
<th>4</th>
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<td>56</td>
<td>28</td>
<td>28</td>
<td>240</td>
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<td>60</td>
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</tr>
<tr>
<td></td>
<td>720</td>
<td>720</td>
<td>720</td>
<td>720</td>
<td>360</td>
<td>240</td>
<td>80</td>
<td>80</td>
<td>180</td>
<td>180</td>
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<tr>
<td></td>
<td>720</td>
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<td>720</td>
<td>360</td>
<td>180</td>
<td>90</td>
<td>180</td>
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<td>120</td>
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<td>56</td>
<td>28</td>
<td>28</td>
<td>56</td>
<td>210</td>
<td>84</td>
<td>84</td>
<td>210</td>
<td>240</td>
<td>60</td>
</tr>
<tr>
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<td>8400</td>
<td>8400</td>
<td>2100</td>
<td>8400</td>
<td>600</td>
<td>400</td>
<td>100</td>
<td>150</td>
<td>630</td>
<td>315</td>
</tr>
<tr>
<td></td>
<td>8400</td>
<td>8400</td>
<td>2100</td>
<td>8400</td>
<td>630</td>
<td>252</td>
<td>126</td>
<td>315</td>
<td>600</td>
<td>150</td>
</tr>
</tbody>
</table>
TABLE A.12b.-PLATE ELEMENT MASS SUBMATRIX $M_b$ IN TERMS OF PLATE DIMENSIONS $a$, $b$, AND $t$.

\[ \begin{array}{cccc}
24178 & 8582 & 24178 \\
8582 & 2758 & 8582 & 24178 \\
2758 & 8582 & 24178 & \\
8582 & 2758 & 8582 & 24178 & \text{SYMMETRIC} \\
3227b & 1918b & 812b & 1393b & 560b^2 \\
-1918b & -3227b & -1391b & -812b & -420b^2 & 560b^2 \\
-812b & -1393b & -3227b & -1919b & -210b^2 & 280b^2 & 560b^2 \\
1393b & 812b & 1918b & 3227b & 280b^2 & -210b^2 & -420b^2 & 560b^2 \\
3227a & 1393a & 812a & 1918a & 441ab & -294ab & -196ab & 294ab & 560a^2 \\
1393a & 3227a & 1918a & 812a & 294ab & -441ab & -294ab & 196ab & 280a^2 & 560a^2 \\
-812a & -1918a & -3227a & -1393a & -196ab & 294ab & 441ab & -294ab & -210a^2 & -420a^2 & 560a^2 \\
-1918a & -812a & -1393a & -3227a & -294ab & 196ab & 294ab & -441ab & -420a^2 & -210a^2 & 280a^2 & 560a^2 \\
\end{array} \]

\[ p = \frac{pabt}{176,400} \]
TABLE A.13a.-PRODUCT MATRIX $\mathbf{m}_a^\top \mathbf{A}^{-1}_b$ IN TERMS OF PLATE DIMENSIONS $a$, $b$, AND $t$.

\[
\begin{bmatrix}
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\
\frac{a}{12} & \frac{a}{12} & \frac{a}{6} & \frac{a}{6} & 0 & 0 & 0 & 0 \\
\frac{ab}{36} & \frac{ab}{18} & \frac{ab}{9} & \frac{ab}{18} & 0 & 0 & 0 & 0 \\
\frac{b}{12} & \frac{b}{6} & \frac{b}{6} & \frac{b}{12} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
0 & 0 & 0 & 0 & \frac{a}{12} & \frac{a}{12} & \frac{a}{6} & \frac{a}{6} \\
0 & 0 & 0 & 0 & \frac{ab}{36} & \frac{ab}{18} & \frac{ab}{9} & \frac{ab}{18} \\
0 & 0 & 0 & 0 & \frac{b}{12} & \frac{b}{6} & \frac{b}{6} & \frac{b}{12}
\end{bmatrix}
\]
TABLE A.13b.-PLATE ELEMENT MASS SUBMATRIX $M_p$

IN TERMS OF PLATE DIMENSIONS $a$, $b$, AND $t$.

$P_1 = \frac{a b t}{36}$

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>4</th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
STIFFENER ELEMENT STIFFNESS AND MASS MATRICES

A stiffener element (with nodal displacements and nodal forces shown at one node only) is given in Fig. 2.b. The nodal displacements of the stiffener element are defined as

\[ \delta^e = \left\{ \begin{array}{c}
\delta_m \\
\delta_h \\
\theta_{xm} \\
\theta_{xn} \\
\theta_{ym} \\
\theta_{yn} \\
\delta_{um} \\
\delta_{vn} \\
\delta_{vm} \\
\delta_{vn} \\
\end{array} \right\} \]

Considering the stiffener to be attached to the plate along the y-axis of a plate element (x=0), then the displacement functions for the stiffener become

\[ \bar{w} = \bar{w}(0,y,0) = w(0,y,0) \text{ of plate element} \]

\[ = \alpha_1 + \alpha_3 y + \alpha_6 y^2 + \alpha_{10} y^3 \]

\[ = \begin{bmatrix} 1, y, y^2, y^3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_3 \\ \alpha_6 \\ \alpha_{10} \end{bmatrix} = \bar{x} \bar{\alpha} \]
\[ \bar{\theta}_x(0,y,0) = \frac{\partial w}{\partial y} = [0,1,2y,3y^2] \begin{bmatrix} \alpha_1 \\ \alpha_3 \\ \alpha_6 \\ \alpha_{10} \end{bmatrix} = \frac{\partial x}{\partial y} \bar{\theta}_x \]

\[ \bar{u}(0,y,0) = \beta_1 + \beta_4 y = [1,y] \begin{bmatrix} \beta_1 \\ \beta_4 \end{bmatrix} = \bar{V} \bar{\beta} \]

\[ \bar{v}(0,y,0) = \gamma_1 + \gamma_4 y = [1,y] \begin{bmatrix} \gamma_1 \\ \gamma_4 \end{bmatrix} = \bar{V} \bar{\gamma} \]

(A.45)  
(A.46)  
(A.47)

Since \( \bar{\theta}_y \) for the stiffener is independent of the lateral deflection, \( w \), it must be treated separately from \( \frac{\partial w}{\partial x} \). It will be assumed here that \( \bar{\theta}_y \) varies linearly, or that

\[ \bar{\theta}_y(0,y,0) = \zeta_1 + \zeta_2 y = [1,y] \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \bar{V} \bar{\zeta} \]

(A.48)

Thus there will also be non-conformity between the slope of the plate element in the \( x \)-direction, \( \theta_y \), along the line \( x=0 \) and the angle of twist of the stiffener element, \( \bar{\theta}_y \).

In terms of the displacement functions, the nodal displacements become
\[
\begin{align*}
\overline{\alpha}^e &= \begin{bmatrix}
(X)_m & 0 & 0 & 0 \\
(X)_n & 0 & 0 & 0 \\
\frac{\partial (X)}{\partial x}_m & 0 & 0 & 0 \\
\frac{\partial (X)}{\partial x}_n & 0 & 0 & 0 \\
0 & (Y)_m & 0 & 0 \\
0 & (Y)_n & 0 & 0 \\
0 & 0 & (Y)_m & 0 \\
0 & 0 & (Y)_n & 0 \\
0 & 0 & 0 & (Y)_m \\
0 & 0 & 0 & (Y)_n
\end{bmatrix}
\begin{bmatrix}
\overline{\alpha} \\
\overline{\zeta} \\
\overline{\beta} \\
\overline{\gamma}
\end{bmatrix} = \overline{A} \overline{\alpha}^e
\end{align*}
\]

or

\[
\overline{\alpha}^e = \overline{A}^{-1} \overline{\delta}^e
\]

The matrices \(\overline{A}\) and \(\overline{A}^{-1}\) are given in terms of the plate dimensions in Tables A.14 and A.15.

The deformations of the y-axis of the stiffener element, \(\overline{\delta}(0,y,0)\), at any point along the y-axis can be found in terms of the nodal displacements, \(\overline{\delta}^e\), by
TABLE A.14.-MATRIX $\bar{A}$ IN TERMS OF STIFFENER LENGTH $b$.

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & b & b^2 & b^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2b & 3b^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & y & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & y & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & y & 0 \\
\end{bmatrix}
$$
TABLE A.15.-MATRIX $\overline{A^{-1}}$ IN TERMS OF STIFFENER LENGTH $b$.

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-3}{b^2} & \frac{3}{b^2} & \frac{-2}{b} & \frac{-1}{b} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{2}{b^3} & \frac{-2}{b^3} & \frac{1}{b^2} & \frac{1}{b^2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{-1}{b} & \frac{1}{b} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{b} & \frac{1}{b} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{b} & \frac{1}{b} & 0 & 0 \\
\end{bmatrix}
$$
The horizontal displacements anywhere in the stiffener element can be found from

\[
\begin{align*}
\vec{\delta}(0,y,0) &= \begin{bmatrix} \vec{w}(0,y,0) \\ \vec{\theta}_x(0,y,0) \\ \vec{\theta}_y(0,y,0) \\ \vec{u}(0,y,0) \\ \vec{v}(0,y,0) \end{bmatrix} = \begin{bmatrix} \chi & 0 & 0 & 0 \\ \frac{\partial \chi}{\partial y} & 0 & 0 & 0 \\ 0 & \bar{Y} & 0 & 0 \\ 0 & 0 & \bar{Y} & 0 \\ 0 & 0 & 0 & \bar{Y} \end{bmatrix} \begin{bmatrix} \vec{\alpha} \\ \vec{\zeta} \\ \vec{\beta} \\ \vec{\gamma} \end{bmatrix} = B A^{-1} \vec{\delta}^e = \vec{a} \vec{\delta}^e
\end{align*}
\]

(A.51)

The shearing stress of the stiffener element in the x-z plane due to torsion will be handled separately.

Thus, strain in the y-direction will be

\[
\begin{align*}
\vec{u}(x,y,z) &= \vec{u} - z \bar{\theta}_y \\
\vec{v}(x,y,z) &= \vec{v} - z \bar{\theta}_x = \vec{v} - z \frac{\partial \vec{w}}{\partial y}
\end{align*}
\]

(A.52a)

(A.52b)

Since \(\partial \vec{u}(x,y,z)/\partial x = 0\), there will be no strain in the x-direction of the stiffener element. Also, even though \(\partial \vec{u}(x,y,z)/\partial y\) does exist, the shear in the x-y plane will be neglected. It is assumed that small relative displacements of the ends of the stiffener element in the x-direction will produce no strains in the element. This assumes, in effect, that the distribution of shearing strains in the x-y plane of the plate structure is not affected by the addition of the stiffeners.
\[
\bar{\varepsilon} = \bar{\varepsilon}_y = \frac{\partial^2 v(x, y, z)}{\partial y^2} = \left[ -z \frac{\partial^2 x}{\partial y^2} \quad 0 \quad 0 \quad \frac{\partial y}{\partial y} \right] \bar{\alpha}^* = \bar{C}_1 \bar{A}^{-1} \bar{\delta}e = b_1 \bar{\delta}e
\]  
(A.53)

The matrix \( \bar{C}_1 \) is given in Table A.16.

For unidirectional stress

\[
\sigma = E \varepsilon
\]  
(A.54)

Thus

\[
\bar{\sigma}_y = E \bar{\varepsilon}_y = E b_1 \bar{\delta}e
\]  
(A.55)

Defining the element nodal force vector as

\[
\bar{F}_e = \begin{pmatrix}
\bar{V}_m \\
\bar{V}_n \\
\bar{T}_{xm} \\
\bar{T}_{xn} \\
\bar{T}_{ym} \\
\bar{T}_{yn} \\
\bar{S}_{xm} \\
\bar{S}_{xn} \\
\bar{S}_{ym} \\
\bar{S}_{yn}
\end{pmatrix}
\]  
(A.56)

and again applying virtual nodal displacements, \( \bar{\delta}^e \), at the element nodes, the external virtual work becomes

\[
W_e = (\bar{\delta}^e)^T \bar{F}
\]  
(A.57)
TABLE A.16.-MATRIX $\mathbf{c}_1$ IN TERMS OF POINT COORDINATES $x$, $y$, AND $z$.

$$
\begin{bmatrix}
0 & 0 & -2z & -6yz & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$
while the internal virtual strain energy becomes

\[ dW_i = (\delta e)^T \int_V (\bar{b}_i^T E \bar{b}_i) \delta e \]  

(A.58)

In the same manner as for the plate element, Eqs. A.57 and A.58 combine to give

\[ \bar{F}^e = \bar{K}_1 \bar{\delta}^e \]  

(A.59)

where

\[ \bar{K}_1 = E (\bar{A}^{-1})^T \int_{t/2}^{t/2} b^T b \int_{(t/2 + h)}^0 (C_1^T C_1) dz dy dx \bar{A}^{-1} = E (\bar{A}^{-1})^T \bar{k}_1 \bar{A}^{-1} \]  

(A.60)

Thus \( \bar{K}_1 \) is the stiffness matrix for bending and in-plane displacements. The matrix \( \bar{k}_1 \) is evaluated in Tables A.17 and A.18.

Due to the complexity of torsion of rectangular sections, the torsional stiffness of the stiffener element will be found using a slightly different approach.

The angle of twist, \( \phi_y \), at a section is the change in \( \phi_y \) with respect to \( y \), or

\[ \bar{\phi}_y = \frac{\partial \phi_y}{\partial y} = [0 \quad \frac{\partial \gamma}{\partial y} \quad 0 \quad 0] \bar{\alpha} = \bar{C}_2 \bar{A}^{-1} \bar{\delta}^e = \bar{b}_2 \bar{\delta}^e \]  

(A.61)

The matrix \( \bar{C}_2 \) is given in Table A.19. The torque at a section is found from

\[ \bar{T}_y = GJ \bar{\phi}_y = GJ \frac{\partial \phi_y}{\partial y} = GJ \bar{b}_2 \bar{\delta}^e \]  

(A.62)
TABLE A.17.-PRODUCT MATRIX $\mathbf{C}_1^T \mathbf{C}_1$ IN TERMS OF POINT COORDINATES $x, y, \text{AND} z$.

\[
\begin{bmatrix}
0 \\
0 & 0 \\
0 & 0 & 43^2 \\
0 & 0 & 12yz^2 & 36y^2z^2 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2z & -6yz & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
TABLE A.1.8.-INTERMEDIATE STIFFNESS MATRIX $k_1$ IN TERMS OF STIFFENER DIMENSIONS $t$, $b$, and $h$.

<table>
<thead>
<tr>
<th>$0$</th>
<th>$0$</th>
<th>$4bI_x$</th>
<th>$6b^2I_x$</th>
<th>$12b^3I_x$</th>
<th>$2bcA_s$</th>
<th>$3b^2cA_s$</th>
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<tr>
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<td>$0$</td>
<td>$2bcA_s$</td>
<td>$3b^2cA_s$</td>
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</table>

$I_x = \frac{th^3}{12} + e_c^2A_s$

$A_s = \bar{t}h$

$e_c = \frac{(t+h)}{2}$

$t =$ plate element thickness
TABLE A.19.-MATRIX $\mathbf{C}_2$ IN TERMS OF POINT COORDINATES $x$, $y$, AND $z$.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$
in which \( G \) is the shearing modulus of elasticity of the stiffener element and \( J \) is the St. Venant's modified polar moment of inertia of the stiffener element cross-section. For a rectangular cross-section (4A),

\[
J = \frac{t^3}{3} (h - 0.63\bar{t}) \tag{A.63}
\]

where

\[ h = \text{long dimension of the rectangular section}, \]
\[ \bar{t} = \text{short dimension of the rectangular section}. \]

Again applying a virtual deflection, \( \vec{\delta}^e \), at the element nodes and equating external and internal virtual work, the torsional stiffness of the stiffener element can be found as

\[
F^e = \overline{K}_2 \vec{\delta}^e \tag{A.64}
\]

where

\[
\overline{K}_2 = JG(A^{-1})^T \left( \int_0^b (C_2^T \bar{C}_2) dy \right) A^{-1} = JG(A^{-1})^T \overline{k}_2 A^{-1} \tag{A.65}
\]

Then \( \overline{K}_2 \) is the stiffness matrix for torsional displacements. The matrix \( \overline{k}_2 \) is evaluated in Tables A.20 and A.21.

The complete stiffness matrix, \( \overline{K} \), for the stiffener element may now be found by adding the stiffness matrix for bending and in-plane displacements to the stiffness matrix for torsional displacements, or
TABLE A.20.-PRODUCT MATRIX $\mathbf{c}_2^T \mathbf{c}_2$ IN TERMS OF POINT COORDINATES x, y, AND z.

\[
\begin{bmatrix}
0 \\
0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

SYMMETRIC
TABLE A.21.-INTERMEDIATE STIFFNESS MATRIX $k_2$ IN TERMS OF STIFFENER DIMENSIONS $t$, $b$, AND $h$. 

\[
\begin{bmatrix}
0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

SYMMETRIC
A complete evaluation of the stiffness matrices, $\bar{K}_1$ and $\bar{K}_2$, for the stiffener element is given in Tables A.22 and A.23.

The mass matrix for the stiffener element is found in a manner similar to that for the plate element. The fact that the plate element was considered to be a thin plate allowed the assumption to be made that the mass was distributed along the middle surface of the plate, i.e., the deflection of the mass of the plate was a function of $x$ and $y$ only. Thus, the deflection of the mass was identical to the deflection of the reference surface (neglecting rotatory inertia). The only limitation upon the stiffener element dimensions was that the stiffener be shallow enough that the shearing strains in the $y$-$z$ plane due to bending are negligible. Thus, the mass of the stiffener cannot be assumed to be concentrated along the stiffener axis and must be assumed to be distributed throughout the entire element. Again neglecting rotatory inertia, the transverse and in-plane motions of the stiffener mass become

\[
\bar{w}_m(x,y,z,\tau) = (\bar{w}(0,y,0) + x\bar{\alpha}_y) \cdot T(\tau)
\]

\[
= T \cdot [\bar{X}, \bar{X}Y, 0, 0] \bar{\alpha}^* \tag{A.67}
\]

\[
\bar{u}_m(x,y,z,\tau) = (\bar{u}(0,y,0) - z\bar{\alpha}_y) \cdot T(\tau)
\]

\[
= T \cdot [0, -z\bar{Y}, \bar{Y}, 0] \bar{\alpha}^* \tag{A.68}
\]
TABLE A.22a.-PRODUCT MATRIX $k_1\bar{A}^{-1}$ IN TERMS OF STIFFENER DIMENSIONS $t$, $b$, AND $h$.

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2I_x & -2I_x & 0 & 0 & 0 & -2e_cA_s & 2e_cA_s & 0 \\
0 & 0 & -6I_x & 6bI_x & 0 & 0 & 0 & -3be_cA_s & 3be_cA_s & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -e_cA_s & e_cA_s & 0 & 0 & 0 & -A_s & A_s & 0
\end{bmatrix}
\]

$I_x = \frac{\bar{t}h^3}{12} + e_c^2A_s$; $A_s = \bar{t}h$; $e_c = \frac{(t+h)}{2}$; $t$ = plate element thickness.
TABLE A.22b.-STIFFENER ELEMENT STIFFNESS MATRIX $\bar{k}_1$ IN TERMS OF STIFFENER DIMENSIONS $\bar{t}$, $\bar{b}$, AND $\bar{h}$.

\[
\begin{bmatrix}
\frac{12I_x}{b^2} & \frac{12I_x}{b^2} & \frac{12I_x}{b^2} \\
-\frac{12I_x}{b^2} & \frac{12I_x}{b^2} & \frac{12I_x}{b^2} \\
6I_x & -6I_x & 4I_x \\
6I_x & -6I_x & 2I_x & 4I_x \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & e_c A_s & -e_c A_s & 0 & 0 & 0 & 0 & A_s \\
0 & 0 & -e_c A_s & e_c A_s & 0 & 0 & 0 & 0 & -A_s & A_s
\end{bmatrix}
\]

$\frac{12I_x}{b^2}$

SYMmetric $\frac{12I_x}{b^2}$

-12I_x \quad 12I_x \quad 12I_x

\[
I_x = \frac{\bar{t}\bar{h}^3}{12} + \frac{e_c^2}{4}A_s
\]

$A_s = \bar{t}\bar{h}$

$e_c = \frac{(t+h)}{2}$

$t = \text{plate element thickness}$
TABLE A.23a.-PRODUCT MATRIX $\bar{k}_2\bar{A}^{-1}$ IN TERMS OF STIFFENER DIMENSIONS $\bar{e}, b, \text{AND} h.$

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
TABLE A.23b.-STIFFENER ELEMENT STIFFNESS MATRIX $k_2$ IN TERMS OF STIFFENER DIMENSIONS $\bar{t}$, $b$, AND $h$.

\[
\begin{bmatrix}
0 \\
0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
J = \frac{\bar{t}^3}{3} (h-0.63\bar{t})
\]
Then

\[ \vec{v}_m(x, y, z, \tau) = \{ \vec{v}(0, y, 0) - z \frac{x}{x} \} \cdot T(\tau) \]

\[ = T \cdot [-z \frac{x}{y}, 0, 0, 0] \alpha^* \]  

(A.69)

Then

\[ \vec{\delta}_m(x, y, z, \tau) = \begin{bmatrix} \vec{w}_m \\ \vec{u}_m \\ \vec{v}_m \end{bmatrix} = T \cdot \begin{bmatrix} x & xy & 0 & 0 \\ 0 & -zY & Y & 0 \\ -z \frac{x}{y} & 0 & 0 & Y \end{bmatrix} \alpha^* \]

\[ = T \cdot \vec{B}_m \alpha^* \]  

(A.70)

and

\[ \vec{\ddot{\delta}}_m = \vec{T} \cdot \vec{B}_m \alpha^* = \vec{B}_m \vec{A}^{-1} \delta E \]  

(A.71)

The matrix \( \vec{B}_m \) is given in Table A.24.

Again applying a virtual nodal displacement at the element nodes and equating the virtual work done by the nodal inertia forces to the virtual work done by the distributed inertia forces yields

\[ \vec{M} = \rho (\vec{A}^{-1}) T \int T/2 \int \begin{bmatrix} t/2 & 0 \\ 0 & t/2 \end{bmatrix} (\vec{B}_m^T \vec{B}_m) dz dy dx \vec{A}^{-1} \]

\[ = \rho (\vec{A}^{-1}) T \vec{m} \vec{A}^{-1} \]  

(A.72)

The matrix \( \vec{m} \) is given in terms of the element dimensions in Table A.25 while an evaluation of \( \vec{M} \) is given in Table A.26.
TABLE A.24.-MATRIX $B_m$ IN TERMS OF POINT COORDINATES $x$, $y$, AND $z$.

$$
\begin{bmatrix}
1 & y & y^2 & y^3 & x & xy & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -z & -zy & 1 & y & 0 & 0 \\
0 & -z & -2yz & -3y^2z & 0 & 0 & 0 & 0 & 1 & y
\end{bmatrix}
$$
TABLE A.25a.-PRODUCT MATRIX $\mathbf{B}_m^T \mathbf{B}_m$ IN TERMS OF POINT COORDINATES $x$, $y$, AND $z$.

$$
\begin{bmatrix}
1 \\
y \\
y^2 \\
y^2 \\
x \\
xy \\
xy \\
x \\
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\end{bmatrix}
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**SYMMETRIC**
TABLE A.25b.-INTERMEDIATE MASS MATRIX $\bar{m}$ IN TERMS OF STIFFENER DIMENSIONS $t$, $b$, AND $h$.

$$ 1 $$

**SYMMETRIC**

<table>
<thead>
<tr>
<th></th>
<th>$\frac{b}{2}$</th>
<th>$\frac{b^2 + I_x}{3 A_s}$</th>
<th>$J_0$</th>
<th>$J_0 = I_x + I_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b^2$</td>
<td>$\frac{b^3 + b I_x}{4 A_s}$</td>
<td>$\frac{b^4 + 4 b^2 I_x}{5 3 A_s}$</td>
<td>$I_x = \frac{t h^3}{12} + e_c^2 A_s$</td>
<td></td>
</tr>
<tr>
<td>$b^3$</td>
<td>$\frac{b^4 + b^2 I_x}{5 A_s}$</td>
<td>$\frac{b^5 + 3 b^3 I_x}{6 2 A_s}$</td>
<td>$I_z = \frac{t h^3}{12}$</td>
<td></td>
</tr>
<tr>
<td>$b^4$</td>
<td>$\frac{b^6 + 9 b^4 I_x}{7 5 A_s}$</td>
<td>$e_c = \frac{(t+h)}{2}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$t =$ plate element thickness

$$ A_s b $$

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>$J_0$</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{b J_0}{2}$</td>
<td>$\frac{b^2 J_0}{3}$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$e_c$</td>
<td>$\frac{b e_c}{2}$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{b e_c}{2}$</td>
<td>$\frac{b^2 e_c}{3}$</td>
</tr>
<tr>
<td>0</td>
<td>$e_c$</td>
<td>$b e_c$</td>
<td>$b^2 e_c$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>$\frac{b e_c}{2}$</td>
<td>$\frac{2 b^2 e_c}{3}$</td>
<td>$\frac{3 b^3 e_c}{4}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
TABLE A.26a.-PRODUCT MATRIX $\mathbf{m}^{-1}$ IN TERMS OF STIFFENER DIMENSIONS $t, b,$ AND $h.$

| $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{b}{12}$ | $-\frac{b}{12}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3$b^{-1}$ | $\frac{7b+1}{20}$ | $\frac{b^2}{30}$ | $-\frac{b^2}{20}$ | 0 | 0 | 0 | 0 | $\frac{e_c}{2}$ | $\frac{e_c}{2}$ |
| $\frac{b^2}{15}$ | $\frac{4b^2}{15}$ | $\frac{b^3-b^{-1}}{60}$ | $-\frac{b^3+b^{-1}}{30}$ | 0 | 0 | 0 | 0 | $\frac{b e_c}{3}$ | $\frac{2b e_c}{3}$ |
| $\frac{b^{-1}}{28}$ | $\frac{3b^3+9b^{-1}}{10}$ | $\frac{b^4-b^{-2}1}{105}$ | $-\frac{b^4+3b^{-2}1}{42}$ | 0 | 0 | 0 | 0 | $\frac{b^2 e_c}{4}$ | $\frac{3b^2 e_c}{4}$ |
| 0 | 0 | 0 | 0 | $\frac{J_0}{2A_s}$ | $\frac{J_0}{2A_s}$ | $\frac{e_c}{2}$ | $\frac{e_c}{2}$ | 0 | 0 |
| 0 | 0 | 0 | 0 | $\frac{b J_0}{6A_s}$ | $\frac{b J_0}{3A_s}$ | $\frac{b e_c}{6}$ | $\frac{b e_c}{3}$ | 0 | 0 |
| 0 | 0 | 0 | 0 | $\frac{e_c}{2}$ | $\frac{e_c}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 |
| 0 | 0 | 0 | 0 | $\frac{b e_c}{6}$ | $\frac{b e_c}{3}$ | $b$ | $\frac{b}{3}$ | 0 | 0 |
| $\frac{-e_c}{b}$ | $\frac{e_c}{b}$ | 0 | 0 | 0 | 0 | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $\frac{-e_c}{2}$ | $\frac{e_c}{2}$ | $\frac{-b e_c}{12}$ | $\frac{b e_c}{12}$ | 0 | 0 | 0 | 0 | $\frac{b}{6}$ | $\frac{b}{3}$ |

$J_0 = I_x + I_z$; $\bar{I} = \frac{I_x}{A_s}$; $I_x = \frac{t h^3}{12} + e^2 A_s$; $I_z = \frac{t^3 h}{12}$; $A_s = \overline{t h}$; $e_c = \frac{(t + h)}{2}$;

t = plate element thickness.
TABLE A.26b.-STIFFENER ELEMENT MASS MATRIX $\bar{M}$ IN TERMS OF STIFFENER DIMENSIONS $t$, $b$, AND $h$.

\[
\begin{array}{cccccc}
\frac{13+6t}{35} & \frac{6b^2}{5b^2} & \frac{9-6t}{70} & \frac{6b^2}{5b^2} & 0 & 0 \\
\frac{13b+6}{210} & \frac{13b-6}{420} & \frac{b^2+2t}{105} & \frac{b^2+2t}{105} & 0 & 0 \\
\frac{-13b+6}{420} & \frac{-11b-6}{210} & \frac{-b^2-6}{140} & \frac{-b^2-6}{140} & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & \frac{J_o}{3A_S} & t = \text{plate element thickness} \\
0 & 0 & 0 & 0 & \frac{J_o}{6A_S} & \frac{J_o}{3A_S} \\
0 & 0 & 0 & 0 & \frac{e_c}{3} & \frac{e_c}{6} & \frac{1}{3} \\
0 & 0 & 0 & 0 & \frac{e_c}{6} & \frac{e_c}{3} & \frac{1}{6} & \frac{1}{3} \\
\frac{-e_c}{2b} & \frac{e_c}{2b} & \frac{e_c}{12} & \frac{-e_c}{12} & 0 & 0 & 0 & 0 & \frac{1}{3} \\
\frac{-e_c}{2b} & \frac{e_c}{2b} & \frac{-e_c}{12} & \frac{e_c}{12} & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{3}
\end{array}
\]

\[\bar{p} = \rho b A_s\]

\[J_o = I_x + I_z\]

\[I = \frac{I_x}{A_s}\]

\[I_x = \frac{\bar{t} h^3}{12} + e_c^2 A_s\]

\[I_z = \frac{\bar{t}^3 h}{12}\]

\[A_s = \bar{t} h\]

\[e_c = \frac{(t+h)}{2}\]
COMPATIBILITY CONDITIONS

The necessary criteria such that the finite element results will converge toward the true state of deformation of the structure have been given in the description of the Finite Element Method. It has already been stated that the finite element solution presented here does not satisfy all of the conditions of compatibility at the element boundaries. However, if, as the mesh size decreases, these compatibility conditions are met, the finite element process will still tend toward the correct solution. This condition is usually reached if (8A):

1. The displacement functions chosen are such that nodal displacements which are consistent with constant strain conditions will cause such constant strain.

2. A constant strain condition insures displacement continuity at the element boundaries.

Using the assumed displacement functions, the strain in the plate element in terms of the nodal displacements, was found to be

\[
\varepsilon = \begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_{xy}
\end{bmatrix} = CA^{-1}\delta_e \quad \text{(A.73)}
\]

The problem now becomes finding the nodal displacements which correspond to constant strain and checking that these nodal displacements yield a constant strain vector from Eq. A.73.
From Eq. A.16a,

\[ \varepsilon_x = \frac{\partial u(x,y,z)}{\partial x} = \frac{\partial}{\partial x}(u(x,y) - z \frac{\partial w(x,y)}{\partial x}) \]

\[ = \beta_2 + \beta_3 y - z(2\alpha_4 + 6\alpha_7 x + 2\alpha_8 y + 6\alpha_{11} x y) \quad \text{(A.74a)} \]

\[ \varepsilon_y = \frac{\partial v(x,y,z)}{\partial y} = \frac{\partial}{\partial y}(v(x,y) - z \frac{\partial w(x,y)}{\partial y}) \]

\[ = \gamma_3 x + \gamma_4 - z(2\alpha_6 + 2\alpha_9 x + 6\alpha_{10} y + 6\alpha_{12} x y) \quad \text{(A.74b)} \]

and

\[ \varepsilon_{xy} = \frac{\partial u(x,y,z)}{\partial y} + \frac{\partial u(x,y,z)}{\partial x} \]

\[ = \beta_3 x + \beta_4 + \gamma_2 + \gamma_3 y - 2z(\alpha_5 + 2\alpha_8 x + 2\alpha_9 y + 3\alpha_{11} x^2 + 3\alpha_{12} y^2) \quad \text{(A.74c)} \]

For constant strain in the x-direction

\[ \frac{\partial \varepsilon_x}{\partial x} = - z(6\alpha_7 + 6\alpha_{11} y) = 0 \quad \text{(A.75a)} \]

and

\[ \frac{\partial \varepsilon_x}{\partial y} = \beta_3 - z(2\alpha_8 + 6\alpha_{11} x) = 0 \quad \text{(A.75b)} \]

In order to satisfy Eqs. A.75 for any value of x, y, and z, the constants \( \alpha_7, \alpha_8, \alpha_{11} \) and \( \beta_3 \) must be zero. Thus, for constant strain in the x-direction, the displacement functions reduce to
The nodal deflections now become

\[ w(x,y,0) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 xy^2 + \alpha_{10} y^3 \]

\[ + \alpha_{12} xy^2 \]

\[ \theta_y(x,y,0) = \frac{\partial w}{\partial x} = \alpha_2 + 2\alpha_4 x + \alpha_5 y + \alpha_7 y^2 + \alpha_{12} y^3 \]

\[ \theta_x(x,y,0) = \frac{\partial w}{\partial y} = \alpha_3 + \alpha_5 x + 2\alpha_6 y + 2\alpha_7 xy + 3\alpha_{10} y^2 + 3\alpha_{12} xy^2 \]

\[ u(x,y,0) = \beta_1 + \beta_2 x + \beta_4 y \]

and

\[ v(x,y,0) = \gamma_1 + \gamma_2 x + \gamma_3 xy + \gamma_4 y \]  \hspace{1cm} (A.76)

The nodal deflections now become

\[ w_i = \alpha_1 \]

\[ w_j = \alpha_1 + b\alpha_3 + b^2\alpha_6 + b^3\alpha_{10} \]

\[ w_k = \alpha_1 + a\alpha_2 + b\alpha_3 + a^2\alpha_4 + ab\alpha_5 + b^2\alpha_6 + ab^2\alpha_9 + b^3\alpha_{10} + ab^3\alpha_{12} \]

\[ w_l = \alpha_1 + a\alpha_2 + a^2\alpha_4 \]  \hspace{1cm} (A.77a)

\[ \theta_{yi} = \alpha_2 \]

\[ \theta_{yj} = \alpha_2 + b\alpha_5 + b^2\alpha_9 + b^3\alpha_{12} \]
\[ \theta_{yk} = a_2 + 2a_4 + b_5 + b^2g + b^3a_{12} \]

\[ \theta_{y1} = a_2 + 2a_4 \quad \text{(A.77b)} \]

and

\[ u_i = \beta_1 \]

\[ u_j = \beta_1 + b\beta_4 \]

\[ u_k = \beta_1 + a\beta_2 + b\beta_4 \]

\[ u_1 = \beta_1 + a\beta_2 \quad \text{(A.77c)} \]

Values for \( \theta_{x1}, \theta_{xj}, \theta_{xk}, \theta_{x1}, v_i, v_j, v_k, \) and \( v_1 \) may also be found in terms of the coefficients and plate dimensions, but the values of these nodal deflections will have no effect upon the strain in the \( x \)-direction.

Multiplication of Eq. A.73 will yield

\[ \varepsilon_x = \frac{6z}{a^2} \left(1 - \frac{2x}{a} - \frac{y}{b} + \frac{2xy}{ab}\right) w_i + \frac{6y^3}{a^2b} \left(1 - \frac{2x}{a}\right) w_j + \frac{6y^3}{a^2b} \left(-1 + \frac{2x}{a} + \frac{y}{b} - \frac{2xy}{ab}\right) w_k + \frac{6y^3}{a^2b} \left(-1 + \frac{2x}{a}\right) w_l \]

\[ + \frac{2z}{a} \left(2 - \frac{3x}{a} - \frac{2y}{b} + \frac{3xy}{ab}\right) \theta_{yi} + \frac{2yz}{ab} \left(2 - \frac{3x}{a}\right) \theta_{yj} + \frac{2yz}{ab} \left(2 - \frac{3x}{a}\right) \theta_{y1} \]

\[ + \frac{1}{a} \left(-1 + \frac{y}{b}\right) u_i + \left(-\frac{y}{ab}\right) u_j + \left(-\frac{y}{ab}\right) u_k + \frac{1}{a} \left(1 - \frac{y}{b}\right) u_l \quad \text{(A.78)} \]
Substituting in the values found for the nodal deflections

\[
\epsilon_x = (0)\alpha_1 + (0)\alpha_2 + (0)\alpha_3 - (2z)\alpha_4 + (0)\alpha_5
\]

\[+ (0)\alpha_6 + (0)\alpha_9 + (0)\alpha_{10} + (0)\alpha_{12}
\]

\[+ (0)\beta_1 + (1)\beta_2 + (0)\beta_4
\]

or

\[
\epsilon_x = -2z\alpha_4 + \beta_2 \tag{A.79}
\]

which is constant except with respect to \(z\). Had the stipulation been made that \(\frac{\partial \epsilon_x}{\partial z}\) also be zero, the constant \(\alpha_4\) would have had to be zero and the above calculations would have given \(\epsilon_x = \beta_2\), a constant value. However, setting \(\frac{\partial \epsilon_x}{\partial z}\) equal to zero corresponds to zero curvature in the element. When the mesh size is decreased indefinitely, the state of strain will have to approach a constant value with respect to \(x\) and \(y\) only, for if it is also constant with respect to \(z\), no element could have any curvature; such a condition could represent only a rotational displacement of the whole structure if compatibility is to be maintained at the element boundaries. Thus, in this context, constant strain refers to strain which is constant with respect to \(x\) and \(y\) only.

For constant strain in the \(y\)-direction

\[
\frac{\partial \epsilon_y}{\partial x} = \gamma_3 - z(2\alpha_9 + 6\alpha_{12}y) = 0 \tag{A.80a}
\]

\[
\frac{\partial \epsilon_y}{\partial y} = -z(6\alpha_{10} + 6\alpha_{12}x) = 0 \tag{A.80b}
\]
\[ \frac{\partial \varepsilon_{xy}}{\partial x} = \beta_3 - 2z(2\alpha_8 + 6\alpha_{11}x) = 0 \quad (A.81a) \]

\[ \frac{\partial \varepsilon_{xy}}{\partial y} = \gamma_3 - 2z(2\alpha_9 + 6\alpha_{12}y) = 0 \quad (A.81b) \]

In order to satisfy Eqs. A.80, the constants \( \alpha_9, \alpha_{10}, \alpha_{12} \) and \( \gamma_3 \) must be equal to zero, while to satisfy Eqs. A.81, the constants \( \alpha_8, \alpha_9, \alpha_{11}, \beta_3 \) and \( \gamma_3 \) must be equal to zero. Using these conditions and the same procedure as used above in the case of strain in the x-direction, it can be shown that the nodal deflections which are consistent with a constant strain in the y-direction and a constant strain in the x-y plane will yield constant values for \( \varepsilon_y \) and \( \varepsilon_{xy} \) respectively.

Equation A.53 gives the strain in the y-direction of the stiffener element as

\[ \varepsilon_y = \bar{\varepsilon}_y = \frac{\partial}{\partial y} (\frac{\partial w(x,y,z)}{\partial y}) = \gamma_4 - z(2\alpha_6 + 6\alpha_{10}y) \quad (A.82) \]

For constant strain in the y-direction

\[ \frac{\partial \varepsilon_y}{\partial y} = -z(6\alpha_{10}) = 0 \quad (A.83) \]

Thus, for constant strain in the y-direction, \( \alpha_{10} \) must be equal to zero, a condition which is also necessary for constant strain in the y-direction for the plate element. For \( \alpha_{10} \) equal to zero,
\[
\bar{w}(x,y,0) = \alpha_1 + \alpha_3 y + \alpha_6 y^2
\]

and

\[
\bar{\varepsilon}_x(x,y,0) = \alpha_3 + 2\alpha_6 y
\]  \hspace{1cm} (A.84)

then

\[
\bar{w}_m = \alpha_1
\]

\[
\bar{w}_n = \alpha_1 + b\alpha_3 + b^2\alpha_6
\]  \hspace{1cm} (A.85a)

\[
\bar{\theta}_{xm} = \alpha_3
\]

\[
\bar{\theta}_{xn} = \alpha_3 + 2b\alpha_6
\]  \hspace{1cm} (A.85b)

and

\[
\bar{v}_m = \gamma_1
\]

\[
\bar{v}_n = \gamma_1 + b\gamma_4
\]  \hspace{1cm} (A.85c)

Substituting into Eq. A.82

\[
\bar{\varepsilon}_y = \frac{6z}{b^2} \left(1 - \frac{2y}{b}\right) \bar{w}_m + \frac{6z}{b^2} \left(-1 + \frac{2y}{b}\right) \bar{w}_n + \frac{2z}{b} \left(2 - \frac{3y}{b}\right) \bar{\theta}_{xm}
\]

\[
+ \frac{2z}{b} \left(1 - \frac{3y}{b}\right) \bar{\theta}_{xn} + \left(-\frac{1}{b}\right) \bar{v}_m + \left(\frac{1}{b}\right) \bar{v}_n
\]  \hspace{1cm} (A.86)
Substituting in values for the nodal deflections

$$\bar{\varepsilon}_y = (0)\alpha_1 + (0)\alpha_3 - 2z\alpha_6 + (0)\gamma_1 + \gamma_4$$

$$= -2z\alpha_6 + \gamma_4$$ \hspace{1cm} (A.87)

which is again constant except with respect to z. Thus, the assumed deflection equations are such that nodal deflections which correspond to constant stress do yield constant values for stresses in both the plate and stiffener elements.

It has already been shown that continuity exists between element boundaries except for the following two cases:

1. Continuity of normal slope across plate element boundaries.
2. Continuity between the normal slope along a plate element boundary parallel to the y-axis and the angle of twist of a beam element attached at that boundary, i.e., continuity of $\theta_y$ at $x = \text{constant}$ and $\bar{\theta}_y$.

For a state of constant strain (except with respect to z) to exist in the x and y directions and in the x-y plane, it is necessary that the constants $\alpha_7$, $\alpha_8$, $\alpha_9$, $\alpha_{10}$, $\alpha_{11}$, $\alpha_{12}$, $\beta_3$, and $\gamma_3$ be equal to zero. This condition reduces the plate element deformation equations to:

$$w(x,y,0) = \alpha_1 + \alpha_2x + \alpha_3y + \alpha_4x^2 + \alpha_5xy + \alpha_6y^2$$

$$\theta_x(x,y,0) = \frac{3w}{3y} = \alpha_3 + \alpha_5x + 2\alpha_6y$$

$$\theta_y(x,y,0) = \frac{3w}{3x} = \alpha_2 + 2\alpha_4x + \alpha_5y$$
Examination of Eq. A.88 reveals that the slope normal to an element boundary will now vary linearly. Thus, the values of the normal slope at the two nodes which define adjacent element boundaries are sufficient conditions to guarantee continuity of normal slope along those boundaries.

The linearity of the normal slope will also guarantee continuity between $\bar{\theta}_y$, the angle of twist of a beam element which varies linearly with respect to $y$, and the normal slope, $\theta_y$, along a boundary where $x = \text{constant}$, which will now also vary linearly with respect to $y$.

Thus, a constant strain condition will insure continuity of all element deformations along the boundaries of the element. The satisfaction of these two conditions should assure that the deformation equations assumed herein cause the finite element procedure to tend toward the correct solution as the number of elements used in the model increases.

\[ u(x,y,0) = \beta_1 + \beta_2 x + \beta_4 y \]

and

\[ v(x,y,0) = \gamma_1 + \gamma_2 x + \gamma_4 y \] (A.88)
REFERENCES


APPENDIX IV.-DESCRIPTION OF COMPUTER PROGRAM

INTRODUCTION

Presented in Appendix IV is a flow chart and program listing of the computer program used to determine the natural frequencies of the finite element model. Also included is the input and output data for finding the natural frequencies of the finite element model corresponding to test plate 5, with 3 equally spaced stiffeners. The output data includes only the 4 lowest frequencies. The shape of the 3,1 mode ($\omega = 596$ Hz.) is the one shown in Fig.7. The program was run on an IBM 360 Model 50 digital computer at the University of Missouri-Rolla, Rolla, Missouri. The program solved for the 52 frequencies corresponding to the 52 degrees of freedom of the system and took approximately 5 minutes to complete.
FLOW CHART

START

Read plate number

Write plate number

Call RDWRT (NS,NSS)

Call MATGEN (PEM,PEK,SME,SEK)

Call STRMAT (SMM,PEM,SEM,NSS,NS)

Call STRMAT (SSM,PEK,SEK,NSS,NS)

Call NROOT (NTUDF,DUM1,DUM2,FREQ,SMODE)

Write natural frequencies and mode shapes

END
RDWRT (NS,NSS)

Read plate properties

Write plate properties

Read finite element model configuration

Write finite element model configuration

Read support conditions

Write support conditions

Calculate indexing constants

RETURN
MATGEN
(PEM, PEK, SEM
SEK)

Generate plate element mass matrix

Generate plate element stiffness matrix

Generate stiffener element mass matrix

Generate stiffener element stiffness matrix

RETURN
STRMAT
(SM,PE,SE,
NSS,NS)

Generate structural (stiffness or mass) matrix elements
N*(1, NX, (NY-1)*NX+1, NX*NY)
N = 1, 5 (corner nodes)

Generate structural matrix elements N*(2 thru NX-1
and (NY-1)*NX+2 thru NX+NY-1)
N = 1, 5 (nodes along y=0 and y=BY)

Generate structural matrix elements N*(M*NX+1 and M*2*NX)
M = 1, NY-2
N = 1, 5 (nodes along x=0 and x=AX)

Generate remaining elements (interior nodes)

NOS>0

Add stiffener properties to structural Matrix beginning at nodes NSS
Add one-half stiffener properties to structural matrix beginning at node NCS

Compact to final structural matrix by eliminating rows and columns corresponding to nodes NS

RETURN
STANDARD IBM SUBROUTINE
Compute eigenvalues and eigenvectors of a real, non-symmetric matrix of the form $M^{-1}K$, where $M$ is positive definite.

RETURN
C S FERRELL--FINITE ELEMENT PROGRAM--STIFFENED PLATE VIBRATION

++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++

+ VECTORS
+ + DUM1, DUM2 = DUMMY STORAGE MATRICES
+ + FREQ = VECTOR OF NATURAL FREQUENCIES
+ + NS = VECTOR OF SUPPORTED DEGREES OF FREEDOM
+ + NSS = VECTOR OF PLATE EDGE NODES TO WHICH STIFFENERS ARE ATTACHED
+ + PEK = PLATE ELEMENT STIFFNESS MATRIX
+ + PEM = PLATE ELEMENT MASS MATRIX
+ + SEK = STIFFENER ELEMENT STIFFNESS MATRIX
+ + SEM = STIFFENER ELEMENT MASS MATRIX
+ + SMODE = MODE SHAPE MATRIX
+ + SMM = STRUCTURE MASS MATRIX
+ + SSM = STRUCTURE STIFFNESS MATRIX

+ SCALARS
+ + AX = PLATE ELEMENT LENGTH IN X-DIRECTION
+ + BY = PLATE AND STIFFENER ELEMENT LENGTH IN Y-DIRECTION
+ + B = STIFFENER WIDTH
+ + EM = MODULUS OF ELASTICITY OF STRUCTURE MATERIAL
+ + H = STIFFENER DEPTH
+ + NCS = O--NO STIFFENER ALONG CENTER LINE
+ + NEXD = NUMBER OF PLATE ELEMENTS IN X-DIRECTION
+ + NEYD = NUMBER OF PLATE ELEMENTS IN Y-DIRECTION
+ + NOEL = TOTAL NUMBER OF ELEMENTS
+ + NOS = NUMBER OF STIFFENERS, EXCLUDING STIFFENERS ALONG CENTER LINE
+ + NOTN = TOTAL NUMBER OF DEGREES OF FREEDOM
+ + NSDF = NUMBER OF SUPPORTED DEGREES OF FREEDOM
+ + NT = TOTAL NUMBER OF NODES
+ + NTBN = NUMBER OF BENDING DEGREES OF FREEDOM
+ + NTUFD = NUMBER OF UNSUPPORTED DEGREES OF FREEDOM
+ + NX = NUMBER OF NODES IN X-DIRECTION
+ + NY = NUMBER OF NODES IN Y-DIRECTION
+ + PN = PLATE IDENTIFICATION NUMBER
+ + PR = POISSON'S RATIO OF STRUCTURE MATERIAL
+ + RHO = MASS DENSITY OF STRUCTURE MATERIAL
+ + T = PLATE THICKNESS
+ + XL = STRUCTURE DIMENSION IN X-DIRECTION
+ + YL = STRUCTURE DIMENSION IN Y-DIRECTION

++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
DIMENSION SSM(120,120), SMM(120,120)

DIMENSION DUM1(52,52), DUM2(52,52), SMODE(52,52), FREQ(52)

DIMENSION NSS(1)

DIMENSION NS(69)

DIMENSION PEM(20,20), PEK(20,20), SEM(10,10), SEK(10,10)

EQUIVALENCE (SSM(1,1),SMM(1,1))
COMMON XL,YL,T,B,H,EM,PR,GHO,NY,NX,NOS,NCS,NSDF,NEXD,NEYD,NT,
1NOTN,AX,BY,NSTP,NTUDF,NTBN,NOEL

+ ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
+ READ AND WRITE INPUT INFORMATION +
+ AND GENERATE INDEXING CONSTANTS +
+ ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++

READ (1,1001) PN
WRITE (3,1002) PN
CALL RDWRT (NS,NSS)

+ ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
+ GENERATE ELEMENT PROPERTY MATRICES +
+ ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++

CALL MATGEN (PEM,PEK,SEM,SEK)

+ ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
+ GENERATE STRUCTURE MASS MATRIX, "SMM" +
+ ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++

CALL STRMAT (SMM,PEM,SEM,NSS,NS)
DO 94 I = 1,NTUDF
DO 94 J = 1,NTUDF
94 DUM2(I,J) = SMM(I,J)

+ ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
+ GENERATE STRUCTURE STIFFNESS MATRIX, "SSM" +
+ ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++

CALL STRMAT (SSM,PEK,SEK,NSS,NS)
DO 501 I = 1,NTUDF
DO 501 J = 1,NTUDF
501 DUM1(I,J) = SSM(I,J)

+ ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
+ FIND AND WRITE NATURAL FREQUENCIES AND MODE SHAPES +
+ ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++

CALL NROOT (NTUDF,DUM1,DUM2,FREQ,SMODE
READ (1,1001) NMWR
NUM = NTUDF + 1
DO 503 I = 1,NMWR
II = NUM-I
FREQ(II) = SQRT(FREQ(II))
C1 = FREQ(II)/6.2832
WRITE (3,1003) FREQ(II),C1
WRITE (3,1004)
SUBROUTINE RDWRT (NS,NSS)
DIMENSION NS(69),NSS(1)
COMMON XL,YL,T,B,H,EM,PR,RHO,NY,NX,NOS,NCS,NSDF,NEXD,NEYD,NT,
1 NOTN,AX,BY,NSTP,NTUDF,NTBN,NOEL
READ(1,1000)XL,YL,T,H,B
WRITE (3,1100) XL,YL,T,H,B
READ(1,1001) EM,PR,RHO
WRITE (3,1111) EM,PR,RHO
READ (1,1002) NX, NY
READ (1,1002) NOS
IF(NOS) 92,93,92
92 READ(1,1002)(NSS(I),I=1,NOS)
WRITE (3,1102) (NSS(I),I=1,NOS)
GO TO 1
93 NSS(1) = 0
1 READ (1,1002)NCS
IF (NCS) 157,158,157
158 WRITE (3,1103)
GO TO 159
157 WRITE (3,1104)
159 CONTINUE
AX = XL/(2.*(NX-1))
BY = YL/(2.*(NY-1))
WRITE (3,1101) AX,BY
NEXD = NX-1
NEYD = NY-1
WRITE (3,1112) NEXD,NEYD
READ (1,1002)NSDF
READ (1,1002)(NS(I), I = 1,NSDF)
NSTP = NSDF-1
WRITE(3,1105) (NS(I),I=1,NSTP)
NT = NX*NY
NOTN = 5*NT
NTBN = 3*NT
NTUDF = NOTN - NSDF + 1
NOEL = (NX-1)*(NY-1)
1000 FORMAT ( 6F10.4)
1001 FORMAT ( 3F15.6)
1002 FORMAT ( 15I4)
SUBROUTINE MATGEN (PEM,PEK,SEM,SEK)
DIMENSION PEM(20,20),PEK(20,20),SEM(10,10),SEK(10,10)
COMMON XL,YL,T,B,H,EM,PR,RHO,RY,NX,NOW,NCS,NSDF,NEXD,NEYD,NT,
1 NOTN,AX,BY,NSTP,NTUDF,NTBN,NOEL

C AX2=AX*AX
BY2=BY*BY
C
C ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
C GENERATE PLATE ELEMENT MASS MATRIX,PEM
C ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ +
C
DO 90 I = 1,20
DO J = 1,20
90 PEM(I,J) = 0.
C1 = RHO*AX*BY*T/176400.
PEM(1,1) = 24178.*C1
PEM(2,1) = 8582.*C1
PEM(2,2) = 24178.*C1
PEM(3,1) = 2758.*C1
PEM(3,2) = 8582.*C1
PEM(3,3) = 24178.*C1
PEM(4,1) = 8582.*C1
PEM(4,2) = 2758.*C1
PEM(4,3) = 8582.*C1
PEM(4,4) = 24178.*C1
PEM(5,1) = 3227.*BY*C1
PEM(5,2) = 1918.*BY*C1
PEM(5,3) = 812.*BY*C1
PEM(5,4) = 1393.*BY*C1
PEM(5,5) = 560.*BY2*C1
PEM(6,1) = -PEM(5,2)
PEM(6,2) = -PEM(5,1)
PEM(6,3) = -PEM(5,4)
PEM(6,4) = -PEM(5,3)
PEM(6,5) = -420.*BY2*C1
<table>
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<tr>
<th>Index</th>
<th>Expression</th>
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<td>$PEM(9,3) = 812.\times AX\times C1$</td>
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<td>$PEM(9,4) = 1918.\times AX\times C1$</td>
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<td>$PEM(9,5) = 441.\times AX\times BY\times C1$</td>
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<td>$PEM(9,6) = -294.\times AX\times BY\times C1$</td>
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<tr>
<td>$12,7$</td>
<td>$PEM(12,7) = -PEM(9,6)$</td>
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PEM(12,8) = -PEM(9,5)
PEM(12,9) = PEM(11,10)
PEM(12,10) = PEM(11,9)
PEM(12,11) = PEM(10,9)
PEM(12,12) = PEM(9,9)
C1 = RHO*AX*BY*T/36.
PEM(13,13) = 4.*C1
PEM(14,13) = 2.*C1
PEM(14,14) = PEM(13,13)
PEM(15,13) = C1
PEM(15,14) = PEM(14,13)
PEM(15,15) = PEM(13,13)
PEM(16,13) = PEM(14,13)
PEM(16,14) = C1
PEM(16,15) = PEM(14,13)
PEM(16,16) = PEM(13,13)
PEM(17,17) = PEM(13,13)
PEM(18,17) = PEM(14,13)
PEM(18,18) = PEM(13,13)
PEM(19,17) = C1
PEM(19,18) = PEM(14,13)
PEM(19,19) = PEM(13,13)
PEM(20,17) = PEM(14,13)
PEM(20,18) = C1
PEM(20,19) = PEM(14,13)
PEM(20,20) = PEM(13,13)
DO 91 I = 1,11
   N1 = I+1
   DO 91 J = N1,12
   91 PEM(I,J) = PEM(J,I)
DO 86 I = 13,19
   N1 = I+1
   DO 86 J = N1,20
   86 PEM(I,J) = PEM(J,I)

C C
C ++++++GENERATE PLATE ELEMENT STIFFNESS MATRIX, PEK+++++
C
C DO 1 I=1,20
   DO 1 J=1,20
   1 PEK(I,J) = 0.
   BET = BY/AX
   BT2 = BET*BET
   BT2I = 1./BT2
   BTC = BT2 + BT2I
   C1 = (EM*T*T*T)/(12.*(1.-PR*PR)*AX*BY)
   C2 = 14.-4.*PR
   C3 = 1. + 4.*PR
   C4 = 1.-PR
   C5 = 1./5.
   C6 = 4./15.
\[ C7 = \frac{4}{3}. \]
\[ C8 = \frac{C7}{2}. \]
\[ PEK(1,1) = (4.*BTC + C5*C2)*C1 \]
\[ PEK(2,1) = (2.*(BT2 - 2.*BT2I) - C5*C2)*C1 \]
\[ PEK(2,2) = PEK(1,1) \]
\[ PEK(3,1) = (-2.*BTC + C5*C2)*C1 \]
\[ PEK(3,2) = (-2.*(2.*BT2-BT2I) - C5*C2)*C1 \]
\[ PEK(3,3) = PEK(1,1) \]
\[ PEK(4,1) = PEK(3,2) \]
\[ PEK(4,2) = PEK(3,1) \]
\[ PEK(4,3) = PEK(2,1) \]
\[ PEK(4,4) = PEK(1,1) \]
\[ PEK(5,1) = (BY*(2.*BT2I + C5*C3))*C1 \]
\[ PEK(5,2) = (-BY*(2.*BT2I + C5*C4))*C1 \]
\[ PEK(5,3) = (BY*(-BT2I + C5*C4))*C1 \]
\[ PEK(5,4) = (BY*(- BT2I - C5*C3))*C1 \]
\[ PEK(5,5) = (BY2*(C7*BT2I + C6*C4))*C1 \]
\[ PEK(5,6) = -PEK(5,2) \]
\[ PEK(6,2) = -PEK(5,1) \]
\[ PEK(6,3) = (BY*(-BT2I + C5*C3))*C1 \]
\[ PEK(6,4) = -PEK(5,3) \]
\[ PEK(6,5) = (BY2*(C8*BT2I - C6*C4/4.))*C1 \]
\[ PEK(6,6) = PEK(5,5) \]
\[ PEK(7,1) = -PEK(5,2) \]
\[ PEK(7,2) = PEK(6,3) \]
\[ PEK(7,3) = -PEK(5,1) \]
\[ PEK(7,4) = -PEK(5,2) \]
\[ PEK(7,5) = PEK(5,5)/4. \]
\[ PEK(7,6) = (BY2*(C8*BT2I - C6*C4))*C1 \]
\[ PEK(7,7) = PEK(5,5) \]
\[ PEK(8,1) = -PEK(7,2) \]
\[ PEK(8,2) = -PEK(7,1) \]
\[ PEK(8,3) = -PEK(6,1) \]
\[ PEK(8,4) = -PEK(6,2) \]
\[ PEK(8,5) = PEK(7,6) \]
\[ PEK(8,6) = PEK(7,5) \]
\[ PEK(8,7) = PEK(6,5) \]
\[ PEK(8,8) = PEK(6,6) \]
\[ PEK(9,1) = (AX*(2.*BT2 + C5*C3))*C1 \]
\[ PEK(9,2) = (AX*(BT2 - C5*C3))*C1 \]
\[ PEK(9,3) = -(AX*(BT2 - C5*C4))*C1 \]
\[ PEK(9,4) = -AX*(2.*BT2 + C5*C4))*C1 \]
\[ PEK(9,5) = (PR*AX*BY)*C1 \]
\[ PEK(9,9) = (AX2*(C7*BT2 + C6*C4))*C1 \]
\[ PEK(10,1) = PEK(9,2) \]
\[ PEK(10,2) = PEK(9,1) \]
\[ PEK(10,3) = PEK(9,4) \]
\[ PEK(10,4) = PEK(9,3) \]
\[ PEK(10,6) = -PEK(9,5) \]
\[ PEK(10,9) = (AX2*(C8*BT2 - C6*C4))*C1 \]
\[ PEK(10,10) = PEK(9,9) \]
\[ PEK(11,1) = -PEK(9,3) \]
\[
\begin{align*}
\text{PEK}(11,2) &= -\text{PEK}(10,3) \\
\text{PEK}(11,3) &= -\text{PEK}(9,1) \\
\text{PEK}(11,4) &= -\text{PEK}(9,2) \\
\text{PEK}(11,7) &= \text{PEK}(9,5) \\
\text{PEK}(11,9) &= \text{PEK}(9,9)/4. \\
\text{PEK}(11,10) &= (AX2*(C8*BT2 - C6*C4/4.))*C1 \\
\text{PEK}(11,11) &= \text{PEK}(9,9) \\
\text{PEK}(12,1) &= \text{PEK}(11,2) \\
\text{PEK}(12,2) &= \text{PEK}(11,1) \\
\text{PEK}(12,3) &= -\text{PEK}(10,1) \\
\text{PEK}(12,4) &= -\text{PEK}(10,2) \\
\text{PEK}(12,8) &= \text{PEK}(10,6) \\
\text{PEK}(12,9) &= \text{PEK}(11,10) \\
\text{PEK}(12,10) &= \text{PEK}(11,9) \\
\text{PEK}(12,11) &= \text{PEK}(10,9) \\
\text{PEK}(12,12) &= \text{PEK}(10,10) \\
C1 &= \left(\frac{EM*T}{12.*(1.-PR*PR)}\right) \\
C2 &= 1.+PR \\
C3 &= 1.-PR \\
C4 &= 1.-3.*PR \\
\text{PEK}(13,13) &= C1*(4.*BET + 2.*C3/BET) \\
\text{PEK}(14,13) &= C1*(2.*BET - 2.*C3/BET) \\
\text{PEK}(14,14) &= \text{PEK}(13,13) \\
\text{PEK}(15,13) &= -\text{PEK}(13,13)/2. \\
\text{PEK}(15,14) &= C1*(-4.*BET + C3/BET) \\
\text{PEK}(15,15) &= \text{PEK}(13,13) \\
\text{PEK}(16,13) &= \text{PEK}(15,14) \\
\text{PEK}(16,14) &= \text{PEK}(15,13) \\
\text{PEK}(16,15) &= \text{PEK}(14,13) \\
\text{PEK}(16,16) &= \text{PEK}(13,13) \\
\text{PEK}(17,13) &= C1*(3.*C2/2.) \\
\text{PEK}(17,14) &= -C1*(3.*C4/2.) \\
\text{PEK}(17,15) &= -\text{PEK}(17,13) \\
\text{PEK}(17,16) &= -\text{PEK}(17,14) \\
\text{PEK}(17,17) &= C1*(4./BET + 2.*C3*BET) \\
\text{PEK}(18,13) &= -\text{PEK}(17,14) \\
\text{PEK}(18,14) &= -\text{PEK}(17,13) \\
\text{PEK}(18,15) &= \text{PEK}(17,14) \\
\text{PEK}(18,16) &= \text{PEK}(17,13) \\
\text{PEK}(18,17) &= C1*(-4./BET + C3*BET) \\
\text{PEK}(18,18) &= \text{PEK}(17,17) \\
\text{PEK}(19,13) &= -\text{PEK}(17,13) \\
\text{PEK}(19,14) &= -\text{PEK}(17,14) \\
\text{PEK}(19,15) &= \text{PEK}(17,13) \\
\text{PEK}(19,16) &= \text{PEK}(17,14) \\
\text{PEK}(19,17) &= -\text{PEK}(17,17)/2. \\
\text{PEK}(19,18) &= C1*(2./BET - 2.*C3*BET) \\
\text{PEK}(19,19) &= \text{PEK}(17,17) \\
\text{PEK}(20,13) &= \text{PEK}(19,14) \\
\text{PEK}(20,14) &= \text{PEK}(17,13) \\
\text{PEK}(20,15) &= -\text{PEK}(17,14) \\
\text{PEK}(20,16) &= -\text{PEK}(17,13) \\
\text{PEK}(20,17) &= \text{PEK}(19,18)
\end{align*}
\]
PEK(20,18) = PEK(19,17)
PEK(20,19) = PEK(18,17)
PEK(20,20) = PEK(17,17)
DO 2 I=1,11
   N1 = I+1
DO 2 J=N1,12
   2 PEK(I,J) = PEK(J,I)
   DO 3 I=13,19
   N1 = I+1
   DO 3 J=N1,20
   3 PEK(I,J) = PEK(J,I)
C
   IF (NOS) 6,6,4
   6 IF (NCS) 5,5,4
C
   +++++++++++++++++++ + GENERATE STIFFENER ELEMENT MASS MATRIX +
   +++++++++++++++++++
C
4 DO 89 I=1,10
   DO 89 J=1,10
   89 SEM(I,J) = 0.
   AS = B*H
   SE = (T+H)/2.
   SI = (B*H*H*H/12.) + AS*SE*SE
   AB = AS*BY
   SS = SE*AS
   IF (H-B) 10,11,11)
   11 SJ = (H-.63*8)*B*B*8/3.
   GO TO 12
   10 SJ = (8-.63*H)*H*H*H/3.
   12 PMMI = SI + B*B*AS/12.
   C1 = RHO*AB
   SEM(1,1) = (13./15. + 6.*SI/(5.*AB*BY))*C1
   SEM(2,1) = (9./70. - 6.*SI/(5.*AB*BY))*C1
   SEM(2,2) = SEM(1,1)
   SEM(3,1) = (11.*BY/210. + SI/(10.*AB))*C1
   SEM(3,2) = (13.*BY/420. - SI/(10.*AB))*C1
   SEM(3,3) = (BY2/105. + 2.*SI/(15.*AS))*C1
   SEM(4,1) = -SEM(3,2)
   SEM(4,2) = -SEM(3,1)
   SEM(4,3) = -(BY2/14C. + SI/(30.*AS))*C1
   SEM(4,4) = SEM(3,3)
   SEM(5,5) = PMMI*C1/(3.*AS)
   SEM(6,5) = SEM(5,5)/2.
   SEM(6,6) = SEM(5,5)
   SEM(7,5) = SE*C1/3.
   SEM(7,6) = SEM(7,5)/2.
   SEM(7,7) = C1/3.
   SEM(8,5) = SEM(7,6)
   SEM(8,6) = SEM(7,5)
   SEM(8,7) = C1/6.
SEM(8,8) = SEM(7,7)
SEM(9,1) = -SEM(1,7)/BY
SEM(9,2) = -SEM(9,1)
SEM(9,3) = SEM(7,5)/4.
SEM(9,4) = -SEM(9,3)
SEM(9,9) = SEM(7,7)
SEM(10,1) = SEM(9,1)
SEM(10,2) = SEM(9,2)
SEM(10,3) = SEM(9,4)
SEM(10,4) = SEM(9,3)
SEM(10,9) = SEM(8,7)
SEM(10,10) = SEM(9,9)
D0 88 I=1,9
N1 = I+1
D0 88 J=N1,10
78 SEM(I,J) = SEM(J,I)

+++ + GENERATE STIFFENER ELEMENT STIFFNESS MATRIX, SEK +

+++ +-----------------------------------------------------------------------------------
C DO 55 I=1,10
D055J=1,10
55 SEK(I,J) = 0.
    ELI = 1./BY
    ELI2 = ELI/BY
    ELI3 = ELI2/BY
    SEK(1,1) = 12.*EM*SI*ELI3
    SEK(2,1) = -SEK(1,1)
    SEK(2,2) = SEK(1,1)
    SEK(3,1) = 6.*EM*SI*ELI2
    SEK(3,2) = -SEK(3,1)
    SEK(3,3) = 4.*EM*SI*ELI
    SEK(4,1) = SEK(3,1)
    SEK(4,2) = SEK(3,2)
    SEK(4,3) = SEK(3,3)/2.
    SEK(4,4) = SEK(3,3)
    SEK(5,5) = SJ*EM*ELI/(2.*(1.+PR))
    SEK(6,5) = -SEK(5,5)
    SEK(6,6) = SEK(5,5)
    SEK(9,3) = EM*SS*ELI
    SEK(9,4) = -SEK(9,3)
    SEK(9,9) = EM*AS*ELI
    SEK(10,3) = SEK(9,4)
    SEK(10,4) = SEK(9,3)
    SEK(10,9) = -SEK(9,9)
    SEK(10,10) = SEK(9,9)
D0 56 I=1,9
N1 = I+1
D0 56 J=N1,10
56 SEK(I,J) = SEK(J,I)
5 CONTINUE
SUBROUTINE STRMAT (SM, PE, SE, NSS, NS)
DIMENSION PE(20,20), SE(10,10), SM(NOTN, NOTN), SNN(1), NS(69)
COMMON XL, YL, T, B, H, EM, PR, RHO, NY, NX, NOS, NCS, NSDF, NEXD, NEYD, NT,
1 NOTN, AX, BY, NSTP, NTUDF, NTBN, NOEL

C
RETURN
END

C
++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
C + GENERATE STRUCTURAL MATRIX +
C ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++

C
DO 141 I = 1, NOTN
DO 141 J = 1, NOTN
141 SM(I, J) = 0.

C
GENERATE ELEMENTS FOR CORNER NODES

C
N1 = 1
N6 = 1
N7 = NX+1
DO 27 I = 1, 2
N8 = N6
N9 = N7
DO 28 II = 1, 2
   IF (I-2) 150, 151, 150
   IF (II-2) 152, 153, 152
150 IF (II-2) 152, 153, 152
152 N2 = 1
   GO TO 156
153 N2 = 4
   GO TO 156
151 IF (II-2) 154, 155, 154
154 N2 = 2
   GO TO 156
155 N2 = 3
156 CONTINUE
DO 29 J = 1, 5
N3 = 1
N4 = N8
N5 = N9
DO 30 JJ = 1, 5
   SM(N1, N4) = PE(N2, N3)
   SM(N1, N4+1) = PE(N2, N3+3)
   SM(N1, N5) = PE(N2, N3+1)
   SM(N1, N5+1) = PE(N2, N3+2)
N3 = N3+4
N4 = N4+NT
30 N5 = N5+NT
N1 = N1+NT
29 N2 = N2 + 4  
N8 = N8 + NX - 2  
N9 = N9 + NX - 2  
28 N1 = N1 + NX - 1 - NOTN  
N6 = N6 + (NY - 2) * NX  
N7 = N7 + (NY - 2) * NX  
27 N1 = N1 + 2 + (NY - 3) * NX

C GENERATE ELEMENTS FOR NODES ALONG Y=0 AND Y=BY

IF(NX-2) 160, 160, 161
161 N7 = 1  
N8 = NX + 1  
N1 = 2  
NX1 = NX - 2  
N9 = 4  
N10 = 1  
DO 31 I=1,2  
DO 32 II=1,NX1  
N2 = N9  
N3 = N10  
DO 33 J=1,5  
N4 = N7  
N5 = N8  
N6 = 1  
DO 34 JJ=1,5  
SM(N1,N4) = PE(N2,N6)  
SM(N1,N4+1) = PE(N2,N6+3) + PE(N3,N6)  
SM(N1,N4+2) = PE(N3,N6+3)  
SM(N1,N5) = PE(N2,N6+1)  
SM(N1,N5+1) = PE(N2,N6+2) + PE(N3,N6+1)  
SM(N1,N5+2) = PE(N3,N6+2)  
N4 = N4 + NT  
N5 = N5 + NT  
34 N6 = N6 + 4  
N1 = N1 + NT  
N2 = N2 + 4  
33 N3 = N3 + 4  
N7 = N7 + 1  
N8 = N8 + 1  
32 N1 = N1 + 1 - NOTN  
N9 = 3  
N10 = 2  
N7 = NT - 2 * NX + 1  
N8 = NT - NX + 1  
31 N1 = NT - NX + 2  
160 CONTINUE

C GENERATE ELEMENTS FOR NODES ALONG X=0 AND X=AX

IF(NY-2) 162, 162, 163
163 N8 = 1  
N9 = NX + 1
N10 = 2*NX+1
N1 = NX+1
NY1 = NY-2
N11 = 2
N12 = 1
DO 35 I=1,2
  DO 36 II=1,NY1
    N2 = N11
    N3 = N12
    DO 37 J=1,5
      N4 = N8
      N5 = N9
      N6 = N10
      N7 = 1
      DO 38 JJ=1,5
        SM(N1,N4) = PE(N2,N7)
        SM(N1,N4+1) = PE(N2,N7+3)
        SM(N1,N5) = PE(N2,N7+1) + PE(N3,N7)
        SM(N1,N5+1) = PE(N2,N7+2) + PE(N3,N7+3)
        SM(N1,N6) = PE(N3,N7+1)
        SM(N1,N6+1) = PE(N3,N7+2)
      N4 = N4+NT
      N5 = N5+NT
      N6 = N6+NT
    38 N7 = N7+4
    N1 = N1+NT
    N2 = N2+4
  37 N3 = N3+4
  N1 = N1+NX-NOTN
  N8 = N8+NX
  N9 = N9+NX
  36 N10 = N10+NX
  N11 = 3
  N12 = 4
  N8 = NX-1
  N9 = 2*NX-1
  N10 = 3*NX-1
  35 N1 = 2*NX
  CONTINUE

C GENERATE ELEMENTS FOR INTERIOR NODES
C
IF(NX-2)164,164,165
  165 IF(NY-2)164,164,166
  166 N10 = 1
  N11 = NX+1
  N12 = 2*NX+1
  N1 = NX+2
  DO 39 I=1,NY1
  DO 40 II=1,NX1
    N2 = 3
    N3 = 2
N4 = 4
N5 = 1
DO 41 J=1,5
   N6 = N10
   N7 = N11
   N8 = N12
   N9 = 1
   DO 42 JJ=1,5
      SM(N1,N6) = PE(N2,N9)
      SM(N1,N6+1) = PE(N2,N9+3) + PE(N3,N9)
      SM(N1,N6+2) = PE(N3,N9+3)
      SM(N1,N7) = PE(N2,N9+1) + PE(N4,N9)
      SM(N1,N7+1) = PE(N5,N9+3)
      SM(N1,N7+2) = PE(N3,N9+2) + PE(N5,N9+3)
      SM(N1,N8) = PE(N4,N9+1)
      SM(N1,N8+1) = PE(N4,N9+2) + PE(N5,N9+1)
      SM(N1,N8+2) = PE(N5,N9+2)
   N6 = N6+NT
   N7 = N7+NT
   N8 = N8+NT
42 N9 = N9+4
   N1 = N1+NT
   N2 = N2+4
   N3 = N3+4
   N4 = N4+4
41 N5 = N5+4
   N10 = N10+1
   N11 = N11+1
   N12 = N12+1
40 N1 = N1+1-NOTN
   N10 = N10+2
   N11 = N11+2
   N12 = N12+2
39 N1 = N1+2
164 CONTINUE
   WRITE (3,200)
200 FORMAT ('BEFORE ADDITION OF STIFFENER '///)
100 FORMAT (/5F25.9)
C
C ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
C + ADD STIFFENER STIFFNESSES TO STRUCTURAL STIFFNESS MATRIX +
C ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
C
IF ( NOS ) 84,85,84
84 CONTINUE
   DO 57 I=1,NOS
      N1 = NSS(I)
      N4 = 1
   DO 58 JJ=1,5
      N2 = NSS(I)
      N3 = NSS(I) + NX
      N5 = 1
DO 59 J = 1, 5
SM(N1, N2) = SM(N1, N2) + SE(N4, N5)
SM(N1, N3) = SM(N1, N3) + SE(N4, N5+1)
N2 = N2 + NT
N3 = N3 + NT
59 N5 = N5 + 2
N1 = N1 + NT
58 N4 = N4 + 2
N1 = NSS(I) + NX
N2 = NSS(I)
N3 = N1
N4 = N1 + NX
DO 60 II = 1, NY1
N5 = 1
N6 = 2
DO 61 JJ = 1, 5
N7 = 1
N8 = 2
DO 62 J = 1, 5
SM(N1, N2) = SM(N1, N2) + SE(N6, N7)
SM(N1, N3) = SM(N1, N3) + SE(N5, N7) + SE(N6, N8)
SM(N1, N4) = SM(N1, N4) + SE(N5, N8)
N2 = N2 + NT
N3 = N3 + NT
N4 = N4 + NT
N7 = N7 + 2
62 N8 = N8 + 2
N1 = N1 + NT
N2 = N2 - NOTN
N3 = N3 - NOTN
N4 = N4 - NOTN
N5 = N5 + 2
61 N6 = N6 + 2
N1 = N1 + NX - NOTN
N2 = N2 + NX
N3 = N3 + NX
N4 = N4 + NX
N1 = NSS(I) + (NY-1)*NX
N4 = 2
DO 63 JJ = 1, 5
N2 = NSS(I) + NY1*NX
N3 = N2 + NX
N5 = 1
DO 64 J = 1, 5
SM(N1, N2) = SM(N1, N2) + SE(N4, N5)
SM(N1, N3) = SM(N1, N3) + SE(N4, N5+1)
N2 = N2 + NT
N3 = N3 + NT
64 N5 = N5 + 2
N1 = N1 + NT
63 N4 = N4 + 2
57 CONTINUE
85 CONTINUE
IF ( NCS ) 83,65,83
83 N1 = NCS
N4 = 1
DO 66 JJ=1,5
N2 = NCS
N3 = NCS + NX
N5 = 1
DO 67 J=1,5
SM(N1,N2) = SM(N1,N2) + SE (N4,N5)/2.
SM(N1,N3) = SM(N1,N3) + SE (N4,N5+1)/2.
N2 = N2 + NT
N3 = N3 + NT
N5 = N5 + 2
N1 = N1 + NT
66 N4 = N4 + 2
N1 = NCS + NX
N2 = NCS
N3 = N1
N4 = N1 + NX
DO 68 II = 1,NY1
N5 = 1
N6 = 2
DO 69 JJ=1,5
N7 = 1
N8 = 2
DO 70 J=1,5
SM(N1,N2) = SM(N1,N2) + SE (N6,N7)/2.
SM(N1,N3) = SM(N1,N3) + SE (N5,N7)/2. + SE (N6,N8)/2.
SM(N1,N4) = SM(N1,N4) + SE (N5,N8)/2.
N2 = N2 + NT
N3 = N3 + NT
N4 = N4 + NT
N7 = N7 + 2
70 N8 = N8 + 2
N1 = N1 + NT
N2 = N2 - NOTN
N3 = N3 - NOTN
N4 = N4 - NOTN
N5 = N5 + 2
69 N6 = N6 + 2
N1 = N1 + NX - NOTN
N2 = N2 + NX
N3 = N3 + NX
N5 = N5 + 2
68 N4 = N4 + NX
N1 = NCS + (NY-1)*NX
N4 = 2
DO 71 JJ=1,5
N2 = NCS + NY1*NX
N3 = N2 + NX
N5 = 1
DO 72 J=1,5
SM(N1,N2) = SM(N1,N2) + SE(N4,N5).2.
SM(N1,N3) = SM(N1,N3) + SE(N4,N5+1)/2.
N2 = N2 + NT
N3 = N3 + NT
72 N5 = N5 + 2
N1 = N1 + NT
71 N4 = N4 + 2
65 CONTINUE

C

C ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
C | REDUCE STIFFNESS MATRIX FOR BOUNDARY CONDITIONS AND MODE |
C | SHAPE TYPE +                                              |
C ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ C

K = 1
II = 1
NSD = NS(II)
DO 73 I=1,NOTN
IF(I-NSD)74,75,74
74 CONTINUE
DO 76 J=1,NOTN
76 SM(K,J) = SM(I,J)
K = K+1
GO TO 77
75 II = II+1
NSD = NS(II)
77 CONTINUE
73 CONTINUE
K = 1
II = 1
NSD = NS(II)
DO 78 I=1,NOTN
IF(I-NSD)79,80,79
79 CONTINUE
DO 81 J=1,NTUDF
81 SM(J,K) = SM(J,I)
K = K+1
GO TO 82
80 II = II+1
NSD = NS(II)
82 CONTINUE
78 CONTINUE
WRITE (3,2002) NTUDF
2002 FORMAT (1514)
8000 FORMAT (/10F12.2)
C
RETURN
END
INPUT DATA

Read PN (Plate Identification Number)
   5

Read XL,YL,T,H,B (Structure Dimension in X-direction, Structure Dimension in Y-direction, Plate Thickness, Stiffener Depth, Stiffener Width)
   11.0 11.0 .0625 .0625 .161

Read EM,PR,RHO (Modulus of Elasticity of Structure Material, Poisson's Ratio of Structure Material, Mass Density of Structure Material)
   10400000. .33 .000262

Read NX,NY (Number of Nodes in X-direction, Number of Nodes in Y-direction)
   8 3

Read NOS (Number of Stiffeners, Excluding Stiffener Along Center Line)
   1

Read NSS (Vector of Plate Edge Nodes to Which Stiffeners Are Attached)
   4

Read NCS (NCS ≠ 0--Stiffener Along Center Line at Node NCS)
   8

Read NSDF (Number of Supported Degrees of Freedom)
   69

Read NS (Vector of Supported Degrees of Freedom)
   1 2 3 4 5 6 7 8 9 17 25 26 27 28 29
   30 31 32 33 41 42 43 44 45 46 47 48 49 50 51
Read NMWR (Number of Mode Shapes About Which Information is to Be Listed)

4
OUTPUT DATA

*** PLATE NUMBER 5 ***

X-DIRECTION PLATE LENGTH = 11.000
Y-DIRECTION PLATE LENGTH = 11.000
PLATE THICKNESS = 0.0625
STIFFENER DEPTH = 0.0625
STIFFENER WIDTH = 0.1610
MODULUS OF ELASTICITY = 10400000.0
POISSON'S RATIO = 0.330
MASS DENSITY = 0.000262
STIFFENERS START AT NODES 4
STIFFENER ALSO ALONG PLATE CENTER LINE
ELEMENT LENGTH IN X-DIRECTION = 0.786
ELEMENT LENGTH IN Y-DIRECTION = 2.750
NUMBER OF ELEMENTS IN X-DIRECTION IN 1/4 PLATE = 7
NUMBER OF ELEMENTS IN Y-DIRECTION IN 1/4 PLATE = 2
THE FOLLOWING DEFLECTIONS ARE ZERO 1 2 3 4 5 6 7 8 9 17 25 26 27 28 29 30 31 32
THE FOLLOWING DEFLECTIONS ARE ZERO 33 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57
THE FOLLOWING DEFLECTIONS ARE ZERO 64 65 72 73 74 75 76 77 78 79 80 81 88 89 96 97 98 99
THE FOLLOWING DEFLECTIONS ARE ZERO 100 101 102 103 104 105 113 114 115 116 117 118 119 120

NATURAL FREQUENCY = 1877.73096 RADIANS/SEC. = 289.84937 CYCLES/SEC.

MODE SHAPE

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NATURAL FREQUENCY = 3747.72095 RADIANS/SEC. = 596.46680 CYCLES/SEC.

MODE SHAPE
NATURAL FREQUENCY = 7606.00000 RADIANS/SEC. = 1210.52979 CYCLES/SEC.

MODE SHAPE

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| 0.0286  | -0.0122 | -0.0289 | -0.1145 | -0.2127 | -0.0142 | 0.1766  | -0.0315 | 0.0225  | 0.1694 |
| 0.2894  | -0.0227 | -0.2872 | -0.0000 | -0.0000 | -0.0000 | -0.0000 | -0.0000 | -0.0000 | 0.0000 |
| 0.0001  | 0.0001  | 0.0000  | 0.0001  | 0.0001  | -0.0000 | -0.0001 | -0.0004 | -0.0002 | -0.0002 |
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