Characterizing topologies by classes of functions and multifunctions

Alexander Hamlin Cramer

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CHARACTERIZING TOPOLOGIES BY CLASSES
OF FUNCTIONS AND MULTIFUNCTIONS

by

ALEXANDER HAMLIN CRAMER, 1940 -

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ABSTRACT

Topological spaces are characterized by the algebraic and topological structures of their classes of continuous selfmaps.

The problem of determining the topology of a set given certain classes of multifunctions or relations is considered. The algebraic structure of the upper semicontinuous multifunctions is shown to determine the topology of $T_1$ spaces.

A partial order for classes of topologies for the real numbers is defined and relationships between various classes are established.
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I. INTRODUCTION

The class $F(X)$ of all mappings from a set $X$ to itself contains the class $Z(X)$ of all constant selfmaps of $X$ and, under the operation of composition, $F(X)$ and $Z(X)$ are semigroups. If $X$ has a topology $t$, then $C(X,t)$ (or $C(X)$ if there is no confusion about the topology) denotes the semigroup of continuous selfmaps on $X$.

If $tr$ and $D$ denote the trivial and discrete topologies respectively, then $C(X, tr)$ and $C(X, D)$ are both equal to $F(X)$. It is therefore clear that the algebraic structure of the semigroup $C(X, t)$ does not, in general, determine the topology for $X$. If $F(X)$ is given a topology, then a topological structure for $C(X, t)$ is induced. The problem of determining the topology for $X$ from the algebraic and topological structure of $C(X, t)$ is considered.

Since the algebraic properties of $C(X, t)$ are not sufficient to determine $t$, then it is desirable to search for semigroups containing $C(X, t)$ that will give information about the topology of $X$. To this end we examine semigroups of multi-valued functions and semigroups of relations on $X$.

Another important problem is to discover new relationships between classes of continuous selfmaps for a set $X$ with various topologies. In particular, if $X$ is the set of real numbers with an arbitrary topology $t$, one might ask if a topology $t'$ exists such that $C(X, t) \subseteq C(X, t') \subseteq F(X)$ or such that $Z(X) \subseteq C(X, t') \subseteq C(X, t)$. Some of these questions are answered for several specific topologies for $X$. 
II. REVIEW OF THE LITERATURE

One of the earliest problems pertaining to characterizing a topological space $X$ by the algebraic properties of a family of mappings defined on $X$ was one concerning the ring of real-valued functions studied by Gelfand and Kolmogoroff (1), in 1939. The well known result was that two compact, Hausdorff spaces $X$ and $Y$ are homeomorphic if and only if their rings of real functions are isomorphic.

The first publication of a similar problem dealing with mappings from a space to itself was given by Everett and Ulam (2), in 1948. They posed the following problem. Given the class of all homeomorphisms $H(X,U)$ of a topological space $(X,U)$ onto itself, what other topologies $V$ exist on $X$ such that $H(X,U) = H(X,V)$?

In 1955, Wechsler (3) exhibited a class of topological spaces that have the property that for any two spaces $X$ and $Y$ in this class, if there is an isomorphism between $H(X)$ and $H(Y)$ which is also a homeomorphism with respect to the point-open topologies for $H(X)$ and $H(Y)$, then $X$ is homeomorphic to $Y$. The class of spaces satisfying this problem was later enlarged and modified by Thomas (4) and Wiginton and Shrader (5).

In 1963, Whittaker (6) proved the following result. If $X$ and $Y$ are compact, locally Euclidean manifolds and if $\phi$ is an isomorphism between $H(X)$ and $H(Y)$, then there exists a homeomorphism $h$ from $X$ to $Y$ such that $\phi(f) = h \circ f \circ h^{-1}$ for each $f$ in $H(X)$. 
Yu-Lee Lee (7) found ways to generate from locally compact spaces and from first countable Hausdorff spaces \((X,U)\), topologies \(V\) for \(X\) that are not homeomorphic to \(U\) and such that \(H(X,U) = H(X,V)\). He also proved in (8) that if \((X,U)\) is the real line with the usual topology, then for any Hausdorff topology \(V\) on \(X\) satisfying any of the properties locally compact, first countable, locally arc-wise connected, locally connected, or semi-locally connected, then \(U = V\).

In 1964, Magill (9) answered the following question. Does there exist a class of spaces such that for any two such spaces \(X\) and \(Y\), the semigroups of continuous selfmaps \(C(X)\) and \(C(Y)\) are isomorphic if and only if \(X\) and \(Y\) are homeomorphic? The class of spaces, called \(S\) spaces, which satisfy the question, include all locally Euclidean spaces and all \(\sigma\)-dimensional Hausdorff spaces. Other classes of spaces for which the problem is true are found in (10, 11, 12) by Magill.

\(M\)-spaces, a collection of spaces which contains the \(S\)-spaces, were found by Hicks and Haddock (13) to satisfy Magill's question where the continuous selfmaps are replaced by arbitrary semigroups of continuous selfmaps which contain \(\tau(X)\), the class of constant selfmaps.

In 1969, Warndof (14) obtained large classes of topologies for a given set with the following property. Within each class a topology is uniquely determined by its class of continuous self-maps.
III. TOPOLOGIES FOR $\alpha(X,t)$

A. EMBEDDING $(X,t)$ IN $\alpha(X,t)$

Let $X$ be a topological space and $F(X)$ be the semigroup of all mappings from $X$ to $X$, under composition.

**DEFINITION A.** A subsemigroup of $F(X)$ which contains the constant selfmaps is called an $\alpha$-semigroup and is denoted by $\alpha(X)$.

A problem that has received considerable attention is the following: If $(X,t_1)$ and $(Y,t_2)$ are topological spaces, when does $\alpha(X,t_1)$ isomorphic to $\alpha(Y,t_2)$ imply that $X$ is homeomorphic to $Y$? By choosing particular $\alpha$-semigroups (such as the continuous selfmaps) and restricting the topologies $t_1$ and $t_2$ in some way, positive results have been obtained by Magill (9, 10, 11, 12), Hicks and Haddock (13), and Rothmann (15).

Wechsler considered a similar problem in (3). Let $H(X)$ denote the group of all homeomorphisms from a topological space $X$ to itself with the point-open topology. Under certain limitations on the topologies of $X$ and $Y$, Wechsler showed that $X$ and $Y$ are homeomorphic if there exists an isomorphism from $H(X)$ onto $H(Y)$ which is also a homeomorphism.

In this chapter we characterize topological spaces by the algebraic and topological properties of $\alpha$-semigroups of selfmaps.

The following definition of topologies for $F(X)$ is well known.

**DEFINITION B.** Let $(X,t)$ be a topological space and $F(X)$ the class of all selfmaps on $X$. The point-open topology for $F(X)$ is generated by a subbase consisting of sets of the form $\{f: f(x) is
in 0} where \( x \) is a point of \( X \) and \( O \) is open in \( X \). The compact-open topology for \( F(X) \) is generated by a subbase consisting of sets of the form \( \{ f: f(C) \subseteq O \} \) where \( C \) is a compact set and \( O \) is an open set in \( X \).

If \( X \) is a uniform space or quasi-uniform space (every topological space is a quasi-uniform space), \( F(X) \) may be topologized as follows.

**DEFINITION C.** Let \((X,\mathcal{U})\) be a uniform (quasi-uniform) space and let \( \mathcal{A} \) be a nonempty family of nonempty subsets of \( X \). For \( A \) in \( \mathcal{A} \) and \( U \) in \( \mathcal{U} \), let \( W(A;U) = \{(f,g) \in F(X) \times F(X): (f(x),g(x)) \text{ is in } U \text{ for all } x \text{ in } A\} \). Then \( \{W(A;U): A \in \mathcal{A} \text{ and } U \in \mathcal{U}\} \) is a subbase for \( W/\mathcal{A} \), the uniformity (quasi-uniformity) of uniform (quasi-uniform) convergence on \( \mathcal{A} \).

The most important special cases are the following:

1. \( \mathcal{A} = \{X\} \), the topology of uniform (quasi-uniform) convergence.
2. \( \mathcal{A} = \{A: A \text{ is compact}\} \), the topology of compact convergence.
3. \( \mathcal{A} = \{\{x\}: x \text{ is in } X\} \), the topology of pointwise convergence.

The following lemma and remark embed \( X \) in \( F(X) \) where \( F(X) \) has any of the topologies of definitions B or C. Let \( Z(X) \) denote the class of constant selfmaps on \( X \) and for each \( x \) in \( X \) define \( \bar{x} \) to be the constant selfmap always equal to \( x \).

**LEMMA 1.** Let \((X,\mathcal{U})\) be a uniform (quasi-uniform) space and \( \mathcal{A} \) a nonempty family of nonempty subsets of \( X \). Let \( \bar{W}/\mathcal{A} = (W/\mathcal{A}) \cap (Z(X)xZ(X)) \). Then \((X,\mathcal{U})\) and \((Z(X),\bar{W}/\mathcal{A})\) are uniformly (quasi-uniformly) isomorphic. In fact if \( h \) is the function from \( X \) onto \( Z(X) \) defined by \( h(x) = \bar{x} \) for all \( x \) in \( X \), and \( \bar{h} \) is the function from
X \times X \text{ onto } Z(X) \times Z(X) \text{ defined by } \tilde{h}(x,y) = (h(X), h(Y)) = (\tilde{x}, \tilde{y}), \text{ for all } x \text{ and } y \text{ in } X, \text{ then } \mathcal{U} = \tilde{h}^{-1}(\tilde{w}/\mathcal{A}), \text{ the weakest uniformity (quasi-uniformity) on } X \text{ such that } h \text{ is uniformly ( quasi-uniformly) continuous.}

PROOF. \tilde{w}/\mathcal{A} \text{ has a subbase consisting of sets of the form } \tilde{w}(A,U) = \{(\tilde{x}, \tilde{y}) : (\tilde{x}(z), \tilde{y}(z)) \text{ is in } U \text{ for all } z \text{ in } A\} = \{(\tilde{x}, \tilde{y}) : (x, y) \text{ is in } U\} = \tilde{h}(U) \text{ for each } U \text{ in } \mathcal{U} \text{ and each } A \text{ in } \mathcal{A}. \text{ Since } h \text{ is one-to-one we have } \tilde{h}^{-1}(\tilde{w}(A,U)) = \tilde{h}^{-1}(\tilde{h}(U)) = U \text{ so } h \text{ is uniformly (quasi-uniformly) continuous with respect to } \mathcal{U} \text{ and } \tilde{h}^{-1}(\tilde{w}/\mathcal{A}) \subseteq \mathcal{U}. \text{ Clearly, } \mathcal{U} \subseteq \tilde{h}^{-1}(\tilde{w}/\mathcal{A}).

REMARK 1. A. The point-open topology and the compact-open topology agree on Z(X).

B. Given a space (X,t), let \mathcal{U} be a compatible quasi-uniform structure for (X,t). The topology of pointwise convergence and the point-open topology agree on Z(X). Hence all five topologies, point-open, compact-open, quasi-uniform convergence, compact convergence, and pointwise agree on Z(X).

PROOF. Let x be in X and O in t. Then \{f \text{ in } F(X) : f(x) \text{ is in } O\} \cap Z(X) = \{\tilde{y} : \tilde{y}(x) \text{ is in } O\} = \{\tilde{y} : y \text{ is in } O\} = h(O). \text{ Therefore } \{h(O) : O \text{ is in } t\} \text{ is a subbase for the point-open topology for } Z(X).

If \mathcal{C} \text{ is a compact subset of } X \text{ then } \{f : f(\mathcal{C}) \subseteq O\} \cap Z(X) = \{\tilde{y} : \tilde{y}(x) \text{ is in } O \text{ for all } x \text{ in } \mathcal{C}\} = \{\tilde{y} : y \text{ is in } O\} = h(O). \text{ Therefore } \{h(O) : O \text{ is in } t\} \text{ is a subbase for the compact-open topology on } Z(X) \text{ and the point-open and compact-open topologies agree on } Z(X).

For U in \mathcal{U} \text{ and } x \text{ in } X \text{ we have } \tilde{w}(x,U) = W(X,U) \cap (Z(X) \times Z(X)) = \{(\tilde{y}, \tilde{z}) : (\tilde{y}(x), \tilde{z}(x)) = (y, z) \text{ is in } U\}. \text{ Sets of this form make up a quasi-uniform subbase for the quasi-uniform structure of pointwise convergence for } Z(X).
Note that for $\tilde{y}$ in $Z(X)$, $(\tilde{W}(x,U))(y) = \{z : z \text{ is in } U(y)\} = h(U(y))$. Then $\{h(0) : y \text{ is in } 0 \text{ and } 0 \text{ is in } t\}$ is a fundamental system of neighborhoods of $\tilde{y}$ in the point-open topology and $(h(U(y)) : U \text{ is in } \mathcal{U})$ is a fundamental system of neighborhoods of $\tilde{y}$ in the topology of pointwise convergence.

Given $0$ in $t$ such that $\tilde{y}$ is in $h(0)$, there exists $U$ in $\mathcal{U}$ such that $U(y) \subseteq 0$. Then $h(U(y)) \subseteq h(0)$. Also, $h(U(y))$ contains $h(\text{interior } U(y))$ and $\text{interior } U(y)$ is in $t$. Hence the topology of pointwise convergence and the point-open topology agree on $Z(X)$.

For an arbitrary topological space $X$, $\alpha(X)$ with one of the several topologies induced from $F(X)$, may or may not be a topological semigroup. The following examples illustrate both possibilities.

EXAMPLE 1. In (16) de-Groot showed the existence of a topological space $X$ with a continuum of points such that $C(X)$, the class of continuous selfmaps on $X$, consists of just the constant maps and identity. Clearly, $C(X)$ is a topological semigroup with any of the previously defined topologies.

EXAMPLE 2. Let $X$ be the interval $[0,1]$ of the real numbers with the usual topology and let $\alpha(X) = F(X)$ with the pointwise topology. Consider the sequences $f_n(x) = x^n$ and $g_n(x) = x^{1/n}$ for $n = 1, 2, \ldots$. Clearly, $f_n$ converges to $\chi_{\{1\}}$ and $g_n$ converges to $X(0,1]$. But $f_n \circ g_n = \text{identity}$ for all $n = 1, 2, \ldots$. Since $\chi_{\{1\}} \circ X(0,1]$ is not the identity, then $F(X)$ is not a topological semigroup.

B. $\alpha(X)$ TOPOLOGICALLY ISOMORPHIC TO $\alpha(Y)$.

It will be established in this section that $\alpha(X)$ topologically
isomorphic to \( \alpha(Y) \) implies \( X \) is homeomorphic to \( Y \).

But first we state a lemma proved by Hicks and Haddock (13).

**LEMMA A.** If \( X \) and \( Y \) are topological spaces and \( \emptyset \) is an isomorphism from \( \alpha(X) \) onto \( \alpha(Y) \), then the restriction of \( \emptyset \) to \( Z(X) \) maps onto \( Z(Y) \).

A topological isomorphism from \( \alpha(X) \) to \( \alpha(Y) \) is defined to be an algebraic isomorphism that is also a homeomorphism with respect to some function space topologies for \( \alpha(X) \) and \( \alpha(Y) \).

**THEOREM 1.** Let \( X \) and \( Y \) be topological spaces and let \( \alpha(X) \) and \( \alpha(Y) \) be arbitrary \( \alpha \)-semigroups with the point-open or compact-open topologies. If \( \alpha(X) \) is topologically isomorphic to \( \alpha(Y) \), then \( X \) is homeomorphic to \( Y \).

**PROOF.** Let \( \emptyset \) be the topological isomorphism from \( \alpha(X) \) onto \( \alpha(Y) \). By lemma A, \( \emptyset \) restricted to \( Z(X) \) maps onto \( Z(Y) \). Therefore \( Z(X) \) and \( Z(Y) \) are homeomorphic with respect to the induced topologies. By Lemma 1, \( X \) and \( Y \) are homeomorphic.

Now consider the case where \( \alpha(X) \) and \( \alpha(Y) \) are uniform (quasi-uniform) spaces. A uniform (quasi-uniform) isomorphism from \( \alpha(X) \) to \( \alpha(Y) \) is defined to be both a uniform (quasi-uniform) space isomorphism and a semigroup isomorphism.

**THEOREM 2.** Let \( (X,\mathcal{U}) \) and \( (Y,\mathcal{V}) \) be uniform (quasi-uniform) spaces and let \( \mathcal{A} \) and \( \mathcal{B} \) be nonempty classes of nonempty subsets of \( X \) and \( Y \) respectively. If \( \alpha \)-semigroups \( \alpha(X) \) and \( \alpha(Y) \) are uniformly (quasi-uniformly) isomorphic with respect to uniform (quasi-uniform) convergence on \( \mathcal{A} \) and \( \mathcal{B} \), then \( X \) and \( Y \) are uniformly (quasi-uniformly) isomorphic.
PROOF. If \( a(X) \) and \( a(Y) \) are uniformly (quasi-uniformly) isomorphic then so are \( Z(X) \) and \( Z(Y) \) with their induced topologies. Lemma 2 gives the result.

For certain classes of \( \alpha \)-semigroups it is possible to obtain partial converses for theorems 1 and 2.

**THEOREM 3.** Let \( X \) and \( Y \) be topological spaces. If \( h \) is a homeomorphism from \( X \) to \( Y \) and \( a(X) \) and \( a(Y) \) are \( \alpha \)-semigroups such that for each \( f \) in \( a(X) \), \( h \circ f \circ h^{-1} \) is in \( a(Y) \) and for each \( g \) in \( a(Y) \), \( h^{-1} \circ g \circ h \) is in \( a(X) \), then \( a(X) \) is topologically isomorphic to \( a(Y) \), where \( a(X) \) and \( a(Y) \) both carry the point-open or compact-open topologies.

**PROOF.** Let \( \emptyset \) be the mapping from \( a(X) \) to \( a(Y) \) such that for each \( f \) in \( a(X) \), \( \emptyset(f) = h \circ f \circ h^{-1} \). Clearly \( \emptyset \) is a semigroup isomorphism.

Now we show that \( \emptyset \) is bicontinuous with respect to the point-open topologies for \( a(X) \) and \( a(Y) \). Let \( (f_n, n \in D) \) be a net in \( a(X) \) converging to \( g \). We must show that \( (\emptyset(f_n), n \in D) \) converges to \( \emptyset(g) \), or \( (\emptyset(f_n)(y), n \in D) \) converges to \( \emptyset(g)(y) \) for each \( y \) in \( Y \).

If \( y \) is in \( Y \) then \( h^{-1}(y) \) is in \( X \) and \( (f_n(h^{-1}(y)), n \in D) \) converges to \( g(h^{-1}(y)) \). Since \( h \) is continuous, \( (h \circ f_n \circ h^{-1}(y), n \in D) \) converges to \( h \circ g \circ h^{-1}(y) = \emptyset(g)(y) \). Therefore \( \emptyset \) is continuous.

A similar argument will show \( \emptyset^{-1} \) is continuous.

If \( a(X) \) and \( a(Y) \) have the compact-open topologies, then a sub-basic open set in \( a(X) \) will be \( U = \{ f \in a(X) : f(A) \subseteq 0 \} \) where \( A \) is compact and \( 0 \) is open in \( X \). Then \( \emptyset(U) = \{ h \circ f \circ h^{-1} : f \in a(X) \} \)
and $f(a) \subset 0 = \{ h \circ f \circ h^{-1} : f \in \alpha(X) \text{ and } (h \circ f \circ h^{-1})(h(A)) \subset h(0) \}$ which will be a subbasic open set for $\alpha(Y)$ since $h(A)$ is compact and $h(0)$ is open in $Y$. Therefore $\emptyset$ is an open mapping. Similarly $\emptyset$ can be shown to be continuous.

COROLLARY. Let $X$ and $Y$ be topological spaces and let $C(X)$ and $C(Y)$ both have the point-open or compact-open topologies. Then $X$ is homeomorphic to $Y$ if and only if $C(X)$ and $C(Y)$ are topologically isomorphic.

PROOF. Necessity follows from theorem 1.

If $h$ is a homeomorphism from $X$ to $Y$ then for $f \in C(X)$, $h \circ f \circ h^{-1}$ is clearly in $C(Y)$ and for $g \in C(Y)$, $h^{-1} \circ g \circ h$ is in $C(X)$. The result now follows from theorem 3.

Several other corollaries to theorem 3 may be obtained by replacing $C(X)$ and $C(Y)$ by the compact selfmaps, connected selfmaps, or even the closed selfmaps if the topologies for $X$ and $Y$ are $T_1$.

THEOREM 4. Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be uniform (quasi-uniform) spaces and $\mathcal{A}$ a nonempty collection of nonempty subsets of $X$. If $h$ is a uniform (quasi-uniform) isomorphism from $X$ to $Y$ and $\alpha(X)$ and $\alpha(Y)$ are $\alpha$-semigroups such that for each $f \in \alpha(X)$, $h \circ f \circ h^{-1}$ is in $\alpha(Y)$ and for each $g \in \alpha(Y)$, $h^{-1} \circ g \circ h$ is in $\alpha(X)$, then $\alpha(X)$ and $\alpha(Y)$ are uniformly (quasi-uniformly) isomorphic where the uniform (quasi-uniform) structures $\mathcal{U}^*$ and $\mathcal{V}^*$ for $\alpha(X)$ and $\alpha(Y)$ respectively are those of convergence on $\mathcal{A}$ and on $h(A) = \{ h(A) : A \in \mathcal{A} \}$ respectively.

PROOF. We again define the mapping $\emptyset$ from $\alpha(X)$ to $\alpha(Y)$ such that for each $f \in \alpha(X)$, $\emptyset(f) = h \circ f \circ h^{-1}$. Clearly $\emptyset$ is an isomorphism.
Let $h$ represent the mapping from $\mathcal{U}$ to $\mathcal{V}$ such that for $U$ in $\mathcal{U}$,
\[ h(U) = \{(h(x), h(y)) : (x, y) \text{ is in } U \}. \]
A subbasic member of $\mathcal{U}^*$ will be a set of the form

\[ \{(f, g) : f, g \in \mathcal{a}(X) \text{ and } (f(x), g(x)) \in U \text{ for every } x \in A\} \]

where $U$ is in $\mathcal{U}$ and $A$ is in $\mathcal{A}$. Denote this set by $U^*$. Since $h$ is one-to-one and onto, then

\[ \mathcal{a}(U^*) = \{(\mathcal{a}(f), \mathcal{a}(g)) : (f, g) \in U^*\} = \]

\[ \{(\mathcal{a}(f), \mathcal{a}(g)) : f \text{ and } g \in \mathcal{a}(X) \text{ and } (f(x), g(x)) \in U \text{ for every } x \in A\} = \]

\[ \{(\mathcal{a}(f), \mathcal{a}(g)) : f, g \in \mathcal{a}(X) \text{ and } (h \circ f(x), h \circ g(x)) \in h(U) \text{ for every } x \in A\} = \]

\[ \{(\mathcal{a}(f), \mathcal{a}(g)) : f, g \in \mathcal{a}(X) \text{ and } (h \circ f \circ h^{-1}(y), h \circ g \circ h^{-1}(y)) \in h(U) \text{ for all } y \in h(A)\} = \]

\[ \{(\mathcal{a}(f), \mathcal{a}(g)) : \mathcal{a}(f), \mathcal{a}(g) \in \mathcal{a}(Y) \text{ and } (\mathcal{a}(f)(y), \mathcal{a}(g)(y)) \in V \text{ for all } y \in h(A)\}, \]

where $V = h(U)$, is a member of $\mathcal{V}^*$. Therefore $\mathcal{a}^{-1}$ is uniformly (quasi-uniformly) continuous.

The proof that $\mathcal{a}$ is uniformly (quasi-uniformly) continuous is similar and is omitted.
COROLLARY. Let \((X, U)\) and \((Y, V)\) be uniform (quasi-uniform) spaces and let \(C(X)\) and \(C(Y)\) have the uniform (quasi-uniform) structures \(\mathcal{U}^\ast\) and \(\mathcal{V}^\ast\) defined in theorem 4. Then \(X\) is uniformly (quasi-uniformly) isomorphic to \(Y\) if and only if \(C(X)\) is uniformly (quasi-uniformly) isomorphic to \(C(Y)\).

Once again, classes of selfmaps other than \(C(X)\) and \(C(Y)\) may be substituted to obtain other corollaries, such as the class of uniformly (quasi-uniformly) continuous selfmaps.
IV MULTIVALUED FUNCTIONS FROM (X,t) TO (X,t)

A. DEFINITIONS AND EXAMPLES

Let (X,t) be a topological space and let $\mathcal{Q}(X)$ denote the class of nonempty subsets of X. The symbol $R(X,t)$ will denote the class of nonempty relations on X. For $r$ in $R(X,t)$ and $x$ in X, we define $r(x) = \{y \in X : (x,y) \text{ is in } r\}$. $M(X,t)$ is defined to be the subclass of $R(X,t)$ consisting of mappings from X to $\mathcal{Q}(X)$. That is, for $m$ in $M(X,t)$ and $x$ in X, $m(x)$ is a nonempty subset of X. The class $M(X,t)$ is called the class of multivalued functions from X to X, or simply multifunctions. Clearly, single valued functions can be considered as special cases of multifunctions such that their ranges are contained in the singleton subsets of X.

Warndof (14) and Hicks and Haddock (13) have shown that among certain classes of topological spaces, if $C(X,t_1) = C(X,t_2)$ then $t_1 = t_2$. However, Rothmann (15) exhibits many topological spaces where $C(X,t_1) = C(X,t_2)$ and $t_1 \neq t_2$. We will show the existence of classes of multifunctions that uniquely determine the topology for X.

In the following definitions (X,t) is an arbitrary topological space.

DEFINITION 1. Let $r$ be a relation on X.

1. For $x$ in X, $r^{-1}(x) = \{y \in X : x \text{ is in } r(y)\}$

2. For $A \subseteq X$, $r(A) = \{r(x) : x \text{ is in } A\}$

3. For $A \subseteq X$, $r^{-1}(A) = \{y \in X : r(y) \cap A \neq \emptyset\}$
DEFINITION 2. Let $r$ be a relation on $X$.

i. $r$ is lower semicontinuous (LSC) if and only if $r^{-1}(A)$ is open in $X$ whenever $A$ is an open set in $X$.

ii. $r$ is upper semicontinuous (USC) if and only if $r^{-1}(A)$ is closed in $X$ whenever $A$ is a closed set in $X$.

iii. $r$ is continuous if and only if it is both upper semicontinuous and lower semicontinuous.

For a number of other definitions of USC and LSC and their origins, see Borges (17).

$UCR(X,t)$, $LCR(X,t)$ and $CR(X,t)$ will denote the classes of USC, LSC, and continuous relations respectively. Since the multifunctions are a subclass of $R(X,t)$, we denote the corresponding classes of multifunctions by $UCM(X,t)$, $LCM(X,t)$, and $CM(X,t)$.

It is easy to construct examples of relations or multifunctions that are USC but not LSC and vice versa.

EXAMPLE 1. Let $X$ be the set of real numbers and $t$ the usual topology for $X$. For any closed nonempty subset $C$ of $X$ such that $C \neq X$, choose distinct points $p$ and $q$ of $X$ and define the multifunction $m$ such that $m(x) = \begin{cases} \{p,q\} & \text{if } x \text{ is in } C \\ \{q\} & \text{if } x \text{ is in } X - C \end{cases}$. Then $m$ is USC since $m^{-1}(S)$ can only be $X$, $C$, or empty for every subset $S$ of $X$.

But for $O$ an open set such that $p$ is in $O$ and $q$ is not in $O$, we have $m^{-1}(O) = C$. Since $t$ is a connected topology for $X$, then $m$ is not LSC.

Let $tr$ denote the trivial topology and $D$ the discrete topology. It is clear that for any set $X$, $LCM(X,tr) = LCM(X,D) = UCM(X,tr) = UCM(X,D) = M(X)$. It is well known that $C(X,t) = F(X)$ if and only if
t is tr or D. It is also clear that $\text{CM}(X,t) = \text{M}(X)$ if and only if $t$ is tr or D. Evidently the above classes of multifunctions will not distinguish between the trivial and discrete topologies on a set $X$.

It will be shown, however, that the USC or LSC multifunctions on a space $X$ will determine the topology uniquely in every case except for the trivial and discrete topologies. Moreover $\text{UCR}(X,t)$ and $\text{LCR}(X,t)$ uniquely determine the topology on $X$ in every case.

B. CLASSES OF RELATIONS ON $(X,t)$

In this section the topology on a space $X$ is characterized by certain classes of relations on $X$.

**Theorem 1.** Let $(X,t)$ be a space. For each subset $S$ of $X$, there exists a relation $r$ such that $S$ is open if and only if $r$ is LSC and $S$ is closed if and only if $r$ is USC.

**Proof.** Let $r = S\times S$. If $0$ is a subset of $X$ then

$$r^{-1}(0) = \begin{cases} S & \text{if } S \cap 0 \neq \emptyset \\ \emptyset & \text{if } S \cap 0 = \emptyset \end{cases}$$

If $S$ is open then clearly $r$ is LSC. If $r$ is LSC then $r^{-1}(X) = S$, so $S$ must be open. The case where $S$ is closed follows similarly.

**Corollary 1.** Let $t_1$ and $t_2$ be topologies for a set $X$. Then:

1. $\text{LCR}(X,t_1) = \text{LCR}(X,t_2)$ if and only if $t_1 = t_2$.
2. $\text{UCR}(X,t_1) = \text{UCR}(X,t_2)$ if and only if $t_1 = t_2$.

The relation $r$ in the proof of theorem 1 is symmetric. Therefore we define $\text{SLCR}(X,t)$ and $\text{SUCR}(X,t)$ to be the classes of symmetric relations that are LSC and USC respectively.
COROLLARY 2. Let $t_1$ and $t_2$ be topologies for a set $X$. Then:

i. $\text{SLCR}(X,t_1) = \text{SLCR}(X,t_2)$ if and only if $t_1 = t_2$.

ii. $\text{SUCR}(X,t_1) = \text{SUCR}(X,t_2)$ if and only if $t_1 = t_2$.

Characterizing the topology on a set $X$ using the class of continuous relations by the method of theorem 1 is not possible, as illustrated by the following theorem.

THEOREM 2. A topological space $(X,t)$ is connected if and only if $\text{CR}(X,t) = \text{CM}(X,t)$.

PROOF. Clearly $\text{CM}(X,t) \subseteq \text{CR}(X,t)$. Suppose $X$ is connected and $r$ is a continuous relation that is not a multifunction. Then $\emptyset \neq r^{-1}(X) \neq X$ and $r^{-1}(X)$ is both open and closed. This is a contradiction since $X$ is connected. Therefore $\text{CR}(X,t) = \text{CM}(X,t)$.

Now suppose $A \cup B$ is a separation of $X$. Define $r$ to be the relation $A \times X$. Then $r^{-1}(O) = \begin{cases} A & \text{if } O \neq \emptyset \\ \emptyset & \text{if } O = \emptyset \end{cases}$ for each subset $O$ of $X$. Since $A$ is both open and closed, $r$ is continuous. Clearly $r$ is not a multifunction.

Now it is apparent that the relation $r = S \times S$ used in theorem 1 would be continuous only if $S$ was both open and closed in $X$. Rather than pursue the special cases of spaces with open and closed base elements, such as zero-dimensional Hausdorff spaces, let us turn to subclasses of the class of relations, the multifunctions.

C. THE CLASSES $\text{UCM}(X,t)$ and $\text{LCM}(X,t)$

If $(X,t)$ is a topological space and $t$ is not the trivial topology,
then the following theorems show that the topology is determined by either of the classes of multifunctions $\text{LCM}(X, t)$ or $\text{UCM}(X, t)$.

**THEOREM 3.** If $(X, t)$ is a space and $t$ is not the trivial topology, then for each subset $S$ of $X$ there is a multifunction $m_S$ such that $m_S$ is LSC if and only if $S$ is open.

**PROOF.** Let $O$ be an open set such that $\emptyset \neq O \neq X$. Choose points $p$ in $O$ and $q$ in $X - O$. For each subset $S$ of $X$ define $m_S$ such that

$$m_S(x) = \begin{cases} 
    \{p, q\} & \text{if } x \in S \\
    \{q\} & \text{if } x \in X - S
\end{cases}.$$  

Then for $U$ in $t$ we have

$$m^{-1}(U) = \begin{cases} 
    X & \text{if } q \in U \\
    S & \text{if } q \in X - U \text{ and } p \in U \\
    \emptyset & \text{if } q \in X - U \text{ and } p \in X - U
\end{cases}.$$  

If $m_S$ is LSC consider $m_S^{-1}(O) = S$. Therefore $S$ is open and the theorem is proved.

**THEOREM 4.** If $t_1$ and $t_2$ are nontrivial topologies for a set $X$ and $\text{LCM}(X, t_1) = \text{LCM}(X, t_2)$, then $t_1 \cap t_2 \neq \text{tr}$.

**PROOF.** Suppose $t_1 \cap t_2 = \text{tr}$ and $\text{LCM}(X, t_1) = \text{LCM}(X, t_2)$. Choose $O_1$ in $t_1$ such that $\emptyset \neq O_1 \neq X$ and choose point $p$ in $O_1$ and point $q$ in $X - O_1$. The multifunction $m_{O_1}$ defined by $m_{O_1}(x) =

$$\begin{cases} 
    \{p, q\} & \text{if } x \in O_1 \\
    \{q\} & \text{if } x \in X - O_1
\end{cases}.$$  

is easily seen to be LSC with respect to $t_1$. Therefore $m_{O_1}$ is in $\text{LCM}(X, t_2)$. Every open set $O_2$ that contains $p$ must also contain $q$, for otherwise $m_{O_1}^{-1}(O_2) = O_1$ and $O_1 \subseteq t_1 \cap t_2$. Since $p$ and $q$ were any points such that $p \in O_1$ and $q \in X - O_1$ then for every $O_2 \in t_2$ such that $O_1 \cap O_2 \neq \emptyset$ we must have $O_2$ containing all of $X - O_1$.
The set \( X - 0_1 \) is \( t_1 \) closed but it may or may not be open.

Case 1. If \( X - 0_1 \) is \( t_1 \) open then for every nonempty \( O_2 \in t_2 \) we must have either \( O_2 \) intersects \( 0_1 \) or the complement of \( 0_1 \). Under either circumstance \( O_2 = X \) and \( t_2 \) is the trivial topology, an obvious contradiction.

Case 2. If \( X - 0_1 \) is not \( t_1 \) open then the multifunction

\[
mx - 0_1 \text{ defined by } mx - 0_1(x) = \begin{cases} (p,q) & \text{if } x \in X - 0_1 \\ (q) & \text{if } x \in 0_1 \end{cases}
\]

is not LSC with respect to \( t_1 \). But it is LSC with respect to \( t_2 \). Again this is a contradiction.

**THEOREM 5.** If \( t_1 \) and \( t_2 \) are nontrivial topologies for a set \( X \), then \( \text{LCM}(X,t_1) = \text{LCM}(X,t_2) \) if and only if \( t_1 = t_2 \).

**PROOF.** Sufficiency is obvious.

If \( \text{LCM}(X,t_1) = \text{LCM}(X,t_2) \) then by theorem 4 there exists a subset \( U \) of \( X \) such that \( U \in t_1 \cap t_2 \) and \( \emptyset \neq U \neq X \). Choose points \( p \) and \( q \) of \( X \) such that \( p \in U \) and \( q \in X - U \). For each subset \( O \) of \( X \) define a multifunction \( m_0 \) such that

\[
m_0(x) = \begin{cases} (p,q) & \text{if } x \in O \\ (q) & \text{if } x \in X - O \end{cases}
\]

Then the following statements are equivalent:

i. \( O \) is in \( t_1 \).

ii. \( m_0 \) is LSC with respect to \( t_1 \).

iii. \( m_0 \) is LSC with respect to \( t_2 \).

iv. \( O \) is in \( t_2 \).

Therefore \( t_1 = t_2 \).

The preceding theorems of this section follow for the case of the USC multifunctions in an entirely analogous way. The theorems are stated without proof.
THEOREM 6. If \((X,t)\) is a space and \(t\) is not the trivial topology, then for each subset \(T\) of \(X\) there is a multifunction \(m_T\) such that \(m_T\) is USC if and only if \(T\) is closed.

THEOREM 7. If \(t_1\) and \(t_2\) are nontrivial topologies for a set \(X\) and \(UCM(X,t_1) = UCM(X,t_2)\), then \(t_1 \cap t_2 \neq tr\).

THEOREM 8. If \(t_1\) and \(t_2\) are nontrivial topologies for a set \(X\), then \(UCM(X,t_1) = UCM(X,t_2)\) if and only if \(t_1 = t_2\).

Let \((X,t)\) be a space with \(t \neq tr\) and let 0 be open in \(X\) such that \(\emptyset \neq 0 \neq X\). Choose points \(p\) and \(q\) in \(X\) such that \(p\) is in 0 and \(q\) is in \(X - 0\). As in theorem 3, let \(m_S\) be the multifunction defined by

\[
m_S(x) = \begin{cases} 
\{p,q\} & \text{if } x \in S \\
\{q\} & \text{if } x \in X - S
\end{cases}
\]

for each subset \(S\) of \(X\). The class \(U_0 = \{m_S : S \in t\}\) has some interesting algebraic properties.

The class \(M(X,t)\) is a semigroup (under composition of multifunctions) where we define \(m \circ n(x) = m(n(x)) = \bigcup \{m(y) : y \in n(x)\}\). \(U_0\) is a subsemigroup of \(M(X,t)\) and in fact \(U_0\) is a semigroup right ideal, i.e. \(m_S \circ n\) is in \(U_0\) for all \(n\) in \(M(X,t)\). Note also that \(m_0\) is a left identity for \(U_0\).

EXAMPLE 2. Let \(X\) be an infinite set and FC the finite complement topology for \(X\). For any open set 0 in \(X\) such that \(\emptyset \neq 0 \neq X\), \(U_0\) consists of the constant function \(\bar{q}\) along with multifunctions \(m\) such that \(m(x) = \{p,q\}\) except for finitely many points \(x\), where \(m(x) = \{p\}\). If \(t\) is any topology for \(X\), then \(U_0 \subseteq LCM(X,t)\) implies that for each \(x\) in \(X\), \(m_X^{-1}(\{x\})\) is in \(U_0\) and \(m_X^{-1}(\{x\})(0) = X - \{x\}\) is open. Therefore \(t\) would be a T1 topology. Furthermore, if \(t\) is a T1 topology then every finite set \(A\) in \(X\) is closed and \(m_X - A\) is in \(LCM(X,t)\). Clearly \(U_0 = \{m_X - A : A \text{ finite}\} \cup \{\bar{q}\}\) so that \(U_0 \subseteq LCM(X,t)\).
Therefore \( U_0 \subseteq LCM(X, t) \) if and only if \( t \) is a \( T_1 \) topology.

For an arbitrary space \((X, t)\) we now define a subclass of \( M(X, t) \) which is related to \( U_0 \). Again we assume that \( t \) is not the trivial topology and that \( 0 \) is a nontrivial open set in \( t \) containing a point \( p \) and not containing a point \( q \). For \( S \) a subset of \( X \) define the multifunction \( m^* \) such that:
\[
\begin{align*}
 m^*_S(x) &= \begin{cases}
 (p, q) & \text{if } x \text{ is in } S \\
 \{p\} & \text{if } x \text{ is in } X - S
\end{cases}
\end{align*}
\]
If \( m^*_S \) is in \( X - S \) we define \( U^*_0 = \{ m^*_S : S \text{ closed in } X \} \) then it is clear that \( U^*_0 \) is a subsemigroup of \( UCM(X, t) \). Even though \( U^*_0 \) and \( U^*_0 \) are closely related semigroups, they are not isomorphic in general.

EXAMPLE 3. Let \( X = \{a, b, c\}, t = \emptyset, \{a\}, \{b\}, \{a, b\}, X \), \( p = a, q = b \) and \( 0 = \{a\} \). Then \( U_0 = \{ m_\emptyset, m_{\{a\}}, m_{\{b\}}, m_{\{a, b\}}, m_X \} \) and the semigroup table for \( U_0 \) follows. Note that only the subscripts of the multifunctions are used as entries.

\[
\begin{array}{c|cccccc}
\hline
 & \emptyset & a & b & ab & X \\
\hline
\emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \\
\hline
a & \emptyset & a & b & ab & X \\
\hline
b & X & X & X & X & X \\
\hline
ab & X & X & X & X & X \\
\hline
X & X & X & X & X & X \\
\hline
\end{array}
\]
We also have $U^*_0 = \{m^*_x, m^*_y, m^*_z, m^*_a, m^*_b, m^*_c, m^*_d, m^*_e\}$ and the semigroup table as follows.

**TABLE 2.**

<table>
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<tr>
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<td>bc</td>
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<td>ac</td>
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</table>

The semigroups $U^*_0$ and $U^*_0$ can be seen to be nonisomorphic since $m^*_x$ appears 17 times in Table 1 but no multifunction appears 17 times in Table 2.

The multifunctions in $U^*_0$ that are USC and the multifunctions in $U^*_0$ that are LSC may indicate which subsets of a topological space are both open and closed.

**REMARK 1.** Let $X$ be a space and $0$ a nontrivial open set in $X$ such that $p$ is in $0$ and $q$ is not in $0$ and suppose that the interior of $X - 0$ is nonempty. Then:

i. $UCM(X,t) \cap U^*_0 = \{m^*_x, m^*_0\}$ if and only if $X$ is connected, and

ii. $LCM(X,t) \cap U^*_0 = \{m^*_x, m^*_0\}$ if and only if $X$ is connected.

**PROOF.** Let $S$ be a subset of $X$ such that $\emptyset \neq S \neq X$. Then $m^*_S$ is in both $UCM(X,t)$ and $U^*_0$ if and only if $S$ is both open and closed, and hence, if and only if $X$ is not connected.
Similarly for part ii.

D. THE CLASS CM(X,t)

A topological space (X,t) is defined to be saturated if the intersection of any collection of open sets is open.

THEOREM 9. If (X,t) is a saturated space and T is the set {0 : X - 0 is in t} then T is a topology for X and LCM(X,t) = UCM(X,T) and UCM(X,t) = LCM(X,T).

PROOF. Obviously T is a topology for X.

If m is in LCM(X,t) and V is closed in T, then V is open in t. Therefore m⁻¹(V) is open in t and m⁻¹(V) is closed in T. Then m is USC with respect to T.

The other parts follow by similar arguments.

COROLLARY. If (X,t) is a saturated space and T = {0 : X - 0 is in t}, then CM(X,t) = CM(X,T).

PROOF. CM(X,t) = LCM(X,t) ∩ UCM(X,t) = UCM(X,T) ∩ LCM(X,T) = CM(X,T).

Since there is an ample supply of saturated spaces (for example any finite space), the problem of determining the topology of a space from the continuous multifunctions has no general solution. However, some partial results can be obtained.

THEOREM 10. Let (X,t) be a space having a point p such that {p} is open but not closed ( {p} is closed but not open), then for each subset S of X there exists a multifunction cm_s such that cm_s is continuous if and only if S is open (closed).
PROOF. Suppose \( \{p\} \) is open but not closed. Let \( S \) be any subset of \( X \) and define \( cm_S \) to be the multifunction such that

\[
cm_S(x) = \begin{cases} 
X & \text{if } x \text{ is in } S \\
X - p & \text{if } x \text{ is in } X - S
\end{cases}
\]

For any closed subset \( C \) of \( X \) then \( C \neq \{p\} \) so that \( cm_S^{-1}(C) = X \) if \( C \) is nonempty and \( cm_S^{-1}(C) = \emptyset \) if \( C \) is empty. Therefore \( cm_S \) is USC for all choices of \( S \). However, if \( 0 \) is an open subset of \( X \) then \( cm_S^{-1}(0) \) is

\[
\begin{cases} 
X & \text{if } 0 \neq \emptyset \text{ and } 0 \neq \{p\} \\
S & \text{if } 0 = \{p\} \\
\emptyset & \text{if } 0 = \emptyset
\end{cases}
\]

Therefore \( cm_S \) is LSC if and only if \( S \) is open.

The proof for the case that \( \{p\} \) is closed and not open is similar to the preceding case and is omitted.

COROLLARY 1. Let \( t \) and \( T \) be topologies for a set \( X \) and let \( p \) be a point of \( X \) such that \( \{p\} \) is open but not closed \((\{p\} \) is closed but not open\) in \( t \) and \( T \). Then \( CM(X,t) = CM(X,T) \) if and only if \( t = T \).

COROLLARY 2. Let \( t \) and \( T \) be topologies for a set \( X \) and let \( p \) be a point of \( X \) such that \( \{p\} \) is open but not closed in \( t \) and \( \{p\} \) is closed but not open in \( T \). Then \( CM(X,t) = CM(X,T) \) if and only if \( t = \{0 : X - 0 \text{ is in } T\} \) and \( t \) and \( T \) are saturated topologies.

COROLLARY 3. Let \( t \) and \( T \) be connected \( T_1 \) topologies for a set \( X \). Then \( CM(X,t) = CM(X,T) \) if and only if \( t = T \).

PROOF. Sufficiency is obvious.

Since both topologies are connected, any point \( \{p\} \) will be closed and not open. The result follows from Corollary 1.
A few examples of spaces that do not satisfy the hypothesis of theorem 10 are listed below.

1. Trivial topology.
2. Discrete topology.
3. The real numbers $\mathbb{R}$ with the topology $\tau = \{(a, \infty) : a \in \mathbb{R}\}$.
4. The partition topology on any set where each member of the partition has more than one point. This is example 5 in Steen and Seebach (18).
V. $\alpha$-SEMIGROUPS OF MULTIFUNCTIONS

A. SOME KNOWN RESULTS

In this chapter some of the results of Magill (9) and Hicks and Haddock (13) will be extended. Several of these known results are listed without proof.

Let $(X, t)$ be an arbitrary space and $\alpha(X)$ an $\alpha$-semigroup of selfmaps on $X$.

**LEMMA A.** Let $f$ be in $\alpha(X)$. Then $f \circ g = f$ for all $g$ in $\alpha(X)$ if and only if $f = \bar{x}$ for some $x$ in $X$.

Suppose $X$ and $Y$ are spaces and $\Phi$ is an isomorphism from $\alpha(X)$ to $\alpha(Y)$. We shall denote the restriction of $\Phi$ to $Z(X)$, the class of constant selfmaps, by $\Phi$ also.

**LEMMA B.** $\Phi$ maps $Z(X)$ onto $Z(Y)$.

Let $x'$ be the one-to-one onto mapping from $X$ to $Z(X)$ such that $x'(z) = \bar{z}$ for each $z$ in $X$ and let $y'$ be the corresponding mapping from $Y$ onto $Z(Y)$. Now consider the following diagram.

**DIAGRAM 1.**

\[
\begin{array}{ccc}
\alpha(X) & \overset{\Phi}{\longrightarrow} & \alpha(Y) \\
\cup & & \cup \\
Z(X) & \overset{\Phi}{\longrightarrow} & Z(Y) \\
\downarrow x' & & \uparrow y' \\
X & \overset{h}{\longrightarrow} & Y
\end{array}
\]

The mapping $h$ is defined by $h(x) = (y')^{-1} \circ \Phi \circ x'(x)$ and is clearly one-to-one and onto from $X$ to $Y$. 
THEOREM A. Suppose \( \beta(X) \) and \( \beta(Y) \) are semigroups of selfmaps such that \( \alpha(X) \subseteq \beta(X) \) and \( \alpha(Y) \subseteq \beta(Y) \). Then \( \emptyset \) can be extended to an isomorphism \( \psi \) from \( \beta(X) \) onto \( \beta(Y) \) if and only if

i. \( h \circ f \circ h^{-1} \) is in \( \beta(Y) \) for every \( f \) in \( \beta(X) \) and

ii. \( h^{-1} \circ g \circ h \) is in \( \beta(X) \) for every \( g \) in \( \beta(Y) \).

Furthermore if \( \emptyset \) can be extended then \( \psi \) is unique and \( \psi(f) = h \circ f \circ h^{-1} \) for every \( f \) in \( \beta(X) \).

B. \( aM(X) \) ISOMORPHIC TO \( aM(Y) \)

In this chapter the notation \( M(X) \) will be used to denote the class of all multifunctions on the set \( X \). Recall that \( M(X) \) is a semigroup under composition of multifunctions. Let \( ZM(X) \) represent the constant multifunctions and \( aM(X) \) will be any subsemigroup of \( M(X) \) that contains \( ZM(X) \). The constant multifunction always equal to \( S \), where \( S \) is a nonempty subset of \( X \), will be denoted by \( S \).

Suppose \( X \) and \( Y \) are spaces and \( \phi^* \) is an isomorphism from \( aM(X) \) to \( aM(Y) \). Then the lemmas of section A will carry over.

LEMMA 1. Let \( m \) be in \( aM(X) \). Then \( m \circ n = m \) for all \( n \) in \( aM(X) \) if and only if \( m \) is a constant multifunction, i.e. \( m = S \) where \( S \) is a nonempty subset of \( X \).

PROOF. If \( m \circ n = m \) for all \( n \) in \( aM(X) \), then for any \( x \) and \( y \) in \( X \), \( m(y) = m(y(x)) = m \circ y(x) = m(x) \). Therefore \( m \) is a constant multifunction.

If \( m \) is constantly equal to \( S \), where \( S \) is a nonempty subset of \( X \), then \( m \circ n = S \circ n = S = m \) for all \( n \) in \( aM(X) \).
LEMMA 2. The restriction of $\emptyset^*$ to $\mathcal{Z}(X)$ maps onto $\mathcal{Z}(Y)$.

PROOF. Let $m$ be in $\mathcal{Z}(X)$ and $n$ in $\mathcal{A}(Y)$. Then $\emptyset_*^{-1}(n)$ is in $\mathcal{A}(X)$ and $m \circ \emptyset_*^{-1}(n) = m$ by lemma 1. Therefore $[\emptyset^*(m)] \circ n = \emptyset_*[m \circ \emptyset_*^{-1}(n)] = \emptyset^*(m)$ for all $n$ in $\mathcal{A}(Y)$. Lemma 1 implies that $\emptyset^*(m)$ is in $\mathcal{Z}(Y)$. Therefore $\emptyset^*$ maps $\mathcal{Z}(X)$ into $\mathcal{Z}(Y)$.

A similar proof shows that $\emptyset_*^{-1}$ maps $\mathcal{Z}(Y)$ into $\mathcal{Z}(X)$ and it follows that $\emptyset^*$ maps $\mathcal{Z}(X)$ onto $\mathcal{Z}(Y)$.

There is a natural one-to-one onto mapping $s$ from $\mathcal{A}(X)$ to $\mathcal{Z}(X)$ such that $s(S) = \bar{S}$ for each $S$ in $\mathcal{A}(X)$. The corresponding mapping from $\mathcal{A}(Y)$ onto $\mathcal{Z}(Y)$ will be designated by $t$. The result of lemma 2 implies the existence of a one-to-one mapping $h^*$ from $\mathcal{A}(X)$ onto $\mathcal{A}(Y)$ such that $h^*(S) = t^{-1} \circ \emptyset^* \circ s(S)$. Then $h^*(S) = t^{-1} \circ \emptyset^*(\bar{S})$ and $\overline{h^*(S)} = t \circ h^*(S) = \emptyset^*(\bar{S})$. The following diagram illustrates these relationships.

Diagram 2.

For each $m$ in $\mathcal{A}(X)$ there is an induced single-valued mapping (which we will also call $m$) from $\mathcal{A}(X)$ to $\mathcal{A}(Y)$ defined by

A result of Hicks and Haddock (13) shows that $\emptyset(g) = h \circ g \circ h^{-1}$ for each $g$ in $\mathcal{A}(X)$ for the case of single-valued functions. However, in this case $h$ was a one-to-one mapping from $X$ onto $Y$. In the present situation $h^* \circ m \circ h_*^{-1}$ has no meaning for $m$ in $\mathcal{A}(X)$.
m(S) = \bigcup \{m(x) : x \text{ is in } S\} \text{ for each } S \text{ in } A(X). \text{ Then } h^* \circ m \circ h^{-1}^* \text{ is a single-valued mapping from } A(Y) \text{ to } A(Y). \text{ If we restrict the domain of } h^* \circ m \circ h^{-1}^* \text{ to singleton subsets of } Y, \text{ then it describes a multifunction from } Y \text{ to } Y. \text{ This, then, is the definition which is given to } h^* \circ m \circ h^{-1}^* \text{ for each } m \text{ in } aM(X). \text{ The relation } h^*(S) = \emptyset^*(S) \text{ is used several times in the following theorem which extends the result of Hicks and Haddock.}

**THEOREM 1.** Let \( m \) be in \( aM(X) \) and \( n \) in \( aM(Y) \). Then \( \emptyset^*(m) = h^* \circ m \circ h^{-1}^* \) and \( \emptyset^{-1}(n) = h^{-1} \circ n \circ h^* \).

**PROOF.** Note that \( m(S) = m(S) \) for each \( S \) in \( A(X) \) and for each \( m \) in \( M(X) \).

For \( y \) in \( Y \) and \( m \) in \( aM(X) \) it follows that \( h^* \circ m \circ h^{-1}^*(\{y\}) = h^*(m(h^{-1}^*(\{y\}))) = h^*(m(h^{-1}^*(\{y\})))(y) = \emptyset^*(m(h^{-1}^*(\{y\}))(y) = \emptyset^*(m(\emptyset^{-1}(\{y\}))(y) = [(\emptyset^*(m)) \circ \bar{y}](y) = [\emptyset^*(m)](y).

Similarly, if \( x \) is in \( X \) and \( n \) is in \( aM(Y) \) we have

\[
h^{-1} \circ n \circ h^*(\{x\}) = h^{-1}[n(h^*(\{x\}))] = h^{-1}[n(h^*(\{x\}))](x) = \\
\emptyset^{-1}[n(h^*(\{x\}))](x) = \emptyset^{-1}[n \circ h^*(\{x\})](x) = \\
[\emptyset^{-1}(n)] \circ [\emptyset^{-1}(h^*(\{x\}))](x) = [\emptyset^{-1}(n)] \circ [h^{-1}(h^*(\{x\}))](x) = \\
[\emptyset^{-1}(n) \circ \bar{x}](x) = [\emptyset^{-1}(n)](x).
\]

**REMARK 1.** If \( T \) is any one-to-one onto mapping from \( A(X) \) to \( A(Y) \) such that \( \emptyset^*(m) = T \circ m \circ T^{-1} \) for every \( m \) in \( M(X) \), then \( T = h^* \).

**PROOF.** If \( S \) is in \( A(X) \) then \( h^*(S)(y) = \emptyset^*(S)(y) = \)
[T o S o T^{-1}](y) = T(S) = T(S)(y) for every y in Y. Hence \( h^*(S) = \frac{T(S)}{T(S)} \) or \( h^* = T \).

**THEOREM 2.** Suppose \(BM(X)\) and \(BM(Y)\) are semigroups such that 
\(aM(X) \subseteq BM(X) \subseteq M(X)\) and \(aM(Y) \subseteq BM(Y) \subseteq M(Y)\). Then \(\phi^*\) can be extended to an isomorphism \(\psi^*\) from \(BM(X)\) onto \(BM(Y)\) if and only if

i. \( h^* \circ m \circ h^*{-1} \) is in \(BM(Y)\) for each \( m \) in \(BM(X)\) and

ii. \( h^*{-1} \circ n \circ h^* \) is in \(BM(X)\) for each \( n \) in \(BM(Y)\).

Furthermore, if \(\phi^*\) can be extended \(\psi^*\) is unique and \(\psi^*(m) = h^* \circ m \circ h^*{-1}\) for every \( m \) in \(BM(X)\).

**PROOF.** The last statement follows from remark 1. If \(\phi^*\) can be extended then i and ii follow from theorem 1. If the conditions hold then \(\psi^*\) maps \(BM(X)\) into \(BM(Y)\), \(\psi^*\) is one-to-one and a homeomorphism. Clearly \(\psi^*(h^*{-1} \circ n \circ h^*) = n\) for each \( n \) in \(BM(Y)\) and \(\psi^*\) is an onto mapping.

Suppose \(X\) and \(Y\) are spaces and \(\phi^*\) is an isomorphism from \(aM(X)\) onto \(aM(Y)\). In some cases the induced mapping \(h^*\) from \(a(X)\) onto \(a(Y)\) will carry singleton subsets into singleton subsets, thereby inducing a one-to-one onto mapping \(h^{**}\) from \(X\) to \(Y\). The following example shows that \(h^{**}\) may or may not be a homeomorphism, depending on the classes \(aM(X)\) and \(aM(Y)\) and the isomorphism \(\phi^*\).

**EXAMPLE 1.** Let \(X = \{a,b\}\) and let \(t\) and \(T\) be topologies for \(\{a,b\}\) such that \(t = \{\emptyset, \{a\}, \{a,b\}\}\) and \(T = \{\emptyset, \{b\}, \{a,b\}\}\). Note that \(t\) and \(T\) are saturated topologies and \(t = \{0 : X-0\ \text{in} \ T\}\). Note that \(C(X,T) = \{a,b,i\} = C(X,t)\), where \(i\) is the identity function for \(X\).
There are only two possible isomorphisms from $C(X,t)$ to $C(X,T)$.

These are $\emptyset_1$, the identity mapping and $\emptyset_2$ such that $\emptyset_2(i) = i$, $\emptyset_2(a) = b$, and $\emptyset_2(b) = a$. The mapping $h_1$ from $(X,t)$ to $(X,T)$ induced by $\emptyset_1$ is clearly the identity mapping and is not a homeomorphism. However $\emptyset_2$ induces the mapping $h_2$ such that $h_2(a) = b$ and $h_2(b) = a$ and is a homeomorphism.

Let $f_1$, $f_2$, $f_3$, and $f_4$ be the multifunctions defined on $X$ whose values at $a$ and $b$ are given in the following table.

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<tr>
<th></th>
<th>$f_1$</th>
<th>$f_2$</th>
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<tr>
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It can be verified that $CM(X,t) = \{i, \tilde{a}, \tilde{b}, X, f_2, f_3\} = CM(X,T)$ and again the only two isomorphisms from $CM(X,t)$ to $CM(X,T)$ are $\emptyset_1^*$ the identity and $\emptyset_2^*$ that maps $\tilde{a}$ into $\tilde{b}$, $\tilde{b}$ into $\tilde{a}$, $f_2$ into $f_3$, and $f_3$ into $f_2$ with $i$ and $X$ being fixed under $\emptyset_2^*$. In both cases the induced mapping $h_1^*$ and $h_2^*$ maps singleton subsets into singleton subsets, but only $h_2^*$ does so homeomorphically. Therefore isomorphisms from $CM(X,t)$ to $CM(X,T)$ may not give any more information about induced mappings between the spaces than isomorphisms between $C(X,t)$ and $C(X,T)$.

Now consider the semigroups $UCM(X,t) = \{i, \tilde{a}, \tilde{b}, X, f_2, f_3, f_4\} = LCM(X,T)$ and $UCM(X,T) = \{i, \tilde{a}, \tilde{b}, X, f_1, f_2, f_3\} = LCM(X,t)$. The tables for these semigroups follow.
### TABLE 2.

**UCM(X,t)**

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### TABLE 3.

**UCM(X,T)**

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</tbody>
</table>
A simple tally of the entries shows that the only possible isomorphism from $\text{UCM}(X,t)$ to $\text{UCM}(X,T)$ must map $a$ into $b$ and $b$ into $\bar{a}$. The induced mapping $h^{**}$ will be a homeomorphism.

**DEFINITION 1.** Let $X$ and $Y$ be spaces. An isomorphism $\phi^*$ from $\alpha M(X)$ onto $\alpha M(Y)$ is said to be faithful if the restriction of $\phi^*$ to $Z(X)$ maps onto $Z(Y)$.

Note that the only possible isomorphism from $\text{UCM}(X,t)$ to $\text{UCM}(X,T)$ in example 1 is faithful.

**REMARK 2.** Let $X$ and $Y$ be spaces and let $\phi^*$ be a faithful isomorphism from $\alpha M(X)$ onto $\alpha M(Y)$. Denote the restriction of $\phi^*$ to $Z(X)$ by $\phi$. Then the mapping $h$ from $X$ to $Y$ induced by $\phi$ and the mapping $h^*$ from $\mathcal{A}(X)$ to $\mathcal{A}(Y)$ induced by $\phi^*$ agree on singleton subsets of $X$ in the sense that $(h(x)) = h^*(\{x\})$ for each $x$ in $X$ and $(h^{-1}(y)) = h^*^{-1}(\{y\})$ for all $y$ in $Y$.

**PROOF.** Since $\phi$ and $\phi^*$ agree on $Z(X)$ then $\phi(\bar{x}) = h \circ \bar{x} \circ h^{-1} = h_* \circ \bar{x} \circ h_*^{-1} = \phi^*(\bar{x})$ for every $\bar{x}$ in $Z(X)$. Since $\bar{x} \circ h^{-1}(y) = \bar{x}(h^{-1}(y)) = x$ and $\bar{x} \circ h_*^{-1}(\{y\}) = \bar{x}(h_*^{-1}(\{y\})) = x$ then $(h(x)) = h \circ \bar{x} \circ h^{-1}(y) = h_* \circ \bar{x} \circ h_*^{-1}(\{y\}) = h^*(\{x\})$.

Further, if $\bar{y}$ is in $Z(Y)$ we have $\phi^{-1}(\bar{y}) = h^{-1} \circ \bar{y} \circ h = h_*^{-1} \circ \bar{y} \circ h^* = \phi^*^{-1}(\bar{y})$ and therefore $(h^{-1}(y)) = h^*^{-1}(\{y\})$.

A stronger hypothesis for the preceding remark will lead to a closer relationship between $h$ and $h^*$.

**THEOREM 3.** Let $X$ and $Y$ be $T_1$ spaces and let $\phi^*$ be a faithful isomorphism from $\text{UCM}(X)$ onto $\text{UCM}(Y)$, where $\phi$ is the restriction of $\phi^*$ to $Z(X)$. If $h$ and $h^*$ are as in remark 2, then $h(S) = h^*(S)$ for all $S$ in $\mathcal{A}(X)$ and $h^{-1}(T) = h^*^{-1}(T)$ for all $T$ in $\mathcal{A}(Y)$.
PROOF. 1. First we show that $h(S) \subseteq h^*(S)$ for each $S$ in $\mathcal{A}(X)$. Let $S$ be in $\mathcal{A}(X)$ and let $p$ be in $S$. Define the multifunction $m$ on $X$ such that $m(x) = \begin{cases} S & \text{if } x = p \\ \{p\} & \text{if } x \neq p \end{cases}$. Clearly $m$ is USC since $X$ is a $T_1$ space. Also note that $m \circ m = \overline{m}$. Let $m^* = \overline{m}(m) = h^* \circ m \circ h^{-1}$. Then for $x$ in $X$ we have $m^*(x) = h^* \circ m \circ h^{-1}(x) = \begin{cases} h^*(S) & \text{if } h^{-1}(x) = p \\ h^*(\{p\}) & \text{if } h^{-1}(x) \neq p \end{cases} = \begin{cases} h^*(S) & \text{if } h^{-1}(x) = p \\ h^*(\{p\}) & \text{if } h^{-1}(x) \neq p \end{cases} = \begin{cases} h^*(S) & \text{if } x = h(p) \\ \{h(p)\} & \text{if } x \neq h(p) \end{cases}$. Since $\overline{m}$ is an isomorphism we must have $m^* \circ m^* = \overline{m}(m) \circ \overline{m}(m) = \overline{m}(S) = h^*(S)$. Now $h(p)$ must be in $h^*(S)$ since, if not, $m^* \circ m^*(h(p)) = m^*(h^*(S)) = \{h(p)\} \neq h^*(S)$, a contradiction. Therefore $h(S) = \bigcup \{h(p) : p \text{ is in } S\} \subseteq h^*(S)$.

2. Now we show that $h^{-1}(T) \subseteq h^{-1}(T)$ for all $T$ in $\mathcal{A}(Y)$. Now choose $T$ in $\mathcal{A}(Y)$ and $q$ a point in $T$ and define the multifunction $n$ on $Y$ such that $n(y) = \begin{cases} T & \text{if } y = q \\ \{q\} & \text{if } y \neq q \end{cases}$, for all $y$ in $Y$. It is clear that $n$ is USC and the $n \circ n = \overline{n}$. Let $n^* = \overline{n}(n) = h^{-1} \circ n \circ h^*$. Then for $y$ in $Y$ we have $n^*(y) = h^{-1} \circ n \circ h^*(y) = \begin{cases} h^{-1}(T) & \text{if } h^*(y) = \{q\} \\ h^{-1}(\{q\}) & \text{if } h^*(y) \neq \{q\} \end{cases} = \begin{cases} h^{-1}(T) & \text{if } y = h^{-1}(q) \\ h^{-1}(\{q\}) & \text{if } y \neq h^{-1}(q) \end{cases}$. Therefore $n^* \circ n^* = \overline{n^*}(n) \circ \overline{n^*}(n) = \overline{n^*}(n) = \overline{n^*}(T) = h^{-1}(T)$. Clearly $h^{-1}(q)$ must be in $h^{-1}(T)$ for otherwise $n^* \circ n^*(h^{-1}(q)) \neq h^{-1}(T)$, a contradiction. Then $h^{-1}(T) = \bigcup \{h^{-1}(q) : q \text{ is in } T\} \subseteq$
For any set $S$ in $\mathcal{A}(X)$ we have $S = h^{-1}(h(S)) \subseteq h^{-1}(h^*(S)) \subseteq h^*(h^*(S)) = S$. Therefore $h^{-1}(h(S)) = h^{-1}(h^*(S))$ or $h(S) = h^*(S)$. Similarly for $T$ in $\mathcal{A}(Y)$, $T = h(h^{-1}(T)) \subseteq h(h^*(T)) \subseteq h^*(h^*(T)) = T$ so that $h^{-1}(T) = h^*(T)$.

The following theorem is a corollary to a later result but the proof is given for completeness.

**THEOREM 4.** If $X$ and $Y$ are $T_1$ spaces, then $X$ is homeomorphic to $Y$ if and only if there exists a faithful isomorphism from $UCM(X)$ onto $UCM(Y)$.

**PROOF.** If $h$ is a homeomorphism from $X$ to $Y$, then define a mapping $\phi^*$ from $UCM(X)$ onto $UCM(Y)$ by $\phi^*(m) = h \circ m \circ h^{-1}$ for each $m$ in $UCM(Y)$. Clearly $\phi^*$ is a faithful isomorphism.

Suppose $\phi^*$ is a faithful isomorphism from $UCM(X)$ onto $UCM(Y)$. By theorem 3 there exists a one-to-one onto mapping $h$ from $X$ to $Y$ such that $\phi^*(m) = h \circ m \circ h^{-1}$ for each $m$ in $UCM(X)$.

Let $C$ be a closed subset of $X$ containing more than one point and let $p$ be a point of $C$. Then the multifunction $m$ defined by

$$m(x) = \begin{cases} C & \text{if } x \in C \\ \{p\} & \text{if } x \in X - C \end{cases}$$

is USC. Now $(\phi^*(m))(y) = h \circ m \circ h^{-1}(y)$. Clearly $h(C)$ contains more than one point and for $q$ in $h(C) - h(p)$, we have $(\phi^*(m))^{-1}(q) = h(C)$. Since $\phi^*(m)$ is USC and $Y$ is a $T_1$ space, then $h(C)$ must be closed in $Y$. 

The fact that $h^{-1}$ is a closed mapping is established in a similar manner. Therefore $h$ is a homeomorphism.

**COROLLARY 1.** Let $X$ and $Y$ be $T_1$ spaces. If there exists isomorphic $\alpha$-semigroups of single-valued functions $\alpha(X)$ and $\alpha(Y)$ such that the isomorphism can be extended to map $UCM(X)$ isomorphically onto $UCM(Y)$, then $X$ and $Y$ are homeomorphic.

**PROOF.** The isomorphism between $UCM(X)$ and $UCM(Y)$ will clearly be faithful.

**COROLLARY 2.** Let $X$ and $Y$ be $T_1$ spaces. An isomorphism $\phi$ from $C(X)$ onto $C(Y)$ is induced by a homeomorphism $h$ (in the sense that $\phi(f) = h \circ f \circ h^{-1}$ for each $f$ in $C(X)$) between $X$ and $Y$ if and only if $\phi$ can be extended to an isomorphism from $UCM(X)$ onto $UCM(Y)$.

**PROOF.** The extension of $\phi$ will clearly be faithful.

The class of all faithful automorphisms of $UCM(X)$ form a group under composition.

**COROLLARY 3.** If $X$ is a $T_1$ space then the group of all faithful automorphisms of $UCM(X)$ is isomorphic to the group, under composition, of all autohomeomorphisms of $X$.

**PROOF.** Let $A$ denote the group of all faithful automorphisms of $UCM(X)$ and let $H(X)$ denote the autohomeomorphism group of $X$. From theorem 4, for each $\phi$ in $A$ there exists a homeomorphism $h$ in $H(X)$ such that $\phi(m) = h \circ m \circ h^{-1}$ for all $m$ in $UCM(X)$. If $h_1$ is any autohomeomorphism of $X$ such that $\phi(m) = h_1 \circ m \circ h_1^{-1}$ for all $m$ in $UCM(X)$, then $h_1 \circ m \circ h_1^{-1} = h \circ m \circ h^{-1}$. In particular, for any $x$ in $X$ we have $h_1 \circ x \circ h_1^{-1} = h \circ x \circ h^{-1}$. Then for any $y$ in $X$, $h_1(x) =$
\[ h_1 \circ \bar{x} \circ h_1^{-1}(y) = h \circ \bar{x} \circ h_1^{-1}(y) = h(x). \] Since \( x \) was arbitrary then \( h = h_1 \) and \( h \) is unique.

Let \( I \) be the mapping from \( A \) to \( H(X) \) defined by \( I(\emptyset) = h \) where \( \emptyset(\bar{x}) = \bar{y} \) if and only if \( h(x) = y \). \( I \) is clearly a homeomorphism and is onto by theorem 4. The kernel of \( I \) consists only of the identity automorphism since if \( I(\emptyset) \) is the identity in \( H(X) \) then \( \emptyset(m) = i \circ m \circ i^{-1} = m \) for each \( m \) in \( UCM(X) \). Therefore \( I \) is one-to-one.

In (16) de-Groot shows the existence of \( 2^c \) nonhomeomorphic subsets of the plane such that their only continuous selfmaps are the identity and the constant maps.

The following remark shows that, in fact, none of the de-Groot topologies \( t \) have \( UCM(X,t) = ZM(X) \cup \{\text{identity}\} \).

**REMARK 3.** If \((X,t)\) is a \( T_1 \) space having at least two points, then \( UCM(X,t) \neq ZM(X) \cup \{\text{identity}\} \).

**PROOF.** Let \( x \) and \( y \) be in \( X \) such that \( x \neq y \). Clearly the multifunction \( m \) defined by \( m(z) = \begin{cases} \{x,y\} & \text{if } z = x \\ \{x\} & \text{if } z \neq x \end{cases} \) for each \( z \) in \( X \) is USC.

**C. NILPOTENT MULTIFUNCTIONS**

**DEFINITION 2.** A multifunction \( m \) from a topological space to itself is nilpotent if and only if \( m \circ m = S \) for some \( S \) in \( \mathcal{A}(X) \).

As in the case of idempotent functions \((f \circ f = f)\) there is a characterization of nilpotence for single-valued functions that does not apply to multifunctions.

**REMARK 4.** A single-valued function \( f \) from a space \( X \) to itself is nilpotent if and only if it is constant on its range.
PROOF. If \( f \) is nilpotent and \( x \) and \( y \) are in \( X \), then \( f(x) \) and \( f(y) \) are elements of the range of \( f \) and \( f(f(x)) = f \circ f(x) = f \circ f(y) = f(f(y)) \). Clearly \( f \) is constant on its range.

Suppose for every \( x \) and \( y \) in \( X \) that \( f(f(x)) = f(f(y)) \). Then \( f \circ f \) is constant and \( f \) is nilpotent.

The following example is a nilpotent multifunction that is not constant on its range.

EXAMPLE 2. Let \( X \) be a space with more than one point and \( A \) and \( B \) nonempty subsets of \( X \) such that \( A \not\subseteq B \subseteq X \). Choose \( p \) in \( A \) and define a multifunction \( m \) from \( X \) to \( X \) such that

\[
m(x) = \begin{cases} 
B & \text{if } x \text{ is in } A \\
\{p\} & \text{if } x \text{ is in } X - A
\end{cases}
\]

Then the range of \( m \) is \( B \), but for \( x \) in \( B - A \), \( m(x) = \{p\} \) and for \( y \) in \( A \), \( m(y) = B \). Clearly \( m \) is nilpotent.

Let \( \text{NM}(X) \) denote the semigroup generated by the nilpotent USC multifunctions on \( X \) and let \( \text{NC}(X) \) denote the semigroup generated by the nilpotent continuous single-valued selfmaps on \( X \). Note that \( \text{NM}(X) \) contains \( \text{ZM}(X) \) and \( \text{NC}(X) \) contains \( \text{Z}(X) \).

THEOREM 5. The class \( U = \{m^{-1}(x) : m \in \text{NM}(X) \text{ and } x \in X \} \) is the class of closed sets for any \( T_1 \) space \( X \).

PROOF. Since \( X \) is a \( T_1 \) space then \( U \) is a collection of closed sets. Clearly \( X \) is in \( U \) since \( \text{Z}(X) \) is contained in \( \text{NM}(X) \).

Let \( p \) be a point in \( X \) and consider the multifunction defined by

\[
m(x) = \begin{cases} 
X & \text{if } x = p \\
\{p\} & \text{if } x \neq p
\end{cases}
\]

Then \( m \) is in \( \text{NM}(X) \) since \( m \circ m = X \) and \( m \) is clearly USC. Furthermore \( m^{-1}(x) = \{p\} \) for any \( x \neq p \). Therefore \( \{p\} \) is a member of \( U \) for each point \( p \) in \( X \).
Now let $C$ be a proper closed subset of $X$ with more than one point. Also let $p$ be a point in $C$ and define a multifunction $n$ such that $n(x) = \begin{cases} C & \text{if } x \text{ is in } C \\ \{p\} & \text{if } x \text{ is in } X - C \end{cases}$. Then $n \circ n = C$ and $n$ is USC.

Choose any point $q$ in $C - p$ and we have $n^{-1}(q) = C$. Therefore $C$ is in $U$ and the theorem is proved.

The next theorem is a stronger form of theorem 3, and the proof follows exactly as that for theorem 3 since the multifunctions used there were nilpotent.

**THEOREM 6.** Let $X$ and $Y$ be $T_1$ spaces and let $\emptyset^*$ be a faithful isomorphism from $NM(X)$ onto $NM(Y)$, where $\emptyset$ denotes the restriction of $\emptyset^*$ to $Z(X)$. If $h$ and $h^*$ are as in remark 1, then $h(S) = h^*(S)$ for all $S$ in $A(X)$ and $h^{-1}(T) = h^*^{-1}(T)$ for all $T$ in $A(Y)$.

**THEOREM 7.** Let $X$ and $Y$ be $T_1$ spaces. Then $X$ and $Y$ are homeomorphic if and only if there exists a faithful isomorphism from $NM(X)$ onto $NM(Y)$.

**PROOF.** If $h$ is a homeomorphism from $X$ to $Y$ then the mapping $\emptyset^*$ from $NM(X)$ onto $NM(Y)$ defined by $\emptyset^*(m) = h \circ m \circ h^{-1}$ for all $m$ in $NM(X)$ is clearly a faithful isomorphism from $NM(X)$ onto $NM(Y)$.

Now let $\emptyset^*$ be a faithful isomorphism from $NM(X)$ onto $NM(Y)$ and let $\emptyset$ denote the restriction of $\emptyset^*$ to $Z(X)$. By theorem 6 the mappings $h$ and $h^*$ induced by $\emptyset$ and $\emptyset^*$ respectively agree on all nonempty subsets of $X$.

Let $m$ be in $NM(X)$ and $x$ a point in $X$. Then all of the following statements are equivalent.
(1) \( y \) is in \( h(m^{-1}(x)) \).
(2) \( z \) is in \( m^{-1}(x) \) and \( h(z) = y \).
(3) \( x \) is in \( m(z) \) and \( h(z) = y \).
(4) \( h(x) \) is in \( h \circ m(z) \) and \( h^{-1}(y) = z \).
(5) \( h(x) \) is in \( h \circ m \circ h^{-1}(y) \).
(6) \( h(x) \) is in \( [\emptyset^*(m)](y) \).
(7) \( y \) is in \( [\emptyset^*(m)]^{-1}(h(x)) \).

Therefore \( h(m^{-1}(x)) = [\emptyset^*(m)]^{-1}(h(x)) \). The result of theorem 6 implies that \( h \) is a closed mapping.

The proof that \( h^{-1} \) is a closed mapping is similar to the preceding and is omitted.

COROLLARY 1. Let \( X \) and \( Y \) be \( T_1 \) spaces. Then \( X \) is homeomorphic to \( Y \) if and only if there exists an isomorphism from \( NC(X) \) onto \( NC(Y) \) which can be extended to an isomorphism from \( NM(X) \) onto \( NM(Y) \).

PROOF. An isomorphism meeting these requirements is clearly faithful since \( Z(X) \) is contained in \( NC(X) \).

COROLLARY 2. If \( t \) and \( T \) are \( T_1 \) topologies for a set \( X \), then \( NM(X,t) = NM(X,T) \) if and only if \( t = T \).

PROOF. The identity isomorphism is faithful.

COROLLARY 3. Let \( X \) and \( Y \) be \( T_1 \) spaces. Then any faithful isomorphism from \( NM(X) \) onto \( NM(Y) \) has a unique extension to an isomorphism from \( UCM(X) \) onto \( UCM(Y) \).

PROOF. Let \( \emptyset \) be a faithful isomorphism from \( NM(X) \) onto \( NM(Y) \). Then there exists a homeomorphism \( h \) from \( X \) to \( Y \) such that \( \emptyset(m) = h \circ m \circ h^{-1} \) for all \( m \) in \( NM(X) \). Clearly for any \( m \) in \( UCM(X) \),
\( \emptyset^*(m) = h \circ m \circ h^{-1} \) is in \( UCM(Y) \) and for any \( n \) in \( UCM(Y) \), then \( \emptyset^*^{-1} = h^{-1} \circ n \circ h \) is in \( UCM(X) \). The result now follows from Theorem 2.

**COROLLARY 4.** If \( X \) is a \( T_1 \) space then the group of all faithful automorphisms of \( NM(X) \) is isomorphic to the group, under composition, of all autohomeomorphisms of \( X \).

**PROOF.** It follows in the same way as the proof of corollary 3, page 35.

We denote the class of nilpotent USC multifunctions on a space \( X \) by \( N(X) \). Note that \( N(X) \) is not necessarily a semigroup and \( N(X) \) is contained in \( NM(X) \).

**THEOREM 8.** Let \( X \) and \( Y \) be spaces and \( \emptyset^* \) an isomorphism from \( UCM(X) \) onto \( UCM(Y) \). Then \( \emptyset^* \) restricted to \( NM(X) \) maps onto \( NM(Y) \).

**PROOF.** If \( m \) is in \( N(X) \) then \( m \circ m = S \) for some \( S \) in \( A(X) \).

Then \( \emptyset^*(m) \circ \emptyset^*(m) = \emptyset^*(m \circ m) = \emptyset^*(S) = h^*(S) \), where \( h^* \) is the mapping from \( A(X) \) to \( A(Y) \) induced by \( \emptyset^* \). Since \( \emptyset^*(m) \circ \emptyset^*(m) \) is constant then \( \emptyset^*(m) \) is in \( N(Y) \).

Now let \( n \) be in \( NM(X) - N(X) \). Then \( n = m_1 \circ m_2 \circ \cdots \circ m_k \) where \( m_i \) is in \( N(X) \) for all \( i = 1, 2, \ldots, k \). Each \( m_i \) need only appear to the first power since, for higher powers, \( n \) would be constant and hence in \( N(X) \). Then \( \emptyset^*(n) = \emptyset^*(m_1) \circ \emptyset^*(m_2) \circ \cdots \circ \emptyset^*(m_k) \) where each \( \emptyset^*(m_i) \) is in \( N(Y) \) for \( i = 1, 2, \ldots, k \). Therefore \( \emptyset^*(n) \) is in \( NM(Y) \).

Similarly \( \emptyset^*^{-1} \) maps \( NM(Y) \) into \( NM(X) \) so \( \emptyset^* \) is an onto mapping.

Theorem 4 of this chapter is now an immediate consequence of theorems 7 and 8. We restate it as a corollary.

**COROLLARY.** Let \( X \) and \( Y \) be \( T_1 \) spaces. Then \( X \) is homeomorphic to \( Y \) if and only if there exists a faithful isomorphism from \( UCM(X) \) onto \( UCM(Y) \).
VI. A PARTIAL ORDER FOR THE TOPOLOGIES OF THE REAL NUMBERS

A. DEFINITIONS AND KNOWN RESULTS

Let $X$ denote the set of real numbers and let $C$ be the class of all topologies for $X$. For $t_1$ and $t_2$ in $C$ we define the relation $t_1 \sim t_2$ if and only if $C(X,t_1) = C(X,t_2)$. Clearly "\sim" is an equivalence relation. The resulting equivalence classes may be partially ordered by "<" where we define $[t_1] < [t_2]$ if and only if $C(X,t_1) \subseteq C(X,t_2)$. Note that "<" is transitive but not reflexive.

If $\text{tr}$ denotes the trivial topology and $D$ denotes the discrete topology then $[\text{tr}] = \{\text{tr}, D\}$ is the largest element of $C/\sim$ since $C(X,\text{tr})$ and $C(X,D)$ both consist of all selfmaps on $X$. The smallest element of the partially ordered array is the equivalence class of those topologies for $X$ having just the constant selfmaps and the identity as continuous selfmaps. In (16) de-Groot shows that at least $2^c$ nonhomeomorphic topologies of this type exist.

Several other topologies whose equivalence classes are known will be examined and certain relationships between these classes will be established.

DEFINITIONS

(1) Let $U$ denote the usual topology for $X$.

(2) Let $X$ be well ordered with smallest element $s$ and linear ordering $\prec$. Then $tw$ and $ctw$ will be defined as follows: $tw = \{\emptyset, \{s\}, X\} \cup \{\{x : s \not\prec x < r\} : r \text{ is in } X\}$ and $ctw = \{0 : X - 0 \text{ is in } tw\}$. $tw$ is called the tower topology.
(3) Let \( a \) be a point in \( X \) and \( S = \{ 0 : 0 \subseteq X \text{ and } a \text{ is in } 0 \} \cup \{ \emptyset \} \). \( S \) is called the "superset of a point" topology for \( X \). Also define \( SC = \{ 0 : X - 0 \text{ is in } S \} \). \( S \) is example 10 and \( SC \) is example 15 in Steen and Seebach (18).

(4) Let \( Z = \{ \emptyset \} \cup \{ 0 : \text{there exists } a \text{ and } b \text{ in } X \text{ such that } \{ (-\infty, a) \cup (b, \infty) \} \subseteq 0 \} \). \( Z \) is called the fuzzy topology. This topology can also be described as one for which a set is closed if and only if it is bounded.

(5) Let \( V = U \cap Z \). This is example 22 in Steen and Seebach (18). Yu-Lee Lee (7) showed that \( H(X,U) = H(X,V) \).

(6) Let \( S^* = S \cap U \).

(7) The finite complement and the countable complement topologies will be denoted by FC and CC respectively.

(8) A space \((X,t)\) is said to be completely homogeneous if and only if every one-to-one onto selfmap is a homeomorphism.

The following results are due to Rothmann (15), Hicks and Haddock (13), and Warndof (14).

THEOREM A. (Hicks and Haddock) \([ FC ] = \{ FC \} \).

THEOREM B. (Rothmann)

i. \([ Z ] = \{ Z \} \).

ii. \([ tw ] = \{ tw, ctw \} \).

iii. \([ S ] = \{ S, SC \} \).

iv. \([ CC ] = \{ CC \} \).

A more general theorem for which theorem A and theorem B, part iv are special cases was obtained independently by Rothmann and
Warndorf. Warndorf also proved several theorems that indicate there are large classes of topologies, including the usual topology, whose equivalence classes are singleton sets.

THEOREM C. (Warndorf) \( [U] = \{U\} \).

THEOREM D. (Rothmann) Let \( X \) be an arbitrary set and let \( \{t_a : a \in A\} \) be a collection of topologies on \( X \), then

i. \( a \in A \ C(X,t_a) \subseteq C(X, \bigcap_{a \in A} t_a) \), and

ii. \( a \in A \ C(X,t_a) \subseteq C(X,T) \) where \( T \) is the least upper bound for the topologies in \( \{t_a : a \in A\} \).

THEOREM E. (Rothmann) There exists no topology \( t \) such that:

i. \( t \not\subseteq S \) and \( C(X,S) \not\subseteq C(X,t) \).

ii. \( t \not\subseteq SC \) and \( C(X,SC) \not\subseteq C(X,t) \).

iii. \( S \subseteq t \not\subseteq D \) and \( C(X,S) \not\subseteq C(S,t) \).

iv. \( t \not\subseteq Z \) and \( C(X,Z) \not\subseteq C(X,t) \).

v. \( tw \not\subseteq t \not\subseteq D \) and \( C(X,tw) \not\subseteq C(X,t) \).

vi. \( ctw \not\subseteq t \not\subseteq D \) and \( C(X,ctw) \not\subseteq C(X,t) \).

THEOREM F. (Rothmann) \( C(X,t) = \{f : f \text{ is a constant function or } f(a) = a\} \) if and only if \( t \) is either \( S \) or \( SC \).

THEOREM G. (Rothmann) \( C(X,t) = \{f : f \text{ is a constant function or } |f(x)| \to \infty \text{ as } |x| \to \infty\} \) if and only if \( t \) is \( Z \).

THEOREM H. (Rothmann) Assuming the continuum hypothesis then \( (X,t) \) is completely homogeneous if and only if \( t \) is \( tr, D, FC, \) or \( CC \).

Larson proved a more general result in (19).

B. A PARTIALLY ORDERED ARRAY OF TOPOLOGIES

THEOREM 1. There exists no topology \( t \) such that \( [S] < [t] < [tr] \).
PROOF. Suppose \( t \) exists. Then \( C(X,t) \) properly contains \( C(X,S) \).

If \( I = t \cap S \) then \( C(X,S) = C(X,S) \cap C(X,t) \subseteq C(X,I) \) by theorem D.

Clearly \( tr \subseteq I \not\subseteq S \). By theorem E, part i, \( I = tr \). Then no open set in \( t \) except \( X \) contains the point \( a \). Therefore \( tr \not\subseteq t \not\subseteq SC \) and \( C(X,t) \) properly contains \( C(X,SC) \). This contradicts theorem E, part ii.

THEOREM 2. There exists no topology \( t \) such that \([Z] < [t] < [tr]\).

PROOF. Suppose \( t \) exists. Then \( C(X,Z) \not\subseteq C(X,t) \). Let \( I = t \cap Z \). Then \( C(X,Z) = C(X,Z) \cap C(X,t) \subseteq C(X,I) \) by theorem D and \( tr \subseteq I \not\subseteq Z \).

The result of theorem E, part iv, gives \( I = tr \). Let \( 0 \) be a set in \( t - tr \). Then \( 0 \) is not in \( Z \) so for each positive integer \( n \), there exists a real number \( b_n \) such that \( |b_n| > n \) and \( b_n \) is in \( X - 0 \). Let "\( a \)" be an arbitrary element in \( 0 \) and define \( f \) as follows:

\[
f(x) = \begin{cases} 
a & \text{for } x = a \\
b_n & \text{for } x \in \{[-n, -n + 1) \cup [n - 1, n)\} - \{a\}.
\end{cases}
\]

Then \( f \) is in \( C(X,Z) \) by theorem G. Therefore \( f \) is in \( C(X,t) \) and \( f^{-1}(0) = \{a\} \) belongs to \( t \). Choose a point \( c \) to be any real number and define \( g_c \)

\[
g_c(x) = \begin{cases} 
a & \text{for } x = c \\
a + 1 & \text{for } x = a \\
x & \text{for } x \in X - \{a,c\}
\end{cases}
\]

such that \( g_c(x) = \begin{cases} 
a & \text{for } x = c \\
a + 1 & \text{for } x = a \\
x & \text{for } x \in X - \{a,c\}
\end{cases} \)

therefore in \( C(X,t) \) and \( g_c^{-1}(\{a\}) = \{c\} \) is in \( t \). Since \( c \) was arbitrary then \( t \) is the discrete topology. This contradicts the hypothesis that \( [tr] < [t] \).

THEOREM 3. There exists no topology \( t \) such that \([tw] < [t] < [tr]\).

PROOF. Suppose \( t \) exists. Then \( C(X,t) \) properly contains \( C(X,tw) \).
Case 1. Suppose there exists a set $0$ in $t - \text{tr}$ such that "s" the smallest element belongs to $0$. Let $x$ be in $X - 0$. Define a function $f$ by $f(z) = \begin{cases} s & \text{if } z = s \\ x & \text{if } z \not< s \end{cases}$. Then $f$ is non-decreasing and $f$ is in $C(X,\text{tw})$ and so is therefore in $C(X,t)$. Therefore $f^{-1}(0) = \{s\}$ belongs to $t$.

For $n \not< s$ define $f_n(z) = \begin{cases} s & \text{if } z \not< n \\ x & \text{if } z \not\geq n \end{cases}$. $f_n$ is non-decreasing so $f_n$ is in $C(X,\text{tw})$ and in $C(X,t)$. Then $f_n^{-1}(\{s\}) = \{z : s \not< z \not< n\}$ is in $t$. Therefore $t \supseteq \text{tw}$ and by theorem E, part v, it is discrete.
This contradicts $[t] < [\text{tr}]$.

Case 2. Suppose for every $0^*$ in $t - \text{tr}$ that $s$ is not in $0^*$. Let $0$ be such a set and let $n$ be the first element of $0$. Let $m$ be an arbitrary element of $X$. Define $f_m$ such that $f_m(z) = \begin{cases} s & \text{for } z \not< m \\ n & \text{for } z \not\geq m \end{cases}$. Then $f_m^{-1}(0) = \{z : z \not\geq m\}$ belongs to $t$. Therefore $ctw \nsubseteq t$ and $t$ is discrete by theorem E, part vi. Again this is a contradiction.

THEOREM 4. There exists no topology $t$ such that $[\text{CC}] < [t] < [\text{tr}]$.

PROOF. Suppose such a topology $t$ exists. Then $C(X,\text{CC}) \nsubseteq C(X,t)$. Since every one-to-one onto selfmap on $X$ is in $C(X,\text{CC})$ then it is in $C(X,t)$. Therefore $(X,t)$ is completely homogeneous. The result of theorem H implies that $t$ is tr, D, CC, or FC. This is clearly a contradiction.
THEOREM 5. There exists no topology such that $[FC] < [t] < [CC]$.

PROOF. The proof of this theorem is similar to the proof of Theorem 4 and is omitted.

THEOREM 6. $[S^*] < [S]$.

PROOF. In (15) Rothmann proved that $C(X,S)$ consists of the constant functions and those functions that fix "a" in the sense that $f(a) = a$. Suppose there exists a function $f$ in $C(X,S^*) - Z(X)$ such that $f(a) = b \neq a$. There exists an open set $O$ in $U$ containing $a$ and not $b$. Then $f^{-1}(O) \neq \emptyset$ and $f^{-1}(O)$ will not contain $a$. Therefore $f^{-1}(O)$ is not in $S^*$. This is a contradiction.

Therefore $C(X,S^*) \subseteq C(X,S)$. Clearly $C(X,S^*) \neq C(X,S)$ since $[S] = \{S, SC\}$.

THEOREM 7. $C(X,U)$ and $C(X,S^*)$ are not comparable by set inclusion.

PROOF. For each non-constant function $f$ in $C(X,S^*)$, $f(a) = a$. The function $g(x) = x + 1$ for all $x$ in $X$ is in $C(X,U)$ but it is not in $C(X,S^*)$. Therefore $C(X,S^*)$ does not contain $C(X,U)$.

Let $O$ be in $U$ such that $a$ is in $O$ and $\emptyset \neq O \neq X$. Choose a point $b$ in $X$ such that $b \neq a$ and define the function $h$ by $h(x) = \begin{cases} a & \text{if } x \text{ is in } O \\ b & \text{if } x \text{ is in } X - O \end{cases}$. Clearly $h$ is not in $C(X,U)$ but $h$ is in $C(X,S^*)$. Therefore $C(X,U)$ does not contain $C(X,S^*)$.

THEOREM 8. $[S^*] = \{S^*\}$.

PROOF. Suppose $t$ is a topology for $X$ such that $C(X,t) = C(X,S^*)$. Then $t$ is not the trivial topology.

Case 1. Suppose there exists a set $O$ in $t - tr$ such that the
point a is in 0. Let q be a point in X - 0 and choose points b and c in X such that \( b < a < c \). Now define a function f on X such that
\[
f(x) = \begin{cases} 
    a & \text{if } x \text{ is not in } (b,c) \\
    q & \text{if } x \text{ is in } (b,c)
\end{cases}
\]
Clearly f is in \( C(X,S^*o) \), so f is in \( C(X,t) \) and \( f^{-1}(0) = (b,c) \) is in t. Since sets of the form (b,c), where \( b < a < c \), form a base for \( S^*o \), then \( S^*o \subseteq t \).

Now define a function g on X such that \( g(x) = \begin{cases} 
    a & \text{if } x \in 0 \\
    c + 1 & \text{otherwise}
\end{cases} \).

Then g is in \( C(X,t) \) and hence g is in \( C(X,S^*) \). Therefore \( g^{-1}((b,c)) = 0 \) is in \( S^*o \) and \( t \subseteq S^*o \). We now have \( t = S^*o \).

Case 2. Suppose no set \( 0^*o \) in \( t - tr \) contains the point a. Clearly t is not discrete. Let 0 be a set in \( t - tr \) and let p be a point in 0. Choose a point b in X such that \( b \neq a \). Then the function h defined by \( h(x) = \begin{cases} 
    p & \text{for } x = b \\
    a & \text{for } x \neq b
\end{cases} \) is in \( C(X,S^*) \). Therefore h is in \( C(X,t) \) and \( h^{-1}(0) = \{b\} \) is in t. Since b was arbitrary but not equal to a, then \( \{\{b\} : b \neq a\} \) is a collection of open sets in t. Therefore \( t = CS \). But this contradicts theorem 6.

**DEFINITION 9.** A topology \( t_o \) for X is called fundamental if and only if for any topology t for X, \( C(X,t_o \land t) \subseteq C(X,t_o) \).

Clearly tr and D are fundamental topologies. In fact they are the only fundamental topologies. To see this let \( t_o \) be a fundamental topology for X. Steiner (20) proved that \( t_o \) will have a complement \( t_o^* \) in the lattice of topologies for X. Then \( C(X, tr) = C(X, t_o \land t_o^*) \) \( \subseteq C(X, t_o) \) and \( t_o \) must be either tr or D.
DEFINITION 10. A topology $t_o$ is P-fundamental if and only if for every topology $t$ for $X$ having property $P$, then $C(X, t \cap t_o) \subseteq C(X, t_o)$.

THEOREM 9. $S$ is a $T_1$-fundamental topology.

PROOF. Let $t$ be a $T_1$ topology for $X$ and suppose there exists a function $f$ in $C(X, t \cap S) - Z(X)$ such that $f(a) = b \neq a$. Let $0 = X - \{b\}$. Clearly $0$ is in $t \cap S$. But $f^{-1}(0) \neq \emptyset$ and $f^{-1}(0)$ does not contain the point $a$ and consequently is not in $t \cap S$. This is a contradiction. Therefore $f(a) = a$ and $C(X, t \cap S) \subseteq C(X, S)$.

THEOREM 10. $SC$ is a $T_1$-fundamental topology.

PROOF. Let $t$ be any $T_1$ topology for $X$ and suppose there exists a function $f$ in $C(X, t \cap SC) - Z(X)$ such that $f(a) = b \neq a$. Let $0 = X - \{a\}$. Then $0$ is in $t \cap SC$ but $f^{-1}(0) \neq X$ and $f^{-1}(0)$ contains the point $a$ and consequently is not in $t \cap SC$. This is a contradiction. Therefore $f(a) = a$ and $C(X, t \cap SC) \subseteq C(X, SC)$.

REMARK 1. If $t$ and $t'$ are $T_1$ topologies such that $S \cap t \neq S \cap t'$, then $C(X, S \cap t)$ and $C(X, S \cap t')$ are not comparable by set inclusion.

PROOF. Let $0$ be a nontrivial set in $t \cap S$ that is not in $t' \cap S$. Choose a point $q$ in $X - 0$. Then the function $f$ defined by

$$f(x) = \begin{cases} q & \text{if } x \text{ is not in } 0 \\ a & \text{if } x \text{ is in } 0 \end{cases}$$

is in $C(X, t \cap S)$. But $f$ is not in $C(X, t' \cap S)$ since $X - \{q\}$ is in $t' \cap S$ but $f^{-1}(X - \{q\}) = 0$. Therefore $C(X, t \cap S)$ is not contained in $C(X, t' \cap S)$.

The argument against set inclusion of $C(X, t' \cap S)$ by $C(X, t \cap S)$ is similar to the above and is omitted.

Recall that $V = U \cap Z$. 
THEOREM 11. \([V] < [Z]\).

PROOF. Let 0 be a closed set in \((X,Z)\) and choose \(f\) in \(C(X,V)\). Then 0 is bounded and there exists real numbers \(a\) and \(b\) such that 0 is contained in \([a,b]\). Then \(f^{-1}([a,b])\) is closed in \(V\) and therefore bounded. Since \(f^{-1}(0) \subseteq f^{-1}([a,b])\) then \(f^{-1}(0)\) is bounded and therefore closed in \(Z\). Hence \(f\) is in \(C(X,Z)\).

REMARK 2. \(C(X,V)\) and \(C(X,U)\) are not comparable by set inclusion.

PROOF. Consider the function \(f\) on \(X\) defined by
\[
f(x) = \begin{cases} 
  x + 1/x & \text{for } x \neq 0 \\
  0 & \text{if } x = 0
\end{cases}
\]
Then \(f\) is in \(C(X,V)\) but \(f\) is not in \(C(X,U)\).

Also \(g(x) = \sin x\) is a function in \(C(X,U)\) but not in \(C(X,V)\).

DEFINITION 11. A topology \(t\) for \(X\) is said to have property \(Q\) if and only if the \(t\)-closure of a bounded set is bounded, where bounded refers to the usual metric for the real numbers.

THEOREM 12. \(Z\) is a \(Q\)-fundamental topology.

PROOF. Let \(t\) be a topology for \(X\) with property \(Q\) and let \(C\) be closed in \(Z\). Choose a function \(f\) in \(C(X,Z \cap t)\). Clearly \(C\) is bounded so the \(t\)-closure of \(C\), denoted by \(C'\) is bounded. Therefore \(C'\) is closed in \(t \cap Z\) and \(f^{-1}(C')\) is closed in \(t \cap Z\) and is consequently bounded. Since \(f^{-1}(C) \subseteq f^{-1}(C')\) then \(f^{-1}(C)\) is bounded and therefore closed in \(Z\). Then \(f\) is in \(C(X,Z)\) and \(Z\) is \(Q\)-fundamental.

A summary of the results of this chapter is given in the following diagram.
The solid lines indicate that there exists no topology whose equivalence class is in this position. The dotted lines indicate possible locations for equivalence classes whose existence is unknown. $T_1$ represents any $T_1$ topology, $Q$ represents any topology with property $Q$, and $T$ represents one of the topologies of de-Groot (16) for which $C(X,T) = Z(X) \cup \{\text{identity}\}$. 
VII. SUMMARY, CONCLUSIONS, AND FURTHER PROBLEMS

Let \((X, t)\) be a topological space and \(S(X, t)\) be a prescribed collection of functions or multifunctions from \(X\) to \(X\) with some algebraic structure relative to composition. The general question is: Given \((X, t_1)\) and \((Y, t_2)\) topological spaces with \(S(X, t_1)\) isomorphic to \(S(Y, t_2)\), is \((X, t_1)\) homeomorphic to \((Y, t_2)\)?

On the last page of this chapter is a table giving a partial survey of published results for this problem along with some known counterexamples.

DEFINITIONS

(1) A space \(X\) is a T-space if and only if it is connected and for every connected subset \(K\) of \(X\) there exists a connected mapping \(f\) from \(X\) into \(X\) such that \(f(X) = K\).

(2) a. A space \(X\) is an S-space if and only if it is Hausdorff and each point has a basis of S-neighborhoods.

b. \(G\) is an S-neighborhood of a point \(x\) in \(X\) if and only if \(G = \{x\}\) or there exists a continuous function \(f\) from \(C(G)\) into \(X\) such that \(f(x) \neq x\) and \(f(y) = y\) for each point \(y\) in \(C(G) - G\), where \(C(G)\) denotes the closure of \(G\).

(3) A space \(X\) is an M-space if and only if \(
\{H(f) : f\ is\ a\ continuous\ selfmap\ on\ X\}\)

\(\) is a base for the closed sets of \(X\), where \(H(f) = \{x : f(x) = x\}\).

(4) A space \(X\) is an S*-space if and only if there exists an \(\alpha\)-semigroup \(\alpha(X)\) such that \(
\{f^{-1}(x) : f \in \alpha(X)\ and\ x\ is\ in\ X\}\)

forms a subbase for the closed sets of \(X\).
(5) A space $X$ is $\omega$-homogeneous if and only if for each positive integer $n$ and for each pair of $n$-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ of points of $X$, there exists an autohomeomorphism $h$ of $X$ such that $h(x_i) = y_i$ for $i = 1, 2, \ldots, n$.

The problem of finding a class of spaces for which the general question is true, where $S(X,t_1)$ and $S(Y,t_2)$ are the LSC multifunctions is a problem which needs further study.

For every example considered in this dissertation, an isomorphism from $UCM(X)$ to $UCM(Y)$ was faithful. It is not known if this must always be true. If it is true then it is likely that the class of spaces for which $UCM(X)$ isomorphic to $UCM(Y)$ implies $X$ is homeomorphic to $Y$ could be enlarged.

It has still not been determined if there exists a nontrivial topology $V$ for the real numbers $\mathbb{R}$ such that $C(\mathbb{R},U) \nsubseteq C(\mathbb{R},V)$ or $C(\mathbb{R},V) \nsubseteq C(\mathbb{R},U)$, where $U$ is the usual topology for $\mathbb{R}$. 
TABLE 1. A partial survey of results

<table>
<thead>
<tr>
<th>($X, t_1), (Y, t_2)$</th>
<th>Homeomorphisms (Group)</th>
<th>Closed Functions (Semigroup)</th>
<th>Connected Functions (Semigroup)</th>
<th>Continuous Functions (Semigroup)</th>
<th>$\alpha$-semi-groups of functions (Semigroup)</th>
<th>UCM($X$) (Semigroup)</th>
<th>NM($X$) (Semigroup)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ALL SPACES</td>
<td>no cofinite and cocountable</td>
<td>no Trivial and discrete</td>
<td>yes * Cramer 1971</td>
<td>yes ** Cramer 1971</td>
<td>yes ** Cramer 1971</td>
<td>yes ** Cramer 1971</td>
<td>yes ** Cramer 1971</td>
</tr>
<tr>
<td>$T_1$ Spaces</td>
<td>no cofinite and cocountable</td>
<td>yes Magill 1966</td>
<td>no de-Groot 1959</td>
<td>yes *(above)</td>
<td>yes ** *(above)</td>
<td>yes ** *(above)</td>
<td>yes ** *(above)</td>
</tr>
<tr>
<td>Locally connected compact Hausdorff T-spaces</td>
<td>yes *(above)</td>
<td>yes Magill 1966</td>
<td>yes *(above)</td>
<td>yes ** *(above)</td>
<td>yes ** *(above)</td>
<td>yes ** *(above)</td>
<td>yes ** *(above)</td>
</tr>
<tr>
<td>S-spaces</td>
<td>yes *(above)</td>
<td>yes Magill 1966</td>
<td>yes *(above)</td>
<td>yes ** *(above)</td>
<td>yes ** *(above)</td>
<td>yes ** *(above)</td>
<td>yes ** *(above)</td>
</tr>
<tr>
<td>M-spaces</td>
<td>yes *(above)</td>
<td>yes *(above)</td>
<td>yes *(above)</td>
<td>yes ** *(above)</td>
<td>yes ** *(above)</td>
<td>yes ** *(above)</td>
<td>yes ** *(above)</td>
</tr>
<tr>
<td>$S^#$-spaces</td>
<td>no Trivial and discrete</td>
<td>no Trivial and discrete</td>
<td>no Trivial and discrete</td>
<td>yes Rothmann 1970</td>
<td>yes *(above)</td>
<td>yes ** *(above)</td>
<td>yes ** *(above)</td>
</tr>
<tr>
<td>Compact locally Euclidean manifolds</td>
<td>yes Whittaker 1963</td>
<td>yes *(above)</td>
<td>yes *(S-space)</td>
<td>yes *(above)</td>
<td>yes ** *(above)</td>
<td>yes ** *(above)</td>
<td>yes ** *(above)</td>
</tr>
</tbody>
</table>

* The isomorphism must also be a homeomorphism with respect to certain function space topologies.
** The isomorphism must be faithful.
*** The isomorphism must be a homeomorphism with respect to the point-open topologies.
REFERENCES


VITA

Alexander Hamlin Cramer was born on April 2, 1940, at Guantanamo Naval Base, Cuba. He graduated from Central Senior High School at Springfield, Missouri in May 1956. That summer he entered Southwest Missouri State College at Springfield, Missouri. In May of 1959 he received a Bachelor of Science degree with a major in Mathematics and a minor in Physics. From September 1959 to May 1960 he attended the University of Missouri at Columbia.

He taught at the high school level for two years and from September 1962 to July 1963 he was a participant in a National Science Foundation Academic Year Institute at Oklahoma State University, Stillwater, Oklahoma, where he received a Master of Science degree in Mathematics.

During the period September 1963 to June 1966 he was an Assistant Professor of Mathematics at the School of the Ozarks at Point Lookout, Missouri. From September 1966 to May 1967 he served as Instructor of Mathematics at Southeast Missouri State College at Cape Girardeau, Missouri.

In September 1967 he began his graduate studies at the University of Missouri at Rolla where he also served as part-time instructor of Mathematics until July 1970. From September 1970 to the present he has served as Assistant Professor of Mathematics at Southwest Missouri State College, Springfield, Missouri.

On February 2, 1962, he was married to the former Linda A. MacLachlan of Springfield, Missouri. They have two children, Tracy and Todd.