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A Limitation to the Use of a Constructive Approach in the Stability Analysis of Fixed-Point Digital Controllers

KELVIN T. ERICKSON

Abstract—A limitation to the application of a constructive algorithm to the stability analysis of fixed-point digital controllers is demonstrated by an example. The particular system analyzed consists of a fixed-point PI digital controller regulating a first-order process.

I. INTRODUCTION

In recent papers, Brayton and Tong [1], [2] established some elegant results which are the basis of a constructive approach in the stability analysis of dynamical systems. This constructive algorithm has recently been applied to the stability analysis of second-order digital filters [4], discrete-time interconnected systems [8], and hybrid composite dynamical systems [9]. The constructive algorithm has also been used to analyze the stability of the recursive part of fixed-point digital controllers [5]. The results obtained by the constructive algorithm have been used to yield conditions (in the parameter plane) under which a system is globally asymptotically stable (g.a.s.), and as such, does not possess zero-input limit cycles.

This note demonstrates a limitation of the constructive algorithm when used to analyze the stability of digital controllers having integral action. The particular system to be analyzed consists of a fixed-point PI controller regulating a first-order process. The remainder of this note is organized as follows. The digital control system to be analyzed is explained in Section II. In Section III, we analyze the stability of the system with a linear digital controller (i.e., no quantization nonlinearities). The stability of the system with fixed-point truncation quantization nonlinearities is examined using the constructive algorithm in Section IV. For comparison purposes, the same system is analyzed in Section V using the Jury–Lee absolute stability criterion [7].

II. SYSTEM TO BE ANALYZED

The system to be analyzed (Fig. 1) consists of a fixed-point digital PI controller $D(z)$ controlling a simple first-order process having no deadtime

$$G(s) = \frac{1}{s+1}.$$  

In order to analyze the stability of this system, the discrete equivalent of the zero-order hold and process is obtained as [3]

$$G(z) = (1 - z^{-1})Z \left[ \frac{G(s)}{s} \right] = \frac{(1 - e^{-T})z^{-1}}{1 - e^{-T}z^{-1}}$$  

where $T$ is the sample period of the controller.

The digital PI controller is the incremental form of the PI digital controller [3] and has a discrete transfer function of

$$D(z) = P \frac{1 + T I}{1 - z^{-1}}$$

where $P$ is the proportional constant, $I$ is the inverse of the integral time constant in $1/s$, and $T$ is the sample period of the digital controller. The transfer function $D(z)$ is obtained by substituting the discrete equivalent of backward rectangular integration in the continuous PI controller.

In practical digital controllers, the representation of signals must necessarily have finite precision. The finite precision, or wordlength, is a consequence of the conversion of the analog process signal to a fixed- or floating-point number and of the storage of these signals in registers which have finite wordlength. Multiplication and addition performed in the controller generally lead to an increase in the wordlength required for the result of the operation. A wordlength reduction may be necessary to prevent the wordlength of the signals from increasing indefinitely.

In this note, we assume that the digital controller uses fixed-point arithmetic. In fixed-point arithmetic, each number is represented by a sign bit and a magnitude. Thus, the magnitude of any number is represented by a string of binary digits of fixed length $B$. When two $B$-bit numbers are multiplied, the result is a $2B$-bit number. A quantization nonlinearity is produced when the $2B$-bit number is reduced in wordlength to $B$ bits. Addition also poses a problem when the sum of two numbers falls outside the representable range. An overflow nonlinearity results when this number is modified so that it falls back within the representable range. In this note we assume that no overflow nonlinearities are present and that only magnitude truncation quantization occurs. In magnitude truncation quantization, the least significant bits of the number are discarded whenever the wordlength needs reduction. The nonlinear characteristic of this quantizer is shown in Fig. 2. Here, $q = 2 - B^{-1}$ is the quantization level.

For the purposes of stability analysis, the magnitude truncation quantization nonlinearity is viewed in this note as belonging to a sector $[k_1, k_2]$ where

$$k_1 \sigma^2 \leq af(\sigma) \leq k_2 \sigma^2$$

for all $\sigma \in R$.

The function $f(\cdot)$ represents the nonlinearity and the only restrictions on $k_1$ and $k_2$ are $-\infty < k_1 \leq k_2 < \infty$. A general sector $[k_1, k_2]$ is represented by the hatched region of Fig. 3. Under this assumption, the truncation quantization nonlinearity in Fig. 2 belongs to the sector $[0, 1]$.

The nonlinear digital system to be analyzed is shown in Fig. 4. Within the controller, calculations are carried out in double precision. Quantizer $Q_1$ in Fig. 4 represents the reduction of the internal numbers to single precision as the final step. The A/D converter is represented by quantizer $Q_2$.

As is customary in the stability analysis of systems, the equilibrium of the system is assumed to be at the origin. If the equilibrium of the system is not at the origin, then we assume that it is translated so the equilibrium is at the origin.

III. LINEAR SYSTEM STABILITY

The stability of the linear system ($Q_1(\cdot) = Q_2(\cdot) = 1$) can be analyzed using the Jury stability test [6, sect. 3.9]. The closed-loop transfer function of the linear system is

$$\frac{Y(z)}{R(z)} = \frac{D(z)G(z)}{1 + D(z)G(z)}.$$  

A necessary and sufficient condition for the linear system to be stable is that all poles of (3) lie within the unit circle. Using (1) and (2), the poles of (3) are the zeros of the polynomial

$$z^2 + \left[ P(1 + T I)(1 - e^{-T}) - 1 - e^{-T} \right] z + \left[ e^{-T} - P(1 - e^{-T}) \right].$$

Using the Jury stability test [6] on the polynomial (4), the linear system

$$z^2 + \left[ P(1 + T I)(1 - e^{-T}) - 1 - e^{-T} \right] z + \left[ e^{-T} - P(1 - e^{-T}) \right].$$

Fig. 1. Digital control system.
then we can rewrite (6) equivalently as
\[(asymptotically stable).\] In the following, we give a brief summary of the results of Brayton and Tong.

Brayton and Tong [1, 2] show that the equilibrium \( x = 0 \) of (7) is stable (globally asymptotically stable) if the set of matrices \( M \) is stable (asymptotically stable). In the following, we give a brief summary of the results of Brayton and Tong.

We call a set \( A \) of \( n \times n \) real matrices stable if for every neighborhood of the origin \( U \subset R^n \), there exists another neighborhood of the origin \( V \subset R^n \) such that for every \( M \in A', \) we have \(MV \leq U\). Here, \( A' \) denotes the multiplicative semigroup generated by a set \( A \) and \( MV = \{u \in R^n : u = Mu, v \in V\} \).

In [1] it is shown that if a set \( A \) of \( n \times n \) matrices is stable, then there exists a vector norm \( |\cdot|_w \) such that \( |Mx|_w \leq |x|_w \) for all \( M \in A \) and \( x \in R^n \). The vector norm \( |\cdot|_w \) defines a positive semi-definite Lyapunov function for \( A \), i.e., it defines a function \( v \) with the property
\[v(Mx) \leq v(x), \quad \text{for all } M \in A \text{ and } x \in R^n.\]

Next, we call a set of matrices \( A \) asymptotically stable if there exists a number \( r > 1 \) such that \( rA \) is stable. (The set \( rA \) is obtained by multiplying every member of \( A \) by \( r \).) Thus, if \( rA \) is stable, then there exists a vector norm \( |\cdot|_w \) such that \( |Mx|_w \leq |x|_w \) for all \( M \in A \) and \( x \in R^n \). The vector norm \( |\cdot|_w \) defines a positive definite Lyapunov function with the property
\[v(Mx) < v(x), \quad \text{for all } M \in A \text{ and } x \in R^n.\]

Thus, given system (7), if \( M \) is asymptotically stable, then the system (7) is globally asymptotically stable. However, if the set \( M \) is unstable, then we can draw no conclusion about the stability of (7).

The set of matrices \( M \) given in (7) consists in general of an infinite number of matrices. However, in [1] it is shown that we need only work with the extreme matrices of the set \( E(M) \) to get an indication of the stability of \( M \). Thus, in those cases where \( E(M) \) is finite, we can use it to analyze the stability of the infinite set \( M \).

### B. Application

With the digital controller (2) implemented as a direct form 1 structure [10], the nonlinear system to be analyzed by the Brayton-Tong method is shown in Fig. 6. The state equations of the system are
\[x_i(k+1) = -Q_1 g_1 x_i(k)
\]
\[x_i(k+1) = d_i x_i(k) + x_k(k) - d_0 Q_1 [g_1 x_i(k)]
\]
\[x_i(k+1) = g_2 x_i(k) + Q_1 [d_i x_i(k) + x_k(k) - d_0 Q_1 [g_1 x_i(k)]]
\]

where \( d_0 = P(1 + TT), d_1 = -P, g_1 = 1 - e^{-T}, \) and \( g_2 = e^{-T}. \) When the state equations (8) are written in the form of (6),
\[M(x(k)) = \begin{bmatrix}
0 & 0 & -g_1 \Phi_1(x) \\
\Phi_1(x) & 1 & -d_0 g_1 \Phi_1(x) \\
d_0 \Phi_1(x) & \Phi_1(x) & g_1 \Phi_1(x)
\end{bmatrix}
\]

where
\[\Phi_1 = Q_1 [d_i x_i + x_k - d_0 Q_1 [g_1 x_i]]
\]
\[\Phi_2 = Q_1 [g_1 x_i]
\]

When the \( M(x(k)) \) given by (9) and (10) is multiplied by \( x(k) = [x_i(k) x_k(k) x_i(k)]^T \), the state equations (8) are obtained.
Since the quantization nonlinearities belong to a sector, the functions $\Phi(x)$ and $\Psi(x)$ are bounded by the constants

$$\alpha_1 \leq \Phi(x) \leq \alpha_2$$

$$\beta_1 \leq \Psi(x) \leq \beta_2$$

where $\alpha_1 = \beta_1 = 0$ and $\alpha_2 = \beta_2 = 1$. The function $\Phi(x)\Psi(x)$ is also bounded by constants

$$\gamma_1 \leq \Phi(x)\Psi(x) \leq \gamma_2$$

where $\gamma_1 = 0$ and $\gamma_2 = 1$. The extreme matrices of the set $M$ are

$$E(M) = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix}, \quad i, j, k = 1, 2$$

where $\alpha_1 = \beta_1 = \gamma_1 = 0$ and $\alpha_2 = \beta_2 = \gamma_2 = 1$.

When the set of extreme matrices (11) is evaluated for asymptotic stability, the constructive algorithm fails to determine any region in the parameter plane where the nonlinear system is globally asymptotically stable. This result is apparent when the eigenvalues are determined for the given digital control system. When the Jury-Lee criterion is applied to the same system, controller parameters $k_i$ are the ith diagonal element of the $m \times m$ matrix $K$. The digital system of Fig. 7 satisfying the above conditions for $\Gamma(z)$ with nonlinearities described by (12) is globally asymptotically stable if

$$\Gamma(z) = \begin{bmatrix} 0 & -D(z) \\ G(z) & 0 \end{bmatrix}$$

The corresponding region in the parameter plane where the system is globally asymptotically stable is shown as the hatched region in Fig. 8. This region is small compared to the region where the linear system is stable (Fig. 5). Nevertheless, the Jury–Lee criterion obtains a result for this problem.

VI. CONCLUSION

Using a simple example, it has been shown that the Brayton-Tong constructive algorithm cannot be used in the stability analysis of a digital control system with a PI controller. When the Jury–Lee criterion is applied to the same system, controller parameters can be found that guarantee the global asymptotic stability of the system. However, it is not known if this result extends to all digital controllers having an integrator. If this result extends to all digital controllers, then this limitation of the constructive algorithm may be serious, since many industrially-useful controllers need integral action to adequately handle persistent process disturbances. However, the constructive algorithm still offers an effective and general approach for the stability analysis of fixed-point digital filters [4], [8].

**REFERENCES**


A General Approach for Constructing the Limit Cycle Locii of Multiple-Nonlinearity Systems

H. C. CHANG, C. T. PAN, C. L. HUANG, AND C. C. WEI

Abstract—This note presents a widely convergent algorithm for finding a limit cycle of systems with multiple nonlinearities. A systematic approach is proposed for constructing the limit cycle loci on the parameter planes. The merits of this approach lie in its simplicity, generality, and ability to provide deeper insight into parameter influences on the limit cycle. Besides, stability of the limit cycle can be predicted with a minimum amount of computations.

I. INTRODUCTION

The sinusoidal input describing function (SIDF) techniques have been used quite successfully to study limit cycles in nonlinear systems with a single nonlinearity [1], [2]. Recently, many new approaches [3]-[6] have been presented for predicting limit cycles of systems with multiple nonlinearities. For example, Taylor [3] used a division technique, Hull [4] applied a corrector technique, Abel [5] adopted a predictor-corrector technique, while Aderibigbe et al. [6] presented a new approach based on harmonic balance or Galerkin’s method. Although these methods have been applied successfully to some systems with multiple nonlinearities, they require lengthy and involved calculations of rather high dimension. Also, other graphical methods [7], [8] have been proposed. A limitation of some of these approaches is that they generally relate to a certain class of systems. In addition, there are many works [9]-[11] dealing with qualitative analysis of nonlinear systems. This approach merits attention by providing a rigorous mathematical justification for using describing functions, and hence will make the designer more confident when using the describing function method to analyze complicated nonlinear systems. However, these general mathematical methods are usually difficult to apply to any but a certain specific configuration. On the other hand, it is frequently desired to investigate the effects of parameter changes on the limit cycle. This poses an even more complicated problem. In view of this, this note presents a systematic approach for plotting limit cycle loci on parameter planes so that one can have a clearer picture about the parameter influences. Significantly, the proposed method can handle any number of general nonlinearities and is applicable to most nonlinear configurations frequently encountered in practice. Besides, stability of the limit cycle can be determined without much additional effort by using the proposed method.

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II. PREDICTION OF A LIMIT CYCLE

For ease of presentation, two typical categories of multiple nonlinearity systems are shown in Figs. 1 and 2, respectively. Fig. 1 is a single-loop system and was previously studied in [12], while Fig. 2 belongs to a multiloop system. As will be seen later, the proposed method, in fact, can be applied to more general multiple nonlinearity systems.

As encountered in most practical systems, it is assumed that the commonly used SIDF can be used to characterize the nonlinear elements of the system of interest. In other words, each nonlinear element considered can be represented approximately as

\[ N(A_i, w) = P_i(A_i, w) + jQ_i(A_i, w) \]

(1)

where \( A_i \) and \( w \) are the amplitude and the frequency of the input sinusoid and \( P_i, Q_i \) are the phase and quadrature gains of the nonlinearity, respectively. Now consider the simple system as shown in Fig. 1. To determine a limit cycle, instead of using one characteristic equation written along the loop, the proposed method separates the characteristic equation into the following equations and terms as “auxiliary characteristic equations” (ACE’s)

\[ X_i(A_i, \theta) + L_i(w)N_i(A_i, w)X_i(A_i, \theta) = 0 \]

(2)

\[ X_i(A_i, \theta) - \omega_i(w)N_i(A_i, w)X_i(A_i, \theta) = 0, \]

\[ i = 2, 3, \ldots, N, \]

(3)

i.e., one equation is written for each nonlinear element. Without loss of generality, \( N_i \) is chosen as reference nonlinear element and the phase \( \theta_i \) of its input sinusoid is assumed zero for convenience. Thus, determination of the limit cycle is reduced to solving the \( N \) complex equations with \( 2N \) unknowns, namely \( A_1, A_2, \ldots, A_N, \theta_N \).

Next consider the multiple-loop system as shown in Fig. 2. The same procedure can be extended to this case. Suppose that there are \( m \) loops for the system and each loop consists of \( r \) nonlinear elements. Then the total number of nonlinear elements is

\[ N = \sum_{i=1}^{m} r_i \]

(4)

Hence, for the \( i \)th loop, one can obtain \( r_i - 1 \) independent ACE’s directly for the uncoupled part of that loop. As to the coupling part, one can use the transfer function matrix \( W(S) \) whose notation is obvious from Fig. 2 to get additional \( m \) equations

\[ \phi + \sum_{j=1}^{m} w_{ij}(w)N_j \psi_j = 0, \]

\[ i = 1, 2, \ldots, m. \]

(5)

Note that in the derivation, the number of equations is equal to that of the nonlinear elements. In general, this dimension is much less than that of using state-space representations. To solve the resulting nonlinear systems is not a simple matter at all. If a good initial guess is available, the Newton–Raphson method may be a good one. Unfortunately, a good initial guess is usually not available in limit cycle analysis. To overcome this difficulty, a recently developed homotopy method [13] which basically possesses the global convergence characteristic may be an ideal algorithm. It has been proved theoretically that the algorithm will eventually lead to the desired solution with probability one [14]. As a rough explanation, consider the problem of solving

\[ f(x) = 0, f: R^n \to R^n, \]

(6)

One can construct a linear homotopy \( H \)

\[ H: R^n \times [0, 1] \to R^n \]

\[ H(x, t) = (1-t)(x - a) + tf(x), \]

(7)