Apr 26th - May 3rd

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Proceedings: First International Conference on Recent Advances in Geotechnical Earthquake Engineering and Soil Dynamics, April 26 - May 3, 1981, St. Louis, Missouri

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SYNOPSIS A rate type constitutive equation with internal parameters is proposed for describing the behaviour of cyclically loaded sands in undrained conditions leading to liquefaction. The mathematical model is checked versus experimental data on undrained cyclic triaxial test on Monterey No.0 sand. General trends of real sand behaviour are captured. The equation is applicable for general stress states too.

INTRODUCTION

The present state of computer technology and the development of numerical methods enable complex analyses of nonlinear soil behaviour under dynamic conditions. Few such analyses have been performed so far (Finn et al.1976, 1978, Liou et al. 1976, Zienkiewicz et al.1978, Szavits-Nossan and Kovačić 1980). Various types of constitutive equations describing stiffness and strength degradation of sand material under undrained cyclic loading were used. While the knowledge of various factors influencing liquefaction potential is well established (e.g.Seed 1976, Castro and Poulos 1976, Townsend 1978), mathematical descriptions of the material behaviour, constitutive equations, are still in their infancy.

The aim of the following study was to develop a proper invariant constitutive equation of rate type with internal parameters which could be constructed on the basis of known features of sand behaviour in static and cyclic conditions with material constants determinable by undrained cyclic triaxial tests. A number of such tests on Montercy No.0 sand were carried out as the experimental basis of the study. The tests were performed on a modified SBEL HX-100 dynamic triaxial apparatus while satisfying standard test quality requirements (Silver 1976). Pore pressure, vertical stress and strain measurements were taken continuously and plotted via two x/y plotters. In this way the effective stress paths and stress-strain curves were obtained (Fig.3).

THE SHEAR STIFFNESS EQUATION IN TRIAXIAL CONDITIONS

As the basis for the relationship between the normalized deviator stress rate $\dot{\eta}$ ($\eta=q/p$, $q=\sigma_1'-\sigma_3'$, $p'=1/3(\sigma_1'+2\sigma_3')$, σ_1' and σ_3' being vertical and lateral effective stresses respectively) and vertical strain rate $\dot{\varepsilon}_1$ in triaxial test conditions, an equation similar to one developed earlier (Szavits-Nossan 1980) will be used:

$$\dot{\eta}=g(p'/p_o)^{\beta-1}\cdot\frac{\dot{\varepsilon}_1-m\eta|\dot{\varepsilon}_1|s_1}{1+m|\eta|s_1} \tag{1}$$

where g is a nondimensional constant dependent on relative sand density D_r, s_1 is a memory parameter

$$s_1=(1-m(\eta_n-|\eta|))^{n_1} \tag{2}$$

where η_n is defined as

$$\eta_n=\int_0^{\eta_n}\frac{d\eta}{d\varepsilon_1}\,d\varepsilon_1 \tag{3}$$

with $s_1=1$, p=const. and $d\varepsilon_1>0$; p_o is an arbitrary reference pressure. Approximating

$$\dot{\eta}_n=g(p/p_o)^{\beta-1}\,(1-m\eta)^2\,\dot{\varepsilon}_1 \tag{4}$$

and defining

$$x_1=\int_0^{x_1}|d\varepsilon_1| \tag{5}$$

the following equation for η_n is obtained:

$$\eta_n\approx\frac{x_1}{\frac{1}{g}(\frac{p'}{p})^{1-\beta}+mx_1} \tag{6}$$

η_n would be equal to η if a sample is sheared for the first time monotonously from $\eta=0$ until the vertical strain ε_1 reaches the value x_1.

The proposed shear stiffness equation (1) has the following features:
- close similarity to the Kondner's equation for fist loading (Kondner and Zalesko 1963)
- irrecoverable strains in unloading-reloading

cycle due to the term $|\dot{\varepsilon}_1|s_1$ in eq.1
- average stress level dependency due to the term $(p'/p_o)^{\beta-1}$ (El-Sohby 1969, Vermeer 1978).

Those features are ilustrated on Fig.1.

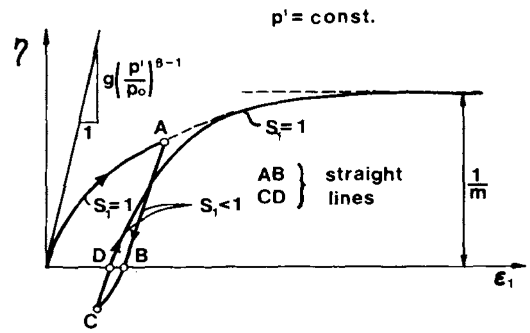

Fig.1. Properties of eq.1.

VOLUME CHANGES IN TRIAXIAL SHEAR

Mean effective stress changes p' in undrained cyclic tests develop due to the tendency of the sand to change it's volume (Seed 1976, Finn et al.1976,1978, Castro and Poulos 1976). The volume changes in drained cyclic shear test happen due to the stress dilatancy effects and the colapse of the grain structure. Most stress dilatancy equations have in common that for some $\eta=1/m_o<1/m$, $dv/d\varepsilon_1=0$, $d\varepsilon_1>0$, v being volumetric strain (Cole 1967, Rowe 1972, Tokue 1977). In the present study the following equation will be adopted for the rate of volume change:

$$\frac{dv_1}{d\varepsilon_1}=F_i(1-m_o\eta),d\varepsilon_1>0,\quad i=\begin{matrix}1 \text{ for } \eta>1/m_o\\ 2 \text{ for } \eta<1/m_o\end{matrix} \tag{7}$$

where the term F_i will describe the colapse of the grain structure and the second term in eq.7. takes account of stress dilatancy effects. In undrained test the described volume change should be compensated by the volume change due to change in mean effective stress p'. This volume change can be described by the following equation (Rowe 1972, El-Sohby 1969).

$$\frac{dp}{dv_2}=K_j(p'/p_o)^{\alpha}\qquad j=\begin{matrix}1 \text{ for } dp'>0\\ 2 \text{ for } dp'<0\end{matrix} \tag{8}$$

In undrained test conditions $dv_1+dv_2=0$ i.e.

$$\frac{\dot{p}'}{p'}=-\frac{F_iK_j}{p_og}\,g\,\left(\frac{p'}{p_o}\right)^{\alpha-1}(1-m_o\eta)\dot{\varepsilon}_1 \tag{9}$$

where eq.9. applies for compression conditions. As always $dp'>0$ for $\eta>1/m_o$ and $dp'<0$ for $\eta<1/m_o$ ($d\varepsilon_1>0$), the following constants will be renamed.

$$\frac{F_1K_1}{p_og}=A_1\quad,\quad\frac{F_2K_2}{p_og}=A_2 \tag{10}$$

Parameter A_1 will be assumed to be a material constant dependent on relative density D_r. As will be later introduced, the parameter A_2 will be taken to be stress path dependent.

INVARIANT CONSTITUTIVE EQUATION FOR TRIAXIAL CONDITIONS

Having developed basic equations of sand deformability in undrained cyclic triaxial tests the following invariant form (independent of sign of η and dq_1) of the constitutive equation for triaxial conditions is proposed:

$$\frac{\dot{q}_1}{p'}=g\left(\frac{p'}{p_o}\right)^{\beta-1}\frac{\dot{\varepsilon}_1-m\eta|\dot{\varepsilon}_1|s_1}{1+m|\eta|s_1} \tag{11}$$

$$\frac{\dot{p}_1}{p'}=-A_1g\left(\frac{p'}{p_o}\right)^{\alpha-1}\cdot\frac{1}{2}\{(|\dot{\varepsilon}_1|-m_o\eta\dot{\varepsilon}_1)-\big||\dot{\varepsilon}_1|-m_o\eta\dot{\varepsilon}_1|\} \tag{12}$$

$$\dot{q}_2=\eta\dot{p}_1' \tag{13}$$

$$\frac{\dot{p}_2'}{p'}=-A_2g\left(\frac{p'}{p_o}\right)^{\alpha-1}\cdot\frac{1}{2}\{|\dot{\varepsilon}_1|-\frac{1}{2}m_o(\eta\dot{\varepsilon}_1+|\eta\dot{\varepsilon}_1|)+\big||\dot{\varepsilon}_1|-\frac{1}{2}m_o(\eta\dot{\varepsilon}_1+|\eta\dot{\varepsilon}_1|)\big|\} \tag{14}$$

where $\dot{q}=\dot{q}_1+\dot{q}_2$, $\dot{p}'=\dot{p}_1'+\dot{p}_2'$. For the case $\eta\dot{\varepsilon}_1<0$

$$\frac{\dot{p}'}{\dot{q}}=A_2\left(\frac{p'}{p_o}\right)^{\alpha-\beta}\frac{|\dot{\varepsilon}_1|}{\dot{\varepsilon}_1} \tag{15}$$

is the inclination of the stress path from A to B and D to E, Fig.2.

Values of the components of the constitutive equation for various section of the cyclic stress path are given in table I.

TABLE I. Values of components of the constitutive equation

stress path sect. from Fig.2	values of $\dot{p}_1$	$\dot{p}_2$	$\dot{q}_1$	$\dot{q}_2$
AC , DE	=0	< 0	≠0	=0
CD , FG	>0	=0	≠0	≠0

It remains to define the stress path dependency of the parameter A_2 from eq.14. Parameter A_2 determines the slope of the stress path when $\dot{p}'<0$ (Fig.2.). It can be seen from two characteristic stress paths for two different stress amplitudes (Fig.3.) that the stress path slope is steeper for smaller stress amplitude test.

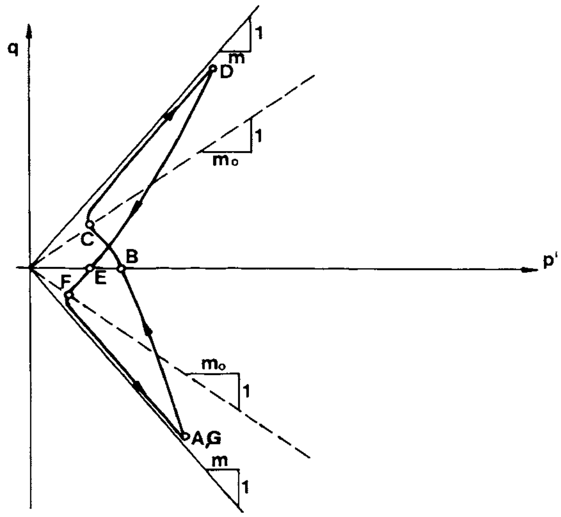

Fig.2. Various sections of the cyclic stress path

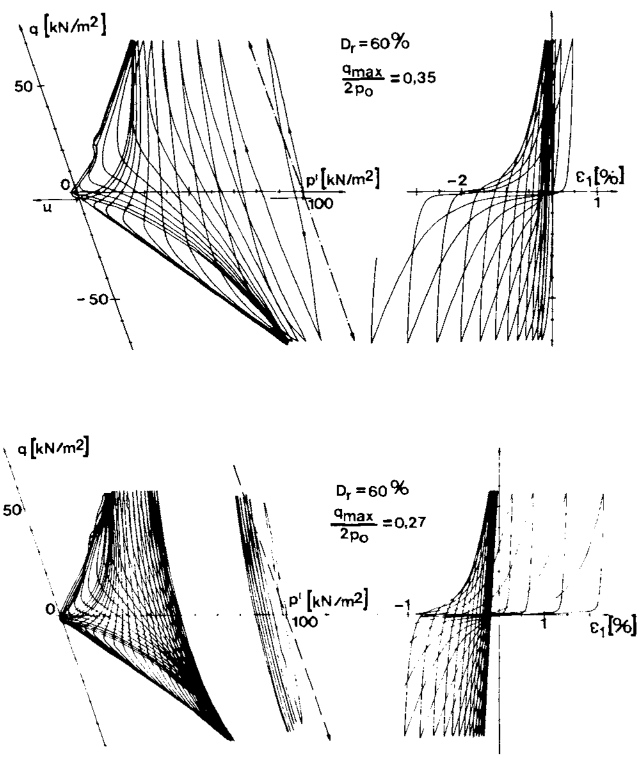

Fig.3. Effective stress pathes and stress-strain curves for two different cyclic stress ratios

To take this feature into account a new internal parameter $\bar{x}$ is introduced:

$$\dot{\bar{x}} = (1-bs_1)\,|\dot{\varepsilon}_1| \qquad (16)$$

where s_1 is defined in eq.2. and b is a material constant. The parameter A_2 will be taken to be dependent on internal parameter $\bar{x}$ by the following relationship:

$$A_2 = A_o \exp(-a\bar{x}) \quad , \quad \bar{x}>0$$

$$A_2 = m-(m-A_o) \exp\left(\frac{A_o}{m-A_o}\, a\bar{x}\right) \quad , \quad \bar{x}\leq 0 \qquad (17)$$

where a is a material constant. The development of the internal parameter $\bar{x}$ during cyclic test with smaller (S) and larger (L) stress amplitudes and it's influence on the magnitude of A_2 is shown on Fig.4.

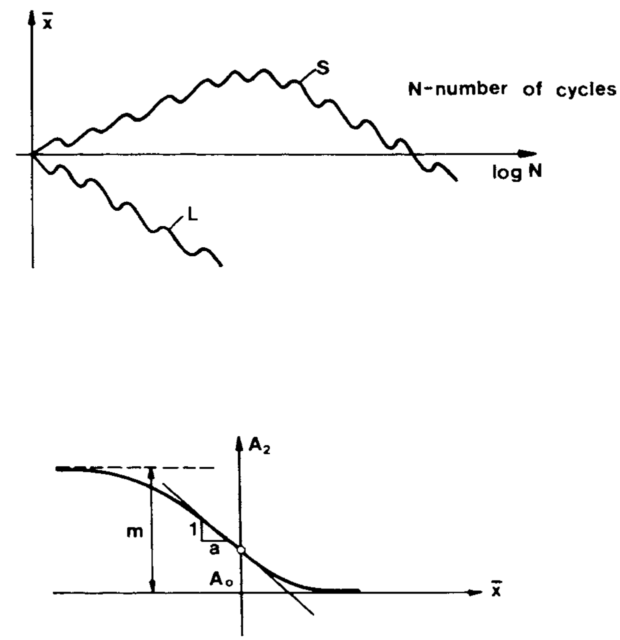

Fig.4. Internal parameter $\bar{x}$ and relationship of A_2 vs. $\bar{x}$

COMPARISON OF THE PROPOSED MODEL WITH TEST RESULTS

The applicability of the proposed equations to describe sand behaviour in undrained cyclic triaxial tests was checked on measured stress-strain and stress path data. Three characteristic undrained cyclic triaxial tests on Monterey No.ø sand of relative density D_r≈60 % were selected. All sand samples were consolidated isotropically to 100 kN/m². Three different deviator stress amplitudes were applied: 55, 70 and

90 kN/m^2. Stress strain curves and stress paths of the first two tests are shown on Fig.3. Best fitting soil constants were adopted but still some improvements are possible. As the proposed constitutive equation has a circular cone as a failure surface in the stress space a compromise cone to approximate Mohr-Coulomb failure surface was used (Humpheson and Naylor 1975). The comparison between computed and measured quantities is shown on Figs.5. and 6. Selected material constants are given in table II.

TABLE II. Best fitting material constants for Monterey No.0 sand, D_r=60 %

m = 0,90	A_1 = 0,324
m_o = 1,11	A_o = 0,03
g = 680	n_1 = 2,0
p_o = 100	a = 165,7
$\alpha=\beta$= 0,5	b = 2,60

Note: All constants are nondimensional except p_o which is given in kN/m^2.

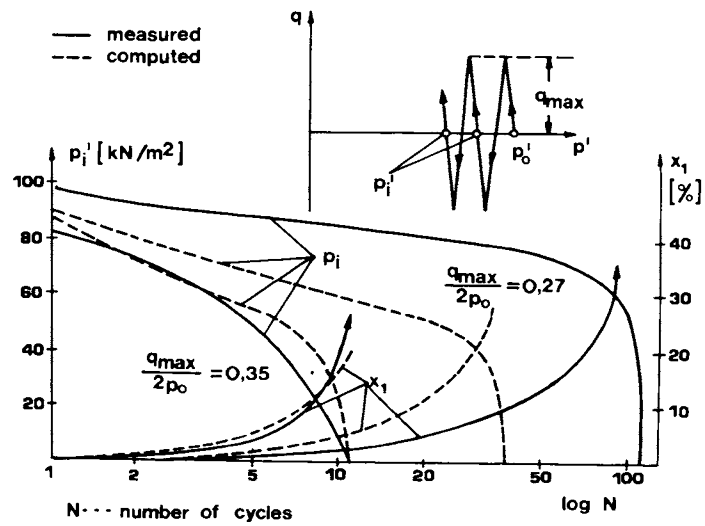

Fig.6. Computed and measured values of x_1 and p_i for two characteristic tests

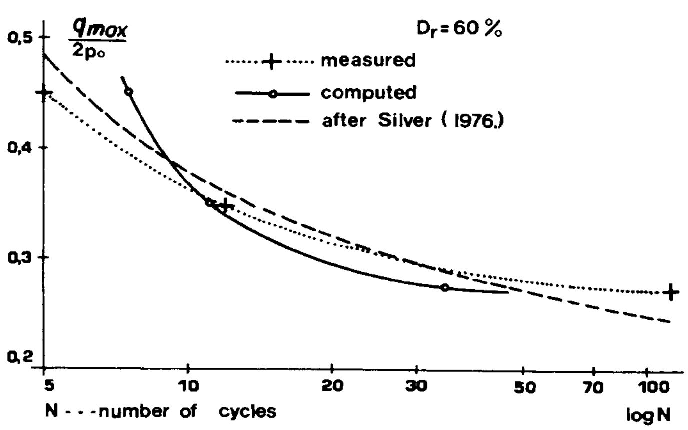

Fig.5. Computed and measured numbers of cycles to liquefaction

Although the computed values do not fit everywhere with a very good accuracy, the general trends of the material behaviour are modeled. The results seem encouraging.

EXTRAPOLATION OF THE PROPOSED MODEL TO GENERAL STRESS STATES

The previously described constitutive equation applies only for undrained triaxial conditions. The equation can be extrapolated to general stress and strain states to be applicable in finite element computer programmes. By interchanging the following scalar stress and strain measures by appropriate stress and strain tensors and their corresponding rates in eqs. 2, 5, 11, 12, 13, 14 and 16:

$$q \to \frac{3}{\sqrt{6}} Q,$$

$$p' \to \frac{1}{3} \text{tr } T'$$

$$\eta \to \frac{9}{6 \text{ tr } T'} Q,$$

$$\dot{\varepsilon}_1 \to \frac{2}{\sqrt{6}} E,$$

$$\dot{x}_1 \to \frac{2}{\sqrt{6}} |E|,$$

$$\eta\dot{\varepsilon}_1 \to \frac{3}{\sqrt{6}} \text{tr } (QE) \qquad (18)$$

where T' is the effective stress tensor,

$$Q = T' - 1/3(\text{tr } T)I, \qquad (19)$$

tr denoting trace, I being the identity tensor,

$$E = D - 1/3(\text{tr } D)I, \qquad (20)$$

D being the strain rate tensor (Truesdell and Noll 1965) the proper invariant constitutive equation in tensorial form is obtained. To incorporate the sand behaviour for deformations with volume changes the procedure used by

Szavits-Nossan (1980) can be adopted.

CONCLUSIONS

The proposed constitutive equation for constant volume deformations has following constants: m, m_o, A_1, g, A_o, n_1, a, b. The first five constants can readily be determined from one undrained triaxial cyclic test at least. The remaining four constants (A_o, n_1, a, b) must be determined by trial and error from one to two other cyclic undrained tests with different stress amplitudes. This is probably the main deficiency of the proposed equation. A computer programme can be devised to perform a best fit determination of those constants. The comparison of computed and measured results seem promising to further develop and study the properties of the proposed constitutive equation.

ACKNOWLEDGEMENT

The authors would like to thank Mr.T.Novosel of the Geotechnical laboratory, Faculty of Civil Engineering, Zagreb, for his support of the research.

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