1-1-2002

Second-Order Statistical Properties of the WSS Jakes' Fading Channel Simulator

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***Abstract*—Recently, an improved Jakes' fading channel simulator was proposed by Pop and Beaulieu to eliminate the stationarity problem occurring in Jakes' original design. In this letter, second-order statistical properties of the improved Jakes' simulator are analyzed. Consistent with Pop and Beaulieu's caution about high-order statistics of the simulator, it is proved that some second-order statistics of both the quadrature components and the envelope do not match the desired ones even if the number of sinusoids approaches infinity. Therefore, care must be taken when the simulator is employed to evaluate algorithms and systems.**

***Index Terms*—Fading channel simulator, fading channels, Rayleigh fading, statistics.**

I. INTRODUCTION

MOBILE radio channel simulators are often used in the laboratory because they allow system tests which are less expensive and more reproducible than field trials. The well known mathematical model due to Clarke [2] and its simplified simulation model due to Jakes [3] have been widely used to simulate Rayleigh fading channels for about three decades. However, recognizing that Jakes' simulator is a deterministic model, some modifications to Jakes' simulator were suggested in the literature [4], [5]. Despite the extensive acceptance and application of the simulator, some important limitations of the simulator were determined and discussed in detail recently [1]. It was shown in [1] that the Jakes' simulator is wide-sense **nonstationary**. Therefore, an improved simulator was proposed in [1] to remove the stationarity problem by introducing random phase shifts in the low-frequency oscillators. However, it was pointed out in [1] that higher order statistics of this improved simulator, which is called the wide-sense stationary (WSS) Jakes' simulator in this letter to distinguish it from other deterministic simulators, may not match the desired ones. In this letter, some second-order statistical properties of the WSS Jakes' simulator are analyzed. The results confirm that the new simulator has the desired complex envelope autocorrelation as previously detailed in [1] but does not realize some other second-order statistical properties. In particular, the autocorrelations and cross-correlations of the quadrature components and the autocorrelation of the squared envelope do not approach those of Clarke's model even as the number of sinusoids approaches infinity.

Paper approved by R. A. Valenzuela, the Editor for Transmission Systems of the IEEE Communications Society. Manuscript received September, 21, 2001.

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Publisher Item Identifier S 0090-6778(02)05541-1.

II. MATHEMATICAL REFERENCE MODEL

Consider a flat fading channel comprised of N propagation paths; the normalized low-pass fading process is given by [3], [6]

$$g(t) = g_c(t) + jg_s(t) \tag{1a}$$

$$g_c(t) = \sqrt{\frac{2}{N}} \sum_{n=1}^{N} \cos(2\pi f_d t \cos\alpha_n + \phi_n) \tag{1b}$$

$$g_s(t) = \sqrt{\frac{2}{N}} \sum_{n=1}^{N} \sin(2\pi f_d t \cos\alpha_n + \phi_n) \tag{1c}$$

where α_n is the angle of the nth incoming wave at the mobile, f_d is the maximum Doppler frequency occurring when $\alpha_n = 0$, and ϕ_n is the random initial phase associated with the nth propagation path. For large N, the central limit theorem justifies that $g_c(t)$ and $g_s(t)$ can be approximated as Gaussian random processes. Assuming that α_n and ϕ_n are mutually independent and uniformly distributed over $[-\pi, \pi]$ for all n, and adopting Clarke's 2-D **isotropic scattering** theory, some desired second-order statistics for fading simulators are given by the autocorrelation and cross-correlation functions [2], [6]

$$R_{g_c g_c}(\tau) = E[g_c(t)g_c(t+\tau)] = J_0(2\pi f_d \tau) \tag{2a}$$

$$R_{g_s g_s}(\tau) = J_0(2\pi f_d \tau) \tag{2b}$$

$$R_{g_c g_s}(\tau) = 0 \tag{2c}$$

$$R_{g_s g_c}(\tau) = 0 \tag{2d}$$

$$R_{gg}(\tau) = E[g(t)g^*(t+\tau)] = 2J_0(2\pi f_d \tau) \tag{2e}$$

$$R_{|g|^2|g|^2}(\tau) = 4 + 4J_0^2(2\pi f_d \tau). \tag{2f}$$

III. STATISTICS OF WSS JAKES' SIMULATOR

The normalized low-pass fading process of the WSS Jakes' simulator proposed in [1] is given by

$$u(t) = u_c(t) + ju_s(t) \tag{3a}$$

$$u_c(t) = \frac{2}{\sqrt{N}} \sum_{n=1}^{M+1} a_n \cos(w_n t + \psi_n) \tag{3b}$$

$$u_s(t) = \frac{2}{\sqrt{N}} \sum_{n=1}^{M+1} b_n \cos(w_n t + \psi_n) \tag{3c}$$

where $N = 4M + 2$, $w_d = 2\pi f_d$, ψ_n are independent random variables uniformly distributed over $[-\pi, \pi]$ for all n, and

$$a_n = \begin{cases} 2\cos\beta_n, & n = 1, 2, \ldots, M \\ \sqrt{2}\cos\beta_{M+1}, & n = M+1 \end{cases} \tag{4a}$$

$$b_n = \begin{cases} 2\sin\beta_n, & n = 1, 2, \ldots, M \\ \sqrt{2}\sin\beta_{M+1}, & n = M+1 \end{cases} \tag{4b}$$

$$\beta_n = \begin{cases} \frac{\pi n}{M}, & n = 1, 2, \ldots, M \\ \frac{\pi}{4}, & n = M+1 \end{cases} \tag{4c}$$

$$w_n = \begin{cases} w_d \cos\frac{2\pi n}{N}, & n = 1, 2, \ldots, M \\ w_d, & n = M+1. \end{cases} \tag{4d}$$

We present the second-order statistics of the fading signal $u(t)$ in the following theorems.

Theorem 1: The autocorrelation and cross-correlation functions of the quadrature components, and the autocorrelation functions of the envelope and the squared envelope of fading signal $u(t)$ are given by

$$R_{u_cu_c}(\tau) = \frac{4}{N}\left[\sum_{n=1}^{M+1} \frac{a_n^2}{2} \cdot \cos(w_n\tau)\right] \tag{5a}$$

$$R_{u_su_s}(\tau) = \frac{4}{N}\left[\sum_{n=1}^{M+1} \frac{b_n^2}{2} \cdot \cos(w_n\tau)\right] \tag{5b}$$

$$R_{u_cu_s}(\tau) = \frac{4}{N}\left[\sum_{n=1}^{M+1} \frac{a_nb_n}{2} \cdot \cos(w_n\tau)\right] \tag{5c}$$

$$R_{u_su_c}(\tau) = R_{u_cu_s}(\tau) \tag{5d}$$

$$R_{uu}(\tau) = \frac{4}{N}\left[2\sum_{n=1}^{M} \cos(w_n\tau) + \cos(w_d\tau)\right] \tag{5e}$$

$$R_{|u|^2|u|^2}(\tau) = 4 + 2R^2_{u_cu_c}(\tau) + 2R^2_{u_su_s}(\tau) + 4R^2_{u_cu_s}(\tau) + \frac{8}{N} J_0(2w_d\tau) + \frac{16(N-1)}{N^2}. \tag{5f}$$

Proof: Equation (5a) is proved first. Since ψ_n and ψ_i are independent when $n \neq i$, and all other terms in the sums are deterministic, one has

$$R_{u_cu_c}(\tau) = \frac{4}{N}\left[\sum_{n=1}^{M+1}\sum_{i=1}^{M+1} a_na_iE\{\cos(w_nt+\psi_n) \cdot \cos[w_i(t+\tau)+\psi_i]\}\right] = \frac{4}{N}\left[\sum_{n=1}^{M+1} a_n^2 \cdot \frac{\cos(w_n\tau)}{2}\right].$$

Similarly, one can prove (5b)–(5e). The proof of (5f) is different and is lengthy. An outline omitting some details is

$$R_{|u|^2|u|^2}(\tau) = E\left[u_c^2(t)u_c^2(t+\tau)\right] + E\left[u_s^2(t)u_s^2(t+\tau)\right] + E\left[u_c^2(t)u_s^2(t+\tau)\right] + E\left[u_s^2(t)u_c^2(t+\tau)\right]$$

$$E\left[u_c^2(t)u_c^2(t+\tau)\right] = \frac{16}{N^2}\left\{\sum_{n=1}^{M+1}\left[\frac{a_n^4}{4} + \frac{a_n^4\cos(2w_n\tau)}{8}\right] + \left[\sum_{n=1}^{M+1}\frac{a_n^2}{2}\right]^2 + 2\left[\sum_{n=1}^{M+1}\frac{a_n^2\cos(w_n\tau)}{2}\right]^2\right\}$$

$$E\left[u_s^2(t)u_s^2(t+\tau)\right] = \frac{16}{N^2}\left\{\sum_{n=1}^{M+1}\left[\frac{b_n^4}{4} + \frac{b_n^4\cos(2w_n\tau)}{8}\right] + \left[\sum_{n=1}^{M+1}\frac{b_n^2}{2}\right]^2 + 2\left[\sum_{n=1}^{M+1}\frac{b_n^2\cos(w_n\tau)}{2}\right]^2\right\}$$

$$E\left[u_c^2(t)u_s^2(t+\tau)\right] = \frac{16}{N^2}\left\{\sum_{n=1}^{M+1}\left[\frac{a_n^2b_n^2}{4} + \frac{a_n^2b_n^2\cos(2w_n\tau)}{8}\right] + \left[\sum_{n=1}^{M+1}\frac{a_n^2}{2}\right] \cdot \left[\sum_{n=1}^{M+1}\frac{b_n^2}{2}\right] + 2\left[\sum_{n=1}^{M+1}\frac{a_nb_n\cos(w_n\tau)}{2}\right]^2\right\}$$

$$E\left[u_s^2(t)u_c^2(t+\tau)\right] = E\left[u_c^2(t)u_s^2(t+\tau)\right].$$

Therefore

$$\begin{aligned} R_{|u|^2|u|^2}(\tau) &= \frac{16}{N^2}\left\{\frac{N^2}{4} + (N-1) + \left[\sum_{n=1}^{M} 2\cos(2w_n\tau) + \frac{1}{2}\cos(2w_d\tau)\right] + 2\left[\sum_{n=1}^{M+1}\frac{a_n^2\cos(w_n\tau)}{2}\right]^2 + 2\left[\sum_{n=1}^{M+1}\frac{b_n^2\cos(w_n\tau)}{2}\right]^2 + 4\left[\sum_{n=1}^{M+1}\frac{a_nb_n\cos(w_n\tau)}{2}\right]^2\right\} \\ &= 4 + 2R^2_{u_cu_c}(\tau) + 2R^2_{u_su_s}(\tau) + 4R^2_{u_cu_s}(\tau) + \frac{8}{N} J_0(2w_d\tau) + \frac{16(N-1)}{N^2}. \end{aligned}$$

■

Theorem 2: When N approaches infinity, the autocorrelation and cross-correlation functions of the quadrature components, the envelope and the squared envelope of fading signal $u(t)$ are given by

$$R_{u_cu_c}(\tau) = J_0(w_d\tau) + J_4(w_d\tau) \tag{6a}$$

$$R_{u_su_s}(\tau) = J_0(w_d\tau) - J_4(w_d\tau) \tag{6b}$$

$$R_{u_cu_s}(\tau) = \frac{2}{\pi}\int_0^{\pi/2} \sin(4\theta)\cos(w_d\tau\cos\theta)\, d\theta \tag{6c}$$

$$R_{u_su_c}(\tau) = R_{u_cu_s}(\tau) \tag{6d}$$

$$R_{uu}(\tau) = 2J_0(w_d\tau) \tag{6e}$$

$$R_{|u|^2|u|^2}(\tau) = 4 + 4J_0^2(w_d\tau) + 4J_4^2(w_d\tau) + 4\left[\frac{2}{\pi}\int_0^{\pi/2} \sin(4x)\cos(w_d\tau\cos x)\, dx\right]^2. \tag{6f}$$

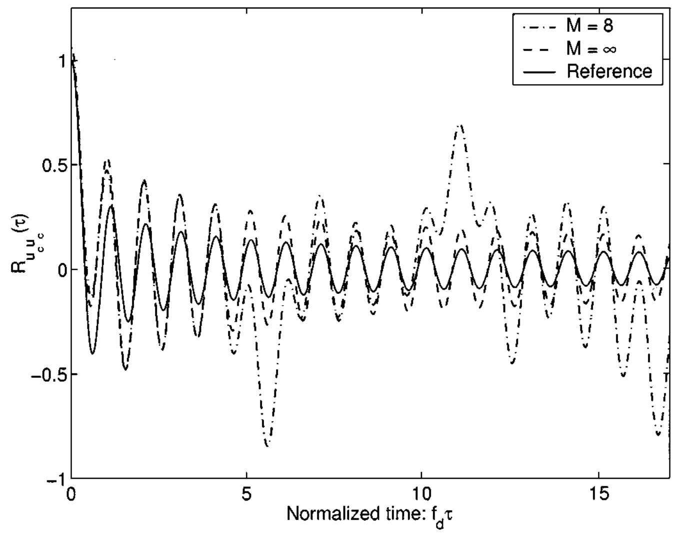

Fig. 1. Comparison of autocorrelations of real parts of simulator output and reference model.

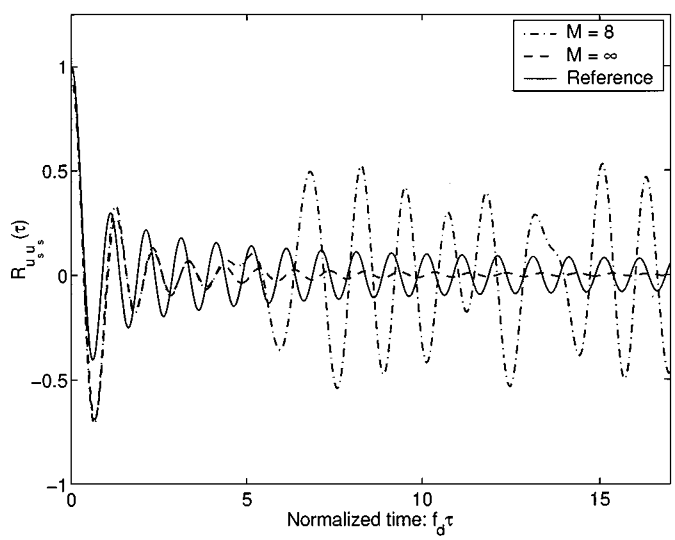

Fig. 2. Comparison of autocorrelations of imaginary parts of simulator output and reference model.

Proof: Since $N = 4M + 2$ and $w_n = w_d \cos(2\pi n/N)$, one has according to Riemann integral theory

$$\sum_{n=1}^{M} \cos(w_n \tau) + \frac{1}{2}\cos(w_d \tau) \rightarrow \frac{N}{4} J_0(w_d \tau) \tag{7a}$$

$$\sum_{n=1}^{M} \cos(2\beta_n)\cos(w_n \tau) + \frac{1}{2}\cos(2\beta_{M+1})\cos(w_d \tau) \rightarrow \frac{N}{4} J_4(w_d \tau) \tag{7b}$$

$$\sum_{n=1}^{M} \sin(2\beta_n)\cos(w_n \tau) + \frac{1}{2}\sin(2\beta_{M+1})\cos(w_d \tau) \rightarrow \frac{N}{2\pi}\int_0^{\pi/2} \sin(4\alpha)\cos(w_d \tau \cos\alpha)\, d\alpha. \tag{7c}$$

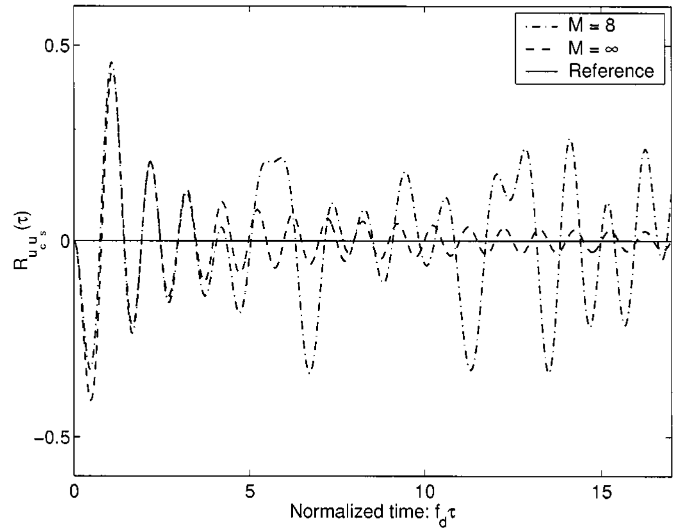

Fig. 3. Comparison of cross-correlations of quadrature components of simulator output and reference model.

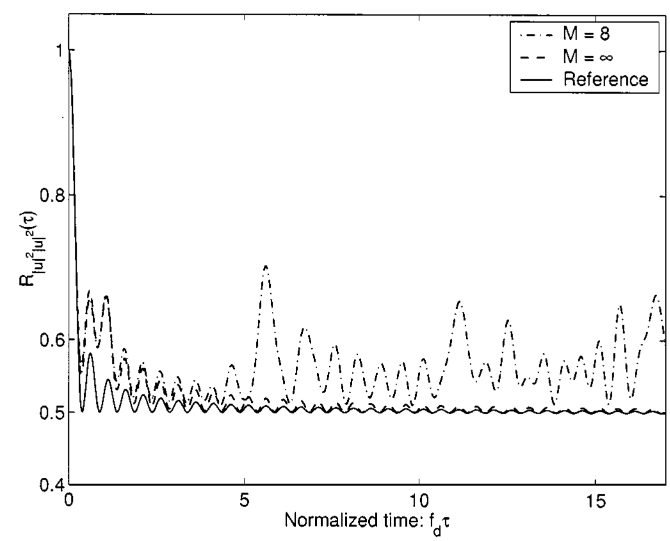

Fig. 4. Comparison of the normalized autocorrelations of the squared envelope of simulator output and reference model.

Using the trigonometric identities $2\cos^2(\beta_n) = 1 + \cos(2\beta_n)$ and $2\sin^2(\beta_n) = 1 - \cos(2\beta_n)$, with (7) and Theorem 1, one can verify (6a)–(6e). Then, based on (6a)–(6e) and (5f), one obtains

$$R_{|u|^2|u|^2}(\tau) = 4 + 4J_0^2(w_d\tau) + 4J_4^2(w_d\tau) + 4\left[\frac{2}{\pi}\int_0^{\pi/2} \sin(4x)\cos(w_d\tau\cos x)dx\right]^2.$$

■

Figs. 1–4 show the second-order statistics of the WSS simulator for $M = 8$, $M = \infty$ and for the reference model. These results are obtained using Theorems 1 and 2, respectively, for the former and (2) for the latter. As discussed in [1], the autocorrelation of the complex envelope $R_{uu}(\tau)$ approaches the desired autocorrelation $R_{gg}(\tau)$ and this second-order statistic is not shown graphically here. However, it is clear from Figs. 1–4 and from comparing eqns. (5) and (6) with eqns. (2) that the autocorrelations of the quadrature components, the cross-correlations of the quadrature components and the autocorrelation of the squared envelope of the WSS simulator do not match the desired ones. Moreover, even in the limit as the number of sinusoids approaches infinity, these particular second-order statistics fail to match the desired second-order statistics. We have verified all the theoretical results shown in Figs. 1–4 with simulations, finding good agreement in all cases.

IV. Conclusion

In this letter, some second-order statistical properties of the improved Jakes' simulator proposed by Pop and Beaulieu [1] have been analyzed. Though the autocorrelation of the complex envelope asymptotically approaches the reference autocorrelation as the number of sinusoids approaches infinity, the autocorrelations and cross-correlations of the quadrature components, and the autocorrelation of the squared envelope do not. Therefore, care must be taken in using the improved Jakes' simulator (and the Jakes' simulator) for evaluating higher-order statistical properties of systems and algorithms.

References

[1] M. F. Pop and N. C. Beaulieu, "Limitations of sum-of-sinusoids fading channel simulators," *IEEE Trans. Commun.*, vol. 49, pp. 699–708, Apr. 2001.

[2] R. H. Clarke, "A statistical theory of mobile-radio reception," *Bell Syst. Tech. J.*, pp. 957–1000, July–Aug. 1968.

[3] W. C. Jakes, *Microwave Mobile Communications*. Piscataway, NJ: IEEE Press, 1994.

[4] M. Patzold, U. Killat, F. Laue, and Y. Li, "On the statistical properties of deterministic simulation models for mobile fading channels," *IEEE Trans. Veh. Technol.*, vol. 47, pp. 254–269, Feb. 1998.

[5] M. Patzold and F. Laue, "Statistical properties of Jakes' fading channel simulator," in *Proc. IEEE VTC'98*, 1998, pp. 712–718.

[6] G. L. Stuber, *Principles of Mobile Communication*, 2nd ed. New York: Kluwer Academic, 2001.