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Nonanalytic Behavior of the Spin Susceptibility in Clean Fermi Systems

Dietrich Belitz

Theodore R. Kirkpatrick

Thomas Vojta
Missouri University of Science and Technology, vojtat@mst.edu

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D. Belitz
Department of Physics and Materials Science Institute, University of Oregon, Eugene, Oregon 97403

T. R. Kirkpatrick
Institute for Physical Science and Technology and Department of Physics, University of Maryland, College Park, Maryland 20742

Thomas Vojta
Department of Physics and Materials Science Institute, University of Oregon, Eugene, Oregon 97403 and Institut für Physik, Technische Universität Chemnitz-Zwickau, D-09107 Chemnitz, Federal Republic of Germany

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The wave vector and temperature-dependent static spin susceptibility, $\chi_s(\mathbf{Q},T)$, of clean interacting Fermi systems is considered in dimensions $1 \leqslant d \leqslant 3$. We show that at zero temperature χ_s is a nonanalytic function of $|\mathbf{Q}|$, with the leading nonanalyticity being $|\mathbf{Q}|^{d-1}$ for $1<d<3$, and $\mathbf{Q}^2 \ln|\mathbf{Q}|$ for $d=3$. For the homogeneous spin susceptibility we find a nonanalytic temperature dependence T^{d-1} for $1<d<3$. We give qualitative mode-mode coupling arguments to that effect, and corroborate these arguments by a perturbative calculation to second order in the electron-electron interaction amplitude. The implications of this, in particular for itinerant ferromagnetism, are discussed. We also point out the relation between our findings and established perturbative results for one-dimensional systems, as well as for the temperature dependence of $\chi_s(\mathbf{Q}=0)$ in $d=3$. [S0163-1829(97)04216-1]

I. INTRODUCTION

It is well known that in fluids—that is, in interacting many-body systems—there are long-range correlations between the particles. For example, in classical fluids in thermal equilibrium there are dynamical long-range correlations that manifest themselves as long-time tails, or power-law decay of equilibrium time correlation functions at large times.[1,2] In frequency space, the analogous effects are nonanalyticities at zero frequency. In an intuitive physical picture, these correlations can be understood as memory effects: the particles ''remember'' previous collisions, and therefore so-called ring collision events, where after a collision the two involved particles move away and later recollide, play a special role for the dynamics of the fluid. Technically, the long-time tails can be described in terms of mode-mode coupling theories. The salient point is that with any quantities whose correlations constitute soft, or gapless, modes (due to conservation laws, or for other reasons), products of these quantities have the same property.[3] In the equations of motion that govern the behavior of time correlation functions this leads to convolutions of soft propagators, which in turn results in nonanalytic frequency dependences. For phase-space reasons, the strength of the effect increases with decreasing dimensionality: while in three-dimensional (3D) classical fluids the long-time tails provide just a correction to the asymptotic hydrodynamic description of the system, in 2D fluids they are strong enough to destroy hydrodynamics.[1,4]

A natural question to ask is whether such long-range correlations also occur in position space. Indeed, in classical fluids in nonequilibrium steady-state effects occur that may be considered as the spatial analogs of long-time tails, but in thermal equilibrium this is not the case.[1,2] This changes, however, if we consider quantum fluids. The quantum nature of a system has two major implications as far as statistical mechanics is concerned. First, temperature enters, apart from occupation numbers, through Matsubara frequencies, which means that the system's behavior as a function of temperature will in general be the same as its behavior as a function of frequency, at least at asymptotically low temperatures. Second, and more importantly, in quantum statistical mechanics statics and dynamics are coupled and need to be considered together. This raises the question of whether in a quantum fluid there might be long-range spatial correlations even in equilibrium.

From studies of systems with quenched disorder, there is evidence that the answer to this question is affirmative. Let us consider interacting fermions in an environment of static scatterers. In dimensions $d>2$, and for a sufficiently small scatterer density, the relevant soft modes in such a system are diffusive, so frequency Ω, or temperature T, scales like the square of the wave vector $\mathbf{Q}$, $\Omega \sim T \sim \mathbf{Q}^2$. Via mode-mode coupling effects that are analogous to those present in classical fluids, dynamical long-range correlations lead to long-time tails in equilibrium time correlation functions. For instance, the electrical conductivity as a function of frequency behaves like $\Omega^{(d-2)/2}$ at small Ω in $d>2$.[5] The dynamical spin susceptibility $\chi_s(\mathbf{Q},\Omega)$ shows no analogous long-time tail at $\mathbf{Q}=0$ for reasons related to spin conservation. However, from the above arguments about the coupling of statics and dynamics in quantum statistical mechanics and the scaling of frequency with wave number, one would expect the *static* spin susceptibility, $\chi_s(\mathbf{Q},\Omega=0)$ at $T=0$, to show a related nonanalyticity at $\mathbf{Q}=0$, namely, $\chi_s \sim |\mathbf{Q}|^{d-2}$. This is indeed the case, as can be seen most easily from perturbative calculations.[6] Schematically, the coupling of two diffusive modes leads to contributions to χ_s of the type

$$\int d\mathbf{q}\int d\omega \frac{1}{\omega+\mathbf{q}^2}\frac{1}{\omega+\Omega+(\mathbf{q}+\mathbf{Q})^2}, \tag{1.1}$$

which leads to the above behavior. One can then invoke renormalization-group arguments to show that this is indeed the leading small-**Q** behavior of χ_s. Similarly, at finite temperature the homogeneous susceptibility behaves as $\chi_s(\mathbf{Q}=0,\Omega=0)\sim T^{(d-2)/2}$. This has interesting consequences for itinerant magnetism in such systems, as has been recently discussed.[6–8]

Somewhat surprisingly, the situation is much less clear in clean Fermi systems. Here the soft modes are density and spin density fluctuations, as well as more general particle-hole excitations. All of these have a linear dispersion relation, i.e., $\Omega\sim|\mathbf{Q}|$. The form of the dispersion relation does not affect the basic physical arguments for nonanalytic frequency and wave-number dependences given above. One might thus expect the spin susceptibility to have mode-mode coupling contributions of a type analogous to those shown in Eq. (1.1), but with ballistic instead of diffusive modes:

$$\int d\mathbf{q}\int d\omega \frac{1}{\omega+|\mathbf{q}|}\frac{1}{\omega+\Omega+|\mathbf{q}+\mathbf{Q}|}, \tag{1.2}$$

which leads to $\chi_s(\mathbf{Q},\Omega=0)\sim \text{const}+|\mathbf{Q}|^{d-1}$ in generic dimensions at $T=0$. In $d=3$, one would expect a $\mathbf{Q}^2\ln|\mathbf{Q}|$ behavior, as convolution integrals tend to yield logarithms in special dimensions. Such a behavior of χ_s would have profound consequences for the critical behavior of itinerant ferromagnets, as has been pointed out recently.[9] It is therefore of importance to unambiguously determine whether or not the above mode-mode coupling arguments do indeed carry over from disordered to clean systems.

Before we start this task, let us discuss the available information concerning long-range correlations in clean Fermi systems. The specific heat is known to be a nonanalytic function of temperature, viz., $C_V/T\sim T^2\ln T$ in $d=3$. This is a consequence of a nonanalytic correction to the linear dispersion relation of the quasiparticles in Fermi-liquid theory, namely, $\Delta\epsilon(p)\sim(p-p_F)^3\ln|p-p_F|$.[10] Such a nonanalyticity signal the presence of a long-range effective interaction between the quasiparticles, and in general it will lead to nonanalytic behavior of both thermodynamic quantities and time correlation functions. The $T^2\ln T$ term in the specific-heat coefficient is an example of such an effect. In $d=2$ the behavior is $C_V/T\sim T$,[11] which is consistent with the behavior $C_V/T\sim T^{d-1}$ in generic dimensions that one would expect from the above arguments. It was natural to look for similar effects in other quantities, in particular in the spin susceptibility. These investigations concentrated on the temperature dependence of χ_s, and several authors indeed reported to have found a $T^2\ln T$ term in the homogeneous static spin susceptibility. However, other investigations did not find such a contribution.[12] The resulting confusion has been discussed by Carneiro and Pethick.[13] These authors used Fermi-liquid theory to show that, while $T^2\ln T$ terms do indeed appear in intermediate stages of the calculation of χ_s as well as of C_V, they cancel in the former.

This somewhat surprising result casts some doubt on the general physical picture painted above, which suggests the qualitative equivalence of disordered and clean systems with respect to the presence of long-range correlations, and resulting nonanalyticities in both the statics and the dynamics of quantum systems. On the other hand, a failure of this general picture would be hard to understand from several points of view. For instance, in $d=1$ the instability of the Fermi liquid with respect to the Luttinger liquid is well known to manifest itself in perturbation theory for χ_s by means of logarithmic singularities.[14,15] This is precisely what one obtains from the mode-mode coupling integral, Eq. (1.2). By continuity one therefore expects $\chi_s(\mathbf{Q}=0,T)\sim T^{d-1}$, and $\chi_s(\mathbf{Q},T=0)\sim|\mathbf{Q}|^{d-1}$, at least in $d=1+\epsilon$. Unless the physics changes qualitatively between $d=1+\epsilon$ and $d=3$, this should still be true in higher dimensions. Also, the corrections to Landau theory we are discussing here can be cast in the language of the renormalization group. In this framework, the Fermi-liquid ground state is described as a stable fixed point,[16] and the effects we are interested in manifest themselves as an irrelevant operator that leads to corrections to scaling near this fixed point.[17] In a system where **Q**, Ω, and T all have a scale dimension of unity, this operator should appear as $|\mathbf{Q}|^{d-1}$, Ω^{d-1}, etc., dependences in various correlation functions. From a general scaling point of view it would be hard to understand if this were not the case, except for the possibility that the prefactors of some nonanalyticities might accidentally vanish in certain dimensions.

It is the purpose of the present paper to clarify this confusing point. We will show that the above general physical picture does indeed hold true, and that it is not violated by the previously found absence of a $T^2\ln T$ term in χ_s in $d=3$, which is accidental. The remainder of this paper is organized as follows. In Sec. II we define our model. In Sec. III we perform an explicit perturbative calculation to second order in the electron-electron interaction. This confirms both our qualitative arguments, and the results of Ref. 13. We explain why there is no contradiction between these results, and we also make contact with established perturbative results in $d=1$. In Sec. IV A we discuss our result in the light of mode-mode coupling arguments that are an elaboration of those given above. In Sec. IV B we make contact with renormalization-group ideas, and argue that the functional forms of the nonanalyticities derived in Sec. III by means of perturbation theory are asymptotically exact. In Sec. IV C we discuss the physical consequences of our results.

II. MODEL, AND THEORETICAL FRAMEWORK

A. The model

Let us consider a system of clean fermions governed by an action[18]

$$S=-\int dx\sum_\sigma \bar{\psi}_\sigma(x)\frac{\partial}{\partial\tau}\psi_\sigma(x)+S_0+S_{\text{int}}. \tag{2.1a}$$

Here we use a four-vector notation, $x\equiv(\mathbf{x},\tau)$, and $\int dx\equiv\int d\mathbf{x}\int_0^\beta d\tau$. $\mathbf{x}$ denotes position, τ imaginary time, $\beta=1/T$, and we choose units such that $\hbar=k_B=1$. σ is the spin label. S_0 describes free fermions with chemical potential μ,

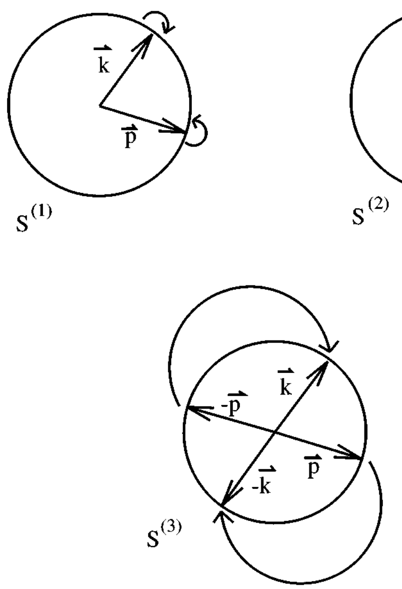

FIG. 1. Typical small-angle (1), large-angle (2), and $2k_F$-scattering processes (3) near the Fermi surface in $d=2$.

$$S_0=\int dx\sum_{\sigma}\bar{\psi}_{\sigma}(x)[\Delta/2m+\mu]\psi_{\sigma}(x), \tag{2.1b}$$

with Δ the Laplace operator, and m the fermion mass. $S_{\rm int}$ describes a two-particle, spin-independent interaction,

$$S_{\rm int}=-\frac{1}{2}\int dx_1 dx_2\sum_{\sigma_1,\sigma_2} v(x_1-x_2)\times\bar{\psi}_{\sigma_1}(x_1)\bar{\psi}_{\sigma_2}(x_2)\psi_{\sigma_2}(x_2)\psi_{\sigma_1}(x_1). \tag{2.1c}$$

The interaction potential $v(x)$ will be specified in Sec. II B below.

We now Fourier transform to wave vectors $\mathbf{k}$ and fermionic Matsubara frequencies $\omega_n=2\pi T(n+1/2)$. Later we will also encounter bosonic Matsubara frequencies, which we denote by $\Omega_n=2\pi Tn$. Using again a four-vector notation, $k\equiv(\mathbf{k},\omega_n)$, $\Sigma_k\equiv T\Sigma_{i\omega_n}\int d\mathbf{k}/(2\pi)^d$, we can write

$$S_0=\sum_{\sigma}\sum_{k}\bar{\psi}_{\sigma}(k)[i\omega_n-\mathbf{k}^2/2m+\mu]\psi_{\sigma}(k), \tag{2.2a}$$

$$S_{\rm int}=\frac{-T}{2}\sum_{\sigma_1,\sigma_2}\sum_{\{k_i\}}\delta_{k_1+k_2,k_3+k_4}v(k_2-k_3)\times\bar{\psi}_{\sigma_1}(k_1)\bar{\psi}_{\sigma_2}(k_2)\psi_{\sigma_2}(k_3)\psi_{\sigma_1}(k_4). \tag{2.2b}$$

For the long-wavelength, low-frequency processes we will be interested in, only the scattering of particles and holes close to the Fermi surface is important. It is customary and convenient to divide these processes into three classes:[19] (1) small-angle scattering, (2) large-angle scattering, and (3) $2k_F$ scattering. These classes are also referred to as the particle-hole channel for classes (1) and (2), and the particle-particle or Cooper channel for class (3), respectively. The corresponding scattering processes are schematically depicted in Fig. 1. For our purposes it is convenient to make the phase-space decomposition that is inherent to this classification explicit by writing the interaction part of the action,

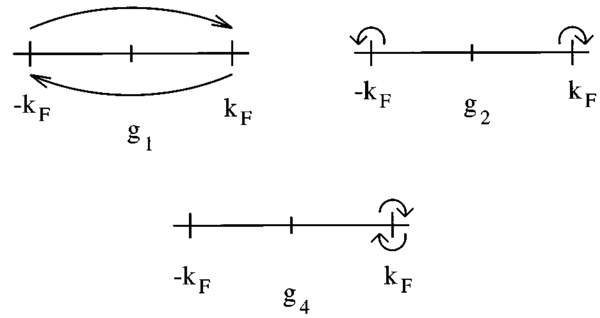

FIG. 2. The three independent scattering processes near the Fermi surface with interaction amplitudes g_1, g_2, and g_4 in $d=1$.

$$S_{\rm int}=S_{\rm int}^{(1)}+S_{\rm int}^{(2)}+S_{\rm int}^{(3)}, \tag{2.3a}$$

where

$$S_{\rm int}^{(1)}=\frac{-T}{2}\sum_{\sigma_1,\sigma_2}\sum_{k,p}\sum_{q}{}' v(\mathbf{q})\bar{\psi}_{\sigma_1}(k)\times\bar{\psi}_{\sigma_2}(p+q)\psi_{\sigma_2}(p)\psi_{\sigma_1}(k+q), \tag{2.3b}$$

$$S_{\rm int}^{(2)}=\frac{-T}{2}\sum_{\sigma_1,\sigma_2}\sum_{k,p}\sum_{q}{}' v(\mathbf{p}-\mathbf{k})\bar{\psi}_{\sigma_1}(k)\bar{\psi}_{\sigma_2}(p+q)\times\psi_{\sigma_2}(k+q)\psi_{\sigma_1}(p), \tag{2.3c}$$

$$S_{\rm int}^{(3)}=\frac{-T}{2}\sum_{\sigma_1\neq\sigma_2}\sum_{k,p}\sum_{q}{}' v(\mathbf{k}+\mathbf{p})\bar{\psi}_{\sigma_1}(k)\bar{\psi}_{\sigma_2}(-k+q)\times\psi_{\sigma_2}(p+q)\psi_{\sigma_1}(-p). \tag{2.3d}$$

Here the prime on the q summation indicates that only momenta up to some cutoff momentum Λ are integrated over. This restriction is necessary to avoid double counting, since each of the three expressions, Eqs. (2.3b)–(2.3d), represents all of $S_{\rm int}$ if all wave vectors are summed over. The long-wavelength physics we are interested in will not depend on Λ.

The above phase-space decomposition is correct in dimensions $d\geqslant 2$. In $d=1$, the Fermi surfaces collapse onto two Fermi points, and the processes we called above large-angle scattering and $2k_F$ scattering become indistinguishable. The three independent scattering processes are usually chosen as the ones shown in Fig. 2, and the corresponding coupling interaction potentials are denoted by g_1, g_2, and g_4.[15] Inspection shows that the action written in Eqs. (2.3) counts each of these processes twice. If $S_{\rm int}^{(3)}$ is dropped, then the g_4 process is still counted twice. However, it is known that g_4 does not contribute to the logarithmic terms we are interested in.[15] For our purposes it therefore is sufficient to just drop the particle-particle channel when we are dealing with $d=1$.

B. Simplifications of the model

The effective interaction potentials that appear in Eqs. (2.3b)–(2.3d) are all given by the basic potential v, taken at different momenta. $S_{\rm int}^{(1)}$ contains the direct scattering contribution, or $v(\mathbf{q})$, with $\mathbf{q}$ the restricted momentum. If v is chosen to be a bare Coulomb interaction, then this leads to singularities in perturbation theory in v that indicate the need for infinite resummations to incorporate screening. For simplicity, we assume that this procedure has already been carried out, and take v to be a statically screened Coulomb interaction. For effects that arise from small values of $|\mathbf{q}|$ it is then sufficient to replace $v(\mathbf{q})$ by the number $\Gamma_1 \equiv v(\mathbf{q}\rightarrow 0)$.[20] In Eqs. (2.3c) and (2.3d) the moduli of $\mathbf{k}$ and $\mathbf{p}$ are equal to k_F for the dominant scattering processes, and one usually expands these coupling constants in Legendre polynomials on the Fermi surface. While all of the terms in this expansion contribute to the processes we want to study, we note that the coefficients in the angular momentum expansion are independent coupling constants. In order to establish the existence of a nonanalytic term in $\chi_s(\mathbf{Q})$, it therefore is sufficient to establish its existence in a particular angular momentum channel. For simplicity we choose the zero angular momentum channel, $l=0$. We then have three coupling constants in our theory, namely, Γ_1, Γ_2, and Γ_3, which are $v(\mathbf{k}-\mathbf{p})$ and $v(\mathbf{k}+\mathbf{p})$, respectively, averaged over the Fermi surface. Instead of Γ_1 and Γ_2 one often uses the particle-hole spin singlet and spin triplet interaction amplitudes Γ_s and Γ_t that are linear combinations of Γ_1 and Γ_2. They are related to the Fermi-liquid parameters F_0^s and F_0^a by

$$\Gamma_s = \Gamma_1 - \Gamma_2/2 = \frac{1}{2N_F}\,\frac{F_0^s}{1+F_0^s}, \tag{2.4a}$$

$$\Gamma_t = \Gamma_2/2 = \frac{-1}{2N_F}\,\frac{F_0^a}{1+F_0^a}, \tag{2.4b}$$

where N_F is the density of states at the Fermi level. Our simplified model is tantamount to taking only F_0^s and F_0^a into account instead of the complete sets of Landau parameters. As explained above, this is sufficient for our purposes. We also define the Cooper channel amplitude,

$$\Gamma_c = \Gamma_3/2, \tag{2.4c}$$

and again we keep only the $l=0$ channel. The particle-particle channel is neglected in Landau theory.

Our model is now defined as Eqs. (2.2) and (2.3), with $v(\mathbf{q})$, $v(\mathbf{p}-\mathbf{k})$, and $v(\mathbf{k}+\mathbf{p})$ replaced by Γ_1, Γ_2, and Γ_3, respectively. We thus have three different interaction vertices, which are shown in Fig. 3. In the following section we will calculate χ_s in perturbation theory with respect to the interaction amplitudes Γ_1, Γ_2, and Γ_3.

III. PERTURBATION THEORY

A. Contributions to second order in the interaction

We now proceed to calculate the spin susceptibility χ_s in perturbation theory with respect to the electron-electron interaction. This can be done by means of standard methods.[21,22,19] We will be interested only in contributions that lead to a nonanalytic wave-number dependence. It is

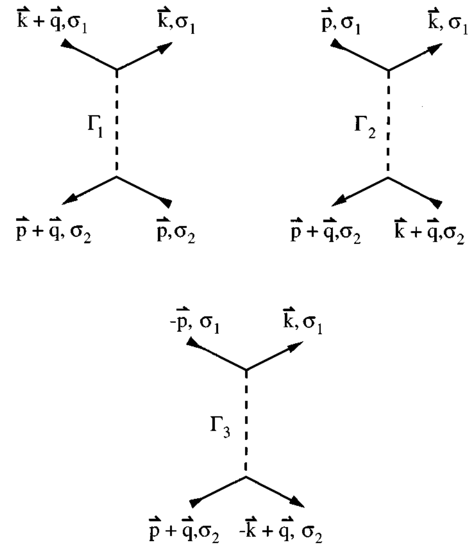

FIG. 3. The three interaction vertices with coupling constants Γ_1, Γ_2, and Γ_3.

easy to see that no nonanalytic behavior can occur at first order in the interaction. At second-order, there is also a large number of diagrams for which this is true, and others vanish due to charge neutrality.[20] There remain seven topologically different second-order diagrams, all shown in Fig. 4, that need to be considered. We thus write

$$\chi_s(Q) = 2\chi_0(Q) + \sum_{i=1}^{7} \chi^{(i)} + (\text{analytic contributions}), \tag{3.1}$$

where χ_0 denotes the Lindhard function, and the correction terms are labeled according to the diagrams in Fig. 4. Here

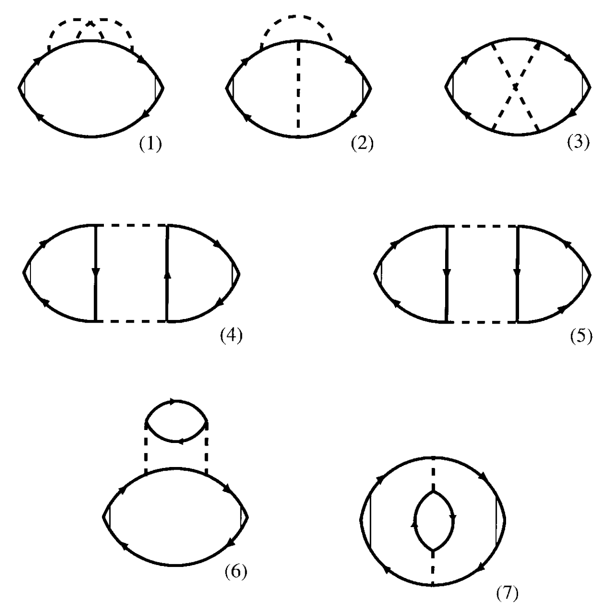

FIG. 4. Second order diagrams that contribute to the nonanalytic behavior of χ_s. The solid vertical line denotes the external spin vertex σ.

and in the remainder of this section we use again the four-vector notation of Sec. II, so $Q \equiv (\mathbf{Q},\Omega_n)$, etc.

These diagrams can be expressed in terms of integrals over electronic Green's functions, or bare electron propagators, that can be read off Eq. (2.2a),

$$G_k \equiv G_{\mathbf{k}}(i\omega_n) = \frac{1}{i\omega_n - \mathbf{k}^2/2m + \mu}. \tag{3.2}$$

In terms of the G_k, we find

$$\chi^{(1)} = -4\Gamma_1\Gamma_2 \sum_\sigma \sigma^2 \sum_q{}' J_1^{(4)}(q,Q)J^{(2)}(q), \tag{3.3a}$$

$$\chi^{(2)} = -2\Gamma_1\Gamma_2 \sum_\sigma \sigma^2 \sum_q{}' \{[J^{(3)}(q,Q)]^2 + J_2^{(4)}(q,Q)J^{(2)}(q)\}, \tag{3.3b}$$

$$\chi^{(3)} = -2\Gamma_1\Gamma_2 \sum_\sigma \sigma^2 \sum_q{}' J^{(3)}(q,Q)J^{(3)}(-q,-Q), \tag{3.3c}$$

$$\chi^{(4)} = (\Gamma_3)^2 \sum_{\sigma_1,\sigma_2} \sigma_1\sigma_2(1-\delta_{\sigma_1\sigma_2}) \sum_q{}' I_1^{(3)}(q,Q)I_2^{(3)}(q,Q), \tag{3.3d}$$

$$\chi^{(5)} = (\Gamma_3)^2 \sum_{\sigma_1,\sigma_2} \sigma_1\sigma_2(1-\delta_{\sigma_1\sigma_2}) \sum_q{}' I_2^{(4)}(q,Q)I^{(2)}(q), \tag{3.3e}$$

$$\chi^{(6)} = 2\sum_{\sigma_1,\sigma_2} \sigma_1^2 \sum_q{}' [(\Gamma_1)^2 J_1^{(4)}(q,Q) + (\Gamma_2)^2 \times J_1^{(4)}(-q,Q)]J^{(2)}(-q) + 2(\Gamma_3)^2 \times \sum_{\sigma_1,\sigma_2} \sigma_1\sigma_2(1-\delta_{\sigma_1\sigma_2}) \sum_q{}' I_1^{(4)}(q,Q)I^{(2)}(q), \tag{3.3f}$$

$$\chi^{(7)} = \sum_{\sigma_1,\sigma_2} \sigma_1^2 \sum_q{}' \{(\Gamma_1)^2 J_2^{(4)}(q,Q)J^{(2)}(q) + (\Gamma_2)^2 \times [J^{(3)}(-q,Q)]^2\} - \chi^{(4)}. \tag{3.3g}$$

Here q is a bosonic frequency-momentum integration variable. In Eqs. (3.3), the following multiplication factors have been taken into account. In diagram (1) of Fig. 4, either one of the interaction lines can be a Γ_1; the other one is then necessarily a Γ_2. This leads to a multiplication factor of 2, and another factor of 2 comes from the existence of an equivalent symmetric diagram. In diagram (2), again either one of the two interaction lines can be a Γ_1, with the other line then being a Γ_2, but here the two expressions one obtains are not identical. Again, there is an overall symmetry factor of 2. The same holds for diagram (3), but without the overall symmetry factor. Diagrams (4) and (5) can be realized only with Γ_3, and they carry no multiplication factors. In diagrams (6) and (7), both interaction lines must be the same, and diagram (6) carries an extra symmetry factor of 2. The spin structures represent the fact that the interaction cannot flip the spin, and that the external vertex carries a factor of σ. The functions in the integrands of Eqs. (3.3) are defined as

$$J^{(2)}(q) = \sum_k G_k G_{k-q}, \tag{3.4a}$$

$$J^{(3)}(q,Q) = \sum_k G_k G_{k-q} G_{k-Q}, \tag{3.4b}$$

$$J_1^{(4)}(q,Q) = \sum_k (G_k)^2 G_{k-q} G_{k-Q}, \tag{3.4c}$$

$$J_2^{(4)}(q,Q) = \sum_k G_k G_{k-q} G_{k-Q} G_{k-q-Q}, \tag{3.4d}$$

$$I^{(2)}(q) = \sum_k G_k G_{-k+q}, \tag{3.4e}$$

$$I_1^{(3)}(q,Q) = \sum_k G_{-k} G_{k+q} G_{-k-Q}, \tag{3.4f}$$

$$I_2^{(3)}(q,Q) = \sum_k G_k G_{-k+q} G_{-k+q-Q}, \tag{3.4g}$$

$$I_1^{(4)}(q,Q) = \sum_k (G_k)^2 G_{-k+q} G_{k-Q}, \tag{3.4h}$$

$$I_2^{(4)}(q,Q) = \sum_k G_k G_{-k+q} G_{k+Q} G_{-k+q-Q}. \tag{3.4i}$$

The information we are interested in is contained in Eqs. (3.1)–(3.4) in terms of integrals. The remaining task is to perform these integrals. While it is easy to see by power counting that all of the above contributions to χ_s do indeed scale like Q^{d-1} for $1<d<3$, and like $O(1)$ and $O(Q^2)$ with logarithmic corrections in $d=1$ and $d=3$, respectively, we have found it impossible to analytically perform the integrals in general, i.e., for a finite external wave number in arbitrary dimensions d. However, for a perturbative confirmation of the expected nonanalyticity such a general analysis is not necessary. Rather, it is sufficient to explicitly obtain the prefactors of the logarithmic singularities in $d=1$ and $d=3$. If they are not zero, then by combining this with power counting and the expected continuity of χ_s as a function of d, it follows that the prefactor of the Q^{d-1} nonanalyticity does not vanish for generic values of d either. For the temperature dependence at $\mathbf{Q}=0$ the integrals can be done in arbitrary d, see Sec. III E below.

In Secs. III B–III D we therefore analyze the above integrals in $d=1$ and $d=3$. In doing so, we treat the particle-hole and particle-particle channel contributions separately, since they have quite different structures. We also anticipate that we will be interested only in the static spin susceptibility, so $Q=(0,\mathbf{Q})$. In $d=1$, we write $\mathbf{Q}$ for the one-dimensional vector, i.e., a real number that can be either positive or negative.

B. $d=1$

Let us first consider $d=1$. We do this mainly to make contact with established results in the literature. As explained above, the particle-particle channel must not be taken into account in $d=1$, so we put $\Gamma_3=0$. Since we are interested in a logarithm that results from an infrared singularity, it suffices to calculate the integrands in the limit of small frequencies and wave numbers. Be performing the integrals in Eqs. (3.4a)–(3.4d) one obtains, with $Q=(0,\mathbf{Q})$ and $q=(\Omega_n,\mathbf{q})$,

$$J^{(2)}(q)=\frac{-N_F}{1+(\Omega_n/v_F\mathbf{q})^2}, \tag{3.5a}$$

$$J^{(3)}(q,Q)=N_F\left[\frac{i\Omega_n\mathbf{q}/\mathbf{Q}}{\Omega_n^2+(v_F\mathbf{q})^2}+\frac{i\Omega_n(\mathbf{Q}-\mathbf{q})/\mathbf{Q}}{\Omega_n^2+[v_F(\mathbf{Q}-\mathbf{q})]^2}\right], \tag{3.5b}$$

$$J_1^{(4)}(q,Q)=N_F\left[\frac{\mathbf{q}}{\mathbf{Q}}\frac{(v_F\mathbf{q})^2-\Omega_n^2}{[\Omega_n^2+(v_F\mathbf{q})^2]^2}-\frac{\mathbf{q}/\mathbf{Q}}{\Omega_n^2+(v_F\mathbf{q})^2}+\frac{\mathbf{q}^2/\mathbf{Q}^2}{\Omega_n^2+(v_F\mathbf{q})^2}-\frac{(\mathbf{Q}-\mathbf{q})^2/\mathbf{Q}^2}{\Omega_n^2+[v_F(\mathbf{Q}-\mathbf{q})]^2}\right], \tag{3.5c}$$

$$J_2^{(4)}(q,Q)=N_F\left[-\frac{2\mathbf{q}^2/\mathbf{Q}^2}{\Omega_n^2+(v_F\mathbf{q})^2}+\frac{(\mathbf{q}-\mathbf{Q})^2/\mathbf{Q}^2}{\Omega_n^2+[v_F(\mathbf{q}-\mathbf{Q})]^2}+\frac{(\mathbf{q}+\mathbf{Q})^2/\mathbf{Q}^2}{\Omega_n^2+[v_F(\mathbf{q}+\mathbf{Q})]^2}\right]. \tag{3.5d}$$

Inserting this into Eqs. (3.3), performing the final integrals, and collecting the results one obtains, apart from analytic terms,

$$\chi_s(\mathbf{Q})=2N_F-4N_F(\Gamma_tN_F)^2\ln(2k_F/|\mathbf{Q}|). \tag{3.6}$$

This result agrees with the well-known one to this order in Γ_t.[14] One would expect that the $\ln|\mathbf{Q}|$ gets replaced by a $\ln\Omega$ or $\ln T$ if one works at $\mathbf{Q}=0$ and finite Ω or T, respectively. Explicit calculations confirm this. Of course the physical content of this perturbative result is limited, since the ground state is not a Fermi liquid.[23] For later reference we also mention that, to logarithmic accuracy, it is not necessary to keep $\mathbf{Q}$ nonzero in the above calculation. If one works at $\mathbf{Q}=0$ and determines the prefactor of the resulting logarithmic divergence, then one obtains the same result as above.

C. Particle-hole channel in $d=3$

In $d=3$, both the particle-hole and the particle-particle channel contribute to the terms we are interested in. Since the structures of the integrals in the two channels are quite different, we first consider the particle-particle channel. In $d=3$, the logarithm appears only at $O(\mathbf{Q}^2)$. Keeping $\mathbf{Q}$ explicitly in the integrals to that order would be hard. However, as was pointed out in the preceding subsection, to logarithmic accuracy this is not necessary. Rather, we can just expand in $\mathbf{Q}$. The prefactor of the $\mathbf{Q}^2$ term will then be logarithmically divergent, and the prefactor of the divergence will be the same as that of the $\mathbf{Q}^2\ln|\mathbf{Q}|$ term whose presence is signaled by the divergence. By expanding Eqs. (3.4b)–(3.4d) to $O(\mathbf{Q}^2)$, and dropping the uninteresting contribution to the homogeneous χ_s, we can express all logarithmic contributions to χ_s in terms of two integrals,

$$J_1=\sum_q{}'\sum_k\left(\frac{\mathbf{k}\cdot\hat{\mathbf{Q}}}{m}\right)^2(G_{k+q})^5G_k\sum_p G_pG_{p-q}=\left(\frac{N_Fv_F}{24}\right)^2\sum_{\mathbf{q}}\frac{1}{(v_F|\mathbf{q}|)^3}, \tag{3.7a}$$

$$J_2=\frac{1}{4}\sum_q{}'\sum_k\left(\frac{\mathbf{k}\cdot\hat{\mathbf{Q}}}{m}\right)^2(G_k)^4G_{k-q}\sum_p(G_p)^2G_{p+q}=-J_1, \tag{3.7b}$$

where we have kept only the most divergent term. We find

$$\chi^{(1)}=-8\Gamma_1\Gamma_2\mathbf{Q}^2J_1, \tag{3.8a}$$

$$\chi^{(2)}=-4\Gamma_1\Gamma_2\mathbf{Q}^2(J_1+J_2)=0, \tag{3.8b}$$

$$\chi^{(3)}=-8\Gamma_1\Gamma_2\mathbf{Q}^2J_2, \tag{3.8c}$$

$$\chi^{(6)}=8(\Gamma_1^2+\Gamma_2^2)\mathbf{Q}^2J_1, \tag{3.8d}$$

$$\chi^{(7)}=8\mathbf{Q}^2(-\Gamma_1^2J_1-\Gamma_2^2J_2). \tag{3.8e}$$

Here we have used the fact that the structure $(J^{(3)})^2$ that appears in $\chi^{(2)}$, $\chi^{(3)}$, and $\chi^{(7)}$, if expanded to order $\mathbf{Q}^2$, yields two terms, one of which gets canceled by parts of the other. The remaining contribution can be expressed in terms of J_2.

We see that in the skeleton diagrams, $\chi^{(1)}$–$\chi^{(3)}$, self-energy contributions and vertex corrections cancel each other. However, in the insertion diagrams, $\chi^{(6)}$ and $\chi^{(7)}$, the same cancellation is effective only in the spin singlet channel, while in the spin triplet channel the two diagrams add up. Interpreting the logarithmic divergence in J_1 as a $\ln|\mathbf{Q}|$ as explained above, we obtain for the particle-hole channel contribution to χ_s,

$$\chi_s^{p\text{-}h}=2N_F+2N_F(\Gamma_tN_F)^2\frac{4}{9}\left(\frac{\mathbf{Q}}{2k_F}\right)^2\ln(2k_F/|\mathbf{Q}|). \tag{3.9}$$

D. Particle-particle channel in $d=3$

We now turn our attention to the particle-particle channel. As can be seen from Sec. II A, diagrams (4)–(7) in Fig. 4 contribute. From Eqs. (3.3d) and (3.3g) it follows that the particle-particle channel contributions of diagrams (4) and (7) cancel each other, so we are left with $\chi^{(5)}$ and $\chi^{(6)}$. Expanding the functions $I_1^{(4)}$ and $I_2^{(4)}$, Eqs. (3.4h) and (3.4i), to order $\mathbf{Q}^2$ and doing the integrals, one finds that the leading logarithmic contributions to both $\chi^{(5)}$ and $\chi^{(6)}$ can be expressed in terms of a single integral,

$$I=\sum_q{}'\sum_k\left(\frac{\mathbf{k}\cdot\hat{\mathbf{Q}}}{m}\right)^2(G_k)^5G_{-k+q}\sum_p G_pG_{-p+q}. \tag{3.10}$$

Inspection of the integrand shows that the leading divergency in I is a logarithm squared, in contrast to the particle-hole channel, where the leading term is a simple logarithm. The reason is that $\Sigma_p G_p G_{-p+q}$ contains a term $\sim \ln|\mathbf{q}|$ for $\mathbf{q}\rightarrow 0$, which is just the usual BCS-type logarithm that is characteristic of the particle-particle channel. It also depends on an ultraviolet cutoff, since $\Sigma_p G_p G_{-p+q}$ does not exist in $d=3$ if the integration is extended to infinity. In conjunction with the other factor in the integrand of I, which is an algebraic function, this gives the leading behavior:

$$I \sim \int d\mathbf{q} \ln|\mathbf{q}| \int_0^\infty d\omega \frac{\mathbf{q}^2 - 3(\omega/v_F)^2}{[\mathbf{q}^2 + (\omega/v_F)^2]^3}. \tag{3.11}$$

While this diverges like $(\ln 0)^2$ by power counting, the prefactor of the divergency turns out to be zero since the frequency integral in Eq. (3.11) vanishes. This leads to the following conclusion for the particle-particle channel contribution to χ_s:

$$\chi_s^{p\text{-}p} = 2N_F + 2N_F(\Gamma_c N_F)^2 \{0 \times [\ln(2k_F/|\mathbf{Q}|)]^2 + O[\ln(2k_F/|\mathbf{Q}|)]\}. \tag{3.12}$$

Our method of expanding in powers of $\mathbf{Q}$, and extracting the prefactor of the ensuing singularity, works only for the *leading* nonanalytic contribution. With this method, therefore, the result that is expressed in Eq. (3.12) is all we can achieve. In order to determine the prefactor of the next-leading $\ln|\mathbf{Q}|$ term, one would have to keep a nonzero external wave number explicitly. As pointed out before in the context of the particle-particle channel, this would be very difficult. However, for our purposes this is not really necessary. We know that the interaction amplitudes in the particle-hole and particle-particle channels, respectively, are independent. Therefore, the particle-particle channel contribution cannot in general cancel the nonzero contribution from the particle-hole channel that we found in Sec. III C. What we have established is that the particle-particle channel is not more singular than the particle-hole channel, and for showing that the leading nonanalyticity in χ_s is $\ln|\mathbf{Q}|$ with a nonzero prefactor this is sufficient.

It should be pointed out that low-order perturbation theory probably overestimates the importance of the particle-particle channel. Usually, singularities in the particle-particle channel are logarithmically weaker than those in the particle-hole channel, since a BCS-type ladder resummation changes a $\ln x$ singularity into a $\ln\ln x$, and a x^y singularity into a $x^y/\ln x$. We expect this mechanism to work in the present problem, so the particle-particle channel singularities are probably in fact asymptotically negligible compared to the particle-hole channel ones. We also note that so far we have not really established that higher-order terms in the perturbation expansion cannot lead to stronger singularities than the ones we found at second order in the interaction amplitudes. This point will be further discussed in Sec. IV below.

E. Temperature dependence of $\chi_s(\mathbf{Q}=0)$

In the last two subsections we have established that χ_s in $d=3$ at $T=0$ does indeed have a nonanalytic contribution proportional to $\mathbf{Q}^2\ln|\mathbf{Q}|$. As we pointed out in the Introduction, in a Fermi liquid the wave number scales like frequency or temperature, and one would therefore naively expect a $T^2\ln T$ contribution to the homogeneous χ_s at $T>0$. This raises the question of whether our results are compatible with those of Carneiro and Pethick,[13] who did not find such a contribution. In order to clarify this, let us calculate $\chi_s(\mathbf{Q}=0,T)$ explicitly within our formalism. For the reasons explained in Sec. III D we restrict ourselves to the particle-hole channel, as did Ref. 13.

To this end, we put $\mathbf{Q}=0$ in Eqs. (3.4a)–(3.4d), and consider the temperature dependence of $\chi^{(1)}$–$\chi^{(3)}$, $\chi^{(6)}$, and $\chi^{(7)}$. The relevant integrals are of the structure,

$$\int d\mathbf{q}\mathbf{q}^2 T \sum_{i\Omega_n} f(\mathbf{q},i\Omega_n) g(\mathbf{q},i\Omega_n), \tag{3.13}$$

which are most conveniently done by using the spectral representation for the causal functions $f(\mathbf{q},i\Omega_n)$ and $g(\mathbf{q},i\Omega_n)$.[22] Simple considerations show that there is no $T^2\ln T$ term if both f and g are algebraic functions; only if at least one of them possesses a branch cut can such a nonanalyticity arise. This immediately rules out $\chi^{(3)}$, and the first and second contribution to $\chi^{(2)}$ and $\chi^{(7)}$, respectively, as sources for a $T^2\ln T$. The reason is that an explicit calculation of $J^{(3)}(q,Q=0)$, Eq. (3.4b), in the limit of small q shows that the only singularities in this function are poles. The same is true for $J_1^{(4)}(q,Q=0)$ and $J_2^{(4)}(q,Q=0)$, but $J^{(2)}(q)$, which is minus the Lindhard function, has a branch cut, and so all of the remaining terms potentially go like $T^2\ln T$.

Since again we are aiming only at logarithmic accuracy, we can replace $J_1^{(4)}(q,Q=0)$ and $J_2^{(4)}(q,Q=0)$ by low-frequency, long-wavelength expressions for which $J_2^{(4)}(q,Q=0) = -2J_1^{(4)}(q,Q=0)$. The contributions from $\chi^{(1)}$ and $\chi^{(2)}$ therefore cancel [remember that diagrams (1) and (2) in Fig. 4 carry multiplication factors 4 and 2, respectively]. The contributions from $\chi^{(6)}$ and $\chi^{(7)}$ can both be expressed in terms of an integral

$$J = \int d\mathbf{q}\mathbf{q}^2 T \sum_{i\Omega_n} J_1^{(4)}(q,Q=0) J^{(2)}(q). \tag{3.14}$$

In doing this integral one may encounter individual terms that go like $T^2\ln T$, but all of those terms cancel, and the leading T dependence of J is T^2. There hence is *no* $T^2\ln T$ contribution to χ_s in $d=3$.

This result agrees with the conclusion of Ref. 13, which reached it on the basis of Fermi-liquid theory. We disagree, however, with the assertion of that reference that within the framework of microscopic perturbation theory the absence of the $T^2\ln T$ is due to cancellations between vertex corrections and self-energies, and is hence a consequence of gauge invariance. What we find instead is that, for all diagrams in Fig. 4, the $T^2\ln T$ terms vanish individually. This is consistent with the result of Ref. 24. These authors calculated χ_s in paramagnon approximation, which in our language corresponds to taking only $\chi^{(6)}$ and $\chi^{(7)}$ into account, plus infinite resummations that contribute to higher orders in the interaction amplitudes. They reported the absence of $T^2\ln T$ terms in their calculation, rather than their cancellation between the two diagrams.

This absence of the expected nonanalytic T dependence in $d=3$ is somewhat accidental. This can be seen from the one-dimensional case, where, as pointed out in Sec. III B, there is a $\ln T$ contribution to the homogeneous spin susceptibility. The technical reason is that in $d=1$, integrands whose only singularities are poles do contribute to the $T^2 \ln T$ terms. Consequently, in $d=1$ T and Q are interchangeable in the logarithmic terms, while in $d=3$ they are not. Furthermore, the same types of integrals that lead to a $\ln T$ term in $d=1$ also contribute to a T^{d-1} nonanalyticity in $1<d<3$. In these dimensions we therefore expect to find

$$\chi_s^{p\text{-}h}(\mathbf{Q}=0)=2N_F+2N_F(\Gamma_t N_F)^2 c_d (T/4\epsilon_F)^{d-1}, \tag{3.15}$$

with c_d a d-dependent, positive number.

We also mention that the absence of a $T^2 \ln T$ term in the self-energy diagrams in $d=3$ does not contradict the presence of such a term in the specific-heat coefficient. The relation between the specific heat and the Green's function is intricate,[22] and the resulting integrals have a different structure from the ones that determine χ_s.

IV. DISCUSSION

A. Our results in a mode-mode coupling theory context

In this section we give a more detailed look at the mode-mode coupling arguments that were presented in the Introduction. We also stress some analogies between classical and quantum fluids, and discuss some important differences between clean and disordered systems.

Let us consider four distinct systems: (1) a classical Lorentz model (i.e., a classical particle moving in a spatially random array of scatterers[25]), (2) a classical fluid, (3) a Fermi liquid with static impurities, and (4) a clean Fermi liquid. These systems represent classical and quantum fluids with and without quenched disorder, respectively. As pointed out in the Introduction, dynamical correlations are ultimately responsible for all of the effects discussed in this paper. However, in classical systems they do not manifest themselves in static equilibrium properties, while in quantum systems they do. In order to discuss the analogies between classical and quantum systems, let us therefore digress and consider an equilibrium time correlation function. A convenient choice is the current-current correlation function, whose Fourier transform determines the frequency-dependent diffusivity $D(\Omega)$. In both of the classical systems, (1) and (2), this correlation function exhibits a long-time tail, so $D(\Omega)$ is nonanalytic at $\Omega=0$. For $\Omega\to 0$ one finds for the classical Lorentz model,

$$D(\Omega)/D(0)=1+ai\Omega+b(i\Omega)^{d/2}, \tag{4.1a}$$

while for the classical real fluid one finds

$$D(\Omega)/D(0)=1-b'(i\Omega)^{(d-2)/2}. \tag{4.1b}$$

The coefficients b and b' in Eqs. (4.1) are positive. The long-time tail in the real fluid is stronger than the one in the Lorentz gas because the former has more soft modes. More importantly, the static scatterers in the Lorentz gas lead to a sign of the effect that is different from the one in the real fluid. All of these features can be understood in terms of the number and the nature of the soft modes in these systems.[2] In disordered Fermi liquids[5] one has

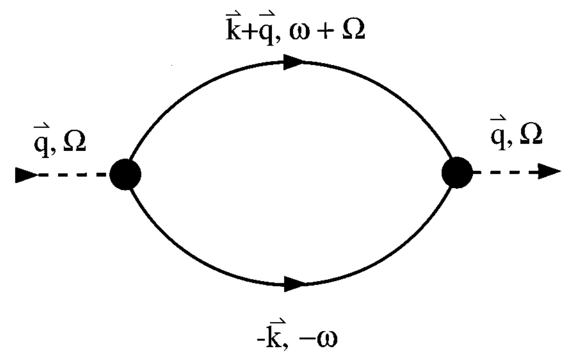

FIG. 5. Mode-mode coupling process describing the decay of a current mode (dashed line) into two sound modes (solid lines).

$$D(\Omega)/D(0)=1+b''(i\Omega)^{(d-2)/2}, \tag{4.1c}$$

with $b''>0$. Here the sign is the same as in the classical Lorentz model, which is due to the quenched disorder in either system. The strength of the long-time tail, however, is equal to that in the classical real fluid. As mentioned in Sec. I, the coupling of statics and dynamics in quantum statistical mechanics leads to a related nonanalyticity in the static spin susceptibility of a disordered Fermi liquid, namely,

$$\chi_s(\mathbf{Q})/\chi_s(0)=1-c|\mathbf{Q}|^{d-2}, \tag{4.2}$$

with $c>0$.

On the basis of these results, it is possible to predict both the strength of the singularity, and the sign of the prefactor, in the $\mathbf{Q}$ dependence of χ_s in a clean Fermi liquid, which is what we are mainly concerned with in this paper. In order to do so, let us recall the origin of the nonanalyticity in the classical fluid, Eq. (4.1b). The density excitation spectrum, i.e., the dynamical structure factor as measured in a light-scattering experiment, in a classical fluid consists of three main features: the Brillouin peaks that describe emission and absorption of sound waves, and the Rayleigh peak that describes heat diffusion. For our purposes, we focus on the former. In the density-density Kubo correlation function, $C(\mathbf{k},\omega)$ [whose spectrum is in a classical system simply proportional to the structure factor $S(\mathbf{k},\omega)$], they manifest themselves as simple poles,[3]

$$C(\mathbf{k},\omega)\sim\frac{1}{\omega-vk+i\gamma k^2/2}+\frac{1}{\omega+vk-i\gamma k^2/2} \equiv C_+(\mathbf{k},\omega)+C_-(\mathbf{k},\omega), \tag{4.3}$$

where v is the speed of sound, and γ is the sound attenuation constant. Now let us consider the simplest possible mode-mode coupling process that contributes to Eq. (4.1b), namely, one where a current mode decays into two sound modes that later recombine; see Fig. 5. Consider a process where one of the internal sound propagators is a C_+, and the other a C_-. At zero external wave number, this leads to a convolution integral,

$$\int d\omega \int d\mathbf{k} C_+(\mathbf{k},\omega) C_-(-\mathbf{k},-\omega+\Omega)$$

$$\sim \int d\mathbf{k} \frac{1}{\Omega + i\gamma k^2} \sim \Omega^{(d-2)/2}. \tag{4.4a}$$

Note that by this mechanism the long-time tail in a system whose low-lying modes have a linear dispersion becomes as strong as the one in a system with diffusive modes. In contrast, if both of the sound propagators are C_+ or C_-, one obtains a weaker singularity,

$$\int d\Omega \int d\mathbf{k} C_+(\mathbf{k},\omega) C_+(-\mathbf{k},-\omega+\Omega)$$

$$\sim \int d\mathbf{k} \frac{1}{\Omega - 2vk + i0} \sim \Omega^{(d-1)}. \tag{4.4b}$$

Now let us consider the corresponding quantum system, i.e., the clean Fermi liquid. Again, the low-lying modes (i.e., particle-hole excitations) have a linear dispersion. However, at zero temperature the structure factor and the Kubo function are no longer proportional to one another. Rather, the fluctuation dissipation theorem shows that they are related by a Bose distribution function that eliminates the pole at $\omega = ck$ from the structure factor. This is simply a consequence of the fact that at zero temperature there are no excitations that could get destroyed in a scattering process. Consequently, the process described by Eq. (4.4a) is not available in this system, and one is left with the weaker singularity of Eq. (4.4b). Since the diffusion coefficient is infinite at $T=0$ in a clean system, we look instead at the spin susceptibility as a function of $\mathbf{Q}$. $\mathbf{Q}$ scales like Ω, so we expect a singularity of the form $|\mathbf{Q}|^{d-1}$, as opposed to the $|\mathbf{Q}|^{d-2}$ in a disordered Fermi liquid, Eq. (4.2). The sign of the prefactor is determined by whether or not the system contains quenched disorder. It should therefore be opposite to the sign in the dirty case. We thus expect for the wave number dependence of the spin susceptibility in a clean Fermi liquid,

$$\chi_s(\mathbf{Q})/\chi_s(0) = 1 + c'|\mathbf{Q}|^{d-1}, \tag{4.5}$$

with $c' > 0$. This is precisely what we found in Sec. III by means of perturbation theory. Notice that the mode-mode coupling arguments suggest that the sign of the prefactor c' will be positive, regardless of the interaction strength, as is the sign of the long-time tail in a classical fluid. We will come back to this point in Sec. IV C below.

B. Our results in a renormalization-group context

Another useful way to look at our results is from a renormalization-group point of view. The Fermi-liquid ground state of interacting fermion systems in $d>1$ has recently been identified with a stable fixed point in renormalization-group treatments of both a basic fermion theory,[16] and a bosonized version of that theory.[26] The instability of the Fermi liquid in $d=1$ is reflected by an infinite number of marginal operators whose scale dimensions are proportional to $d-1$, i.e., they all become relevant in $d<1$, and are irrelevant in $d>1$. In the present context, the Fermi-liquid nature of the ground state in $d>1$ is reflected by the fact that the homogeneous spin susceptibility is finite in perturbation theory. The nonanalytic corrections at finite wave number that we are interested in correspond to the leading correction to scaling in the vicinity of the Fermi-liquid fixed point, i.e., to an irrelevant operator with respect to that fixed point. Among the irrelevant operators, there thus must be one whose scale dimension determines the leading wave-number dependence of the spin susceptibility.

An identification of this operator within the framework of a renormalization-group analysis would not only provide another derivation of our result, but would also establish that the behavior we have found in perturbation theory constitutes the leading $\mathbf{Q}$ dependence to *all* orders in the interaction amplitudes. This program has not been carried out yet, although preliminary results are encouraging.[17] This will provide a connection between the mode-mode coupling arguments presented in the previous subsection and renormalization-group arguments that will be analogous to a corresponding connection in classical fluids that has been known to exist for some time.[4]

In this context it should also be mentioned that there is no universal agreement that the ground state of a weakly interacting Fermi system in $d>1$ is a Fermi liquid. It has been proposed that there exists a relevant operator that makes the Fermi-liquid fixed point unstable, and leads to a non-Fermi-liquid ground state.[27] In order to destroy the Fermi liquid in d dimensions, this would require a long-range effective interaction that falls off more slowly than $1/r^d$ at large distances. While we do find an effective long-range interaction between the spin degrees of freedom, it falls off like $1/r^{2d-1}$, and hence leaves the Fermi-liquid fixed point intact. The same conclusion was reached in Ref. 11 from studying the specific heat in $d=2$.

C. Summary, and physical consequences of our result

We finally turn to a summary of our results, and to a discussion of their physical consequences. By means of explicit perturbative calculations to second order in the interaction, we have found that the wave-number-dependent spin susceptibility in $d=3$ has the form

$$\chi_s(\mathbf{Q})/\chi_s(\mathbf{Q}=0) = 1 + c_3(\mathbf{Q}/2k_F)^2 \ln(2k_F/|\mathbf{Q}|) + O(\mathbf{Q}^2). \tag{4.6a}$$

We have calculated the particle-hole channel contribution to the constant c_3, and have found it to be positive. More generally, it follows from our analysis that in d-dimensional systems, the spin susceptibility has a nonanalyticity of the form

$$\chi_s(\mathbf{Q})/\chi_s(\mathbf{Q}=0) = 1 + c_d(|\mathbf{Q}|/2k_F)^{d-1} + O(\mathbf{Q}^2), \tag{4.6b}$$

where the particle-hole channel contribution to c_d is again positive.

A very remarkable feature of Eqs. (4.6) is the sign of the leading $\mathbf{Q}$ dependence: For $d \leqslant 3$, χ_s *increases* with increasing $|\mathbf{Q}|$ like $|\mathbf{Q}|^{d-1}$. For any physical system for which this were the true asymptotic behavior at small $\mathbf{Q}$, this would have remarkable consequences for the zero-temperature phase transition from the paramagnetic to the ferromagnetic state as a function of the exchange coupling. One possibility is that the ground state of the system will not be ferromagnetic, irrespective of the strength of the spin triplet interaction, since the functional form of χ_s leads to the instability of any homogeneously magnetized ground state.[28] Instead, with increasing interaction strength, the system would undergo a transition from a paramagnetic Fermi liquid to some other type of magnetically ordered state, most likely a spin density wave. While there seems to be no observational evidence for this, let us point out that in $d=3$ the effect is only logarithmic, and would hence manifest itself only as a phase transition at exponentially small temperatures, and exponentially large length scales, that might well be unobservable. For $d\leqslant 2$, on the other hand, there is no long-range Heisenberg ferromagnetic order at finite temperatures, and the suggestion seems less exotic. Furthermore, any finite concentration of quenched impurities will reverse the sign of the leading nonanalyticity, and thus make a ferromagnetic ground state possible again.

Another possibility is that the zero-temperature paramagnet-to-ferromagnet transition is of first order. It has been shown in Ref. 9 that the nonanalyticity in $\chi_s(\mathbf{Q})$ leads to a similar nonanalyticity in the magnetic equation of state, which takes the form

$$tm - v_d m^d + u m^3 = h, \tag{4.7}$$

with m the magnetization, h the external magnetic field, and $u>0$ a positive coefficient. If the soft mode mechanism discussed above is the only mechanism that leads to nonanalyticities, then the sign of the remaining coefficient v in Eq. (4.7) should be the same as that of c_d in Eq. (4.6b), i.e., $v_d>0$. This would imply a first-order transition for $1<d<3$. In this case the length scale that in the previous paragraph would have been attributed to a spin density wave would instead be related to the critical radius for nucleation at the first-order phase transition. Further work will be necessary to decide between these possibilities.

The conclusion that there is no continuous zero-temperature paramagnet-to-ferromagnet transition is inescapable for any system with a particle-hole channel interaction that is sufficiently weak for our perturbative treatment to be directly applicable. An important question is now whether or not it holds more generally for systems whose interactions are in general not weak. There are four obvious mechanisms by which the sign of the leading $\mathbf{Q}$ dependence of χ_s could be switched from positive to negative: (1) higher-order contributions could lead to a sign of c_d for realistic interaction strengths that is different from the one for weak interactions, or (2) they might lead to a stronger singularity with a negative prefactor that constitutes the true long-wavelength asymptotic behavior, or (3) the particle-particle channel contribution might have a negative sign that overcompensates the positive contribution from the particle-hole channel, or (4) the higher angular momentum channels that we neglected might lead to a different sign. At this point, none of these possibilities can be ruled out mathematically. However, from a physical point of view none is very likely to occur. As we have explained in Sec. IV A, both the functional form and the sign of the nonanalyticity found in perturbation theory are in agreement with what one would expect on the basis of a suggestive analogy with classical fluids. Also, the renormalization-group arguments sketched in Sec. IV B make it appear likely that Eqs. (4.6) constitute the actual asymptotic small-$\mathbf{Q}$ behavior of χ_s, although an actual renormalization-group proof of this is still missing. This makes the first two possibilities appear unlikely. The third possibility is unappealing for two reasons. First, the effective interaction in the particle-particle channel is typically much weaker than the one in the particle-hole channel. The reason is the characteristic ladder resummation that occurs in the particle-particle channel if one goes to higher orders in perturbation theory. This leads to an effective interaction of the ''Coulomb pseudopotential'' type that is much weaker (typically by a factor of 5–10) than what low-order perturbation theory seems to suggest.[29] Second, that same resummation weakens any singularity (cf. the discussion at the end of Sec. III D), which probably makes the particle-particle channel singularity subleading. Finally, the higher angular momentum Fermi-liquid parameters are usually substantially smaller than the ones at $l=0$, which makes possibility (4) unlikely, except possibly in particular systems.

If the sign of the nonanalyticity is, for some reason, negative at the coupling strength necessary for a ferromagnetic transition to occur, at least in some systems, then in these systems the quantum phase transition from a paramagnet to a ferromagnet at zero temperature as a function of the interaction strength will be a conventional continuous quantum phase transition with an interesting critical behavior. This is because the nonanalyticity in χ_s leads to an effective long-range interaction between spin fluctuations, which in turn leads to critical behavior that is not mean-field-like, yet exactly solvable. This has been discussed recently in some detail.[9]

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[1] For a recent review, see, e.g., J. R. Dorfman, T. R. Kirkpatrick, and J. V. Sengers, Annu. Rev. Phys. Chem. **45**, 213 (1994).

[2] For a brief review see, e.g., T. R. Kirkpatrick and D. Belitz, J. Stat. Phys. (to be published).

[3] For an elementary discussion, see, e.g., D. Forster, *Hydrodynamic Fluctuations, Broken Symmetry, and Correlation Functions* (Benjamin, New York, 1975).

[4] D. Forster, D. R. Nelson, and M. J. Stephen, Phys. Rev. A **16**, 732 (1977).

[5] For a review, see, D. Belitz and T. R. Kirkpatrick, Rev. Mod. Phys. **66**, 261 (1994).

[6] T. R. Kirkpatrick and D. Belitz, Phys. Rev. B **53**, 14 364 (1996), and references therein.

[7] D. Belitz and T. R. Kirkpatrick, Europhys. Lett. **35**, 201 (1996).

[8] D. Belitz and T. R. Kirkpatrick, J. Phys. Condens. Matter **8**, 1 (1996).
[9] T. Vojta, D. Belitz, R. Narayanan, and T. R. Kirkpatrick, Europhys. Lett. **36**, 191 (1996); (to be published).
[10] See, e.g., G. Baym and C. Pethick, *Landau Fermi-Liquid Theory* (Wiley, New York, 1991).
[11] D. Coffey and K. S. Bedell, Phys. Rev. Lett. **71**, 1043 (1993).
[12] See the discussion and references in Ref. 13.
[13] G. M. Carneiro and C. J. Pethick, Phys. Rev. B **16**, 1933 (1977).
[14] I. E. Dzyaloshinskii and A. I. Larkin, Zh. Éksp. Teor. Fiz. **61**, 791 (1971) [Sov. Phys. JETP **34**, 422 (1972)].
[15] J. Sólyom, Adv. Phys. **28**, 201 (1979).
[16] R. Shankar, Rev. Mod. Phys. **66**, 129 (1994).
[17] T. R. Kirkpatrick and D. Belitz (unpublished).
[18] We use a field theoretic language in terms of Grassmann valued fields and their expectation values, see, e.g., Ref. 21. Equivalently, one can formulate the same theory in terms of operators and time ordered Green's functions, see, e.g., Ref. 22.
[19] See, e.g., A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinskii, *Methods of Quantum Field Theory in Statistical Physics* (Dover, New York, 1975), Sec. 18. Notice that the Cooper channel is neglected in Fermi-liquid theory.
[20] For charge neutrality, we assume a homogeneous, positively charged background. Therefore $v(\mathbf{q}=0)=0$.
[21] J. W. Negele and H. Orland, *Quantum Many-Particle Systems* (Addison-Wesley, New York, 1988).
[22] A. L. Fetter and D. L. Walecka, *Quantum Theory of Many-Particle Systems* (McGraw-Hill, New York, 1971).
[23] See, e.g., H. J. Schulz, in *Fermi Liquids and Non-Fermi Liquids*, in Proceedings of Les Houches Summer School LXI, edited by E. Akkermans, G. Montambaux, J. Pichard, and J. Zinn-Justin (Elsevier, Amsterdam, 1995), p. 533.
[24] M. T. Beal-Monod, S. K. Ma, and D. R. Fredkin, Phys. Rev. Lett. **20**, 929 (1968).
[25] For a discussion of Lorentz models, see, E. H. Hauge, in *Transport Phenomena*, Lecture Notes in Physics No. 31, edited by G. Kirczenow and J. Marro (Springer, New York, 1974), p. 337.
[26] H. J. Kwon, A. Houghton, and J. B. Marston, Phys. Rev. B **52**, 8002 (1995).
[27] P. W. Anderson, Rev. Math. Phys. **6**, 1085 (1995).
[28] One might argue that this conclusion cannot really be drawn, since critical fluctuations near a ferromagnetic phase transition might change the **Q** dependence of the two-point vertex function. However, it has been shown in Ref. 9 that any ferromagnetic transition in a system whose spin susceptibility far from criticality is given by Eq. (4.6b) is not driven by fluctuations, but characterized by a Gaussian fixed point. The functional form of the bare two-point vertex therefore is preserved even at criticality.
[29] P. Morel and P. W. Anderson, Phys. Rev. **125**, 1263 (1962).