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Transition to Turbulence, Small Disturbances, and Sensitivity Analysis II: The Navier-Stokes Equations

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Abstract

Recent research has shown that small disturbances in the linearized Navier-Stokes equations cause large energy growth in solutions. Although many researchers believe that this interaction triggers transition to turbulence in flow systems, the role of the nonlinearity in this process has not been thoroughly investigated. This paper is the second of a two part work in which sensitivity analysis is used to study the effects of small disturbances on the transition process. In the first part, sensitivity analysis was used to predict the effects of a small disturbance on solutions of a motivating problem, a highly sensitive one dimensional Burgers' equation. In this paper, we extend the analysis to study the effects of small disturbances on transition to turbulence in the three dimensional Navier-Stokes equations. We show that the change in a laminar flow with respect to small variations in the initial flow or small forcing acting on the system is large when the linearized operator is stable yet non-normal. In this case, the solution of the disturbed problem can be very large (and potentially turbulent) even if the disturbances are extremely small. We also give bounds on the disturbed flow in terms of certain constants associated with the linearized operator.

Key words: Navier-Stokes equations, non-normality, transition to turbulence, small disturbances, Fréchet differentiability, sensitivity analysis, sensitivity equations, semigroup theory

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1 Introduction

Predicting transition to turbulence is an important problem in fluid mechanics. As is well known, linear stability analysis often fails to predict transition for many simple flows [1]. Recent "mostly linear" transition scenarios have emphasized the importance of small disturbances in the transition process. Small disturbances were discovered to cause large energy growth in linearized flow systems [2–7]. Many researchers believe this interaction triggers transition even though the effects of small disturbances on the full nonlinear Navier-Stokes equations has not been fully investigated. The main reason for the focus on the linearized problem is that the nonlinear term in the Navier-Stokes equations conserves energy in many types of flow problems; thus, the linearized operator is solely responsible for any increase in energy in the flow system.

This is the second paper in a two part work on the use of sensitivity analysis to study the effects of small disturbances on transition to turbulence in flow systems. In the first part of this work ([8], hereafter referred to as Part I), sensitivity analysis was used to study "transition" in a model flow problem, a highly sensitive one dimensional Burgers' equation. Solutions of that problem are known to move an order of magnitude if there is a small disturbance in the boundary conditions. We used the continuous sensitivity equation method to differentiate the solution of the Burgers' equation with respect to the disturbance parameter. The derivatives (or sensitivities) were shown to predict the large change in the solution.

In this work, we use sensitivity analysis to study the effects of small disturbances on the transition process in the three dimensional Navier-Stokes equations. As discussed in more detail in Part I, researchers have found that small disturbances in the initial conditions and also small forcing can cause large energy amplification in the linearized equations. Therefore, in this paper, we examine these types of disturbances on solutions of the Navier-Stokes equations. We use sensitivity analysis to measure the change in a laminar flow with respect to these small disturbances. Specifically, we use the continuous sensitivity equation method to differentiate the laminar flow state with respect to the disturbance parameters. This method leads to linear differential equations for the sensitivities which can be used to gain information about the disturbed flow problem. We show that the change in the laminar flow with respect to small variations in the initial flow or small forcing acting on the system is large when the linearized operator is stable yet non-normal. The change can also be large when the linearized operator has spectrum near the imaginary axis. Furthermore, we show that a laminar flow state is more sensitive with respect to small forcing than a small deviation in the initial flow.

Expanding the flow in a Taylor series in the disturbance parameters shows that very small disturbances have the ability to cause transition in the full nonlinear flow system. In particular, we use the sensitivities to obtain rigorous estimates for the fluctuations $w(t; w_0, f)$ about a laminar flow as a function of a small initial fluctuation w_0 and a small forcing f that take the form

$$\|w(t; w_0, 0)\|_\alpha \leq e^{-\omega t} \sum_{n=1}^{\infty} \frac{1}{n!}\, c_n(t; \alpha, \omega) M^{2n-1} \|w_0\|_\alpha^n ,$$

$$\|w(t; 0, f)\|_\alpha \leq \sum_{n=1}^{\infty} \frac{1}{n!}\, d_n(t; \alpha, \omega) M^{2n-1} \|f\|_{H_\sigma}^n .$$

Here, the constant M is large if A is non-normal and the coefficients c_n and d_n are large if A has spectrum near the imaginary axis. The analysis extends the "mostly linear" transition scenarios by showing that very small disturbances have the potential to cause large energy growth in the velocity fluctuations in the full nonlinear Navier-Stokes system.

Remark: We do not prove that small variations to the laminar flow or small forcing trigger transition in every flow system. It is entirely possible that transition in a certain flow system is triggered by some other phenomenon not considered here. Rather, we demonstrate that these two disturbances studied here have the ability to cause transition in a flow system. Also, our results do not indicate the most likely form of disturbance that has the most potential to cause transition. However, we will comment later on using sensitivity analysis to predict whether a specific disturbance will trigger transition.

Although our main interest lies in the Navier-Stokes equations, these sensitivity analysis techniques can be used to study the effects of small disturbances on many other nonlinear systems. In particular, in [9, Chapter III] various equations of fluid dynamics are placed in a similar form to the fluctuation Navier-Stokes equations (14). The results obtained here may extend to these other flow scenarios and other nonlinear equations with a non-normal linearized operator.

Our analysis of the Navier-Stokes equations proceeds in a similar fashion to the study of Burgers' equation in Part I. We begin in Section 2 with an abstract semigroup formulation of the fluctuation Navier-Stokes equations. This formulation is used to prove the differentiability of the laminar flow with respect to the disturbance parameters in Section 3; equations for the sensitivities are also derived. The sensitivities are used to prove estimates on the size of the disturbed problem in Section 4. We close with conclusions and applications.

2 An Abstract Formulation of the Navier-Stokes Equations

As is standard, we take the incompressible Navier-Stokes equations as our model for fluid flow:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\,\vec{v} = -\nabla p + \frac{1}{R}\nabla^2 \vec{v} + f, \tag{1}$$

$$\nabla \cdot \vec{v} = 0. \tag{2}$$

Here, $\vec{v}$ is the flow velocity vector, p is the pressure, f is a forcing function, and the constant R is the Reynolds number. For simplicity, we consider the equations on a bounded open domain Ω in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$ and nonhomogeneous Dirichlet boundary conditions

$$\vec{v}(t, \vec{x}) = \vec{g}(\vec{x}), \quad \vec{x} \in \partial\Omega, \tag{3}$$

and a given initial flow

$$\vec{v}(0, \vec{x}) = \vec{v}_0(\vec{x}). \tag{4}$$

We suppose there exists a steady (i.e., time independent) flow $\vec{U}$ and a pressure state P that satisfy the Navier-Stokes equations exactly. For the remainder of this work, $\vec{U}$ will be referred to as the base flow. Relatively simple base flows such as Poiseuille, Couette, and Hagen-Poiseuille (pipe) flow are often studied in the literature.

Define the velocity and pressure fluctuations, $\vec{u}$ and q, by $\vec{v} = \vec{U} + \vec{u}$ and $p = P + q$. Substituting these relationships into the Navier-Stokes equations gives the fluctuation Navier-Stokes equations

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\nabla q + \frac{1}{R}\nabla^2 \vec{u} - (\vec{U} \cdot \nabla)\vec{u} - (\vec{u} \cdot \nabla)\vec{U}, \tag{5}$$

$$\nabla \cdot \vec{u} = 0, \tag{6}$$

$$\vec{u}(t, \vec{x}) = \vec{0}, \quad \vec{x} \in \partial\Omega, \tag{7}$$

$$\vec{u}(0, \vec{x}) = \vec{u}_0(\vec{x}) := \vec{v}_0(\vec{x}) - \vec{U}(\vec{x}). \tag{8}$$

Due to the homogeneous (or no-slip) boundary conditions for the fluctuations (7), the zero state is a solution to this problem for zero initial data. Also, the zero solution directly corresponds to the base flow solution $\vec{U}$ since $\vec{v} = \vec{U} + \vec{u}$. Thus, if the fluctuations do not remain small then the flow has transitioned to another state away from the base flow which could be turbulent.

The goal of this work is to determine if small variations to the base flow or small forcing acting on the flow system can cause the velocity fluctuations to become large. In order to do this, we examine the change in the base flow with respect to small initial fluctuations and small forcing. Since the base flow corresponds to the zero solution of the fluctuation Navier-Stokes equations, we differentiate the zero solution with respect to these small disturbances. To prove the differentiability of the zero solution, we rewrite the fluctuation Navier-Stokes equations as an abstract differential equation over an infinite dimensional Hilbert space of the form

$$\dot{w}(t) = Aw(t) + B(w(t), w(t)), \quad w(0) = w_0. \tag{9}$$

Here, A is the linearized operator and $B(w, w)$ is the quadratic nonlinear term. We use this formulation to compute the derivatives of the zero solution and obtain bounds on the solutions of the disturbed flow system.

The presentation given here primarily follows the abstract formulation of the Navier-Stokes equations given in [10, p. 79]. For other formulations and more complete details, see [11–14]. From now on, we drop the vector notation $(\vec{\ })$ unless needed for clarity.

We begin by setting notation. Let $L^2(\Omega)$ be the Hilbert space of square integrable vector functions over Ω with standard inner product

$$(u, v)_{L^2} = \int_\Omega u \cdot v \, dx$$

and corresponding energy norm $\|u\|_{L^2} = (u, u)_{L^2}^{1/2}$. Let H_σ be the Hilbert space of divergence free functions (with the L^2 inner product and norm) given by

$$H_\sigma = \left\{ u \in L^2(\Omega) \,:\, \nabla \cdot u = 0 \text{ in } \Omega, \;\; u \cdot n = 0 \text{ on } \partial\Omega \right\},$$

where n is the outward normal to the boundary. Define $H^m(\Omega)$ to be the Hilbert space of functions in $L^2(\Omega)$ with m distributional derivatives that are all square integrable. Let V be the Hilbert space

$$V = \left\{ u \in H_\sigma \,:\, u \in H^1(\Omega), \; u = 0 \text{ on } \partial\Omega \right\},$$

with inner product $(u, v)_V = \sum(\nabla u_i, \nabla v_i)$ and norm $\|u\|_V = (u, u)_V^{1/2}$.

Let Π be the orthogonal projection from $L^2(\Omega)$ onto the divergence free space H_σ. Formally projecting the fluctuation Navier-Stokes equations (5)-(8) onto H_σ eliminates the pressure gradient term and gives the ordinary differential equation (9) where A is the linear operator given by

$$Aw = \Pi\{R^{-1}\nabla^2 w - (U \cdot \nabla)w - (w \cdot \nabla)U\}, \tag{10}$$

and B is the bilinear operator

$$B(u, v) = -\Pi\{(u \cdot \nabla)v\}. \tag{11}$$

To make this formulation complete, we must specify the domains of the operators. It can be shown that the domain of A is given by

$$D(A) := \{\, u \in H_\sigma \; : \; Au \in H_\sigma \} = H^2 \cap V.$$

Furthermore, the linear operator A can be used to define fractional powers H_σ^α of the state space for $0 \le \alpha \le 1$ (see [10,15]) that satisfy

$$H_\sigma^0 = L^2(\Omega), \quad H_\sigma^{1/2} = V, \quad H_\sigma^1 = D(A).$$

We assume the base flow U is in $H^3(\Omega)$, so that the linear operator $-A$ is sectorial and A generates an analytic C_0-semigroup [13], which we denote e^{At}. This allows the nonlinear abstract differential equation (9) to be rewritten using the "variation of parameters" formula

$$w(t) = e^{At} w_0 + \int_0^t e^{A(t-\tau)} B(w(\tau), w(\tau))\, d\tau. \tag{12}$$

It is known that e^{At} maps H_σ into H_σ^α for any $\alpha \in [0, 1]$ and that

$$\|B(u, v)\| \le C \, \|u\|_\alpha \, \|v\|_\alpha \,, \tag{13}$$

for all $u, v \in H_\sigma^\alpha$ whenever $\alpha \in (3/4, 1)$. Here, $\|\cdot\|_\alpha$ is the norm on H_σ^α. Therefore, for α in this range, B is a continuous bilinear mapping from $H_\sigma^\alpha \times H_\sigma^\alpha$ into H_σ and the integral equation (12) holds in H_σ^α. Furthermore, the nonlinear differential and integral equations are equivalent formulations of the problem.

Thus, the fluctuation Navier-Stokes equations (5)-(8) is reformulated as the abstract differential equation (9) over H_σ^α, with $3/4 < \alpha < 1$, and the initial data w_0 is also taken in H_σ^α. Here, Aw is defined in (10) for any $w \in D(A) = H^2 \cap V$ and $B(w, w)$ is defined in (11) for any $w \in H_\sigma^\alpha$. The solution is given in terms of the above integral equation (12). Later, we consider the above fluctuation equation with small forcing, i.e.,

$$\dot{w}(t) = Aw(t) + B(w(t), w(t)) + f, \quad w(0) = w_0 \in H_\sigma^\alpha, \tag{14}$$

where $f \in H_\sigma$ is independent of time. A standard result in the theory of semilinear parabolic equations [10, Theorem 3.3.3] gives local existence of a unique solution.

Theorem 2.1 *Let $\alpha \in (3/4, 1)$, $w_0 \in H_\sigma^\alpha$, and $f \in H_\sigma$. There exists a $T = T(w_0, f) > 0$ such that the fluctuation Navier-Stokes equations (14) has a unique solution on $(0, T)$.*

Remark: In certain cases, it is known that the Navier-Stokes equations have a globally defined unique solution (i.e., $T = \infty$) when w_0 and f are small enough [12,13]. In particular, if A is stable and $f = 0$, then the zero solution is asymptotically stable and solutions must exist for all time and approach zero whenever w_0 is small enough. There are similar results when f is nonzero yet approaches zero sufficiently fast. In order to simplify the analysis in this work, we only consider the case where the forcing is independent of time. Therefore, whenever A is stable but f is nonzero, the solution may not exist for all time even if f is small in H_σ. It may be possible to estimate $T(w_0, f)$ for this case, but this is not the focus of the present work.

As mentioned in the introduction, the nonlinear term is conservative in the sense that

$$(B(u, u), u)_{H_\sigma} = 0$$

for all $u \in V$. This can be obtained by integrating by parts and using the no-slip boundary conditions (7) and the divergence-free condition (6). If we take the inner product of the differential equation (9) with the solution, we formally obtain

$$\frac{d}{dt}\frac{1}{2}\|w(t)\|_{H_\sigma}^2 = (Aw(t), w(t))_{H_\sigma}.$$

Therefore, the change in the velocity fluctuation energy is completely governed by the linear operator. If the operator A is non-normal (i.e., $AA^* \neq A^*A$, where A^* is the adjoint operator), then the quantity (Aw, w) can be positive and large even when A is stable (i.e., the spectrum of A is bounded away from the imaginary axis in the left half plane). Thus, even if the linear operator is stable, the velocity fluctuation energy has the potential to undergo significant transient growth and possibly transition to turbulence. This idea is the basic foundation of the "mostly linear" transition theories mentioned in the introduction. In this work, we use sensitivity analysis to extend these mostly linear transition scenarios by showing that small disturbances can cause large energy growth and possibly trigger transition in the full Navier-Stokes equations.

3 The Differentiability of the Base Flow With Respect to Initial Data and Forcing

Our method of using sensitivity analysis to study transition is similar to the approach used on the fluctuation Burgers' equation in Part I. In this case, we think of the solution $w(t)$ of the above disturbed fluctuation Navier-Stokes problem (14) as a function of the initial data w_0 and forcing f, i.e., $w(t) = w(t; w_0, f)$. If there is no initial flow (i.e., $w_0 = 0$) and no forcing (i.e., $f = 0$), then $w(t; 0, 0) = 0$ is the unique solution to this problem. The zero solution corresponds to the base flow of interest. We use sensitivity analysis to take Fréchet derivatives of the zero solution with respect to the initial data w_0 and the forcing f. This is done using the continuous sensitivity equation method; the fluctuation problem is differentiated with respect to the disturbance parameters leading to equations for the sensitivities. This procedure is made rigorous using the parameter differentiability theory summarized in Part I. We recall some results in this paper for convenience; however, for complete details and more background information, the reader is referred to Part I.

Our main tool is the sensitivity theory for semilinear parabolic problems found in Henry's book [10, Theorem 3.4.4 and Corollary 3.4.5].

Theorem 3.1 (Henry) *Suppose X and Y are Banach spaces, $-A$ is sectorial on X, $\alpha \in (0, 1)$, U is open in X^α, and Q is open in Y. Suppose also that $F : U \times Q \to X$ is k times continuously Fréchet differentiable or analytic over $U \times Q$. For $x_0 \in U$ and $q \in Q$, let $x = x(t; x_0, q)$ be the solution of*

$$\dot{x}(t) = Ax(t) + F(x(t); q), \quad x(0) = x_0, \tag{15}$$

on the interval $0 < t < T(x_0, q)$. Then on the interval $0 < t < T(x_0, q)$, $x(t; x_0, q)$ is k times continuously Fréchet differentiable or analytic with respect to x_0 and q as a mapping from $X^\alpha \times Q$ into X^α.

Since the solution $x(t; x_0, q)$ of the differential equation (15) satisfies the integral equation

$$x(t; x_0, q) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}F(x(\tau; x_0, q); q)\, d\tau, \tag{16}$$

an immediate consequence of this theorem is that equations can be derived for the derivatives of the solution with respect to the initial data, x_0, and parameter, q. Define the first order sensitivity operators $S_1(t) = D_{x_0}x(t; x_0, q)$ and $S_2(t) = D_q x(t; x_0, q)$, and sensitivities $s_1(t) = S_1(t)x_0$ and $s_2(t) = S_2(t)q$.

Corollary 3.1 (Henry) *Under the assumptions of Theorem 3.1, the sensitivities $s_1(t)$ and $s_2(t)$ satisfy the integral equations*

$$s_1(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}[D_xF(x(\tau;q);q)]s_1(\tau)\,d\tau,$$
$$s_2(t) = \int_0^t e^{A(t-\tau)}\Big([D_xF(x(\tau;q);q)]s_2(\tau) + [D_qF(x(\tau;q);q)]q\Big)\,d\tau,$$

and are mild solutions of the linear initial value problems

$$\dot{s_1}(t) = As_1(t) + [D_xF(x(t;q);q)]s_1(t), \quad s_1(0) = x_0.$$
$$\dot{s_2}(t) = As_2(t) + [D_xF(x(t;q);q)]s_2(t) + [D_qF(x(t;q);q)]q, \quad s_2(0) = 0.$$

Differentiating the integral equation (16) with respect to x_0 and q gives integral equations for higher order sensitivities.

Each higher order sensitivity satisfies an integral equation that directly corresponds to a linear differential equation; these differential sensitivity equations can be obtained by formally differentiating the original differential equation (15) with respect to the parameter (either x_0 or q), interchanging the order of differentiation, and using the chain rule.

Since the nonlinear term in the Navier-Stokes equations is derived from a continuous bilinear form, the nonlinear term $F(w) = B(w,w)$ is analytic (see Lemma 5.1 in Part I). An application of Theorem 3.1 shows that the solution of the fluctuation Navier-Stokes equations is analytic with respect to the initial data w_0 and forcing f.

Theorem 3.2 *Let $\alpha \in (3/4, 1)$ and suppose the assumptions in Section 2 are satisfied. Then there exists an open neighborhood $U \subset H^\alpha_\sigma \times H_\sigma$ about $w_0 = 0$ and $f = 0$ such that for any $(w_0, f) \in U$, the solution $w(t; w_0, f)$ of the fluctuation Navier-Stokes problem (14) is analytic as a function of the initial data w_0 and forcing f as long as it exists.*

As mentioned earlier, we have assumed the forcing $f \in H_\sigma$ is independent of time for simplicity. We note that Henry also used Theorem 3.1 to obtain the analyticity of the solution with respect to the initial data w_0 [10, p. 81] (see also [13, Theorem 5.2]).

The solution $w(t; w_0, f)$ of the disturbed fluctuation Navier-Stokes problem

(14) can be expressed as the solution of the integral equation

$$w(t; w_0, f) = e^{At} w_0 + \int_0^t e^{A(t-\tau)} \{B(w(\tau; w_0, f), w(\tau; w_0, f)) + f\} \, d\tau. \quad (17)$$

Corollary 3.1 can now be used to derive the sensitivity equations by differentiating through this integral equation with respect to w_0 and f. We use the following lemma to obtain the precise form of the derivatives of the nonlinear term with respect to w_0 and f. The lemma is an extension of the Leibniz rule (or generalized product rule)

$$\frac{d^n}{dx^n} f(x)g(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x), \qquad \binom{n}{k} = \frac{n!}{(n-k)!\,k!},$$

to the current situation. This result has also been used in [16, page 1428] to derive higher order sensitivity equations for the Korteweg-de Vries equation.

Lemma 3.1 *Let Q, U, and Y be Banach spaces and suppose $B : U \times U \to Y$ is a continuous bilinear form on U. If $F : Q \to Y$ is defined by $F(q) = B(u(q), u(q))$ and $u(q)$ is N times Fréchet differentiable, then F is N times Fréchet differentiable with respect to q, and*

$$[D_q^n F(q)] p^n = \sum_{k=0}^{n} \binom{n}{k} B(s_k, s_{n-k}) = \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} B'(s_k, s_{n-k})$$

for $n = 1, \ldots, N$. Here, $B'(u, v) = B(u, v) + B(v, u)$, p^n is the n-vector $(p, \ldots, p)$, $s_0 = u(q)$, and $s_k = [D_q^k u(q)] p^k$ for $k = 1, \ldots, N$.

Proof: It follows directly from the definition of the Fréchet derivative (see Part I) that $[D_u B(u, u)] v = B(u, v) + B(v, u) = B'(u, v)$. The proof follows by induction, the chain rule for Fréchet derivatives [17], and the identity

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

This lemma allows us to obtain the precise form of the sensitivity equations.

Corollary 3.2 *Assume the hypotheses of Theorem 3.2 are satisfied and suppose $w_0 \in H_\sigma^\alpha$ and $f \in H_\sigma$ are small enough. Let $w(t; w_0, f)$ be the solution of the fluctuation Navier-Stokes problem (14) on $0 < t < T(w_0, f)$. For any n, let w_0^n be the n-vector $(w_0, \ldots, w_0)$ and similarly for f^n. Then the sensitivities*

$$s_n(t) = [D_{w_0}^n w(t; 0, 0)] w_0^n, \quad p_n(t) = [D_f^n w(t; 0, 0)] f^n,$$

are defined on the intervals $(0, T(w_0, 0))$ and $(0, T(0, f))$, respectively, and they *are given by*

$$
\begin{aligned}
s_1(t) &= e^{At} w_0, \\
s_n(t) &= \int_0^t e^{A(t-\tau)} \sum_{k=1}^{n-1} \binom{n}{k} B(s_k(\tau), s_{n-k}(\tau))\, d\tau, \quad n \geq 2 \\
p_1(t) &= \int_0^t e^{A(t-\tau)} f\, d\tau, \\
p_n(t) &= \int_0^t e^{A(t-\tau)} \sum_{k=1}^{n-1} \binom{n}{k} B(p_k(\tau), p_{n-k}(\tau))\, d\tau, \quad n \geq 2.
\end{aligned}
$$

In particular, the first order sensitivity operators are given by

$$
D_{w_0} w(t;0,0) = e^{At}, \quad D_f w(t;0,0) = \int_0^t e^{A(t-\tau)}\, d\tau. \tag{18}
$$

The sensitivities are also mild solutions of the linear differential sensitivity equations

$$
\begin{aligned}
\dot{s}_1(t) &= As_1(t), \quad s_1(0) = w_0, \\
\dot{s}_n(t) &= As_n(t) + \sum_{k=1}^{n-1} \binom{n}{k} B(s_k(t), s_{n-k}(t)), \quad s_n(0) = 0, \quad n \geq 2, \\
\dot{p}_1(t) &= Ap_1(t) + f, \quad p_1(0) = 0, \\
\dot{p}_n(t) &= As_n(t) + \sum_{k=1}^{n-1} \binom{n}{k} B(p_k(t), p_{n-k}(t)), \quad p_n(0) = 0, \quad n \geq 2.
\end{aligned}
$$

Proof: Apply Corollary 3.1 and the Lemma to the integral equation (17). Note that the sums all begin at $k = 1$ and end at $k = n-1$ (instead of $k = 0$ and $k = n$, respectively) since $s_0 = p_0 = w(t;0,0) = 0$. □

This form of the first order sensitivity operators in (18) shows that the change in the zero solution (or the base flow) with respect to small variations in initial data or forcing can be large if e^{At} is large over some period of time. Also, if the spectrum of A is "close" to the imaginary axis, e^{At} will decay to zero very slowly as $t \to \infty$. In this case, the sensitivity to the initial data could be small, but the integral term could cause the sensitivity to forcing to become large. This is why the fluctuation Burgers' equation considered in Part I is extremely sensitive to small disturbances; the linearized operator is known to have an eigenvalue which is exponentially small compared to the constant μ [18]. Also, as with Burgers' equation, these first order sensitivities do not depend on the bilinear term B. Therefore, the linearized operator completely determines how the zero solution changes with respect to small initial data and forcing.

The higher order sensitivities provide more information as they did with the fluctuation Burgers' equation. Here are the first three initial data integral sensitivity equations:

$$
\begin{aligned}
s_1(t) &= e^{At} w_0, \\
s_2(t) &= 2 \int_0^t e^{A(t-\tau)} B(s_1(\tau), s_1(\tau))\, d\tau, \\
s_3(t) &= 3 \int_0^t e^{A(t-\tau)} \{B(s_1, s_2) + B(s_2, s_1)\}\, d\tau.
\end{aligned}
$$

The sensitivity equations for $p_n(t)$ are identical for $n \geq 2$ except they depend on the previous p sensitivities. As with Burgers' equation, the higher order sensitivities have the potential to be quite large since they depend on the operator e^{At} as well as the previous sensitivities (which also depend on e^{At}). The nonlinear term B also appears in these equations and "mixes" the previous sensitivities to form time dependent forcing functions in the linear differential sensitivity equations. This may be precisely the nonlinear mixing of energy thought to cause turbulence in the mostly linear transition scenarios discussed earlier. Since the sensitivities can be large, small variations in the initial flow and small forcing have the potential to trigger transition even when the linear operator is stable.

4 Estimating the Fluctuation Energy of the Disturbed Problem

The sensitivities can be used to obtain rough estimates of the size of the solution $w(t; w_0, f)$ of the disturbed fluctuation Navier-Stokes equations

$$\dot{w}(t) = Aw(t) + B(w(t), w(t)) + f, \quad w(0) = w_0.$$

We separate the effects of the initial data and forcing for comparison purposes. Since $w(t; w_0, f)$ is analytic in w_0 and f (see Theorem 3.2), we can expand the solution of the fluctuation problem in a Taylor series.

Corollary 4.1 *Assume the hypotheses of Theorem 3.2 and let $w_0 \in H_\sigma^\alpha$ and $f \in H_\sigma$. Also let $w(t; w_0, 0)$ and $w(t; 0, f)$ be the solutions of the disturbed fluctuation Navier-Stokes equations (14) with $f = 0$ and $w_0 = 0$, respectively. If w_0 and f are small enough, then*

$$
\begin{aligned}
w(t;w_0,0) &= w(t;0,0) + [D_{w_0}w(t;0,0)]w_0 + \frac{1}{2!}[D^2_{w_0}w(t;0,0)](w_0,w_0) + \cdots \\
&= s_1(t) + \frac{1}{2!}s_2(t) + \frac{1}{3!}s_3(t) + \cdots \\
w(t;0,f) &= w(t;0,0) + [D_f w(t;0,0)]f + \frac{1}{2!}[D^2_f w(t;0,0)](f,f) + \cdots \\
&= p_1(t) + \frac{1}{2!}p_2(t) + \frac{1}{3!}p_3(t) + \cdots
\end{aligned}
$$

as long as the solutions exist. The sensitivities $s_n(t)$ and $p_n(t)$ are defined in Corollary 3.2.

Note that we used $w(t;0,0) = 0$ above. Below, we estimate the size of the sensitivities in order to estimate the magnitude of the disturbed flows.

Fix α in $(3/4,1)$ and assume the operator A is stable. It is known [10,13] that there exist constants $M \geq 1$ and $\omega > 0$ such that

$$
\|e^{At}v\|_\alpha \leq Me^{-\omega t}\|v\|_\alpha, \quad \text{for any } v \in H^\alpha_\sigma, \tag{19}
$$

$$
\|e^{At}v\|_\alpha \leq Mt^{-\alpha}e^{-\omega t}\|v\|_{H_\sigma}, \quad \text{for any } v \in H_\sigma. \tag{20}
$$

Note that the norm of $e^{At}v$ must tend to zero since A is stable. The constant ω satisfies $0 < \omega < -\mathrm{Re}(\lambda)$ where λ is any point in the spectrum of A. Therefore, if the spectrum of A is near the imaginary axis, the constant ω will be small and therefore the magnitude of $e^{At}v$ may tend to zero very slowly. Also, it was discussed earlier that e^{At} may undergo significant transient growth before tending to zero. Therefore, the constant M may be quite large. Studies of the Orr-Sommerfeld/Squire operator (a transformation of the linearized Navier-Stokes operator) suggest that M is on the order of the Reynolds number for certain flow configurations [19,20]; the constant ω also becomes small as the Reynolds number increases [21].

If the constants M and ω can be accurately estimated, then one can bound the sensitivities and thus arrive at a bound for the solution of the disturbed fluctuation problem. See Chapter IV in [22] for various methods for estimating the magnitude of e^{At}. The following result shows that if M is large or if ω is small, then the sensitivities can be very large even though A is stable.

Theorem 4.1 *Assume the hypotheses of Theorem 3.2, fix $\alpha \in (3/4,1)$, and suppose A is stable so that there exist constants $M \geq 1$ and $\omega > 0$ such that (19) and (20) are satisfied. Let C be a positive constant satisfying $\|B(u,v)\|_{H_\sigma} \leq C\|u\|_\alpha \|v\|_\alpha$ for all $u, v \in H^\alpha_\sigma$.*

For $n \geq 1$, the sensitivities $s_n(t)$ and $p_n(t)$ defined in Corollary 3.2 satisfy

$$\|s_n(t)\|_\alpha \le e^{-\omega t} c_n(t;\alpha,\omega) M^{2n-1} \|w_0\|_\alpha^n, \quad \textit{for any } t \in (0,\infty),$$
$$\|p_n(t)\|_\alpha \le d_n(t;\alpha,\omega) M^{2n-1} \|f\|_{H_\sigma}^n, \quad \textit{for any } t \in (0,T),$$

where the solution $w(t;0,f)$ exists on the interval $(0,T)$. The positive coefficients $c_n(t;\alpha,\omega)$ and $d_n(t;\alpha,\omega)$ are defined recursively by

$$c_1(t;\alpha,\omega) = 1,$$
$$c_n(t;\alpha,\omega) = C \sum_{k=1}^{n-1} \binom{n}{k} \int_0^t (t-\tau)^{-\alpha} e^{-\omega\tau} c_k(\tau;\alpha,\omega) c_{n-k}(\tau;\alpha,\omega)\, d\tau, \quad n \ge 2,$$
$$d_1(t;\alpha,\omega) = \int_0^t (t-\tau)^{-\alpha} e^{-\omega(t-\tau)}\, d\tau,$$
$$d_n(t;\alpha,\omega) = C \sum_{k=1}^{n-1} \binom{n}{k} \int_0^t (t-\tau)^{-\alpha} e^{-\omega(t-\tau)} d_k(\tau;\alpha,\omega) d_{n-k}(\tau;\alpha,\omega)\, d\tau, \quad n \ge 2.$$

The coefficients can be bounded independently of t and T by

$$\sup_{t\ge 0} c_n(t;\alpha,\omega) \le \frac{(2n-2)!}{(n-1)!} C^{n-1} K_1(\alpha,\omega)^{n-1}, \quad n \ge 1,$$
$$K_1(\alpha,\omega) = \sup_{t\ge 0} \int_0^t (t-\tau)^{-\alpha} e^{-\omega\tau}\, d\tau < \infty,$$
$$\sup_{t\ge 0} d_n(t;\alpha,\omega) \le \frac{(2n-2)!}{(n-1)!} C^{n-1} K_2(\alpha,\omega)^{2n-1}, \quad n \ge 1,$$
$$K_2(\alpha,\omega) = \sup_{t\ge 0} \int_0^t (t-\tau)^{-\alpha} e^{-\omega(t-\tau)}\, d\tau < \infty.$$

Proof: Since A is stable, for small enough $w_0 \in H_\sigma^\alpha$ the solution $w(t;w_0,0)$ must exist for all $t > 0$ and tend to zero as $t \to \infty$ (see [10, Chapter 5] and [15, Sections 6.4 and 6.6]). Corollary 3.2 then shows that the sensitivities $s_n(t)$ exist for all $t > 0$.

The time dependent bounds on the sensitivities follow by induction using the integral sensitivity equations in Corollary 3.2, the inequalities (19) and (20), and the continuity of the bilinear form B. The time independent bounds on the coefficients c_n and d_n are also proved by induction. First, one can prove that $K_1(\alpha,\omega)$ and $K_2(\alpha,\omega)$ are finite using the change of variable $u = t - \tau$ and splitting the integrals up over the intervals $(0,1)$ and $(1,t)$. Then, it is easily shown that for $n \geq 1$,

$$c_n(t;\alpha,\omega) \leq \gamma_n C^{n-1} K_1(\alpha,\omega)^{n-1}, \quad d_n(t;\alpha,\omega) \leq \gamma_n C^{n-1} K_2(\alpha,\omega)^{2n-1},$$

where the constants γ_n satisfy the recursion

$$\gamma_1 = 1, \quad \gamma_n = \sum_{k=1}^{n-1} \binom{n}{k} \gamma_k \gamma_{n-k}, \quad n \geq 2.$$

One can prove that $\gamma_n = (2n-2)!/(n-1)!$ by relating this sequence to the Catalan numbers which are known to satisfy a similar recursion formula (see sequences A001813 and A000108 in [23]). □

In Corollary 4.1, the solutions $w(t;w_0,0)$ and $w(t;0,f)$ of the disturbed fluctuation Navier-Stokes problem (14) were expanded in a Taylor series of sensitivities. With these bounds on the sensitivities, the Taylor series representation gives the following estimates of the magnitude of the disturbed flows.

Corollary 4.2 *Assume the hypotheses of Theorem 4.1 and fix* α *in* $(3/4, 1)$. *If* $w_0 \in H_\sigma^\alpha$ *and* $f \in H_\sigma$ *are small enough, then*

$$\|w(t;w_0,0)\|_\alpha \leq e^{-\omega t} \sum_{n=1}^{\infty} \frac{1}{n!}\, c_n(t;\alpha,\omega) M^{2n-1} \|w_0\|_\alpha^n, \quad \text{for any } t \in (0,\infty),$$

$$\|w(t;0,f)\|_\alpha \leq \sum_{n=1}^{\infty} \frac{1}{n!}\, d_n(t;\alpha,\omega) M^{2n-1} \|f\|_{H_\sigma}^n, \quad \text{for any } t \in (0,T).$$

The constants M *and* ω *and the coefficients* c_n *and* d_n *are defined in Theorem 4.1.*

These bounds are worst-case estimates as it is unknown whether a certain disturbance w_0 or f will cause the norm of $w(t;w_0,0)$ or $w(t;0,f)$, respectively, to attain these bounds. However, the solutions of the disturbed problem still have the potential to be quite large when M is large or ω is small. If we ignore the coefficients $c_n(t;\alpha,\omega)/n!$ and $d_n(t;\alpha,\omega)/n!$, it seems reasonable to conjecture that disturbances w_0 and f may cause transition if their norms are larger than $O(M^{-\gamma})$ for some $\gamma > 2$. This is similar to a conjecture made in [20]. Again, the constant M has been estimated to be on the order of the Reynolds number for certain flow configurations. Therefore, when the Reynolds number is large, extremely small disturbances have the potential to trigger transition.

Notice however that Theorem 4.1 shows the coefficients c_n and d_n may not be negligible if ω is small. In particular, if ω is small the $e^{-\omega t}$ terms in the integrals will decay to zero very slowly. This will cause the coefficients to be large. This shows that the zero solution of the fluctuation Navier-Stokes equations will become more sensitive to disturbances as the spectrum of the linearized operator approaches the imaginary axis. Thus, flows that are near to being unstable in the classical linearized (spectrum) sense can be highly sensitive to disturbances.

The constants M and $K_2(\alpha,\omega)$ are critical in the bounds on the sensitivities and the disturbed flows. Both of these constants appear to the power of $2n-1$ while the norms of the disturbances w_0 and f are raised to the n^{th} power. Therefore, if one or both of M and K_2 are much larger than one, the terms M^{2n-1} and K_2^{2n-1} will dominate the n^{th} powers of the small disturbances. Thus, even extremely small disturbances could cause large growth in the fluctuations and trigger transition.

Remark: A comparison of the bounds for the solutions $w(t;w_0,0)$ and $w(t;0,f)$ shows that the main difference is in the coefficients $c_n(t;\alpha,\omega)$ and $d_n(t;\alpha,\omega)$ and the factor of $e^{-\omega t}$ in the bound for $w(t;w_0,0)$. The time independent bounds on these coefficients in Theorem 4.1 show that d_n will be larger than c_n due to the factor of $K_2(\alpha,\omega)^{2n-1}$ appearing in the bound for d_n as opposed to $K_1(\alpha,\omega)^{n-1}$ for c_n. The factor of $e^{-\omega t}$ will also cause the bound for $w(t;w_0,0)$ to be smaller than the bound for $w(t;0,f)$. Therefore, even if the disturbances w_0 and f have the same magnitude, the solution $w(t;0,f)$ of the problem with small forcing has the potential to be larger than the solution $w(t;w_0,0)$ of the problem with small initial data. In this way, the flow is more sensitive to small forcing acting on the flow system than small variations in the base flow.

4.1 Possible Extensions

In this paper, we made certain assumptions on the Navier-Stokes problem to simplify the analysis. For instance, we assumed the base flow is independent of time, the flow domain is bounded with smooth boundary, and that the disturbances in initial data $w_0 \in H_\sigma^\alpha$ and forcing $f \in H_\sigma$ are somewhat smooth. The differentiability of the zero solution of the fluctuation Navier-Stokes equations can be extended to some other cases with relative ease. For example, if the forcing function or base flow is time dependent (leading to a time dependent linear operator in the latter case), one can apply the more general parameter differentiability theory in [10,24] to establish similar results to those obtained here. For more general disturbances such as $w_0 \in H_\sigma$ and $f \in L^2(0,\infty;V')$, where V' is the dual of V, it would be necessary to examine weaker solutions of the fluctuation Navier-Stokes equations. The author is not aware of any differentiability theory for this case. However, if we proceed in a formal manner, we obtain weak versions of the differential sensitivity equations derived above. Therefore, at least formally, the same reasoning presented here would apply to this situation.

5 Conclusion

In this work, we used sensitivity analysis to study the effects of small disturbances on the Navier-Stokes equations. We showed that a small variation in the base flow or a small forcing disturbance acting on the flow system has great potential to trigger transition when the linearized operator is stable but non-normal. More specifically, the disturbed flow can be very different from the base flow whenever the C_0-semigroup e^{At} is large or the spectrum of the linearized operator A is near the imaginary axis. In this way, the results presented here agree with the recently proposed "mostly linear" transition scenarios that depend on the non-normality of the linearized operator. We also used sensitivity analysis to give estimates of the magnitude of the disturbed flows in terms of bounds on e^{At}. If these bounds on e^{At} can be accurately computed, one could estimate the magnitude of disturbances that can trigger transition in the fully nonlinear flow system.

Since the C_0-semigroup e^{At} exercises great influence on the solution of the disturbed flow system, linear feedback control has the potential to be an effective method of delaying transition in a nonlinear flow system. A standard linear feedback control such as LQG or H^∞ control would reduce the size of e^{At} thus reducing the magnitude of the disturbed flow state (see also the recent theoretical results in [25–27]). It may be beneficial to design the control in a specific way in order to minimize the size of the disturbed fluctuations. Furthermore, it is also possible that a linear feedback control could be given more control authority over the nonlinear system by somehow using sensitivity information in the control design.

Since small disturbances have the potential to trigger transition, work must be done to identify and model realistic disturbances in flow systems. The disturbances to the system could be physical disturbances (such as small wall roughness) or neglected dynamics ("small" terms discarded in the modeling process). If this is done, one can efficiently estimate the effect of a particular disturbance on the flow by computing the sensitivities of the flow with respect to that disturbance. This procedure was shown to be effective in part one of this work with Burgers' equation. This method could be used to investigate whether a certain small disturbance is essential to the flow model. If the specific cause of transition is identified, it is possible that controllers could be developed to minimize the effects of the disturbance and therefore delay or possibly even eliminate transition.

As mentioned in the introduction, our approach to studying the effects of small disturbances on fluid flow does not give an indication of a particular shape of a disturbance that is "most likely" to trigger transition. Again, one may be able to use information about such a disturbance to design flow control strategies.

Finding this type of disturbance is an interesting open question and will be the subject of future research.

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