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# Periodic Solutions of Functional Dynamic Equations with Infinite Delay

Li Bi

Meng Fan

Martin Bohner
*Missouri University of Science and Technology*, bohner@mst.edu

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Li Bi*, Martin Bohner**, and Meng Fan*

* School of Mathematics and Statistics, KLAS, and KLVE

Northeast Normal University, Changchun, Jilin, 130024, P. R. China

** Department of Mathematics and Statistics

University of Missouri–Rolla, Rolla, MO 65401, USA

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**Abstract**

In this paper, sufficient criteria are established for the existence of periodic solutions of some functional dynamic equations with infinite delays on time scales, which generalize and incorporate as special cases many known results for differential equations and for difference equations when the time scale is the set of the real numbers or the integers, respectively. The approach is mainly based on the Krasnosel′skiĭ fixed point theorem, which has been extensively applied in studying existence problems in differential equations and difference equations but rarely applied in dynamic equations on time scales. This study shows that one can unify such existence studies in the sense of dynamic equations on general time scales.

**Key words:** periodic solution, time scale, functional dynamic equation, infinite delay

## 1 Introduction

In the real world, some processes vary continuously while others vary discretely. These processes can be modeled by differential equations and difference equations, respectively. There are also many processes that vary both continuously and discretely. Thus an interesting and challenging problem arises for mathematicians: How can we model these mixed processes? The erratic stop-start of the real world has long defeated mathematicians. Now the theory of time scale calculus and dynamic equations on time scales provides us a powerful tool to attack such mixed processes [20]. For example, time scales are believed to be a good way to understand and control the West Nile virus since time scale calculus and dynamic equations on time scales bridge the divide between discrete and continuous aspects of West Nile [20]. In addition, the choice of time scale is very important in real world applications (see

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e.g., [7, 8, 9]). The calculus on time scales (see [4, 5] and references cited therein) was initiated by Stefan Hilger in his 1988 PhD dissertation [12] in order to unify continuous and discrete analysis, and it has a tremendous potential for applications and has recently received much attention since this foundational work. The two main features of the calculus on time scales are unification and extension.

The global existence of periodic solutions of differential equations and difference equations is a very basic and important problem, which plays a similar rôle as a globally stable equilibrium does in an autonomous model. Thus, it is reasonable to seek conditions under which the resulting periodic nonautonomous system would have a periodic solution. Much progress has been seen in this direction and many criteria are established based on different approaches (e.g., differential equations [6, 15, 17, 19, 22, 23, 24], difference equations [10, 14, 16, 18, 25]). Careful investigation reveals that it is similar to explore the existence of periodic solutions for nonautonomous differential equations and their discrete analogue in the approaches, the methods and the main results. For example, extensive research shows that many results concerning the existence of periodic solutions of differential equations can be carried over to their discrete analogues ([2, 3] and references cited therein, [14, 15, 16, 17, 24, 25]). It is natural to ask whether we can explore such an existence problem in a unified way and offer more general conclusions.

Scholars in the fields of differential equations and difference equations have long been aware of the startling similarities and intriguing differences between the two fields. Many results concerning differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be completely different from their continuous counterparts. The study of dynamic equations on general time scales unifies and extends the fields of differential and difference equations, highlighting the similarities and providing insight into some of the differences. To prove a result for a dynamic equation on a time scale is not only related to the set of real numbers or set of integers but those pertaining to more general time scales. For example, extensive research reveals that many results concerning the existence of periodic solutions of predator–prey systems modeled by differential equations can be carried over to their discrete analogues based on coincidence theory. Based on this fact, with the help of the theory of dynamic equations on time scales and a continuation theorem in coincidence degree theory, Bohner, Fan and Zhang [2, 3] systematically unified the existence of periodic solutions of population models modeled by ordinary differential equations and their discrete analogues in form of difference equations and extended these results to more general time scales.

Only a few papers have studied the existence of solutions of periodic boundary value problems of some dynamic equations on time scales (see, e.g., [1, 21]). Among the known results on the existence of periodic solutions of differential equations and difference equations, many are achieved based on the Krasnosel′skiĭ fixed point theorem. The theoretical evidence suggests that many results of the discrete systems are similar to those of the corresponding continuous systems based on the Krasnosel′skiĭ fixed point theorem (e.g., [14, 15, 16, 17, 24, 25]), which motivates us to consider that whether we can unify these results and extend them to more general time scales. Although the Krasnosel′skiĭ fixed point theorem has been proved to be powerful and effective in dealing with existence problems and has been applied widely to study the existence of periodic solutions of differential equations and difference equations, it is rarely used to explore the existence of periodic solutions of dynamic equations on time scales, especially those with infinite delays. In this paper, we will systematically investigate the existence of periodic solutions of some dynamic equations with infinite delay on time scales, which will unify some related studies of such problems for differential equations and difference equations. The approach is based on the Krasnosel′skiĭ fixed point theorem.

The setup of this paper is as following. In the coming section, we present some preliminary results on the calculus on time scales. In Section 3, we establish a $C_h$ space for the functional dynamic equations with infinite delay on time scales. Then, we present some preliminary results and the Krasnosel′skiĭ fixed point theorem. In the rest of this paper, we systematically explore the existence of periodic solutions of some dynamic equations with infinite delay on time scales, which incorporate as special cases many well-known models in population dynamics, hematopoiesis, etc. This study reveals that, when we deal with the existence of positive periodic solutions of differential equations and difference equations, it is unnecessary to prove results for differential equations and separately again for their discrete analogues (difference equations). One can unify such problems in the framework of dynamic equations on time scales.

## 2 Preliminaries on Time Scales

In this section, first we will mention without proof several foundational definitions and results from the calculus on time scales so that the paper is self-contained. For more details, one can see [4, 5].

A *time scale*, which is a special case of a measure chain, is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Throughout this paper, we will denote a time scale by the symbol $\mathbb{T}$, which has the topology inherited from the real numbers with the standard topology.

Let $\mathbb{R}^+ = [0, \infty)$, $\mathbb{R}^- = (-\infty, 0]$ and $a, b \in \mathbb{T}$, and define the intervals in $\mathbb{T}$ by

$$\mathbb{T}^- = \mathbb{T} \cap \mathbb{R}^-, \quad \mathbb{T}^+ = \mathbb{T} \cap \mathbb{R}^+, \quad (a, b) = \{t \in \mathbb{T} : \ a < t < b\}, \quad [a, b] = \{t \in \mathbb{T} : \ a \leq t \leq b\}.$$

Other intervals are defined accordingly.

**Definition 2.1.** Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$, we define the *forward and backward jump operators* $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : \ s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} : \ s < t\}.$$

If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then $t$ is called *right-dense* (otherwise: *right-scattered*), and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then $t$ is called *left-dense* (otherwise: *left-scattered*). The graininess function $\mu : \mathbb{T} \to \mathbb{R}^+$ is defined by $\mu(t) = \sigma(t) - t$.

**Definition 2.2.** A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{\rm rd} = C_{\rm rd}(\mathbb{T}) = C_{\rm rd}(\mathbb{T}, \mathbb{R})$. The set of all bounded rd-continuous functions is denoted by $\mathrm{BC_{rd}}$. A function $f : \mathbb{T} \to \mathbb{R}$ is called *regulated* provided its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.

**Lemma 2.1.** *Assume $f : \mathbb{T} \to \mathbb{R}$.*

(i) *If $f$ is continuous, then $f$ is rd-continuous.*

(ii) *If $f$ is rd-continuous, then $f$ is regulated.*

(iii) *Assume $f$ is continuous. If $g : \mathbb{T} \to \mathbb{R}$ is regulated or rd-continuous, then $f \circ g$ has that property too.*

(iv) *Every regulated function on a compact interval is bounded.*

**Definition 2.3.** Assume that $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that, for any given $\varepsilon > 0$, there is a neighborhood $U$ of $t$ (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)|\sigma(t) - s|| \le \varepsilon|\sigma(t) - s| \quad \text{for all} \quad s \in U.$$

We call $f^\Delta(t)$ the *delta* (or *Hilger*) *derivative* of $f$ at $t$. If $F^\Delta(t) = f(t)$, then we define the *Cauchy integral* by

$$\int_r^s f(\tau)\Delta\tau = F(s) - F(r) \quad \text{for} \quad r, s \in \mathbb{T}.$$

**Lemma 2.2.** *If $t \in \mathbb{T}^\kappa$ and $f : \mathbb{T} \to \mathbb{R}$ is delta differentiable at $t$, then*

$$f^\sigma(t) = f(t) + \mu(t) f^\Delta(t), \quad \textit{where} \quad f^\sigma = f \circ \sigma.$$

**Lemma 2.3.** *If $f \in \mathrm{C_{rd}}$ and $t \in \mathbb{T}^\kappa$, then* $\displaystyle\int_t^{\sigma(t)} f(s)\Delta s = \mu(t) f(t).$

**Lemma 2.4.** *If $a, b, c \in \mathbb{T}$ and $f \in \mathrm{C_{rd}}$, then*

(i) $\displaystyle\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t;$

(ii) *if $|f(t)| \le g(t)$ for all $t \in [a, b)$, then* $\displaystyle\left|\int_a^b f(t)\Delta t\right| \le \int_a^b g(t)\Delta t;$

(iii) *if $f(t) \ge 0$ for all $a \le t < b$, then* $\displaystyle\int_a^b f(t)\Delta t \ge 0.$

**Definition 2.4.** If $a \in \mathbb{T}$, $\sup \mathbb{T} = \infty$, and $f$ is rd-continuous on $[a, \infty)$, then we define the *improper integral* by

$$\int_a^\infty f(t)\Delta t := \lim_{b\to\infty} \int_a^b f(t)\Delta t$$

provided this limit exists, and we say that the improper integral *converges* in this case. If this limit does not exist, then we say that the improper integral *diverges*.

**Lemma 2.5.** *If $f, g : \mathbb{T} \to \mathbb{R}$ are delta differentiable at $t \in \mathbb{T}^\kappa$, then*

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g^\sigma(t).$$

**Lemma 2.6.** *Every rd-continuous function has an antiderivative. In particular, if $t_0 \in \mathbb{T}$, then $F$ defined by $F(t) := \int_{t_0}^t f(\tau)\Delta\tau$ for all $t \in \mathbb{T}$ is an antiderivative of $f$, i.e., $F^\Delta = f$.*

**Definition 2.5.** A function $r : \mathbb{T} \to \mathbb{R}$ is called *regressive* provided

$$1 + \mu(t)r(t) \neq 0 \quad \text{for all} \quad t \in \mathbb{T}^\kappa.$$

The set of all regressive and rd-continuous functions will be denoted by $\mathcal{R}$.

**Definition 2.6.** We define the set $\mathcal{R}^+$ of all positively regressive elements of $\mathcal{R}$ by

$$\mathcal{R}^+ = \{p \in \mathcal{R} :\ 1 + \mu(t)p(t) > 0 \text{ for all } t \in T\}.$$

**Definition 2.7.** If $p \in \mathcal{R}$, then the delta exponential function $e_p(\cdot, s)$ is defined as the unique solution of the initial value problem

$$y^\Delta = p(t)y, \quad y(s) = 1,$$

where $s \in \mathbb{T}$. Furthermore, for $p, q \in \mathcal{R}$, we also define

$$p \oplus q = p + q + \mu pq \quad \text{and} \quad p \ominus q = -\frac{p - q}{1 + \mu q}.$$

**Lemma 2.7.** *If $p, q \in \mathcal{R}$, then*

$$e_p(t,t) = 1, \quad e_p(t,s) = 1/e_p(s,t), \quad e_p(t,u)e_p(u,s) = e_p(t,s),$$
$$e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t,s), \quad e_p(s, \sigma(t)) = \frac{e_p(s,t)}{1 + \mu(t)p(t)}$$
$$e_p^\Delta(\cdot, s) = pe_p(\cdot, s), \quad e_p^\Delta(s, \cdot) = (\ominus p)e_p(s, \cdot),$$
$$e_{p\oplus q} = e_p e_q, \quad e_{p\ominus q} = \frac{e_p}{e_q}.$$

**Lemma 2.8.** *If $p \in \mathcal{R}^+$ and $t_0 \in \mathbb{T}$, then $e_p(t, t_0) > 0$ for all $t \in \mathbb{T}$.*

**Definition 2.8.** Let $\omega > 0$. A time scale $\mathbb{T}$ is called *$\omega$-periodic* if $t + \omega \in \mathbb{T}$ whenever $t \in \mathbb{T}$. We also write $\mathbb{T} + \omega \subset \mathbb{T}$, i.e., $\{t + \omega : t \in \mathbb{T}\} \subset \mathbb{T}$. A function $p$ is said to be *$\omega$-periodic* on $\mathbb{T}$ if $p(t + \omega) = p(t)$ for all $t \in \mathbb{T}$.

**Theorem 2.1.** *Suppose $\mathbb{T}$ is $\omega$-periodic, $p \in \mathrm{C_{rd}}(\mathbb{T})$ is $\omega$-periodic, and $a, b \in \mathbb{T}$. Then*

$$\sigma(t+\omega) = \sigma(t) + \omega, \quad \rho(t+\omega) = \rho(t) + \omega, \quad \mu(t+\omega) = \mu(t),$$
$$\int_{a+\omega}^{b+\omega} p(t)\Delta t = \int_a^b p(t)\Delta t, \quad e_p(b,a) = e_p(b+\omega, a+\omega) \text{ if } p \in \mathcal{R},$$
$$k_p := e_p(t+\omega, t) - 1 \quad \text{is independent of} \quad t \in \mathbb{T} \quad \text{whenever} \quad p \in \mathcal{R}.$$

*Proof.* The first two statements follow from $\mathbb{T} + \omega = \mathbb{T}$. Next,

$$\mu(t+\omega) = \sigma(t+\omega) - (t+\omega) = \sigma(t) + \omega - t - \omega = \sigma(t) - t = \mu(t).$$

If $p \in \mathrm{C_{rd}}$ is $\omega$-periodic on $\mathbb{T}$, then we use [4, Theorem 1.98] with $\nu(t) = t + \omega$ so that $\nu$ is strictly increasing and satisfies $\nu^\Delta(t) = 1$ and $\nu(\mathbb{T}) = \mathbb{T} + \omega = \mathbb{T} =: \tilde{\mathbb{T}}$. Hence, using [4, Theorem 1.98],

$$\begin{aligned} \int_a^b p(t)\Delta t &= \int_a^b p(t)\nu^\Delta(t)\Delta t = \int_{\nu(a)}^{\nu(b)} (p \circ \nu^{-1})(s)\tilde{\Delta} s \\ &= \int_{a+\omega}^{b+\omega} p(s-\omega)\Delta s = \int_{a+\omega}^{b+\omega} p(s)\Delta s. \end{aligned}$$

Next, if $p \in \mathcal{R}$, we use the representation of the exponential function in terms of the cylinder transform $\xi$ [4, Definition 2.30] to conclude

$$e_p(b,a) = \exp\left\{\int_a^b \xi_{\mu(t)}(p(t))\Delta t\right\} = \exp\left\{\int_{a+\omega}^{b+\omega} \xi_{\mu(t)}(p(t))\Delta t\right\} = e_p(b+\omega, a+\omega).$$

Finally, let $p \in \mathcal{R}$ and define $f(t) := e_p(t+\omega, t)$ for $t \in \mathbb{T}$. Let $t_0 \in \mathbb{T}$. Then

$$f(t) = e_p(t+\omega, t_0)e_p(t_0, t) = e_p(t, t_0 - \omega)e_p(t_0, t)$$

so that $f = e_p(\cdot, t_0 - \omega)e_p(t_0, \cdot)$ and hence, using Lemma 2.7,

$$\begin{aligned} f^\Delta &= e_p^\Delta(\cdot, t_0 - \omega)e_p(t_0, \cdot) + e_p^\sigma(\cdot, t_0 - \omega)e_p^\Delta(t_0, \cdot) \\ &= pe_p(\cdot, t_0 - \omega)e_p(t_0, \cdot) + (1 + \mu p)e_p(\cdot, t_0 - \omega)(\ominus p)e_p(t_0, \cdot) \\ &= pf - \frac{p}{1+\mu p}(1+\mu p)f = pf - pf = 0. \end{aligned}$$

Therefore $k_p = e_p(t+\omega, t) - 1$ does indeed not depend on $t \in \mathbb{T}$. □

**Lemma 2.9.** *Let $\mathbb{T}$ be $\omega$-periodic and suppose $f : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ satisfies the assumptions of [4, Theorem 1.117]. Define*

$$g(t) = \int_t^{t+\omega} f(t,s)\Delta s.$$

*If $f^\Delta(t,s)$ denotes the derivative of $f$ with respect to $t$, then*

$$g^\Delta(t) = \int_t^{t+\omega} f^\Delta(t,s)\Delta s + f(\sigma(t), t+\omega) - f(\sigma(t), t).$$

*Proof.* Let $t_0 \in \mathbb{T}$ and use a proof similar to that of the fourth part of Theorem 2.1 to obtain

$$g(t) = \int_t^{t_0+\omega} f(t,s)\Delta s + \int_{t_0+\omega}^{t+\omega} f(t,s)\Delta s = \int_t^{t_0+\omega} f(t,s)\Delta s + \int_{t_0}^{t} f(t,s+\omega)\Delta s.$$

Now we apply [4, Theorem 1.117] to arrive at

$$\begin{aligned} g^{\Delta}(t) &= \int_t^{t_0+\omega} f^{\Delta}(t,s)\Delta s - f(\sigma(t),t) + \int_{t_0}^{t} f^{\Delta}(t,s+\omega)\Delta s + f(\sigma(t),t+\omega) \\ &= \int_t^{t+\omega} f^{\Delta}(t,s)\Delta s + f(\sigma(t),t+\omega) - f(\sigma(t),t). \end{aligned}$$

This concludes the proof. □

## 3 The $\mathrm{C}_h$ Space

It is well known that the development of the theory of functional differential equations with infinite delay primarily depends on the choice of a phase space. Many authors have been actively devoted to this topic, and various phase spaces have been proposed (see [13, 23] and references cited therein). The phase space for initial functions plays a very important rôle in the study of functional dynamic equations with infinite delay on time scales. However, no attempts have been made to construct a phase space for functional dynamic equations with infinite delay on time scales. In this section, we establish a phase space for functional dynamic equations with infinite delay on time scales.

Suppose $\inf \mathbb{T} = -\infty$ and $t_1, t_2 \in \mathbb{T}$ imply $t_1 + t_2 \in \mathbb{T}$. Let $h \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}^-, \mathbb{R}^+)$, $h(s) > 0$ for all $s \in \mathbb{T}^-$, and $\int_{-\infty}^{0} h(s)\Delta s = 1$. Define

$$\mathrm{C}_h = \left\{ \varphi \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}^-, \mathbb{R}^n) : \int_{-\infty}^{0} h(s)|\varphi|^{[s,0]}\Delta s < \infty \right\}, \quad \text{where} \quad |\varphi|^{[s,0]} = \sup_{s \le \theta \le 0} |\varphi(\theta)|.$$

One can easily verify that $\mathrm{C}_h$ is a linear subspace of $\mathrm{C}_{\mathrm{rd}}$ and $\mathrm{BC}_{\mathrm{rd}}$ is a linear subspace of $\mathrm{C}_h$. For $\varphi \in \mathrm{C}_h$, define $|\varphi|_h = \int_{-\infty}^{0} h(s)|\varphi|^{[s,0]}\Delta s < \infty$, then $(\mathrm{C}_h, |\cdot|_h)$ is a normed space. For simplicity, we denote it by $\mathrm{C}_h$.

**Lemma 3.1.** *For any $\varepsilon > 0$ and $K > 0$, there exists $\delta = \delta(\varepsilon, K) > 0$ such that, for any $\varphi_1, \varphi_2 \in \mathrm{C}_h$, if $|\varphi_1 - \varphi_2|_h \le \delta$, then $|\varphi_1 - \varphi_2|^{[-K,0]} \le \varepsilon$.*

*Proof.* We complete the proof by contradiction. Suppose that there exist $\varepsilon^* > 0$ and $K^* > 0$ such that, for any $\delta > 0$, there exist $\varphi_1^{\delta}, \varphi_2^{\delta} \in \mathrm{C}_h$, such that $|\varphi_1^{\delta} - \varphi_2^{\delta}|_h \le \delta$ but $|\varphi_1^{\delta} - \varphi_2^{\delta}|^{[-K^*,0]} > \varepsilon^*$.

Let $\delta^* = \frac{\varepsilon^*}{2} \int_{-\infty}^{-K^*} h(s)\Delta s > 0$ and put $\varphi_1^* = \varphi_1^{\delta^*}$ and $\varphi_2^* = \varphi_2^{\delta^*}$. Then $|\varphi_1^* - \varphi_2^*|_h \le \delta^*$ but

$|\varphi_1^* - \varphi_2^*|^{[-K^*,0]} > \varepsilon^*$. Whence

$$\begin{aligned}
\delta^* \ \geq\ |\varphi_1^* - \varphi_2^*|_h &= \int_{-\infty}^{0} h(s)|\varphi_1^* - \varphi_2^*|^{[s,0]}\Delta s \\
&= \int_{-\infty}^{-K^*} h(s)|\varphi_1^* - \varphi_2^*|^{[s,0]}\Delta s + \int_{-K^*}^{0} h(s)|\varphi_1^* - \varphi_2^*|^{[s,0]}\Delta s \\
&\geq \int_{-\infty}^{-K^*} h(s)|\varphi_1^* - \varphi_2^*|^{[s,0]}\Delta s \\
&\geq \int_{-\infty}^{-K^*} h(s)|\varphi_1^* - \varphi_2^*|^{[-K^*,0]}\Delta s \ > \ \varepsilon^* \int_{-\infty}^{-K^*} h(s)\Delta s \ = \ 2\delta^*,
\end{aligned}$$

a contradiction. The proof is complete. □

**Lemma 3.2.** *Suppose that* $\{\varphi_n\} \subset \mathrm{C_{rd}}(\mathbb{T}^-, \mathbb{R}^n)$ *is uniformly bounded. Then* $\lim_{n\to\infty} |\varphi_n - \varphi_0|_h = 0$ *if and only if for any* $K \in \mathbb{N}$*, we have* $\lim_{n\to\infty} |\varphi_n - \varphi_0|^{[-K,0]} = 0$.

*Proof.* We prove necessity first. By Lemma 3.1, for any $\varepsilon > 0$ and $K > 0$, there exists $\delta = \delta(\varepsilon, K)$ such that, for any $\varphi_1, \varphi_2 \in \mathrm{C}_h$ with $|\varphi_1 - \varphi_2|_h \leq \delta$, we have $|\varphi_1 - \varphi_2|^{[-K,0]} < \varepsilon$. Since $\lim_{n\to\infty} |\varphi_n - \varphi_0|_h = 0$, there exists $N \in \mathbb{N}$ such that, for any $n > N$, we have $|\varphi_n - \varphi_0|_h \leq \delta$ for all $n \geq N$. Hence

$$|\varphi_n - \varphi_0|^{[-K,0]} < \varepsilon, \quad \text{i.e.,} \quad \lim_{n\to\infty} |\varphi_n - \varphi_0|^{[-K,0]} = 0.$$

Now we show sufficiency. Suppose that $\{\varphi_n\}$ is uniformly bounded, i.e., there exists $H > 0$ such that $|\varphi_n| \leq H$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$. Since $\int_{-\infty}^{0} h(s)\Delta s = 1 < \infty$, there exists $K \in \mathbb{N}$ such that $\int_{-\infty}^{-K} h(s)\Delta s < \varepsilon$. Moreover, for all $k \in \mathbb{N}$, there exists $N_k \in \mathbb{N}$ such that if $n \geq N_k$, then $|\varphi_n - \varphi_0|^{[-k,0]} \leq \varepsilon$, whence $|\varphi_0|^{[-k,0]} \leq H + \varepsilon$, so $|\varphi_0| \leq H + \varepsilon$. Therefore, for $n \geq N_K$, we have

$$\begin{aligned}
|\varphi_n - \varphi_0|_h &= \int_{-\infty}^{0} h(s)|\varphi_n - \varphi_0|^{[s,0]}\Delta s \\
&= \int_{-\infty}^{-K} h(s)|\varphi_n - \varphi_0|^{[s,0]}\Delta s + \int_{-K}^{0} h(s)|\varphi_n - \varphi_0|^{[s,0]}\Delta s \\
&\leq (2H + \varepsilon)\varepsilon + \varepsilon = (2H + \varepsilon + 1)\varepsilon.
\end{aligned}$$

This completes the proof. □

**Lemma 3.3.** *The space* $(\mathrm{C_{rd}}[a,b], \mathbb{R}^k)$ *is complete when endowed with the supremum norm.*

*Proof.* Clearly $\mathrm{C}_{\mathrm{rd}}[a,b]$ is a linear space. By Lemma 2.1, every rd-continuous function is bounded.

Let $\{f_n\} \subset \mathrm{C}_{\mathrm{rd}}[a,b]$ be a Cauchy sequence. Let $\varepsilon > 0$. There exists $N_\varepsilon \in \mathbb{N}$ such that for any $m,n \geq N_\varepsilon$, we have $|f_m - f_n|^{[a,b]} = \sup_{t\in[a,b]} |f_m(t) - f_n(t)| < \varepsilon$. Now let $m,n \geq N_\varepsilon$ and $t \in [a,b]$. Then $|f_m(t) - f_n(t)| \leq \sup_{t\in[a,b]} |f_m(t) - f_n(t)| < \varepsilon$. Thus $\{f_n(t)\} \subset \mathbb{R}^k$ is a Cauchy sequence and hence convergent to, say, $f(t)$. Let $n \geq N_\varepsilon$. Then

$$|f_n(t) - f(t)| = \lim_{m\to\infty} |f_n(t) - f_m(t)| \leq \frac{\varepsilon}{3}.$$

We will show that $f \in \mathrm{C}_{\mathrm{rd}}[a,b]$.

Suppose $t^* \in [a,b]$ is right-dense. Let $\varepsilon > 0$. Since $f_{N_{\varepsilon/3}} \in \mathrm{C}_{\mathrm{rd}}[a,b]$, there exists a neighborhood $U_1$ of $t^*$ such that for any $t \in U_1$, we have $|f_{N_{\varepsilon/3}}(t^*) - f_{N_{\varepsilon/3}}(t)| < \frac{\varepsilon}{3}$. Let $t \in U_1$. Then

$$|f(t^*) - f(t)| \leq |f(t) - f_{N_{\varepsilon/3}}(t^*)| + |f_{N_{\varepsilon/3}}(t^*) - f_{N_{\varepsilon/3}}(t)| + |f_{N_{\varepsilon/3}}(t) - f(t)| < \varepsilon.$$

It follows that $f$ is continuous at the right-dense point $t^*$.

Now suppose $t^* \in [a,b]$ is left-dense. Let $\varepsilon > 0$. Since $f_{N_{\varepsilon/2}} \in \mathrm{C}_{\mathrm{rd}}[a,b]$, we can conclude that there exist $\delta > 0$ and $\alpha \in \mathbb{R}$ such that

$$|f_{N_{\varepsilon/2}}(t) - \alpha| < \frac{\varepsilon}{2} \quad \text{for all} \quad t \in U_2 = (t^* - \delta, t^*) \cap \mathbb{T}.$$

Let $t \in U_2$. Then

$$|f(t) - \alpha| \leq |f(t) - f_{N_{\varepsilon/2}}(t)| + |f_{N_{\varepsilon/2}}(t) - \alpha| < \varepsilon.$$

Thus $f$ has the finite limit $\alpha$ at the left-dense point $t^*$.

Therefore, $f \in \mathrm{C}_{\mathrm{rd}}[a,b]$. This implies that $\mathrm{C}_{\mathrm{rd}}[a,b]$ is complete. □

**Theorem 3.1.** *$(\mathrm{C}_h, |\cdot|_h)$ is a Banach space.*

*Proof.* Let $\{\varphi_n\} \subset \mathrm{C}_h$ be a Cauchy sequence. Thus $\{\varphi_n\}$ is bounded, i.e., there exists $M > 0$ such that $|\varphi_n|_h \leq M$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$. Let $K > 0$ be such that

$$\int_{-\infty}^{-K} h(s)\Delta s < \frac{\varepsilon}{4M + \varepsilon}.$$

Since $\lim_{m,n\to\infty} |\varphi_n - \varphi_m|_h = 0$, by Lemma 3.2, we have $\lim_{m,n\to\infty} |\varphi_n - \varphi_m|^{[-K,0]} = 0$, so $\{\varphi_n\}$ is a Cauchy sequence in $\mathrm{C}_{\mathrm{rd}}([-K,0])$. Hence, by Lemma 3.3, there exists $\varphi \in \mathrm{C}_{\mathrm{rd}}([-K,0])$ such that $\varphi_n \to \varphi$ with respect to the supremum norm on $[-K,0]$. Therefore there exists $N \in \mathbb{N}$ such that

$$|\varphi_n - \varphi|^{[-K,0]} < \frac{\varepsilon}{2} \quad \text{for all} \quad n \geq N.$$

Hence for $t \geq -K$,

$$|\varphi(t)| \leq |\varphi_n(t) - \varphi(t)| + |\varphi_n(t)| \leq |\varphi_n - \varphi|^{[-K,0]} + M < \frac{\varepsilon}{2} + M.$$

Now define

$$\varphi(t) = \varphi(-K) \quad \text{for all} \quad t < -K.$$

Then $\varphi \in \mathrm{C_{rd}}(\mathbb{T}^-)$ and for $n \geq N$ we now have

$$\begin{aligned}
|\varphi_n - \varphi|_h &= \int_{-\infty}^{0} h(s)|\varphi_n - \varphi|^{[s,0]}\Delta s \\
&\leq \int_{-\infty}^{-K} h(s)\left(|\varphi_n|^{[s,0]} + |\varphi|^{[s,0]}\right)\Delta s + \int_{-K}^{0} h(s)|\varphi_n - \varphi|^{[-K,0]}\Delta s \\
&\leq \int_{-\infty}^{-K} h(s)\left(M + \frac{\varepsilon}{2} + M\right)\Delta s + \int_{-K}^{0} h(s)\frac{\varepsilon}{2}\Delta s \\
&< \left(2M + \frac{\varepsilon}{2}\right)\int_{-\infty}^{-K} h(s)\Delta s + \frac{\varepsilon}{2}\int_{-\infty}^{0} h(s)\Delta s \\
&< \left(2M + \frac{\varepsilon}{2}\right)\frac{\varepsilon}{4M+\varepsilon} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Hence $\varphi_n \to \varphi$ in $\mathrm{C}_h$. $\square$

**Theorem 3.2.** *Suppose that* $\varphi \in \mathrm{C}_h$ *and* $x_t(\theta) = x(t+\theta)$ *for* $\theta \in (-\infty, 0]$.

(i) *Let* $A \in (0,\infty)$. *Suppose that* $x : (-\infty, A) \to \mathbb{R}^n$ *is rd-continuous on* $[0, A]$ *and satisfies* $x_0 = \varphi$. *Then for any* $t \in [0, A]$, *we have* $x_t \in \mathrm{C}_h$ *and* $x_t$ *is rd-continuous with respect to* $t$.

(ii) *There exists* $K_1$ *such that* $|\varphi(0)| \leq K_1|\varphi|_h$.

*Proof.* We first show (i). In fact, for any $t \in [0, A]$,

$$\begin{aligned}
\int_{-\infty}^{0} h(s)|x_t|^{[s,0]}\Delta s &= \int_{-\infty}^{-t} h(s)|x_t|^{[s,0]}\Delta s + \int_{-t}^{0} h(s)|x_t|^{[s,0]}\Delta s \\
&\leq \int_{-\infty}^{-t} h(s)\max\left(|x|^{[s+t,0]}, |x_t|^{[-t,0]}\right)\Delta s + \int_{-t}^{0} h(s)|x|^{[0,t]}\Delta s \\
&\leq \int_{-\infty}^{0} h(s)|x|^{[s+t,0]}\Delta s + \int_{-\infty}^{0} h(s)|x|^{[0,t]}\Delta s + \int_{-\infty}^{0} h(s)|x|^{[0,t]}\Delta s \\
&\leq \int_{-\infty}^{0} h(s)|\varphi|^{[s,0]}\Delta s + 2\int_{-\infty}^{0} h(s)|x|^{[0,A]}\Delta s \\
&= \int_{-\infty}^{0} h(s)|\varphi|^{[s,0]}\Delta s + 2|x|^{[0,A]} < \infty
\end{aligned}$$

so that $x_t \in \mathrm{C}_h$ for $t \in [0, A]$.

Next we show that $x_t$ is rd-continuous with respect to $t$. We prove that $x_t$ is continuous at right-dense points (in a similar way it can be shown that $x_t$ has finite left-sided limits at left-dense points, so the details are omitted here). Let $t \in [0, A]$ be right-dense. If $t$ is also left-dense (again, in a similar way, we can show the corresponding conclusion for the case that $t$ is left-scattered, and hence the details are omitted here), let $t_0 \in [0, t)$. Since $x_{t_0} \in \mathrm{C}_h$, for any $\varepsilon > 0$, there exists $M(t_0, \varepsilon) > 0$ such that

$$\int_{-\infty}^{-M} h(s)|x_{t_0}|^{[s,0]}\Delta s < \frac{\varepsilon}{4} \quad \text{and} \quad \int_{-\infty}^{-M} h(s)\Delta s < \frac{\varepsilon}{4L}.$$

By Lemma 2.1 (iv), we know that $x$ is bounded on $[0, A]$, say $|x|^{[0,A]} \le L$. Since $\mathrm{C}_h$ is a subspace of $\mathrm{C}_{\mathrm{rd}}$ and $x_t \in \mathrm{C}_h$, we have $x_t \in \mathrm{C}_{\mathrm{rd}}$. Assume $-M \le \theta \le 0$. If $\theta$ is right-dense, since $x_t \in \mathrm{C}_{\mathrm{rd}}$, we can choose sufficiently small $\delta_1$ such that for any $t_0 \in (t-\delta_1, t+\delta_1) \cap \mathbb{T}$, we have

$$|x_t(\theta) - x_{t_0}(\theta)| = |x_t(\theta) - x_t(\theta + t_0 - t)| < \frac{\varepsilon}{4}.$$

If $\theta$ is right-scattered and left-scattered, it is obvious that there exists $\delta_2 > 0$ such that if $\theta^* \in (\theta - \delta_2, \theta + \delta_2) \cap \mathbb{T}$, then $\theta^* = \theta$. So, when $|t_0 - t| < \delta_2$, we have $\theta - \delta_2 < \theta + t_0 - t < \theta + \delta_2$, i.e., $\theta + t_0 - t \in (\theta - \delta_2, \theta + \delta_2)$. Hence $\theta + t_0 - t = \theta$. Therefore

$$|x_t(\theta) - x_{t_0}(\theta)| = |x_t(\theta) - x_t(\theta + t_0 - t)| = 0 < \frac{\varepsilon}{4}.$$

Assume $\theta$ is right-scattered and left-dense. Note that since $x_t \in \mathrm{C}_{\mathrm{rd}}$, by definition, $x_t$ has a finite left-sided limit at $\theta$. Hence there exist $\delta_3 > 0$ and $\alpha \in \mathbb{R}$ such that

$$|x_t(s) - \alpha| < \frac{\varepsilon}{8} \quad \text{for any} \quad s \in (\theta - \delta_3, \theta + \delta_3) \cap \mathbb{T}.$$

Then

$$|x_t(s) - x_t(\theta)| < |x_t(s) - \alpha| + |x_t(\theta) - \alpha| < \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \frac{\varepsilon}{4}.$$

Let $|t - t_0| < \delta_3$. Then $\theta + t_0 - t \in (\theta - \delta_3, \theta + \delta_3) \cap \mathbb{T}$. Thus

$$|x_{t_0}(\theta) - x_t(\theta)| = |x_t(\theta + t_0 - t) - x_t(\theta)| < \frac{\varepsilon}{4}.$$

Therefore, for any $\theta \in [-M, 0]$, there exists $\delta > 0$ such that, if $t_0 \in (t - \delta, t + \delta) \cap \mathbb{T}$, we have $\max_{-M \le \theta \le 0} |x_t(\theta) - x_{t_0}(\theta)| < \frac{\varepsilon}{4}$. Thus

$$\begin{aligned}
|x_t - x_{t_0}|_h &= \int_{-\infty}^{0} h(s)|x_t - x_{t_0}|^{[s,0]}\Delta s \\
&\le \int_{-\infty}^{-M} h(s)(|x_t|^{[s,0]} + |x_{t_0}|^{[s,0]})\Delta s + \int_{-M}^{0} h(s)|x_t - x_{t_0}|^{[s,0]}\Delta s \\
&\le \int_{-\infty}^{-M} h(s)\left[\max\left\{|x|^{[t_0,t]}, |x_{t_0}|^{[s,0]}\right\} + |x_{t_0}|^{[s,0]}\right]\Delta s + \int_{-M}^{0} h(s)|x_t - x_{t_0}|^{[s,0]}\Delta s \\
&\le \int_{-\infty}^{-M} h(s)(L + 2|x_{t_0}|^{[s,0]})\Delta s + |x_t - x_{t_0}|^{[-M,0]} \ \le\ L\frac{\varepsilon}{4L} + 2\frac{\varepsilon}{4} + \frac{\varepsilon}{4} \ =\ \varepsilon.
\end{aligned}$$

To conclude, we have proved that $x_t$ is rd-continuous with respect to $t$ on $[0, A]$.

For (ii), note that $|\varphi|_h = \int_{-\infty}^{0} h(s)|\varphi|^{[s,0]}\Delta s \ge \int_{-\infty}^{0} h(s)|\varphi(0)|\Delta s = |\varphi(0)|$. □

*Remark* 3.1. If $\mathbb{T} = \mathbb{R}$, then the $\mathrm{C}_h$ space is the phase space established in [23] for functional differential equations with infinite delay.

## 4 Krasnosel′skiĭ's Fixed Point Theorem and Preliminary Results

Let $\omega \in \mathbb{T}$ be positive. Assume that $\mathbb{T}$ is an $\omega$-periodic time scale (see Definition 2.8). In the next section, we will first explore the existence and nonexistence of positive $\omega$-periodic solutions of the functional dynamic equation with infinite delay on $\mathbb{T}$ of the form

$$x^{\Delta}(t) = -a(t, x(t))x(\sigma(t)) + f(t, x_t), \quad t \in \mathbb{T}, \tag{4.1}$$

where $x_t \in \mathrm{C}_h$ and $x_t(\theta) = x(t + \theta)$ for $\theta \in (-\infty, 0]$. Throughout this paper, we assume:

($\mathrm{H}_1$) $a(t, x)$ is rd-continuous and $\omega$-periodic in $t$ and is continuous in $x$. There exist $\omega$-periodic $\alpha, \beta \in \mathrm{C}_{\mathrm{rd}}$ such that $\alpha(t) \leq a(t, x) \leq \beta(t)$ and $k_\alpha > 0$ (where $k_\alpha$ is defined as in Theorem 2.1).

($\mathrm{H}_2$) $f(t, \varphi)$ is rd-continuous in $t$ and is continuous in $\varphi$ with $f(t + \omega, \varphi) = f(t, \varphi)$ for $\varphi \in \mathrm{C}_h$, and $f(t, \varphi)$ maps bounded sets into bounded sets and $f(t, \varphi) \geq 0$ for $\varphi \in \mathrm{C}_h$ with $\varphi(\theta) \geq 0$, $\theta \in \mathbb{T}^-$.

In order to explore the periodicity of (4.1), we first introduce the essential tool to be applied throughout this paper and prove some preliminary results.

**Definition 4.1.** Let $(X, \|\cdot\|)$ be Banach space. A nonempty closed subset $K \subset X$ is called a *cone* if it satisfies the following two conditions:

(i) For any $u, v \in K, \alpha, \beta > 0$, we have $\alpha u + \beta v \in K$;

(ii) $u, -u \in K$ implies $u = 0$.

Now let us introduce the famous Krasnosel′skiĭ fixed point theorem [11], which will come into play soon.

**Lemma 4.1.** *Let $X$ be a Banach space and let $K \subset X$ be a cone. Assume that $\Omega_1, \Omega_2$ are bounded open subsets of $X$ with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $F : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that one of*

(i) *$\|Fu\| \leq \|u\|$ for any $u \in K \cap \partial\Omega_1$ and $\|Fu\| \geq \|u\|$ for any $u \in K \cap \partial\Omega_2$;*

(ii) *$\|Fu\| \geq \|u\|$ for any $u \in K \cap \partial\Omega_1$, and $\|Fu\| \leq \|u\|$ for any $u \in K \cap \partial\Omega_2$.*

*holds. Then $F$ has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.*

Let

$$P_w = \{u \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}) : \ u(t + \omega) = u(t)\}$$

and define

$$\|u\| = \max_{t \in [0, \omega]} |u(t)| \quad \text{for} \quad u \in P_w.$$

**Lemma 4.2.** *Suppose that* $\{u^n\} \subset P_\omega$, $u \in P_\omega$, *and* $u^n \to u$ *as* $n \to \infty$. *Then* $\{u_t^n\}$ *converges uniformly to* $u_t \in \mathrm{C}_h$ *with respect to* $t$.

*Proof.* Let $\varepsilon > 0$. Since $\|u^n - u\| \to 0$ as $n \to \infty$, there exists $N \in \mathbb{N}$ such that for $n \geq N$, we have $\|u^n - u\| < \varepsilon$. Let $n \geq N$. Then

$$|u_t^n - u_t|_h = \int_{-\infty}^{0} h(s)|u_t^n - u_t|^{[s,0]}\Delta s = \int_{-\infty}^{0} h(s)|u^n - u|^{[s+t,t]}\Delta s \leq \|u^n - u\| \int_{-\infty}^{0} h(s)\Delta s < \varepsilon.$$

This completes the proof. □

**Lemma 4.3.** *Let* $b, p \in P_\omega$. *Then*

$$x^\Delta(t) = -b(t)x(\sigma(t)) + p(t) \tag{4.2}$$

*has a unique* $\omega$*-periodic solution* $x$ *given by*

$$x(t) = \frac{1}{k_b}\int_{t}^{t+\omega} p(s)e_b(s,t)\Delta s. \tag{4.3}$$

*Proof.* First we show that $x$ defined by (4.3) is a $\omega$-periodic solution of (4.2). Using Theorem 2.1, we find

$$\begin{aligned} x(t+\omega) &= \frac{1}{k_b}\int_{t+\omega}^{t+2\omega} p(s)e_b(s,t+\omega)\Delta s = \frac{1}{k_b}\int_{t}^{t+\omega} p(s+\omega)e_b(s+\omega,t+\omega)\Delta s \\ &= \frac{1}{k_b}\int_{t}^{t+\omega} p(s)e_b(s,t)\Delta s = x(t) \end{aligned}$$

so that $x$ is $\omega$-periodic. Next, we use Lemma 2.9, Theorem 2.1, Theorem 2.7, and Lemma 2.3 to calculate

$$\begin{aligned} k_b\left\{x^\Delta(t) + b(t)x(\sigma(t))\right\} &= \int_{t}^{t+\omega} p(s)(\ominus b)(t)e_b(s,t)\Delta s + p(t+\omega)e_b(t+\omega,\sigma(t)) - p(t)e_b(t,\sigma(t)) \\ &\quad + b(t)\left\{\int_{t}^{t+\omega} p(s)e_b(s,\sigma(t))\Delta s - \int_{t}^{\sigma(t)} p(s)e_b(s,\sigma(t))\Delta s + \int_{t+\omega}^{\sigma(t+\omega)} p(s)e_b(s,\sigma(t))\Delta s\right\} \\ &= -b(t)\int_{t}^{t+\omega} p(s)e_b(s,\sigma(t))\Delta s + p(t)e_b(t+\omega,\sigma(t)) - p(t)e_b(t,\sigma(t)) \\ &\quad + b(t)\left\{\int_{t}^{t+\omega} p(s)e_b(s,\sigma(t))\Delta s - \mu(t)p(t)e_b(t,\sigma(t)) + \mu(t+\omega)p(t+\omega)e_b(t+\omega,\sigma(t))\right\} \\ &= p(t)\left\{e_b(t+\omega,\sigma(t)) - e_b(t,\sigma(t)) - \mu(t)b(t)e_b(t,\sigma(t)) + \mu(t)b(t)e_b(t+\omega,\sigma(t))\right\} \\ &= p(t)\left\{(1+\mu(t)b(t))e_b(t+\omega,\sigma(t)) - (1+\mu(t)b(t))e_b(t,\sigma(t))\right\} \\ &= p(t)\left\{e_b(t+\omega,t) - e_b(t,t)\right\} = k_b p(t) \end{aligned}$$

so that $x$ solves (4.2).

Now we assume that $x$ is any $\omega$-periodic solution of (4.2). Let $t_0 \in \mathbb{T}$. By Lemma 2.5 and Lemma 2.7, we have

$$[xe_b(\cdot,t_0)]^\Delta = x^\Delta e_b(\cdot,t_0) + x^\sigma e_b^\Delta(\cdot,t_0) = e_b(\cdot,t_0)(x^\Delta + bx^\sigma) = e_b(\cdot,t_0)p.$$

Integrating both sides of this equation from $t$ to $t+\omega$ produces

$$\begin{aligned}\int_t^{t+\omega} e_b(s,t_0)p(s)\Delta s &= x(t+\omega)e_b(t+\omega,t_0) - x(t)e_b(t,t_0) \\ &= x(t)[e_b(t+\omega,t_0) - e_b(t,t_0)] \;=\; x(t)e_b(t,t_0)k_b\end{aligned}$$

so that indeed $x$ is equal to the right-hand side of (4.3). $\square$

For any $u \in P_\omega$, consider the dynamic equation

$$x^\Delta(t) = -a(t,u(t))x(\sigma(t)) + f(t,u_t). \tag{4.4}$$

From ($\mathrm{H}_2$), Lemma 2.1 and Theorem 3.2, it follows that $f(t,u_t) \in P_\omega$. Lemma 4.3 tells us that the unique $\omega$-periodic solution of (4.4) is given by

$$x_u(t) = \int_t^{t+\omega} G(t,s)f(s,u_s)\Delta s,$$

where

$$G(t,s) = \frac{e_a(s,t)}{k_a} \quad \text{with} \quad a(t) = a(t,u(t)) \text{ for } u \in P_\omega.$$

Define the operator $F : P_\omega \to P_\omega$ by

$$(Fu)(t) = \int_t^{t+\omega} G(t,s)f(s,u_s)\Delta s \quad \text{for} \quad u \in P_\omega \quad \text{and} \quad t \in \mathbb{T}. \tag{4.5}$$

One can easily show that $x$ is an $\omega$-periodic solution of (4.1) if and only if $x$ is a fixed point of $F$ in $P_\omega$. Define

$$\gamma_1 = \inf_{0\le t\le s\le\omega} e_\alpha(s,t), \quad \gamma_2 = \sup_{0\le t\le s\le\omega} e_\beta(s,t), \quad \delta = \frac{\gamma_1 k_\alpha}{\gamma_2 k_\beta}.$$

From ($\mathrm{H}_1$), Definition 2.7, and the definition of $G(t,s)$, we can conclude the following.

**Lemma 4.4.** *The function $G(t,s)$ satisfies*

(i) $G(t,s) = G(t+\omega,s+\omega)$ *for any* $s \in [t,t+\omega]$;

(ii) $A := \dfrac{\gamma_1}{k_\beta} \le G(t,s) \le \dfrac{\gamma_2}{k_\alpha} := B$ *for any* $s \in [t,t+\omega]$.

It is clear that $\delta = A/B$ and $0 < \delta \le 1$. Define

$$K_\delta = \{x \in P_\omega : \; x(t) \ge \delta\,\|x\| \; \text{ for all } t \in \mathbb{T}\}.$$

It is trivial to show that $K_\delta$ is a cone.

**Lemma 4.5.** *$F(K_\delta) \subset K_\delta$.*

*Proof.* For any $u \in K_\delta$, we show that $Fu \in K_\delta$. It is easy to see that $Fu \in P_\omega$ by the definition of $F$. Since $u \in K_\delta$, we have $u_s(\theta) = u(s+\theta) \ge 0$ for $\theta \in \mathbb{T}^-$. By ($H_2$) and Lemma 4.4, we have $f(s, u_s) \ge 0$, $(Fu)(t) \ge 0$, and

$$(Fu)(t) = \int_t^{t+\omega} G(t,s) f(s,u_s)\Delta s \le B \int_0^\omega f(s,u_s)\Delta s.$$

Whence $\|Fu\| \le B \int_0^\omega f(s,u_s)\Delta s$. Moreover

$$(Fu)(t) = \int_t^{t+\omega} G(t,s) f(s,u_s)\Delta s \ge A \int_0^\omega f(s,u_s)\Delta s \ge \frac{A}{B}\|Fu\| = \delta \|Fu\|,$$

which implies $Fu \in K_\delta$. □

**Lemma 4.6.** *Let $\eta > 0$ and $\Omega = \{x \in P_\omega : \ \|x\| < \eta\}$. Then $F : K_\delta \cap \overline{\Omega} \to K_\delta$ is completely continuous.*

*Proof.* First we show that $F$ is continuous. Suppose that

$$u^n, u \in K_\delta \cap \overline{\Omega}, \quad \|u^n - u\| \to 0, \quad (Fu^n)(t) = \int_t^{t+\omega} G_n(t,s) f(s,u_s^n)\Delta s,$$

where

$$G_n(t,s) = \frac{e_{a_n}(s,t)}{k_{a_n}} \quad \text{with} \quad a_n(t) := a(t, u^n(t)).$$

Now, using Lemma 2.9, we find

$$\begin{aligned} (Fu^n)^\Delta(t) &= \int_t^{t+\omega} (\ominus a_n)(t) G(t,s) f(s,u_s^n)\Delta s + G(\sigma(t), t+\omega) f(t,u_t^n) - G(\sigma(t),t) f(t,u_t^n) \\ &= (\ominus a_n)(t)(Fu^n)(t) + \frac{f(t,u_t^n)}{k_a}\left\{\frac{1}{1+\mu(t)a_n(t)} e_{a_n}(t+\omega,t) - \frac{1}{1+\mu(t)a_n(t)}\right\} \\ &= (\ominus a_n)(t)(Fu^n)(t) + \frac{f(t,u_t^n)}{1+\mu(t)a_n(t)}. \end{aligned}$$

Hence for $v^n = Fu^n - Fu$, we have

$$\begin{aligned} (v^n)^\Delta(t) &= (Fu^n)^\Delta(t) - (Fu)^\Delta(t) \\ &= (\ominus a_n)(t)(Fu^n)(t) + \frac{f(t,u_t^n)}{1+\mu(t)a_n(t)} - (\ominus a)(t)(Fu)(t) - \frac{f(t,u_t)}{1+\mu(t)a(t)} \\ &= (\ominus a_n)(t) v^n(t) + f_n(t), \end{aligned}$$

where

$$f_n(t) = -\left\{(\ominus a_n)(t) - (\ominus a)(t)\right\}(Fu)(t) + \left\{\frac{f(t,u_t^n)}{1+a_n(t)\mu(t)} - \frac{f(t,u_t)}{1+a(t)\mu(t)}\right\}.$$

By Lemma 4.2 and ($H_2$), one has $u_t^n \to u_t$ and $f(t,u_t^n) \to f(t,u_t)$ as $n \to \infty$. Moreover, since $a(t,x)$ is continuous with respect to $x$ on $\{(t,x) : \ 0 \le t \le \omega, \ |x| \le \eta\}$ and $a(t,u^n(t)) \to a(t,u(t))$ (i.e., $a_n \to a$

in brief), we have $f_n(t) \to 0$ as $n \to \infty$. Since $f$ maps bounded sets into bounded sets by ($H_2$), there exists $M_1 > 0$ such that $|f(s, x_s)| \le M_1$ for any $x \in \overline{\Omega}$ and $s \in [t, t+\omega]$. In addition, there exists $M_2 > 0$ such that $|a(t,x)| \le M_2$. Whence,

$$|f_n(t)| \le 2M_2|(Fu)(t)| + 2M_1 \le 2M_2 B\omega M_1 + 2M_1.$$

Consequently, $\|v^n\| \le B^* \int_0^\omega f_n(s)\Delta s$, where

$$B^* = \sup_{0\le t,s\le\omega} G_n^*(t,s), \quad G_n^*(t,s) = \frac{e_{a_n^*}(s,t)}{k_{a_n^*}}, \quad a_n^* = \ominus a_n.$$

By the dominated convergence theorem, we have $\|v^n\| = \|Fu^n - Fu\| \to 0$ as $n \to \infty$, which shows that $F$ is continuous.

Next, we prove that $F(K_\delta \cap \overline{\Omega})$ is compact. In fact, for any $u \in K_\delta \cap \overline{\Omega}$, we have

$$|(Fu)(t)| \le B \int_t^{t+\omega} f(s, u_s)\Delta s \le B\omega M_1$$

and

$$|(Fu)^\Delta(t)| = \left|(\ominus a)(t)(Fu)(t) + \frac{f(t,u_t)}{1 + a(t)\mu(t)}\right| \le B\omega M_1 + M_1.$$

So $F(K_\delta \cap \overline{\Omega})$ is uniformly bounded and equicontinuous, and hence, by the Arzelà–Ascoli theorem, $F(K_\delta \cap \overline{\Omega})$ is compact. Therefore $F : K_\delta \cap \overline{\Omega} \to K_\delta$ is completely continuous. □

# 5 Existence and Nonexistence of Positive Periodic Solutions

For convenience in the subsequent discussion, we introduce the following notation:

$$f^0 = \lim_{|\varphi|_h \to 0} \max_{t\in[0,\omega]} \frac{f(t,\varphi)}{|\varphi|_h}, \quad f_0 = \lim_{|\varphi|_h \to 0} \min_{t\in[0,\omega]} \frac{f(t,\varphi)}{|\varphi|_h},$$
$$f^\infty = \lim_{|\varphi|_h \to \infty} \max_{t\in[0,\omega]} \frac{f(t,\varphi)}{|\varphi|_h}, \quad f_\infty = \lim_{|\varphi|_h \to \infty} \min_{t\in[0,\omega]} \frac{f(t,\varphi)}{|\varphi|_h},$$
$$\mathrm{D}_\delta = \{\varphi \in \mathrm{C}_h : \ \varphi(\theta) \ge \delta|\varphi|_h \text{ for all } \theta \in \mathbb{T}^-\}.$$

## 5.1 Positive periodic solutions of the system (4.1)

Assume that

($H_3$) There exists $K_1 > 0$ such that for any $\varphi \in \mathrm{D}_\delta$ with $|\varphi|_h \in [\delta K_1, K_1]$, we have $f(t,\varphi) > K_1/(A\omega)$.

($H_4$) There exists $K_2 > 0$ such that for any $\varphi \in \mathrm{D}_\delta$ with $|\varphi|_h \le K_2$, we have $f(t,\varphi) < K_2/(B\omega)$.

**Theorem 5.1.** (i) *If ($H_3$) holds and $f^0 = f^\infty = 0$, then (4.1) has at least two positive $\omega$-periodic solutions $u_1$ and $u_2$ with $0 < \|u_1\| < K_1 < \|u_2\|$.*

(ii) *If* $(H_4)$ *holds and* $f_0 = f_\infty = \infty$, *then* (4.1) *has at least two positive* $\omega$*-periodic solutions* $u_1$ *and* $u_2$ *with* $0 < \|u_1\| < K_2 < \|u_2\|$.

*Proof.* We only prove (i) since the proof of (ii) is very similar. From $f^0 = 0$, it follows that, for any $\varepsilon$ with $0 < \varepsilon \le 1/(B\omega)$, there exists $r_0 < K_1$ such that

$$f(t,\varphi) \le \varepsilon|\varphi|_h \quad \text{for} \quad \varphi \in \mathrm{D}_\delta \quad \text{with} \quad 0 < |\varphi|_h \le r_0. \tag{5.1}$$

Let $\Omega_{r_0} = \{u \in P_\omega : \ \|u\| < r_0\}$. Then for any $u \in K_\delta \cap \partial\Omega_{r_0}$, we have

$$\|u\| = r_0 \quad \text{and} \quad \delta|u_t|_h \le \delta\,\|u\| \le u_t(\theta).$$

Hence $u_t \in \mathrm{D}_\delta$ and $\delta r_0 \le |u_t|_h \le r_0$. By (4.5) and (5.1), we have

$$\|Fu\| \le \int_0^\omega f(s,u_s)\Delta s \le B\varepsilon \int_0^\omega |u_s|_h \Delta s \le B\varepsilon\omega\,\|u\| \le \|u\|\,,$$

that is, $\|Fu\| \le \|u\|$ for $u \in K_\delta \cap \partial\Omega_{r_0}$.

On the other hand, we know from $f^\infty = 0$ that, for any $\varepsilon$ with $0 < \varepsilon < 1/(2B\omega)$, there is $N_1 > K_1$ such that

$$|f(t,\varphi)| \le \varepsilon|\varphi|_h \quad \text{for} \quad \varphi \in \mathrm{D}_\delta \quad \text{with} \quad |\varphi|_h \ge N_1. \tag{5.2}$$

Let $\Omega_{r_1} = \{u \in P_\omega : \ \|u\| < r_1\}$, where $r_1$ is chosen such that

$$r_1 > N_1 + 1 + 2B\omega \sup_{\substack{t\in[0,\omega] \\ |\varphi|_h \le N_1,\ \varphi\in \mathrm{D}_\delta}} f(t,\varphi). \tag{5.3}$$

Then, for any $u \in K_\delta \cap \partial\Omega_{r_1}$, we have $\delta|u_t|_h \le \delta\,\|u\| \le u_t(\theta)$. Hence

$$u_t \in \mathrm{D}_\delta \quad \text{and} \quad \delta r_1 \le |u_t|_h \le r_1 = \|u\|\,.$$

It follows from (4.5), (5.2), and (5.3) that

$$\begin{aligned} \|Fu\| &\le B\int_t^{t+\omega} f(s,u_s)\Delta s \;=\; B\int_{I_1} f(s,u_s)\Delta s + B\int_{I_2} f(s,u_s)\Delta s \\ &\le B\int_0^\omega \frac{r_1}{2B\omega}\Delta s + B\int_0^\omega \varepsilon|u_s|_h\Delta s \;\le\; \frac{r_1}{2} + Br_1\varepsilon\omega < r_1 = \|u\|\,, \end{aligned}$$

where

$$I_1 = \{s \in [0,\omega] : \ |u_s|_h \le N_1\} \quad \text{and} \quad I_2 = \{s \in [0,\omega] : \ |u_s| > N_1\}.$$

This means

$$\|Fu\| \le \|u\| \quad \text{for} \quad u \in K_\delta \cap \partial\Omega_{r_1}.$$

Let $\Omega_{K_1} = \{u \in P_\omega : \ \|u\| < K_1\}$ with $K_1 > r_0$. Then, for any $u \in K_\delta \cap \partial\Omega_{K_1}$, we obtain $K_1 \geq |u_t|_h \geq \delta K_1$. By (H$_3$), $f(t, x_t) > K_1/(A\omega)$. By Lemma 4.4,

$$\|Fu\| \geq A \int_t^{t+\omega} f(s, u_s)\Delta s > \frac{AK_1\omega}{A\omega} = K_1 = \|u\|.$$

This shows that $\|Fu\| \geq \|u\|$ for $u \in K_\delta \cap \partial\Omega_{K_1}$. It follows from Lemma 4.5 and Lemma 4.6 that

$$F : K_\delta \cap (\overline{\Omega}_{r_1} \setminus \Omega_{K_1}) \to K_\delta \quad \text{and} \quad F : K_\delta \cap (\overline{\Omega}_{K_1} \setminus \Omega_{r_0}) \to K_\delta \quad \text{are completely continuous.}$$

Thus, by Lemma 4.1, $F$ has fixed points $u_1$ and $u_2$ in $K_\delta \cap (\overline{\Omega}_{K_1} \setminus \Omega_{r_0})$ and $K_\delta \cap (\overline{\Omega}_{r_1} \setminus \Omega_{K_1})$, respectively. That is to say, (4.1) has at least two positive $\omega$-periodic solutions $u_1$ and $u_2$ with $0 < \|u_1\| < K_1 < \|u_2\|$. □

The following theorem is crucial in our subsequent arguments.

**Lemma 5.1.** *If ($H_3$) and ($H_4$) hold, then (4.1) has at least one positive $\omega$-periodic solution $u$ with $\|u\|$ lying between $K_1$ and $K_2$, where $K_1$ and $K_2$ are defined in ($H_3$) and ($H_4$), respectively.*

*Proof.* Without any loss of generality, we can assume that $K_2 < K_1$. Otherwise, we can employ Lemma 4.1 (ii) to prove this lemma.

Let $\Omega_{K_2} = \{u \in P_\omega : \ \|u\| < K_2\}$. Then for any $u \in K_\delta \cap \partial\Omega_{K_2}$, it follows from (4.5) and (H$_4$) that

$$\|Fu\| \leq B \int_t^{t+\omega} f(s, u_s)\Delta s < \frac{B\omega K_2}{B\omega} = K_2 = \|u\|.$$

This means that $\|Fu\| \leq \|u\|$ for any $u \in K_\delta \cap \partial\Omega_{K_2}$.

Let $\Omega_{K_1} = \{u \in P_\omega : \ \|u\| < K_1\}$. Then for any $u \in K_\delta \cap \partial\Omega_{K_1}$, we have $|u_t|_h \leq K_1$. It follows from (4.5) and (H$_3$) that

$$\|Fu\| \geq A \int_t^{t+\omega} f(s, u_s)\Delta s > \frac{A\omega K_1}{A\omega} = K_1 = \|u\|.$$

This means that $\|Fu\| > \|u\|$ for $u \in K_\delta \cap \partial\Omega_{K_1}$. Hence the proof is complete by Lemma 4.1. □

**Theorem 5.2.** (i) *If $f^0 \in [0, 1/(B\omega))$ and $f_\infty \in (1/(A\delta\omega), \infty)$, then (4.1) has at least one positive $\omega$-periodic solution.*

(ii) *If $f^\infty \in [0, 1/(B\omega))$ and $f_0 \in (1/(A\delta\omega), \infty)$, then (4.1) has at least one positive $\omega$-periodic solution.*

*Proof.* We first show (i). Assume that $f^0 = \alpha_1 \in [0, 1/(B\omega))$ and $f_\infty = \beta_1 \in (1/(A\delta\omega), \infty)$. For $\varepsilon = 1/(B\omega) - \alpha_1 > 0$, there exists a sufficiently small $R_1 > 0$ such that for $|\varphi|_h \leq R_1$, we have

$$\max_{t\in[0,\omega]} \frac{f(t, \varphi)}{|\varphi|_h} < \alpha_1 + \varepsilon = \frac{1}{B\omega},$$

i.e., when $|\varphi|_h \le R_1$ and $t \in [0,\omega]$, we have $f(t,\varphi) < |\varphi|_h/(B\omega) \le R_1/(B\omega)$. Hence ($H_4$) is satisfied.

For $\varepsilon = \beta_1 - 1/(A\delta\omega) > 0$, there exists a sufficiently large $R_2 > 0$ such that, when $|\varphi|_h \ge \delta R_2$, we have

$$\min_{t\in[0,\omega]} \frac{f(t,\varphi)}{|\varphi|_h} > \beta_1 - \varepsilon = \frac{1}{A\delta\omega}.$$

Therefore, when $|\varphi|_h \in [\delta R_2, R_2]$ and $t \in [0,\omega]$, we have $f(t,\varphi) > R_2\delta/(A\delta\omega) = R_2/(A\omega)$, i.e., ($H_3$) is satisfied. Lemma 5.1 tells us that the claim (i) is true.

Now we show (ii). Assume that $f_0 = \alpha_2 \in (1/(A\delta\omega), \infty)$ and $f^\infty = \beta_2 \in [0, 1/B\omega)$. For any $\varepsilon = \alpha_2 - 1/(A\delta\omega) > 0$, there exists a sufficiently small $R_3 < \delta R_2$ such that, when $0 < |\varphi|_h \le R_3$, we have

$$\min_{t\in[0,\omega]} \frac{f(t,\varphi)}{|\varphi|_h} > \alpha_2 - \varepsilon = \frac{1}{A\delta\omega}.$$

This implies that, when $|\varphi|_h \in [\delta R_3, R_3]$ and $t \in [0,\omega]$, we have $f(t,\varphi) > \delta R_3/(A\delta\omega) = R_3/(A\omega)$, which shows that ($H_3$) holds. For $\varepsilon = 1/(B\omega) - \beta_2 > 0$, $f^\infty = \beta_2$ implies that there exists a sufficiently large $R_4 > R_1$ such that, when $|\varphi|_h > R_4$, we have

$$\max_{t\in[0,\omega]} \frac{f(t,\varphi)}{|\varphi|_h} < \beta_2 + \varepsilon = \frac{1}{B\omega}. \tag{5.4}$$

In order to prove that ($H_4$) is satisfied, we consider two cases.

Case 1. If $\max_{t\in[0,\omega]} f(t,\varphi)$ is unbounded, then there exist $\varphi^* \in \mathrm{D}_\delta$ with $|\varphi^*|_h = R_5 > R_4$ and $t_0 \in [0,\omega]$ such that

$$f(t,\varphi) \le f(t_0,\varphi^*) \quad \text{for} \quad 0 < |\varphi|_h \le |\varphi^*|_h = R_5. \tag{5.5}$$

Since $|\varphi^*|_h = R_5 > R_4$, by (5.4) and (5.5), we obtain

$$f(t,\varphi) \le f(t_0,\varphi^*) < \frac{|\varphi^*|_h}{B\omega} = \frac{R_5}{B\omega} \quad \text{for} \quad 0 < |\varphi|_h \le R_5 \quad \text{and} \quad t \in [0,\omega],$$

i.e., ($H_4$) holds.

Case 2. If $\max_{t\in[0,\omega]} f(t,\varphi)$ is bounded, then there exists $M_5 > 0$ such that

$$f(t,\varphi) \le M_5 \quad \text{for} \quad (t,\varphi) \in [0,\omega] \times \mathrm{D}_\delta. \tag{5.6}$$

Here we can choose $R_5$ such that $R_5 > M_5 B\omega$. When $0 < |\varphi|_h \le R_5, t \in [0,\omega]$, by (5.6), one has $f(t,\varphi) \le M_5 < R_5/(B\omega)$, i.e., ($H_4$) holds.

Lemma 5.1 implies the conclusion (ii). □

**Theorem 5.3.** (i) *If ($H_4$) is satisfied and $f_0, f_\infty \in (1/(A\delta\omega), \infty)$, then (4.1) has at least two positive $\omega$-periodic solutions $u_1$ and $u_2$ with $0 < \|u_1\| < K_2 < \|u_2\|$, where $K_2$ is defined in ($H_4$).*

(ii) *If ($H_3$) is satisfied and $f^0, f^\infty \in [0, 1/(B\omega))$, then (4.1) has at least two positive $\omega$-periodic solutions $u_1$ and $u_2$ with $0 < \|u_1\| < K_1 < \|u_2\|$, where $K_1$ is defined in ($H_3$).*

*Proof.* We only show (i) since the proof of (ii) is very similar. From $f_\infty \in (1/(A\delta\omega), \infty)$ and the proof of Theorem 5.2 (i), we know that there exists a sufficiently large $R_2 > K_2$ such that

$$f(t,\varphi) > \frac{R_2}{A\omega} \quad \text{for} \quad \varphi \in \mathrm{D}_\delta \quad \text{with} \quad |\varphi|_h \in [\delta R_2, R_2].$$

From $f_0 \in (1/(A\delta\omega), \infty)$ and the proof of Theorem 5.2 (ii), we know that there exists a sufficiently small $R_2^* \in (0, K_2)$ such that

$$f(t,\varphi) > \frac{R_2^*}{A\omega} \quad \text{for} \quad \varphi \in \mathrm{D}_\delta \quad \text{with} \quad |\varphi|_h \in [\delta R_2^*, R_2^*].$$

By Lemma 5.1, (4.1) has at least two positive $\omega$-periodic solutions $u_1$ and $u_2$, which satisfy

$$R_2^* < \|u_1\| < K_2 < \|u_2\| < R_2.$$

The proof is complete. □

**Theorem 5.4.** *If one of the following conditions is satisfied,*

(i) $f_0 = \infty$ *and* $f^\infty \in [0, 1/(B\omega))$, (ii) $f_\infty = \infty$ *and* $f^0 \in [0, 1/(B\omega))$,
(iii) $f^0 = 0$ *and* $f_\infty \in (1/(A\delta\omega), \infty)$, (iv) $f^\infty = 0$ *and* $f_0 \in (1/(A\delta\omega), \infty)$,

*then* (4.1) *has at least one positive $\omega$-periodic solution.*

*Proof.* Assume that the condition in (i) is satisfied. We can choose $M_6$ so that $M_6 > 1/(A\delta\omega)$. Since $f_0 = \infty$, there exists a constant $r_4$ such that

$$f(t,\varphi) \geq M_6|\varphi|_h \quad \text{for} \quad \varphi \in \mathrm{D}_\delta \quad \text{with} \quad 0 < |\varphi|_h \leq r_4. \tag{5.7}$$

Let $\Omega_{r_4} = \{u \in P_\omega : \|u\| < r_4\}$. Then, for any $u \in K_\delta \cap \partial\Omega_{r_4}$, we have $\delta|u_t|_h \leq \delta\|u\| \leq u_t(\theta)$. It follows that $u_t \in \mathrm{D}_\delta$ and $\delta r_4 \leq |u_t|_h \leq r_4$. By (4.5) and (5.7), we obtain

$$\|Fu\| \geq A \int_t^{t+\omega} f(s,u_s)\Delta s \geq AM_6 \int_0^{\omega} |u_s|_h \Delta s \geq AM_6\delta r_4\omega \geq r_4 = \|u\|,$$

i.e., for $u \in K_\delta \cap \partial\Omega_{r_4}$, we have $\|Fu\| \geq \|u\|$.

On the other hand, the fact that $f^\infty \in [0, 1/(B\omega))$ and Theorem 5.2 (ii) imply that there exists $R_5 > r_4 > 0$ such that $f(t,\varphi) < R_5/(B\omega)$ for $\varphi \in \mathrm{D}_\delta$ with $|\varphi|_h \leq R_5$. Let $\Omega_{R_5} = \{u \in P_\omega : \|u\| < R_5\}$. Then, for any $u \in K_\delta \cap \partial\Omega_{R_5}$, we have

$$\|Fu\| \leq B \int_t^{t+\omega} f(s,u_s)\Delta s < \frac{B\omega R_5}{B\omega} = R_5 = \|u\|,$$

i.e., for $u \in K_\delta \cap \partial\Omega_{R_5}$, we have $\|Fu\| \leq \|u\|$. Conclusion (i) is valid by Lemma 4.1. By carrying out similar arguments as above, we can prove the claim for the cases (ii)–(iv). The details are omitted here. □

*Remark* 5.1. From Theorem 5.4, one knows that if $f_0 = \infty$, $f^\infty = 0$ or $f^0 = 0$, $f_\infty = \infty$, then (4.1) has at least one positive $\omega$-periodic solution. When $\mathbb{T} = \mathbb{R}$, this conclusion reduces to the main result of [19].

**Theorem 5.5.** *Assume* $(H_4)$. *If* $f_0 = \infty$, $f_\infty \in (1/(A\delta\omega), \infty)$ *or* $f_\infty = \infty$, $f_0 \in (1/(A\delta\omega), \infty)$ *holds, then* (4.1) *has at least two positive* $\omega$*-periodic solutions* $u_1$ *and* $u_2$ *with* $0 < \|u_1\| < K_2 < \|u_2\|$, *where* $K_2$ *is defined in* $(H_4)$.

*Proof.* We only prove the case that $f_0 = \infty$, $f_\infty \in (1/(A\delta\omega), \infty)$, since the other case is similar. Let $\Omega_{r_4} = \{u \in P_\omega : \ \|u\| < r_4\}$ and $f_\infty = \alpha_3$, where $r_4 < r_2$. It follows from $f_0 = \infty$ and the proof of Theorem 5.4 (i) that $\|Fu\| \geq \|u\|$ for $u \in K_\delta \cap \partial\Omega_{r_4}$. Let $\Omega_{R_2} = \{u \in P_\omega : \ \|u\| < R_2\}$ with $R_2 > K_2$. It follows from $f_\infty = \alpha_3 \in (1/A\delta\omega, \infty)$ and Theorem 5.2 (i) that

$$f(t, \varphi) > \frac{R_2}{A\omega} \quad \text{for} \quad \varphi \in \mathrm{D}_\delta \quad \text{with} \quad |\varphi|_h \in [\delta R_2, R_2].$$

By $(\mathrm{H}_4)$ and the proof of Lemma 5.1, (4.1) has at least two positive $\omega$-periodic solutions $u_1$ and $u_2$ with $0 < \|u_1\| < K_2 < \|u_2\|$. □

**Theorem 5.6.** *Assume that* $(H_3)$ *holds. If* $f^0 = 0$, $f^\infty \in [0, 1/(B\omega))$ *or* $f^\infty = 0$, $f^0 \in [0, 1/(B\omega))$, *then* (4.1) *has at least two positive* $\omega$*-periodic solutions* $u_1$ *and* $u_2$ *with* $0 < \|u_1\| < K_1 < \|u_2\|$, *where* $K_1$ *is defined in* $(H_3)$.

So far, we have deliberately explored the existence and the multiplicity of positive periodic solutions. To conclude, we have proved the following theorem.

**Theorem 5.7.** *If* $f_0 \in (1/(A\delta\omega), \infty]$, $f^\infty \in [0, 1/(B\omega))$ *or* $f^0 \in [0, 1/(B\omega))$, $f_\infty \in (1/(A\delta\omega), \infty]$ *or* $(H_3)$, $(H_4)$ *holds, then* (4.1) *has at least one positive* $\omega$*-periodic solution. If* $(H_3)$, $f^0, f^\infty \in [0, 1/(B\omega))$ *or* $(H_4)$, $f_0, f_\infty \in (1/(A\delta\omega), \infty]$ *is satisfied, then* (4.1) *has at least two positive* $\omega$*-periodic solutions.*

Now we continue with the nonexistence of periodic solutions of (4.1).

**Theorem 5.8.** *Let* $R_i$, $i \in \{1, 2, 3, 4\}$, *be as in the proof of Theorem* 5.2. *If*

$$f_0, f_\infty, \min_{R_3 \leq |\varphi|_h \leq \delta R_2} \frac{f(t, \varphi)}{|\varphi|_h} \in \left(\frac{1}{A\delta\omega}, \infty\right)$$

*or*

$$f^0, f^\infty, \max_{R_1 \leq |\varphi|_h \leq R_4} \frac{f(t, \varphi)}{|\varphi|_h} \in \left[0, \frac{1}{B\omega}\right)$$

*is satisfied, then* (4.1) *has no positive* $\omega$*-periodic solution.*

*Proof.* We only prove the first claim. From the fact that $f_0, f_\infty \in (1/(A\delta\omega), \infty)$ and the proof of Theorem 5.2, it follows that there exist $R_3, R_2 > 0$ such that

$$f(t, \varphi) > \frac{1}{A\delta\omega}|\varphi|_h \text{ for } 0 < |\varphi|_h \leq R_3 \quad \text{and} \quad f(t, \varphi) > \frac{1}{A\delta\omega}|\varphi|_h \text{ for } |\varphi|_h \geq \delta R_2.$$

Then by $\min_{R_3 \le |\varphi|_h \le \delta R_2} \frac{f(t,\varphi)}{|\varphi|_h} > \frac{1}{A\delta\omega}$, we have

$$f(t,\varphi) > \frac{1}{A\delta\omega}|\varphi|_h \quad \text{for any} \quad |\varphi|_h \in (0,\infty).$$

If (4.1) has a positive $\omega$-periodic solution, say $v$, then $Fv = v$. Hence

$$\begin{aligned} \|v\| &= \|Fv\| = \left\| \int_t^{t+\omega} G(t,s) f(s, v_s) \Delta s \right\| > \int_t^{t+\omega} A \frac{1}{A\delta\omega} |v_s|_h \Delta s \\ &\ge \int_t^{t+\omega} \frac{1}{\omega\delta} |v_s(0)| \Delta s = \int_t^{t+\omega} \frac{1}{\omega\delta} |v(s)| \Delta s \ge \int_t^{t+\omega} \frac{1}{\omega\delta} \delta \|v\| \Delta s = \|v\|, \end{aligned}$$

which is a contradiction. The claim (i) is valid. The proof of (ii) will be omitted since it is similar to that of (i). □

Although we have established sufficient criteria for the nonexistence of a periodic solution of (4.1), the criteria depend on the parameters in the proof of Theorem 5.2. As in [22], consider the system

$$x^{\Delta}(t) = -a(t, x(t))x(\sigma(t)) + \lambda f(t, x_t), \quad t \in \mathbb{T}, \tag{5.8}$$

where a parameter $\lambda$ is isolated from the functional $f$. By carrying out similar arguments as above, one can easily reach the following claim.

**Theorem 5.9.** *If $f_0 > 0$, $f_\infty > 0$ or $f^0$, $f^\infty < \infty$, then (5.8) has no positive $\omega$-periodic solution for sufficiently large or small $\lambda > 0$, respectively.*

*Remark* 5.2. By carrying out similar arguments as above, one can also derive sufficient criteria for the existence of at least one or two periodic solutions for the functional dynamic equations with infinite delay on time scales of the form (5.8). Usually, sufficient criteria for the existence or nonexistence of periodic solutions of the equations of form (5.8) involves the statements "... for sufficiently large or small $\lambda > 0$ ..." (one can see this from Theorem 5.9 and [16, 22]), which are a little bit less concrete. So we prefer to study the functional dynamic equations of form (4.1) instead of form (5.8) in this paper. One can also observe that the equations of form (5.8) can facilitate the study of the nonexistence of periodic solutions as we have seen in Theorem 5.8 and 5.9.

*Remark* 5.3. In this subsection, we have systematically studied the existence and the nonexistence of positive periodic solutions of (4.1). In fact, by carrying out exactly the same arguments, one can easily establish the corresponding (same) criteria for the existence and nonexistence of positive periodic solutions of

$$x^{\Delta}(t) = a(t, x(t))x(\sigma(t)) - f(t, x_t). \tag{5.9}$$

The only difference is that $G(t,s)$ should be replaced by

$$G(t,s) = \frac{e_a(t+\omega, s)}{k_a}.$$

*Remark* 5.4. As promised in the introduction, one of our principle aims is to unify the existence of periodic solutions of some differential equations and their corresponding discrete analogues. If $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, then (4.1) reduces to

$$x'(t) = -a(t, x(t))x(t) + f(t, x_t), \quad t \in \mathbb{R} \tag{5.10}$$

and

$$x(t+1) - x(t) = -a(t, x(t))x(t+1) + f(t, x_t), \quad t \in \mathbb{Z},$$

respectively. Another discrete analogue of (5.10) reads

$$x(t+1) - x(t) = -a(t, x(t))x(t) + f(t, x_t), \quad t \in \mathbb{Z}.$$

In order to include this equation in our study, it suffices to explore the existence and nonexistence of periodic solutions of the system

$$x^{\Delta}(t) = a(t, x(t))x(t) - f(t, x_t), \quad t \in \mathbb{T}. \tag{5.11}$$

By carrying out exactly the same arguments as those for (4.1), it is not difficult to see that Theorems 5.7 and 5.8 are valid for (5.11). The only difference is that $G, \gamma_1, \gamma_2, A, B$ here should be replaced by

$$G(t, s) = \frac{e_a(t + \omega, s)}{k_a}, \quad \gamma_1 = \inf_{0 \le t \le s \le \omega} e_{\alpha}(t + \omega, s), \quad \gamma_2 = \sup_{0 \le t \le s \le \omega} e_{\beta}(t + \omega, s), \quad A = \frac{\gamma_1}{k_{\beta}}, \quad B = \frac{\gamma_2}{k_{\alpha}}.$$

In addition, based on the exact same arguments, one can also derive the same criteria for the existence and nonexistence of periodic solutions of

$$x^{\Delta}(t) = -a(t, x(t))x(t) + f(t, x_t) \tag{5.12}$$

provided that $1/(1 - \mu(s)a(s))$ is positive and bounded. For brevity, details are omitted here.

### 5.2 Periodic solutions of higher-dimensional dynamic systems

In the previous subsection, we focused on scalar dynamic equations with infinite delay on time scales. In this subsection, we turn to investigate the $n$-dimensional system

$$X^{\Delta}(t) = -A(t)X(\sigma(t)) + G(t, X_t), \tag{5.13}$$

where $A(t) = \text{diag}[a_1(t), a_2(t), \dots, a_n(t)]$ with $a_i \in \mathrm{C}_{\mathrm{rd}}$, $a_i(t + \omega) = a_i(t)$, and $k_{a_i} > 0$ for all $i \in \{1, \dots, n\}$. $G = (g_1, g_2, \dots, g_n)^T$ is defined on $\mathbb{R} \times \mathrm{C}_h$, and $G(t, \phi)$ is rd-continuous in $t$ and is continuous in $\phi$ with $G(t + \omega, \phi) = G(t, \phi)$. In addition, $g_i(t, \varphi)$ maps bounded sets into bounded sets and $g_i(t, \phi) \ge 0$ for $\phi \in \mathrm{C}_h$ with $\phi_i(\theta) \ge 0$ and $\theta \in \mathbb{R}^-$. For convenience, we introduce the following notation:

$$G_i(t,s)=\frac{e_{a_i}(s,t)}{k_{a_i}},\quad \delta_i=\frac{1}{e_{a_i}(\omega,0)},\quad \delta=\min_{1\le i\le n}\{\delta_i\},\quad A_i=\frac{\delta_i}{1-\delta_i},\quad B_i=\frac{1}{1-\delta_i},$$

$$A_0=\sum_{i=1}^n A_i,\quad B_0=\sum_{i=1}^n B_i,\quad P=\{u\in \mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^n):\ u(t+\omega)=u(t),\ t\in\mathbb{R},\ u_i\in \mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R})\},$$

$$\|u\|=\max_{t\in[0,\omega]}\sum_{i=1}^n|u_i(t)|\quad \text{for}\quad u\in P,$$

$$E_\delta=\{\phi\in \mathrm{C}_h:\ \phi_i(\theta)\ge\delta|\phi_i|_h,\ \theta\in\mathbb{R}^-\},\quad K=\{x\in P:\ x_i(t)\ge 0,\ x_i(t)\ge\delta|x_i|\},$$

$$g_i^0=\lim_{|\varphi|_h\to 0}\max_{t\in[0,\omega]}\frac{g_i(t,\varphi)}{|\varphi|_h},\quad g_0^i=\lim_{|\varphi|_h\to 0}\min_{t\in[0,\omega]}\frac{g_i(t,\varphi)}{|\varphi|_h},$$

$$g_i^\infty=\lim_{|\varphi|_h\to \infty}\max_{t\in[0,\omega]}\frac{g_i(t,\varphi)}{|\varphi|_h},\quad g_\infty^i=\lim_{|\varphi|_h\to \infty}\min_{t\in[0,\omega]}\frac{g_i(t,\varphi)}{|\varphi|_h},$$

$$G^0=\max_{1\le i\le n}g_i^0,\quad G^\infty=\max_{1\le i\le n}g_i^\infty,\quad G_0=\min_{1\le i\le n}g_0^i,\quad G_\infty=\min_{1\le i\le n}g_\infty^i.$$

We first list below some conclusions without proof since the proofs are very similar to those in the above section.

**Lemma 5.2.** *Let $A,Q\in P$, $A(t)=\mathrm{diag}[a_1(t),a_2(t),\dots,a_n(t)]$, $Q(t)=(q_1(t),q_2(t),\dots,q_n(t))^T$. Then*

$$X^\Delta(t)=-A(t)X(\sigma(t))+Q(t)$$

*has a unique $\omega$-periodic solution given by*

$$X(t)=(x_1(t),x_2(t),\dots,x_n(t)),\quad \textit{where}\quad x_i(t)=\int_t^{t+\omega}G_i(t,s)q_i(s)\Delta s.$$

For any $u\in P$, consider the equation

$$X^\Delta(t)=-A(t)X(\sigma(t))+G(t,u_t). \tag{5.14}$$

Lemma 5.2 tells us that the unique $\omega$-periodic solution of (5.14) is given by

$$X_u(t)=(x_1(t),x_2(t),\dots,x_n(t)),\quad \text{where}\quad x_i(t)=\int_t^{t+\omega}G_i(t,s)g_i(s,u_s)\Delta s.$$

Define the operator $F:K\to P$ with components $(F_1,F_2,\dots,F_n)$ by

$$(F_iu)(t)=\int_t^{t+\omega}G_i(t,s)g_i(s,u_s)\Delta s\quad \text{for}\quad u\in P_\omega\quad\text{and}\quad t\in\mathbb{T}.$$

One can easily show that $X$ is an $\omega$-periodic solution of (5.13) if and only if $X$ is a fixed point of $F$ in $K$. It is not difficult to show that $F(K)\subset K$. Let $\eta$ be a positive constant and $\Omega=\{x\in P:\ |x|\le\eta\}$. Then $F:K\cap\overline{\Omega}\to K$ is completely continuous.

Based on $G_0,G_\infty,G^\infty,G^0$, one can establish exactly the same sufficient criteria (Theorems 5.7 and 5.8) for the existence and nonexistence of $\omega$-periodic solutions of (5.13). For simplicity, as an example, we only present the details of the proof for the case that $G^0=0$, $G_\infty=\infty$.

**Theorem 5.10.** *If $G^0 = 0$ and $G_\infty = \infty$, then (5.13) has at least one positive $\omega$-periodic solution.*

*Proof.* Since $G^0 = 0$, we have $g_i^0 = 0$. Choose $\varepsilon > 0$ such that $\varepsilon B_0 \omega < 1$. Then there exists a constant $s_1$ such that $g_i(t, \phi) \le \varepsilon |\phi|_h$, $0 < |\phi|_h \le s_1$, $\phi \in E_\delta$. Let $\Omega_1 = \{x \in P : \ |x| < s_1\}$. Then for any $u \in K \cap \partial\Omega_1$, we have $u_t^i(\theta) \ge \delta |u_t^i|_h$ and $\delta |u| \le |u_t|_h \le |u|$, so $u_t \in E_\delta$. Therefore

$$\begin{aligned} \|Fu\| &= \sum_{i=1}^{n} \int_{t}^{t+\omega} G_i(t,s) g_i(s, u_s) \Delta s \ \le\ \sum_{i=1}^{n} B_i \int_0^\omega g_i(s, u_s)\Delta s \\ &\le \sum_{i=1}^{n} B_i \int_0^\omega \varepsilon |u_s|_h \Delta s \ \le\ \varepsilon \sum_{i=1}^{n} B_i \int_0^\omega |u| \Delta s \ <\ |u|. \end{aligned}$$

On the other hand, suppose $G_\infty = \infty$. We may choose $M$ such that $M\delta\omega A_0 > 1$ and we can observe that there exists a constant $s_0$ with $s_0 > s_1$ such that $g_i(t, \varphi) > M|\varphi|_h$, $|\varphi|_h \ge s_0$, $\varphi \in E_\delta$. Let $s_2 = s_0/\delta$ and $\Omega_2 = \{x \in P : \ |x| < s_2\}$. Then, for any $u \in K \cap \partial\Omega_2$, we have $u_t \in E_\delta$ and

$$|u_t|_h = \int_{-\infty}^{0} h(s) \sup_{s \le \theta \le 0} \sum_{i=1}^{n} |u_t^i(\theta)|^{[s,0]} \Delta s \ge \int_{-\infty}^{0} h(s) \sup_{s \le \theta \le 0} \sum_{i=1}^{n} \delta |u_t^i| \Delta s = \delta |u| = \delta s_2 = s_0.$$

Whence

$$\begin{aligned} \|Fu\| &= \sum_{i=1}^{n} \int_{t}^{t+\omega} G_i(t,s) g_i(s, u_s) \Delta s \ \ge\ \sum_{i=1}^{n} A_i \int_0^\omega g_i(s, u_s)\Delta s \\ &\ge \sum_{i=1}^{n} A_i \int_0^\omega M |u_s|_h \Delta s \ \ge\ M\delta |u| \omega A_0 \ \ge\ |u|. \end{aligned}$$

It follows from (4.1), (5.5), and (5.6) that $F$ has a fixed point $u_1 \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ such that $(Fu_1)(t) = u_1(t)$ and $u_1^i(t) \ge \delta \left\| u_1^i \right\| \ge \delta s_1 > 0$. Hence $u_1$ is an $\omega$-periodic solution of (5.13). ☐

*Remark* 5.5. In this paper, we have systematically explored the existence of periodic solutions of some dynamic equations with infinite delay on time scales, which incorporate as special cases the differential and the difference equations investigated in [6, 10, 14, 15, 16, 17, 18, 19, 22, 24, 25], and hence the dynamic equations investigated here also incorporate as special cases many well-known models in population dynamics, hematopoiesis, etc. and provide sufficient criteria for the existence of positive periodic solutions of those models.

*Remark* 5.6. The explorations in this paper reveal that, when one deals with the existence of positive periodic solutions of differential equations and difference equations, especially by using the method of Krasnosel′skiĭ's fixed point theorem, it is unnecessary and useless to prove results for differential equations and separately again for their discrete analogues (difference equations). One can unify such problems in the framework of dynamic equations on time scales.

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