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On a Sub-supersolution Method for the Prescribed Mean Curvature Problem

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SUB-SUPERSOLUTION THEOREMS FOR QUASILINEAR ELLIPTIC PROBLEMS: A VARIATIONAL APPROACH

VY KHOI LE, KLAUS SCHMITT

Dedicated to Hans Knobloch with much admiration and appreciation

Abstract. This paper presents a variational approach to obtain sub - supersolution theorems for a certain type of boundary value problem for a class of quasilinear elliptic partial differential equations. In the case of semilinear ordinary differential equations results of this type were first proved by Hans Knobloch in the early sixties using methods developed by Cesari.

1. Introduction - Problem setting

We are interested here in a variational sub-supersolution approach to a quasilinear elliptic boundary value problem which, in the one-space dimensional and semilinear case, is a boundary value problem for a second order scalar ordinary differential equation subject to periodic boundary conditions. The latter problem was first studied by Hans Knobloch [5] and later by many other authors using various kinds of nonlinear analysis methods (see, e.g., [10, 9, 6, 2]). The present paper is a continuation of and extends the results of the recent note [11]. In the case of boundary value problems subject to Dirichlet boundary conditions, sub-supersolution results, in a variational setting, were first obtained by Peter Hess [3]. We here follow closely the approach used in [7], where Dirichlet boundary value problems for degenerate elliptic equations were studied.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. We consider the following boundary value problem:

$$-\operatorname{div}[A(x,\nabla u)] + f(x,u) = 0, \quad x \in \Omega, \tag{1.1}$$

$$u(x) = \text{constant}, \quad x \in \partial\Omega, \tag{1.2}$$

$$\int_{\partial\Omega} A(x,\nabla u)\cdot n, S = 0. \tag{1.3}$$

(Note that in condition (1.2) it is understood that the trace of u is a constant function, with the constant not being fixed.) Here, $A : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying the following conditions:

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- There exist $p \in (1,\infty)$, $a_1 \in L^{p'}(\Omega)$ (p' is the conjugate of p), and $b_1 > 0$ such that
$$|A(x,\xi)| \le a_1(x) + b_1|\xi|^{p-1}, \tag{1.4}$$
for a.e. $x \in \Omega$, all $\xi \in \mathbb{R}^N$.
- $A(x,\xi)$ is monotone in ξ, that is
$$[A(x,\xi) - A(x,\xi')] \cdot (\xi - \xi') \ge 0, \quad \text{for a.e. } x \in \Omega, \text{ all } \xi, \xi' \in \mathbb{R}^N. \tag{1.5}$$
- A has the following coercivity property: There exist $a_2 \in L^1(\Omega)$ and $b_2 > 0$ such that
$$A(x,\xi) \cdot \xi \ge b_2|\xi|^p - a_2(x), \quad \text{for a.e. } x \in \Omega, \text{ all } \xi \in \mathbb{R}^N. \tag{1.6}$$

Remark 1.1. (a) When $N = 1$ and $\Omega = (a,b)$, the boundary condition (1.2)-(1.3) becomes the boundary condition on (a,b):
$$u(a) = u(b),\ A(a,u'(a)) = A(b,u'(b)),$$
which, when $A(x,v) = v$ is the usual set of periodic boundary conditions
$$u(a) = u(b), \quad u'(a) = u'(b).$$
(b) An example of the operator A above is the p-Laplacian, i.e.,
$$A(x,\nabla u) = |\nabla u|^{p-2}\nabla u, \quad p > 1.$$
It is easy to check that A satisfies conditions (1.4), (1.5), and (1.6) above. In this case, the boundary condition (1.3) becomes
$$\int_{\partial\Omega} |\nabla u|^{p-2}\frac{\partial u}{\partial n}\, dS = 0\,.$$

Assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with some appropriate growth condition to be specified later. We denote by $W^{1,p}(\Omega)$ the usual Sobolev space, equipped with the norm
$$\|u\| = \|u\|_{W^{1,p}(\Omega)} = \Big(\|u\|^p_{L^p(\Omega)} + \||\nabla u|\|^p_{L^p(\Omega)}\Big)^{1/p}. \tag{1.7}$$
Let $\mathcal{A}, F : W^{1,p}(\Omega) \to [W^{1,p}(\Omega)]^*$ be defined by
$$\langle Fu, v\rangle = \int_\Omega f(x,u)v\, dx,$$
and
$$\langle \mathcal{A}u, v\rangle = \int_\Omega A(x,\nabla u) \cdot \nabla v\, dx, \quad \forall u, v \in W^{1,p}(\Omega).$$
From (1.4)-(1.6), we see that $\mathcal{A}$ is continuous, bounded, monotone, and coercive in the following sense:
$$\langle \mathcal{A}u, u\rangle \ge b_2\||\nabla u|\|^p_{L^p(\Omega)} - \|a_2\|_{L^1(\Omega)}, \quad \forall u \in W^{1,p}(\Omega). \tag{1.8}$$
Let
$$V_c = \{u \in W^{1,p}(\Omega) : u\big|_{\partial\Omega} = \text{constant}\}.$$
Then V_c is a closed subspace of $W^{1,p}(\Omega)$ and thus a reflexive Banach space with the restricted norm of (1.7). The weak (variational) formulation of the boundary value problem (1.1)-(1.3) is the following variational equality:
$$\begin{gathered} \int_\Omega A(x,\nabla u) \cdot \nabla v\, dx + \int_\Omega f(x,u)v\, dx = 0, \quad \forall v \in V_c \\ u \in V_c. \end{gathered} \tag{1.9}$$

To check this, note that if u satisfies (1.1)-(1.3) and $v \in V_c$ then

$$\begin{aligned}
0 &= -\int_\Omega \operatorname{div} A(x,\nabla u) v\,dx + \int_\Omega f(x,u)v\ dx \\
&= \int_\Omega A(x,\nabla u)\cdot\nabla v\,dx - (v|_{\partial\Omega})\int_{\partial\Omega} A(x,\nabla u)\cdot n\,dS + \int_\Omega f(x,u)v\,dx \\
&= \int_\Omega A(x,\nabla u)\cdot\nabla v\,dx + \int_\Omega f(x,u)v\,dx\,.
\end{aligned}$$

Hence, we have (1.9). Conversely, if $u \in V_c$ is a solution of (1.9) then by choosing $v \in C_0^\infty(\Omega) \subset V_c$ in (1.9) and applying the divergence theorem as above, we see that (1.1) holds. Choosing $v = 1$ in (1.9), we have $\int_\Omega f(x,u)\,dx = 0$. On the other hand, integrating (1.1) over Ω and using once more the Divergence theorem yield

$$0 = -\int_\Omega \operatorname{div} A(x,\nabla u)\,dx + \int_\Omega f(x,u)\,dx = -\int_{\partial\Omega} A(x,\nabla u)\cdot n\,dS.$$

Hence, we have the boundary condition (1.3).

2. Sub-Supersolutions

We shall study the existence of solutions of (1.9) by first defining appropriate concepts of sub- and supersolutions.

Definition 2.1. A function $\underline{u}$ (resp. $\overline{u}$) in V_c is called a subsolution (resp. supersolution) of (1.9) if

$$\int_\Omega A(x,\nabla\underline{u})\cdot\nabla v\,dx + \int_\Omega f(x,\underline{u})v\,dx \le 0 \quad (\text{resp.} \ge 0), \tag{2.1}$$

for all $v \in V_c$, $v \ge 0$ a.e. in Ω.

Remark 2.2. When A is the Laplacian, i.e., $A(x,\nabla u) = \nabla u$ and $p = 2$, or when $N = 1$ (ODE case), the above definition of sub- and supersolutions is the variational form of that given in [11], without imposing additional smoothness assumptions.

As is the case with solutions satisfying additional smoothness conditions, sub- and supersolutions, when smooth enough, satisfy additional boundary conditions. Let us see this in the case of the $p-$Laplacian. For assume that $\alpha \in V_c \cap W^{2,p}(\Omega)$ satisfies (cf. (17) of [11]):

$$\int_\Omega |\nabla\alpha|^{p-2}\nabla\alpha\cdot\nabla\phi\,dx + \int_\Omega f(x,\alpha)\phi\,dx \le 0, \quad \forall\phi \in C_0^\infty(\Omega), \phi \ge 0, \tag{2.2}$$

and

$$\int_{\partial\Omega} |\nabla\alpha|^{p-2}\nabla\alpha\cdot n\,dS \le 0. \tag{2.3}$$

Since $\alpha \in W^{2,p}(\Omega)$, Green's theorem (or the Divergence theorem) implies that

$$\int_\Omega [-\operatorname{div}\big(|\nabla\alpha|^{p-2}\nabla\alpha\big) + f(x,\alpha)]\phi\,dx \le 0, \quad \forall\phi \in C_0^\infty(\Omega),\ \phi \ge 0,$$

i.e., (in the sense of distributions),

$$-\operatorname{div}\big(|\nabla\alpha|^{p-2}\nabla\alpha\big) + f(x,\alpha) \le 0 \quad \text{a.e. on } \Omega. \tag{2.4}$$

Let $v \in V_c$, $v \geq 0$ a.e. on Ω. It follows from (2.4) that

$$\begin{aligned} 0 &\geq \int_\Omega [-\operatorname{div}\left(|\nabla\alpha|^{p-2}\nabla\alpha\right) + f(x,\alpha)]v\,dx \\ &= \int_\Omega |\nabla\alpha|^{p-2}\nabla\alpha\cdot\nabla v\,dx - \int_{\partial\Omega}|\nabla\alpha|^{p-2}\frac{\partial\alpha}{\partial\nu}v\,dS + \int_\Omega f(x,\alpha)v\,dx. \end{aligned}$$

Hence,

$$\int_\Omega |\nabla\alpha|^{p-2}\nabla\alpha\cdot\nabla v\,dx + \int_\Omega f(x,\alpha)v\,dx \leq (v|_{\partial\Omega})\int_{\partial\Omega}|\nabla\alpha|^{p-2}\frac{\partial\alpha}{\partial\nu}\,dS \leq 0\,,$$

that is, α satisfies (2.1). Conversely, assume $\alpha \in V_c \cap W^{2,p}(\Omega)$ satisfies (2.1). Since $C_0^\infty(\Omega) \subset V_c$, we have (2.2). To prove that α satisfies (2.3), we choose a sequence $\{\Omega_n\}$ of subdomains of Ω such that

$$\overline{\Omega_n} \subset \Omega_{n+1},\ \forall n, \quad \text{and} \quad \Omega = \bigcup_{n=1}^\infty \Omega_n. \tag{2.5}$$

For each $n \in \mathbb{N}$, choose $\phi_n \in C_0^\infty(\Omega)$ such that $0 \leq \phi_n(x) \leq 1$, $\forall x \in \Omega$, and $\phi_n(x) = 1$, $\forall x \in \Omega_n$. Let $v_n = 1 - \phi_n$ $(n \in \mathbb{N})$. Then $v_n \in V_c$, $v_n = 1$ on $\partial\Omega$, and $0 \leq v_n \leq 1$ on Ω. Letting $v = v_n$ in (2.1), one gets

$$\begin{aligned} 0 &\geq \int_\Omega |\nabla\alpha|^{p-2}\nabla\alpha\cdot\nabla v_n\,dx + \int_\Omega f(x,\alpha)v_n\,dx \\ &= \int_\Omega [-\operatorname{div}\left(|\nabla\alpha|^{p-2}\nabla\alpha\right) + f(x,\alpha)]v_n\,dx + \int_{\partial\Omega}|\nabla\alpha|^{p-2}\frac{\partial\alpha}{\partial\nu}v_n\,dS \\ &= \int_\Omega [-\operatorname{div}\left(|\nabla\alpha|^{p-2}\nabla\alpha\right) + f(x,\alpha)]v_n\,dx + \int_{\partial\Omega}|\nabla\alpha|^{p-2}\frac{\partial\alpha}{\partial\nu}\,dS. \end{aligned} \tag{2.6}$$

Because $v_n = 0$ on Ω_n, from (2.5) and the Dominated convergence theorem, one obtains

$$\lim_{n\to\infty}\int_\Omega [-\operatorname{div}\left(|\nabla\alpha|^{p-2}\nabla\alpha\right) + f(x,\alpha)]v_n\,dx = 0.$$

Letting $n \to \infty$ in (2.6), we obtain $\int_{\partial\Omega}|\nabla\alpha|^{p-2}\frac{\partial\alpha}{\partial\nu}\,dS \leq 0$.

3. Existence Results

Our main existence result is the following theorem.

Theorem 3.1. *Assume there exists a pair of sub- and supersolution $\underline{u}$ and $\overline{u}$ of (1.9) such that $\underline{u} \leq \overline{u}$ and that f satisfies the following growth condition:*

$$|f(x,u)| \leq a_3(x), \tag{3.1}$$

for a.e. $x \in \Omega$, all $u \in [\underline{u}(x), \overline{u}(x)]$, with $a_3 \in L^{p'}(\Omega)$. Then, (1.9) has a solution $u \in V_c$ such that $\underline{u} \leq u \leq \overline{u}$.

Proof. We define

$$b(x,u) = \begin{cases} [u - \overline{u}(x)]^{p-1} & \text{if} \quad u > \overline{u}(x) \\ 0 & \text{if} \quad \underline{u}(x) \leq u \leq \overline{u}(x) \\ -[\underline{u}(x) - u]^{p-1} & \text{if} \quad u < \underline{u}(x), \end{cases} \tag{3.2}$$

for $x \in \Omega$, $u \in \mathbb{R}$, and

$$(Tu)(x) = \begin{cases} \overline{u}(x) & \text{if} \quad u(x) > \overline{u}(x) \\ u(x) & \text{if} \quad \underline{u}(x) \le u(x) \le \overline{u}(x) \\ \underline{u}(x) & \text{if} \quad u(x) < \underline{u}(x), \end{cases} \tag{3.3}$$

for $x \in \Omega$ and $u \in W^{1,p}(\Omega)$. Straightforward calculations show that

$$|b(x,u)| \le a_4(x) + b_4|u|^{p-1},$$

for a.e. $x \in \Omega$, all $u \in \mathbb{R}$, where $b_4 > 0$ and $a_4 \in L^{p'}(\Omega)$. Therefore, the operator $B : W^{1,p}(\Omega) \to [W^{1,p}(\Omega)]^*$ given by

$$\langle Bu, v\rangle = \int_\Omega b(x,u)v\,dx \ (u,v \in W^{1,p}(\Omega))$$

is well defined, completely continuous, and bounded. Moreover, there are $a_5, b_5 > 0$ such that

$$\langle Bu, u\rangle \ge b_5\|u\|^p_{L^p(\Omega)} - a_5, \ \forall u \in W^{1,p}(\Omega). \tag{3.4}$$

Let us consider the following variational equality in V_c:

$$\begin{gathered} \langle \mathcal{A}u + Bu + F(Tu), v\rangle = 0, \ \forall v \in V_c \\ u \in V_c. \end{gathered} \tag{3.5}$$

It follows from (3.1) that $F \circ T$ is well defined and completely continuous from $W^{1,p}(\Omega)$ to its dual space. Because $\mathcal{A}$ is monotone, $\mathcal{A} + B + F \circ T$ is pseudo-monotone. Next, let us show that $\mathcal{A} + B + F \circ T$ is coercive on $W^{1,p}(\Omega)$ in the following sense:

$$\lim_{\|u\|\to\infty} \frac{\langle \mathcal{A}u + Bu + F(Tu), u\rangle}{\|u\|} = \infty. \tag{3.6}$$

In fact, from (3.3) and (3.1),

$$|\langle F(Tu), u\rangle| = \Big|\int_\Omega f(x,Tu)u\,dx\Big| \le \int_\Omega a_3|u|\,dx \le \|a_3\|_{L^{p'}(\Omega)}\|u\|_{L^p(\Omega)}. \tag{3.7}$$

Combining (3.7) with (3.4) and (1.8), we get

$$\begin{aligned} &\langle \mathcal{A}u + Bu + F(Tu), u\rangle \\ &\ge b_2\||\nabla u|\|^p_{L^p(\Omega)} - \|a_2\|_{L^1(\Omega)} + b_5\|u\|^p_{L^p(\Omega)} - a_5 - \|a_3\|_{L^{p'}(\Omega)}\|u\|_{L^p(\Omega)} \\ &\ge \min\{b_2, b_5\}(\|u\|^p_{L^p(\Omega)} + \||\nabla u|\|^p_{L^p(\Omega)}) - \|a_3\|_{L^{p'}(\Omega)}\|u\| - \|a_2\|_{L^1(\Omega)} - a_5 \\ &= b_6\|u\|^p - a_6\|u\| - a_7, \quad \forall u \in W^{1,p}(\Omega), \end{aligned}$$

with $a_6, a_7, b_6 > 0$. Because $p > 1$, this estimate implies (3.6).

Since V_c is a closed subspace of $W^{1,p}(\Omega)$, the existence of solutions of (3.5) follows from classical existence theorems for elliptic variational inequalities (cf. e.g. [8]). Assume that u is any solution of (3.5). We prove that

$$\underline{u} \le u \le \overline{u} \quad \text{a.e. in } \Omega, \tag{3.8}$$

and thus u is also a solution of (1.9). Let us verify the first inequality in (3.8). Since $u, \underline{u} \in W^{1,p}(\Omega)$, we have $(\underline{u} - u)^+ \in W^{1,p}(\Omega)$. Moreover, since $u|_{\partial\Omega}$ and $\underline{u}|_{\partial\Omega}$ are constants,

$$[(\underline{u} - u)^+]|_{\partial\Omega} = (\underline{u}|_{\partial\Omega} - u|_{\partial\Omega})^+ = \text{constant},$$

i.e.,

$$(\underline{u} - u)^+ \in V_c. \tag{3.9}$$

Choosing $v = (\underline{u} - u)^+$ in (3.5), one obtains

$$\int_\Omega A(x, \nabla u) \cdot \nabla[(\underline{u} - u)^+]\, dx + \int_\Omega [b(x, u) + f(Tu)](\underline{u} - u)^+\, dx = 0. \tag{3.10}$$

On the other hand, letting $v = (\underline{u} - u)^+ (\geq 0)$ in (2.1) gives us

$$\int_\Omega A(x, \nabla \underline{u}) \cdot \nabla[(\underline{u} - u)^+]\, dx + \int_\Omega f(\underline{u})(\underline{u} - u)^+\, dx \leq 0. \tag{3.11}$$

Subtracting (3.10) from (3.11) yields

$$\begin{aligned} &\int_\Omega [A(x, \nabla \underline{u}) - A(x, \nabla u)] \cdot \nabla[(\underline{u} - u)^+]\, dx + \int_\Omega [f(\underline{u}) - f(Tu)](\underline{u} - u)^+\, dx \\ &\leq \int_\Omega b(x, u)(\underline{u} - u)^+\, dx. \end{aligned} \tag{3.12}$$

Note that from (1.5) and Stampacchia's theorem (cf. e.g. [4, 1]), we have

$$\begin{aligned} &\int_\Omega [A(x, \nabla \underline{u}) - A(x, \nabla u)] \cdot \nabla[(\underline{u} - u)^+]\, dx \\ &= \int_{\{x\in\Omega:\underline{u}(x)>u(x)\}} [A(x, \nabla \underline{u}) - A(x, \nabla u)] \cdot (\nabla \underline{u} - \nabla u)\, dx \\ &\geq 0. \end{aligned} \tag{3.13}$$

From the definition of Tu in (3.3), we have $Tu(x) = \underline{u}(x)$ on $\{x \in \Omega : \underline{u}(x) > u(x)\}$ and thus

$$\int_\Omega [f(\underline{u}) - f(Tu)](\underline{u} - u)^+\, dx = \int_{\{x\in\Omega:\underline{u}(x)>u(x)\}} [f(\underline{u}) - f(Tu)](\underline{u} - u)\, dx = 0. \tag{3.14}$$

Using (3.13) and (3.14) in (3.12), one obtains

$$0 \leq \int_\Omega b(x, u)(\underline{u} - u)^+\, dx = -\int_{\{x\in\Omega:\underline{u}(x)>u(x)\}} (\underline{u} - u)^p\, dx \leq 0.$$

This implies that

$$\int_{\{x\in\Omega:\underline{u}(x)>u(x)\}} (\underline{u} - u)^p\, dx = 0,$$

i.e., $\underline{u} - u = 0$ a.e. on $\{x \in \Omega : \underline{u}(x) > u(x)\}$, or, $\{x \in \Omega : \underline{u}(x) > u(x)\}$ has measure 0. This shows the first inequality in (3.8). The other inequality there is established in the same way. From (3.8) and (3.2)-(3.3), we immediately have $b(x, u(x)) = 0$ and $Tu(x) = u(x)$ for a.e. $x \in \Omega$. (3.5) thus becomes (1.9). □

Remark 3.2. By modifying the proof of Theorem 3.1 appropriately, we can extend that theorem to the existence of solutions of (1.9) between a finite number of sub- and supersolutions. In fact, we can show that if $\underline{u}_1, \dots, \underline{u}_k$ (resp. $\overline{u}_1, \dots, \overline{u}_m$) are subsolutions (resp. supersolutions) of (1.9) such that

$$\max\{\underline{u}_1, \dots, \underline{u}_k\} \leq \min\{\overline{u}_1, \dots, \overline{u}_m\},$$

and that f satisfies an appropriate growth condition between these sub- and supersolutions, then there exists a solution u of (1.9) such that $\max\{\underline{u}_1, \dots, \underline{u}_k\} \leq u \leq \min\{\overline{u}_1, \dots, \overline{u}_m\}$ (see, for example, [7] for more details).

Remark 3.3. We note that in order for our method of proof of Theorem 3.1 to work the important property of the subspace V_c that was needed was that $u^+ \in V_c$ for any $u \in V_c$. We therefore see that Theorem 3.1 remains valid, if V_c is replaced by any subpsce V which has this property (and, of course, the definitions of sub- and supersolutions are appropriately modified). This, more general theorem, for example, contains the sub-supersolution existence result for boundary-value problems subject to Neumann boundary conditions.

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