Elementary length topologies constructed using pseudo-norms with values in Tikhohov semi-fields

Jackie Ray Hamm

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ELEMENTARY LENGTH TOPOLOGIES CONSTRUCTED USING PSEUDO-NORMS WITH VALUES IN TIKHONOV SEMI-FIELDS

by

JACKIE RAY HAMM, 1939-

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ABSTRACT

Elementary length topologies defined on normed and pseudo-normed linear spaces are studied. It is shown that elementary length topologies constructed with different pseudo-norms are never equivalent.

Elementary length topologies are constructed on certain topological spaces and some of their properties are investigated. It is shown that certain "measuring devices" (i.e., norms, pseudo-norms, semi-norms, and pseudo-metrics) which take their values in Tikhonov semifields may be used to construct elementary length topologies on any topological linear space. Relationships between two elementary length topologies generated with different measuring devices are considered.

Let (X,t) be a topological linear space such that t is determined from a convex functional, p-norm or quasi-norm. The function q may be used to construct an elementary length topology on X. Let X_r denote X with this elementary length topology and let C(X_r,k) be the set of all bounded, continuous, complex valued functions defined on X_r. Some of the properties of C(X_r,k) which may be used in a quantum mechanical scattering analysis involving elementary length are considered.
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I. INTRODUCTION

The term "elementary length" is used to denote a fundamental physical constant $\lambda > 0$ such that certain "length measurements" are integer multiples of $\lambda$. The existence of such a constant has been contemplated by Pythagoras, Werner Heisenberg, Bertrand Russell, Poincare [1] and many others. George Gamow [2] has conjectured that the introduction of $\lambda$ into theoretical physics would initiate a "complete" understanding of physical phenomena. Although an adequate theory based on the existence of $\lambda$ has not yet emerged, some recent attempts have been made with some apparent success [3]. These attempts involve topologies which are derived from the assumption of the existence of $\lambda$.

Gudder [1] has shown that each norm on a linear space may be used to generate an elementary length topology. The topology of a normed linear space may be generated by many distinct (but possibly equivalent) norms. It is therefore important to investigate the relationship of elementary length topologies which are generated with different norms. It is shown, in fact, that distinct but equivalent norms will generate inequivalent elementary length topologies.

Furthermore, the above questions involving norms are generalized to questions involving, in place of norms, other "positive" functions taking their values in Tikhonov semi-fields, and answers similar to those mentioned above are obtained.

The problem of determining those linear spaces which allow the construction of an elementary length topology on them is considered.
When such a construction is possible some of the properties of the topologies are investigated.

The set of all bounded, continuous, complex-valued functions defined on a space with an elementary length topology is important in attempts to set up a quantum mechanics involving elementary length. Some of the properties of this set and one of its self-maps are considered.
II. REVIEW OF THE LITERATURE

In 1966 a backward, large-angle scattering which could not be explained by the existing quantum electrodynamic theory was reported by Blumenthal, et al. [4]. Atkinson and Halpern [3] suggested that this unexplained scattering may be caused by a "topological potential" due to the existence of an elementary length. The elementary length topologies presented in [3] are introduced in an ad hoc fashion and it is explained that there may be many such "non-usual" topologies which involve an elementary length. In [1] Gudder gave definitions of "length" and "elementary length" and showed that the elementary length topologies introduced by Atkinson and Halpern can be derived with the aid of these concepts. It will be necessary to refer to the results obtained by Gudder throughout this work. The following is an outline of those results.

Let $(X, ||\cdot||)$ be a normed linear space over the real or complex numbers.

DEFINITION G1. A length on $X$ is a function $f : X \rightarrow \{n\lambda : n = 0, 1, 2, \cdots\}$ where $\lambda > 0$ and

(L1) if $||x|| = n\lambda$ then $f(x) = n\lambda$, $n = 0, 1, 2, \cdots$;

(L2) if $||x_1|| < ||x_2||$ then $f(x_1) < f(x_2)$;

(L3) if $||x|| > 0$ then $f(x + \frac{\lambda x}{||x||}) = f(x) + \lambda$.

LEMMA G1. If $||y|| > \lambda$ then $f(y) = f(y - \frac{\lambda y}{||y||}) + \lambda$.

DEFINITION G2. If $r$ is a real number let $[r]$ be the smallest integer which is not less than $r$, and let ${r}$ be the smallest integer which is strictly greater than $r$. 
THEOREM G1. A function $f$ on $X$ is a length if and only if $f$ has one of the two forms:

1. $f(x) = \lambda \left( \frac{|x|}{\lambda} - a \right)$ where $\lambda > 0$ and $0 \leq a < \lambda$;
2. $f(x) = \lambda \left( \frac{|x|}{\lambda} - a \right)$ where $\lambda > 0$ and $0 < a \leq \lambda$.

THEOREM G2. Let $A$ be a subset of $(0, \infty)$ with $0$ as a limit point. If $f_\lambda$ is a length for $\lambda \in A$, then $\lim_{\lambda \to 0} f_\lambda(x) = |x|$ for every $x \in X$.

DEFINITION G3. An elementary length on $X$ is a function $f : X \to \{ n\lambda : n = 0, 1, 2, \cdots \}$ for $\lambda > 0$ which satisfies (L1), (L2) and

- (L3') $f(nx) \leq nf(x)$ for $n = 0, 1, 2, \cdots$ and every $x \in X$.

THEOREM G3. A function $f$ on $X$ is an elementary length if and only if $f(x) = \lambda \left( \frac{|x|}{\lambda} \right)$ for every $x \in X$.

COROLLARY G1. An elementary length is a length.

THEOREM G4. A length $f$ is an elementary length if and only if it is sub-additive.

COROLLARY G2. If $f$ is an elementary length on $X$, then $\rho(x, y) = f(x-y)$ is a metric on $X$.

Gudder shows that the metric topology obtained from the metric $\rho$ of corollary G2 is the discrete topology. He then explains that the discrete topology is physically unsatisfactory and obtains a non-discrete topology by proceeding as follows.

If $f$ is an elementary length on $X$ it is clear that $d(x, y) = |f(x) - f(y)|$ is a pseudo-metric on $X$. 

DEFINITION G4. For each real number \( r > 0 \) let 
\[ B(x, r) = \{ y : d(x, y) < r \} \]
The set \( B(x, r) \) will be called a simple ball and a simple ball will be denoted by \( B(x) \).

LEMMA G2. Two simple balls are either disjoint or equal.

LEMMA G3. Every ball is a union of a finite number of simple balls.

THEOREM G5. The balls \( B(x, n\lambda) \) are a base for a non-discrete topology on \( X \). Every (bounded) open set is a (finite) countable union of simple balls. If \( x \neq 0 \) then every simple ball has the form 
\[ B(x + \frac{j\lambda x}{|x|}), j = 0, \pm 1, \pm 2, \ldots \]

Let \( X_c \) denote \( X \) with the norm topology and let \( X_r \) be \( X \) with the elementary length topology described in theorem G5. Let 
\[ T : X \times X \to X_c \times X_r \]
be defined by 
\[ T(x, y) = \left( \frac{x+y}{2}, y-x \right) \]
Then the weakest topology on \( X \times X \) for which \( T \) is continuous is the elementary length topology on \( X \times X \) which was derived in [4].
III. CONVEX FUNCTIONALS

In this chapter elementary length topologies generated with the aid of norms, pseudo-norms, quasi-norms and Minkowski functionals will be considered. Let \((Y, \tau)\) be a topological space and \(T : X \rightarrow Y\). If \(T\) is one-to-one and onto, and if \(t\) is the weakest topology on \(X\) for which \(T\) is continuous, then \((X, t)\) and \((Y, \tau)\) are homeomorphic. Thus if \(t\) is the elementary length topology on \(X \times X\) as defined in chapter II then \((X \times X, t)\) is homeomorphic to \(X_c \times X_r\). In the results to be presented in this work the particular measuring device under consideration may be used to determine the topological space \(X_c\). Thus the specification of the topological space \(X_r\) determines the elementary length topology \(t\) on \(X \times X\). This topology will be derived by carrying over Gudder's approach in the various cases.

Let \(X\) be a linear space over the real or complex numbers.

DEFINITION 3.1. A real-valued function \(p\) on \(X\) is a convex functional if it satisfies

1) \(p(x) \geq 0\) for every \(x \in X\);
2) \(p(\alpha x) = \alpha p(x)\) for every \(\alpha \geq 0\);
3) \(p(x + y) \leq p(x) + p(y)\).

DEFINITION 3.2. A real-valued function \(p\) on \(X\) is a pseudo-norm if it satisfies

1) \(p(x) \geq 0\) for every \(x \in X\);
2) \(p(\alpha x) = |\alpha|p(x)\) for every \(x \in X\) and every scalar \(\alpha\);
3) \(p(x + y) \leq p(x) + p(y)\).
DEFINITION 3.3. A real-valued function \( p \) on \( X \) is a quasi-norm if it satisfies
1) \( p(x) \geq 0 \) for every \( x \in X \);
2) if \( p(x) = 0 \) then \( x = 0 \);
3) \( p(\alpha x) = |\alpha|p(x) \) for every \( x \in X \) and every scalar \( \alpha \);
4) there is a \( k \geq 1 \) for which \( p(x + y) \leq k(p(x) + p(y)) \) for every \( x, y \in X \).

DEFINITION 3.4. A pseudo-metric on a set \( X \) is a mapping \( d : X \times X \to [0, \infty) \) which satisfies
1) \( d(x, x) = 0 \);
2) \( d(x, y) = d(y, x) \);
3) \( d(x, y) \leq d(x, z) + d(z, y) \).

DEFINITION 3.5. A subset \( U \) of \( X \) is absorbing if for each \( x \in X \) there is a real number \( r(x) > 0 \) such that \( x \in \alpha U \) for every scalar \( \alpha \) such that \( |\alpha| \geq r(x) \).

DEFINITION 3.6. The Minkowski functional of an absorbing set \( U \) is a mapping \( p : X \to [0, \infty) \) defined by \( p(x) = \inf\{\alpha > 0 : x \in \alpha U\} \).

It is shown in Taylor [5] that the Minkowski functional of a convex, absorbing set which contains \( 0 \) is a convex functional.

THEOREM 3.1. All of the proofs in [1] remain valid if norms are replaced by convex functionals.

The details of the proofs are essentially unchanged so they will not be repeated here. We note also that, with the exception of theorem G4 and corollary G2, the proofs in [1] remain valid if norms are replaced by the Minkowski functionals of absorbent sets containing \( 0 \) which are not convex. This allows the construction of elementary
length topologies with the aid of any non-negative, positive homogeneous function, and in particular, a quasi-norm.

Two norms on a linear space $X$ are equivalent if they generate the same topology. It seems natural to ask if equivalent norms generate the same elementary length topology. Before answering this question, some results which are used repeatedly in this work will be given.

If $p$ is a non-negative, positive homogeneous function defined on $X$ then $d(x,y) = \lambda \left[ \frac{p(x)}{\lambda} - \frac{p(y)}{\lambda} \right]$ is a pseudo-metric on $X$ which generates the elementary length topology determined from $p$. By theorem G5, a base for this topology consists of simple balls of the form $B(x) = \{ y : d(x,y) < \lambda \} = \left\{ y : \left[ \frac{p(x)}{\lambda} \right] = \left[ \frac{p(y)}{\lambda} \right] \right\}$.

**Proposition 3.1.** If $x \in X$ and $z \in B(x)$ then $B(z) = B(x)$.

**Proof.** If $z \in B(x)$ then $\left[ \frac{p(z)}{\lambda} \right] = \left[ \frac{p(x)}{\lambda} \right]$ so $B(z) = \left\{ y : \left[ \frac{p(y)}{\lambda} \right] = \left[ \frac{p(z)}{\lambda} \right] = \left[ \frac{p(x)}{\lambda} \right] \right\} = B(x)$.

Let $q_1$ and $q_2$ be non-negative, positive homogeneous functions on $X$ such that $q_1 \neq q_2$ and $q_1$ and $q_2$ are not identically zero. Let $\tau_1$ and $\tau_2$ be the elementary length topologies determined from $q_1$ and $q_2$ respectively.

**Lemma 3.1.** $\tau_1 \subseteq \tau_2$ if and only if for every $x \in X$, $z \in B_1(x)$ implies that $B_2(z) \subseteq B_1(x)$.

**Proof.** $\tau_1 \subseteq \tau_2$ if and only if for every $x \in X$, $z \in B_1(x)$ implies $z \in B_2(w) \subseteq B_1(x)$ for some $w \in X$. If $z \in B_2(w)$ then $B_2(z) = B_2(w)$ by proposition 3.1.

**Lemma 3.2.** If $\tau_1 \subseteq \tau_2$ then $q_1^{-1}(0) = q_2^{-1}(0)$. 
PROOF. For $i = 1, 2$, \( B_i(0) = \{ y : \left[ \frac{q_i(y)}{\lambda} \right] = \left[ \frac{q_i(0)}{\lambda} \right] = 0 \} = q_i^{-1}(0) \) and \( 0 \in B_1(0) \) so \( B_2(0) \subseteq B_1(0) \) by lemma 3.1, hence \( q_2^{-1}(0) \subseteq q_1^{-1}(0) \).

Suppose \( z \in X \) such that \( q_1(z) = 0 \) and \( q_2(z) \neq 0 \). Then \( z \in B_1(0) \) so \( B_2(z) \subseteq B_1(0) \). Let \( \left[ \frac{q_2(z)}{\lambda} \right] = m \) and let \( x \in X \) such that \( q_2(x) \neq 0 \).

Then \( q_2(\frac{\lambda x}{q_2(x)}) = m \lambda \) so \( \frac{\lambda x}{q_2(x)} \in B_2(z) = \{ y : \left[ \frac{q_2(y)}{\lambda} \right] = \left[ \frac{q_2(z)}{\lambda} \right] = m \} \).

But \( B_2(z) \subseteq B_1(0) \) so \( q_1(\frac{\lambda x}{q_2(x)}) = 0 \) hence \( q_1(x) = 0 \). We then have \( q_2(x) = 0 \) implies \( q_1(x) = 0 \), since \( q_2^{-1}(0) \subseteq q_1^{-1}(0) \), and \( q_2(x) \neq 0 \) implies \( q_1(x) = 0 \) so taking both of these results, this implies that \( q_1 \) is identically zero which is a contradiction. Therefore

\( q_1^{-1}(0) = q_2^{-1}(0) \).

**THEOREM 3.2.** \( \tau_1 \preceq \tau_2 \) if and only if there is an integer \( k > 1 \) such that \( q_2(x) = kq_1(x) \) for every \( x \in X \).

**PROOF.** Let \( x \in X \) such that \( q_1(x) = \lambda \). If \( \left[ \frac{q_2(x)}{\lambda} \right] = k \) then \( \frac{q_2(x)}{\lambda} \leq k \) and, by lemma 3.2, \( k \neq 0 \). Suppose that \( \frac{q_2(x)}{\lambda} < k \). Then \( q_2(\frac{\lambda x}{q_2(x)}) = \lambda k \) so \( \frac{\lambda x}{q_2(x)} \in B_2(x) \). But \( q_1(\frac{\lambda x}{q_2(x)}) = q_1(x) = \frac{\lambda k}{q_2(x)} \lambda > \lambda \) implies that \( \frac{\lambda k x}{q_2(x)} \notin B_1(x) \). This is a contradiction since \( B_2(x) \subseteq B_1(x) \). Thus \( q_1(x) = \lambda \) implies that \( q_2(x) = k\lambda \) for some integer \( k \geq 1 \). It will now be shown that the integer \( k \) does not depend on the particular \( x \) chosen. Suppose \( z_1, z_2 \in X \) such that \( q_1(z_1) = q_1(z_2) = \lambda, \ q_2(z_1) = \lambda k_1, \ q_2(z_2) = \lambda k_2 \) and \( k_1 \neq k_2 \).
If \( k_2 > k_1 \) then \( q_2 \left( \frac{k_2 z_1}{k_1} \right) = \lambda k_2 \) hence \( \frac{k_2 z_1}{k_1} \in B_2(z_2) \). But
\[
q_1 \left( \frac{k_2 z_1}{k_1} \right) = \frac{k_2}{k_1} \lambda > \lambda \text{ implies that } \frac{k_2 z_1}{k_1} \notin B_1(z_2).
\]
This is a contradiction since \( B_2(z_2) \subseteq B_1(z_2) \). If \( k_1 > k_2 \) then \( q_2 \left( \frac{k_1 z_2}{k_2} \right) = k_1 \lambda \) implies that \( \frac{k_1 z_2}{k_2} \in B_2(z_1) \). But
\[
q_1 \left( \frac{k_1 z_2}{k_2} \right) = \frac{k_1}{k_2} \lambda > \lambda \text{ implies that } \frac{k_1 z_2}{k_2} \notin B_1(z_1) \text{ and this is a contradiction since } B_2(z_1) \subseteq B_1(z_1).
\]
Hence \( q_1(x) = \lambda \) implies that \( q_2(x) = k \lambda \) for some integer \( k \). Suppose that \( z \in X \) such that \( q_1(z) \neq 0 \). Then \( q_2(z) \neq 0 \) and \( q_1 \left( \frac{\lambda z}{q_1(z)} \right) = \lambda \) hence \( q_2 \left( \frac{\lambda z}{q_1(z)} \right) = k \lambda \) and therefore \( q_2(z) = k q_1(z) \).

Now suppose that \( q_2(x) = k q_1(x) \) for every \( x \in X \) and some integer \( k > 1 \). \( \tau_1 \subseteq \tau_2 \) iff for every \( x \in X, z \in B_1(x) \) implies that
\( B_2(z) \subseteq B_1(x) \) and this is true iff
\[
\left( \frac{q_2(z)}{\lambda} \right) - 1 < \frac{q_2(w)}{\lambda} \leq \left( \frac{q_2(z)}{\lambda} \right) \Rightarrow m_1 - 1 < \frac{q_1(w)}{\lambda} \leq m_1
\]
where
\[
m_1 = \left\lfloor \frac{q_1(x)}{\lambda} \right\rfloor.
\]
Suppose that \( m_1 - 1 < \frac{q_1(z)}{\lambda} \leq m_1 \) and \( \left[ \frac{q_2(z)}{\lambda} \right] - 1 < \frac{q_2(w)}{\lambda} \leq \left[ \frac{q_2(w)}{\lambda} \right] \). Since \( q_2(x) = k q_1(x) \) for every \( x \in X \), we have
\[
\left[ \frac{q_2(z)}{\lambda} \right] = \left[ k q_1(z) \right] \leq k \left[ \frac{q_1(z)}{\lambda} \right] = km_1 \text{ and } \frac{q_1(w)}{\lambda} = \frac{q_2(w)}{\lambda} \leq \left[ \frac{q_2(z)}{\lambda} \right] \leq km_1
\]
so \( \frac{q_1(w)}{\lambda} \leq m_1 \). Now, \( m_1 - 1 < \frac{q_1(z)}{\lambda} \leq \frac{q_2(z)}{\lambda} \) implies that \( k(m_1 - 1) < \frac{k q_1(z)}{\lambda} = \frac{q_2(z)}{\lambda} \leq \left[ \frac{q_2(z)}{\lambda} \right] \). Then \( k(m_1 - 1) < \left[ \frac{q_2(z)}{\lambda} \right] \) implies that \( k(m_1 - 1) \leq \left[ \frac{q_2(z)}{\lambda} \right] - 1 < \frac{q_2(w)}{\lambda} = \frac{k q_1(w)}{\lambda} \) hence \( m_1 - 1 < \frac{q_1(w)}{\lambda} \).
COROLLARY 3.1. \( \tau_1 = \tau_2 \) if and only if \( q_1 = q_2 \).

It is important to note here that theorem 3.2 holds if \( q_1 \) and \( q_2 \) are chosen to be any of the following measuring devices.

1) norm
2) pseudo-norm
3) quasi-norm
4) convex functional
5) any non-negative, positive homogeneous function.

Thus if the measuring devices 1) - 5) are used, theorem 3.2 states that different measuring devices always generate different elementary length topologies.
IV. LOCALLY BOUNDED TOPOLOGICAL VECTOR SPACES

In this chapter elementary length topologies for locally bounded topological vector spaces will be considered. The following definitions and theorems concerning these spaces are given in [6].

DEFINITION 4.1. A topological vector space is said to be locally bounded if it has a bounded neighborhood of 0.

THEOREM 4.1. The topology of a topological vector space can be given by a quasi-norm if it is locally bounded. Conversely, a quasi-normed space is always locally bounded.

DEFINITION 4.2. A p-norm, $0 < p \leq 1$, on $X$ is a real-valued function $q$ on $X$ such that
\begin{enumerate}
  \item [P1)] $q(x) \geq 0$;
  \item [P2)] $q(x) = 0 \Rightarrow x = 0$;
  \item [P3)] $q(\alpha x) = |\alpha|^p q(x)$;
  \item [P4)] $q(x+y) \leq q(x) + q(y)$.
\end{enumerate}

THEOREM 4.2. A p-normed space is locally bounded. Conversely, every locally bounded space is p-normable.

A construction of elementary length topologies with the aid of p-norms will now be given.

DEFINITION 4.3. Let $q$ be a p-norm on $X$. A length on $X$ corresponding to $q$ is a function $f : X \to \{n\lambda : n = 0, 1, 2, \ldots\}$ such that
\begin{enumerate}
  \item [M1)] $q(x) = n\lambda \Rightarrow f(x) = n\lambda$;
  \item [M2)] $q(x_1) \leq q(x_2) \Rightarrow f(x_1) \leq f(x_2)$;
  \item [M3)] $f((1 + \frac{\lambda}{q(x)})^p x) = f(x) + \lambda$ for $x \neq 0$.
\end{enumerate}
LEMMA 4.1. If $q(y) \geq \lambda$ then $f(y) = f\left((1 - \frac{\lambda}{q(y)})^{1/p} \right) + \lambda$.

PROOF. If $q(y) = \lambda$ the result follows from M1) so suppose that $q(y) > \lambda$. Let $x = (1 - \frac{\lambda}{q(y)})^{1/p} y$. Then $q(x) = q(y) - \lambda$ and

$$(1 + \frac{\lambda}{q(x)})^{1/p} x = (1 + \frac{\lambda}{q(y) - \lambda})^{1/p} (1 - \frac{\lambda}{q(y)})^{1/p} y = y.$$  

By M3),

$$f(y) = f\left((1 + \frac{\lambda}{q(x)})^{1/p} x\right) = f(x) + \lambda = f\left((1 - \frac{\lambda}{q(y)})^{1/p} y\right) + \lambda.$$

For any real number $r$ let $[r]$ and $\{r\}$ be defined as in definition G2 of chapter II.

THEOREM 4.3. A function $f$ is a length on $X$ corresponding to a $p$-norm $q$ if and only if $f$ has one of the two forms:

1) $f(x) = \lambda \left[\frac{q(x) - a}{\lambda}\right]$ where $\lambda > 0$ and $0 \leq a < \lambda$;

2) $f(x) = \lambda \left[\frac{q(x) - a}{\lambda}\right]$ where $\lambda > 0$ and $0 < a \leq \lambda$.

PROOF. Suppose that $f$ is an elementary length on $X$ corresponding to a $p$-norm $q$ and let $A = \{x : f(x) = 0\}$, $a = \mbox{sup}\{q(y) : y \in A\}$, $B(a) = \{x : q(x) < a\}$ and $\overline{B}(a) = \{x : q(x) \leq a\}$. It follows from M1) that $0 \leq a \leq \lambda$. It is clear that $A \subset \overline{B}(a)$. If $x \in B(a)$ then there is a $y \in A$ such that $q(x) < q(y)$ hence, by M2), $x \in A$. Thus $B(a) \subset A \subset \overline{B}(a)$. Suppose that $A \neq B(a)$. Then there is an $x \in A$ such that $q(x) = a$. If $q(y) = a$ then $q(y) = q(x)$ and $x \in A$ so it follows from M2) that $y \in A$. Thus $A = \overline{B}(a)$ or $A = B(a)$. Suppose $A = B(a)$. Then $a \neq 0$ since $B(0) = \phi$ and $0 \in B(a)$ by M1). Thus $0 < a \leq \lambda$. If $a \leq q(x) \leq \lambda$ then $f(x) \neq 0$ and $q(x) \leq q\left((\frac{\lambda}{q(x)})^{1/p} x\right) = \lambda$ so
\[ f(x) \leq f\left(\left(\frac{\lambda}{q(x)}\right)^p x\right) = \lambda \text{ by M2)}. \] If \( \lambda \leq q(x) < \lambda + a \) then

\[ q\left(1 - \frac{\lambda}{q(x)}\right)^p x) = q(x) - \lambda < a \text{ so applying lemma 4.1 we have} \]

\[ f(x) = f\left(\left(1 - \frac{\lambda}{q(x)}\right)^p x\right) + \lambda = \lambda . \] Hence \( f(x) = \lambda \) for \( a \leq q(x) < \lambda + a \).

Suppose that \( \lambda + a \leq q(x) < 2\lambda + a \). Then \( a \leq q\left(1 - \frac{\lambda}{q(x)}\right)^p x) = q(x) - \lambda < \lambda + a \) so \( f(x) = f\left(\left(1 - \frac{\lambda}{q(x)}\right)^p x\right) + \lambda = 2\lambda . \) It follows by induction that

\[ f(x) = \lambda \left\{ \frac{q(x) - a}{\lambda} \right\} . \] If \( A = \overline{B}(a) \) then \( 0 \leq a < \lambda . \) If \( a < q(x) \leq \lambda \)

then \( f(x) \neq 0 \) and \( q(x) \leq q\left(\left(1 - \frac{\lambda}{q(x)}\right)^p x\right) = \lambda \) so \( f(x) \leq f\left(\left(1 - \frac{\lambda}{q(x)}\right)^p x\right) = \lambda \)

by M2). If \( \lambda \leq q(x) \leq \lambda + a \) then \( q\left(1 - \frac{\lambda}{q(x)}\right)^p x) = q(x) - \lambda \leq a \)

and by lemma 4.1, \( q(x) = q\left(\left(1 - \frac{\lambda}{q(x)}\right)^p x\right) + \lambda = \lambda . \) By induction, if

\[ n\lambda + a < q(x) \leq (n + 1)\lambda + a \] then \( f(x) = (n + 1)\lambda \) for \( n = 0, 1, 2, \ldots \) hence \( f \) has form 1).

Suppose that \( f \) is a function which has form 1) or form 2). M1)

and M2) are clear and M3) follows from the fact that

\[ q\left(1 + \frac{\lambda}{q(x)}\right)^p x) = q(x) + \lambda . \]
THEOREM 4.4. Let $A$ be a subset of $(0,\infty)$ with $0$ as a limit point. If, for $\lambda \in A$, $f_\lambda$ is a length on $X$ corresponding to a $p$-norm $q$ then
\[ \lim_{\lambda \to 0} f_\lambda(x) = q(x) \]
for every $x \in X$.

PROOF. The proof of this theorem is identical to the proof of theorem G2 of chapter II with the norm in that proof replaced by $q$.

DEFINITION 4.4. An elementary length on $X$ corresponding to a $p$-norm $q$ is a function $f : X \to \{n_\lambda : n = 0, 1, 2, \cdots \}$ which satisfies M1), M2) and

\[ f((n)^P x) \leq nf(x). \]

THEOREM 4.5. A function $f$ is an elementary length on $X$ corresponding to a $p$-norm $q$ if and only if $f(x) = \lambda \left[ \frac{q(x)}{\lambda} \right]$ for every $x \in X$.

PROOF. Let $f$ be an elementary length. It follows from M3') that
\[ f(0) = 0. \]
Let $n\lambda < q(x) \leq (n + 1)\lambda$. Then $q(x) \leq q((\frac{(n+1)\lambda}{q(x)})^P x) = (n + 1)\lambda$. Let $L$ and $m$ be integers such that $m < L$ and $q(x) > \frac{Ln\lambda}{m}$. Then, by M2) and M3'),
\[ f(x) \geq \frac{m}{m} f((\frac{Ln\lambda}{m q(x)})^P x) \geq \frac{1}{m} f((\frac{Ln\lambda}{q(x)})^P x) = \frac{Ln\lambda}{m} \geq n\lambda. \]
Thus $f(x) \geq (n + 1)\lambda$ so $f(x) = (n + 1)\lambda$.

Conversely, suppose that $f(x) = \lambda \left[ \frac{q(x)}{\lambda} \right]$. M1) and M2) are clear and $f((n)^P x) = \lambda \left[ \frac{1}{\lambda} q((n)^P x) \right] = \lambda \left[ \frac{nq(x)}{\lambda} \right]$. It is shown in [1] that
\[ \left[ n \frac{q(x)}{\lambda} \right] \leq n \left[ \frac{q(x)}{\lambda} \right] \]
so $f$ is an elementary length.
COROLLARY 4.1. An elementary length is a length.

THEOREM 4.6. A length $f$ is an elementary length if and only if it is sub-additive.

PROOF. Gudder's proof of this theorem carries over without change if norms are replaced by $p$-norms.

If $f$ is a length or an elementary length then
\[ \rho(x,y) = |f(x) - f(y)| \]
is a pseudo-metric on $X$. Let $B(x,r) = \{y : \rho(x,y) < r\}$. $B(x,\lambda) = \{y : f(x) = f(y)\}$ will be called a simple ball.

LEMMA 4.2. Two simple balls are either disjoint or equal.

PROOF. Gudder's proof of this lemma carries over unchanged if norms are replaced by $p$-norms.

LEMMA 4.3. Every ball is a union of a finite number of simple balls.

PROOF. If $x \neq 0$ and $B(x,n\lambda) \neq \emptyset$ we show that $B(x,n\lambda) = \bigcup_{j=-(n-1)}^{j=n-1} B((1 + \frac{1}{q(x)})^p x, \lambda)$. Let $y \in B(x,n\lambda)$. Then there is an integer $m$ such that $0 \leq m \leq n-1$ and $\rho(x,y) = m\lambda$. Suppose $q(y) \geq q(x)$.

Then $m\lambda = \rho(x,y) = f(y) - f(x)$ and $y \in B((1 + \frac{m\lambda}{q(x)})^p x, \lambda)$ since
\[
\rho(y,(1 + \frac{m\lambda}{q(x)})^p x) = |f(y) - f((1 + \frac{m\lambda}{q(x)})^p x)| = |f(y) - f(x) - m\lambda| = 0.
\]
If $q(y) \leq q(x)$ then $m\lambda = f(x) - f(y)$ and
\[
\rho(y,(1 - \frac{m\lambda}{q(x)})^p x) = |f(y) - f((1 - \frac{m\lambda}{q(x)})^p x)| = |f(y) - f(x) + m\lambda| = 0.
\]
so $y \in B((1 - \frac{m\lambda}{q(x)})^p x, \lambda)$. Hence $B(x, n\lambda) \subseteq \bigcup_{j=-n+1}^{j=n-1} B((1 + \frac{j\lambda}{q(x)})^p x, \lambda)$.

If $y \in B((1 + \frac{j\lambda}{q(x)})^p x, \lambda)$ for $-(n-1) \leq j \leq n-1$ then

$$\rho(y,x) \leq |f(y) - f((1 + \frac{j\lambda}{q(x)})^p x)| + |f((1 + \frac{j\lambda}{q(x)})^p x) - f(x)| = |f((1 + \frac{j\lambda}{q(x)})^p x) - f(x)| = |j|\lambda < n\lambda$$

so $y \in B(x, n\lambda)$.

If $x = 0$ a similar argument shows that $B(0, n\lambda) = \bigcup_{j=0}^{j=n-1} B(z_j, \lambda)$ where $f(z_j) = j\lambda$.

**THEOREM 4.7.** The balls $B(x, n\lambda)$ generate a non-discrete topology on $X$. Every (bounded) open set is a (finite) countable union of simple balls. If $x \neq 0$ then every simple ball has the form

$$B(x + \frac{j\lambda y}{q(x)}, \lambda), \ j = 0, \pm 1, \pm 2, \ldots.$$ 

It is natural to ask if different $p$-norms can generate the same elementary length topology. Some preliminary results are needed to answer this question. We first note that if $B(x) = B(x, \lambda)$ is a simple ball defined with the aid of an elementary length $f$ defined in terms of a $p$-norm $q$ then $B(x) = \{y : f(x) - f(y)\} = \{y : [\frac{q(x)}{\lambda}] = [\frac{q(y)}{\lambda}]\}$.

**PROPOSITION 4.1.** If $x \in X$ and $z \in B(x)$ then $B(z) = B(x)$.

**PROOF.** If $z \in B(x)$ then $[\frac{q(x)}{\lambda}] = [\frac{q(z)}{\lambda}]$. Hence

$$B(z) = \{y : [\frac{q(y)}{\lambda}] = [\frac{q(z)}{\lambda}] = [\frac{q(x)}{\lambda}]\} = B(x).$$
Let \( q_1 \) and \( q_2 \) be \( p \)-norms on \( X \) which are not identically zero and are such that, for every scalar \( \alpha \) and every \( x \in X \),

\[
q_1(\alpha x) = |\alpha|^{p_1} q_1(x) \text{ and } q_2(\alpha x) = |\alpha|^{p_2} q_2(x).
\]

**LEMMA 4.4.** If \( S_1 = \{x : q_1(x) \leq \lambda \} \) and \( S_2 = \{x : q_2(x) \leq \lambda \} \)
then \( S_1 \cap S_2 \neq \{0\} \).

**PROOF.** The function \( q_1 \) is not identically zero on \( X \) so there is

\[
a z \in X \text{ such that } q_1(z) \neq 0. \text{ It follows that } q_1(1) = 1 \text{ hence } S_1 \neq \{0\}. \text{ Let } y \in S_1 \text{ such that } y \neq 0. \text{ If } q_2(y) \leq \lambda \text{ we are finished so suppose } q_2(y) > \lambda. \text{ Let } \beta \text{ be a real number such that }
\]

\[
0 < \beta < \left(\frac{\lambda}{q_2(y)}\right)^{p_2} < 1. \text{ Then } q_1(\beta y) = \beta^{p_1} q_1(y) < q_1(y) \leq \lambda \text{ and }
\]

\[
q_2(\beta y) = \beta^{p_2} q_2(y) < \left(\frac{\lambda}{q_2(y)}\right)^{p_2} q_2(y) = \lambda \text{ so } 0 \neq \beta y \in S_1 \cap S_2.
\]

Let \( \tau_1 \) and \( \tau_2 \) be the elementary length topologies determined with the aid of \( q_1 \) and \( q_2 \), respectively. As in the proof of lemma 3.1 of chapter III, the proof of the next lemma follows from proposition 4.1.

**LEMMA 4.5.** \( \tau_1 \subset \tau_2 \) if and only if for every \( x \in X \), \( z \in B_1(x) \)
implies that \( B_2(z) \subset B_1(x) \).

**THEOREM 4.8.** If \( q_1 \neq q_2 \), then \( \tau_1 \neq \tau_2 \).

**PROOF.** CASE 1. Suppose \( \left(\frac{\lambda}{q_1(y)}\right)^{p_1} \neq \left(\frac{\lambda}{q_2(y)}\right)^{p_2} \) for some \( y \in X \).

If \( \left(\frac{\lambda}{q_1(y)}\right)^{p_1} < \left(\frac{\lambda}{q_2(y)}\right)^{p_2} \) let \( \alpha \) be a real number such that
\[
\left(\frac{\lambda}{q_1(y)}\right)^{p_1} < \alpha < \left(\frac{\lambda}{q_2(y)}\right)^{p_2}.
\]
Then \(q_1(\alpha y) = \alpha^{p_1} q_1(y) > \frac{\lambda}{q_1(y)} q_1(y) = \lambda\)
and \(q_2(\alpha y) = \alpha^{p_2} q_2(y) < \frac{\lambda}{q_2(y)} q_2(y) = \lambda\).
By lemma 4.4 there is a
\[
z \in S_1 \cap S_2 \text{ such that } z \neq 0.
\]
Then \(\alpha y \in B_2(z) = \left\{ w : \left[ \frac{q_2(w)}{\lambda} \right] = 1 \right\} \text{ and } \alpha y \notin B_1(z) \text{ since } q_1(\alpha y) > \lambda.
\]
Thus \(z \in B_1(z) \not\subseteq B_2(z) \subseteq B_1(z) \text{ so } \tau_1 \not\subseteq \tau_2 \text{ by lemma 4.5. If}
\]
\[
\left(\frac{\lambda}{q_2(y)}\right)^{p_2} < \left(\frac{\lambda}{q_1(y)}\right)^{p_1}
\]
a similar argument shows that \(\tau_2 \not\subseteq \tau_1\).

**CASE 2.** Suppose that \(x \neq 0\) implies that
\[
\left(\frac{\lambda}{q_1(x)}\right)^{p_1} = \left(\frac{\lambda}{q_2(x)}\right)^{p_2}.
\]
If \(p_1 = p_2\) then \(q_1 = q_2\) so assume that \(p_2 < p_1\). It is easy to verify
that \(\left(\frac{\lambda}{q_1(x)}\right)^{p_1} = \left(\frac{\lambda}{q_2(x)}\right)^{p_2}\) if and only if \(\left(\frac{q_1(x)}{\lambda}\right)^{p_1} = \left(\frac{q_2(x)}{\lambda}\right)^{p_2}\).
It is also true that \(\left(\frac{q_1(x)}{\lambda}\right)^{p_1} = \left(\frac{q_2(x)}{\lambda}\right)^{p_2}\) if and only if \(\left(\frac{q_1(x)}{\lambda}\right)^{p_2} = \frac{q_2(x)}{\lambda}\).
It follows that if \(m > 1\) then \(m - 1 < \frac{q_1(x)}{\lambda} \leq m\) if and only if
\[
\frac{p_2}{p_1} \leq \frac{q_2(x)}{\lambda} \leq \frac{p_2}{p_1}.
\]
Since \(q_1\) is not identically zero there is a
\[
y \in X \text{ such that } q_1(y) \neq 0.
\]
If \(z = \left(\frac{2\lambda}{q_1(y)}\right)^{p_1} y\) then \(q_1(z) = 2\lambda\).
Hence \( 1 < \frac{q_1(z)}{\lambda} \leq 2 \) and this implies that \( 1 < \frac{q_2(z)}{\lambda} \leq 2 \). If
\[
\alpha = \frac{2\lambda}{q_2(z)} \quad \text{then} \quad \frac{q_2(\alpha z)}{\lambda} = 2 = \left[ \frac{q_2(z)}{\lambda} \right] \quad \text{and this implies that}
\]
\[\alpha z \in B_2(z). \quad \text{But} \quad \frac{q_2(\alpha z)}{\lambda} = 2 \geq 2 \quad \text{and this implies that} \quad \alpha z \notin B_1(z)
\]
since \( 1 < \frac{q_1(x)}{\lambda} \leq 2 \) if and only if \( 1 < \frac{q_2(x)}{\lambda} \leq 2 \). Hence \( B_2(z) \not\subset B_1(z) \) and it follows from lemma 4.5 that \( \tau_1 \not\subset \tau_2 \). If \( p_1 < p_2 \) a similar argument shows that \( \tau_2 \not\subset \tau_1 \).

The topology of a locally bounded topological linear space may be given by a \( p \)-norm or a quasi-norm. It is therefore natural to ask if a \( p \)-norm and a quasi-norm can generate the same elementary length topology. The answer is contained in the next theorem which deals with a more general case.

**THEOREM 4.9.** Let \( q_1 \) be a \( p \)-norm, \( q \) a non-negative, absolutely homogeneous function and let \( \tau_1 \) and \( \tau \) be the elementary length topologies on \( X \) determined by \( q_1 \) and \( q \) respectively. If \( q \neq q_1 \), then \( \tau_1 \neq \tau \).

**PROOF.** We consider two cases:

**CASE 1.** Suppose \( \{0\} = q_1^{-1}(0) \neq q^{-1}(0) \) and let \( y \in q^{-1}(0) \) such that \( y \neq 0 \). If \( z = \left( \frac{1}{q_1(\lambda)} \right)^\frac{1}{p_1} y \) then \( q_1(z) = \lambda \). Now, \( q(z) = 0 \) so \( 0 \in B(z) \) but \( 0 \notin B_1(z) \) hence \( B(z) \not\subset B_1(z) \) and it follows that \( \tau_1 \not\subset \tau \).
CASE 2. Suppose $q_1^{-1}(0) = q^{-1}(0)$. Since the proof of theorem 4.8 did not require sub-additivity the proof of theorem 4.8 with $p_2 = 1$ may be used for this case.
V. LOCALLY CONVEX TOPOLOGICAL VECTOR SPACES

A. LOCALLY CONVEX SPACES AND TIKHONOV SEMI-FIELDS

The following definition and theorems are given in [5].

DEFINITION 5.1. A topological vector space is locally convex if it has a base at 0 consisting of convex sets.

THEOREM 5.1. The topology of a locally convex space X can always be defined by a family of pseudo-norms on X.

THEOREM 5.2. If X is a locally convex topological linear space with topology generated by a family \( P = (p_\alpha)_{\alpha \in \Delta} \) of pseudo-norms, then X is a \( T_1 \) space if and only if to each \( x \neq 0 \) corresponds \( p_\alpha \in P \) such that \( p_\alpha(x) \neq 0 \).

We now state some definitions and results which are given in [7].

Let \( \Delta \) be an arbitrary set and let \( R^\Delta \) be the set of all real functions \( f: \Delta \rightarrow R \). For \( q \in \Delta \) and \( a, b \in R \) let \( U^q_{a,b} \) be the set of all \( f \in R^\Delta \) that satisfy the condition \( a < f(q) < b \). Taking the sets \( U^q_{a,b} \) as a subbase we obtain a topology for \( R^\Delta \) and \( R^\Delta \) with this topology is called a Tikhonov semi-field. In what follows, for any set \( \Delta \), \( R^\Delta \) will always denote a Tikhonov semi-field.

Let \( K^\Delta \) be the set of all \( f \in R^\Delta \) such that \( f(q) > 0 \) for every \( q \in \Delta \). The set \( K^\Delta \) is called the cone of strictly positive elements and its closure \( \overline{K^\Delta} \) consists of all functions \( f \in R^\Delta \) for which \( f(q) > 0 \) for all \( q \in \Delta \). Addition and multiplication in \( R^\Delta \) are defined coordinatewise: for \( f, g \in R^\Delta \) and \( q \in \Delta \), \( (f+g)(q) = f(q) + g(q) \), \( (fg)(q) = f(q)g(q) \). Thus an order may be defined on \( R^\Delta \) by \( f \geq g \) iff \( f - g \in K^\Delta \).
DEFINITION 5.2. Let $X$ be an arbitrary set. A mapping $\rho : X \times X \to K^\Delta$ is a metric on $X$ over $R^\Delta$ if the following conditions are satisfied:

1) $\rho(x,y) = 0$ if and only if $x = y$;
2) $\rho(x,y) = \rho(y,x)$;
3) $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$.

A set $X$ with a metric $\rho$ over $R^\Delta$ is called a metric space over $R^\Delta$.

THEOREM 5.3. Let $(X,\rho)$ be a metric space over $R^\Delta$, $U$ a neighborhood of the origin in $R^\Delta$, and $x \in X$. If $\Omega(x,U) = \{y \in X : \rho(x,y) \in U\}$ then the family of sets $\Omega(x,U)$, as $U$ varies over the neighborhoods of the origin in $R^\Delta$, can be taken as neighborhoods of $x$ which introduces a topology on $X$. This is called the natural topology of $X$.

For $f \in R^\Delta$ define $|f|$ by $|f|(q) = |f(q)|$.

THEOREM 5.4. Let $\Delta$ be an arbitrary set. The mapping $\rho : R^\Delta \times R^\Delta \to K^\Delta$ defined by $\rho(f,g) = |f-g|$ makes $R^\Delta$ into a metric space over $R^\Delta$. The natural topology of this metric space coincides with the initial topology on $R^\Delta$.

The following definition is given in [8].

DEFINITION 5.3. Let $X$ be a linear space over the real or complex numbers. A mapping $|| \ || : X \to K^\Delta$ is a norm on $X$ over $R^\Delta$ if it satisfies the following conditions:

1) $||x|| = 0$ if and only if $x = 0$;
2) $||\alpha x|| = |\alpha| ||x||$ for every scalar $\alpha$ and every $x \in X$;
3) $||x+y|| \leq ||x|| + ||y||$.

If $|| \ ||$ is a norm on $X$ over $R^\Delta$ it is easy to see that $\rho(x,y) = ||x-y||$ is a metric on $X$ over $R^\Delta$. If the topology of a
A topological vector space $X$ is the natural topology of $X$ as a metric space over $\mathbb{R}^\Delta$ and the metric is defined in terms of a norm over $\mathbb{R}^\Delta$ then $X$ is said to be normable over $\mathbb{R}^\Delta$.

The function $\| \| \| \|$ of definition 5.3 will be called a pseudo-norm over $\mathbb{R}^\Delta$ if it satisfies 2) and 3) but may be zero at some $x \neq 0$. It is clear that the definitions of pseudo-metric over $\mathbb{R}^\Delta$ and pseudo-normable over $\mathbb{R}^\Delta$ may be defined by making the appropriate alterations of the definitions of metric over $\mathbb{R}^\Delta$ and normable over $\mathbb{R}^\Delta$.

By theorem 5.1, the topology of a locally convex topological vector space $X$ can be defined by a family $(p_{\alpha})_{\alpha \in \Delta} = P$ of pseudo-norms defined on $X$. As in [8], we define $\| x \| : X \to \mathbb{K}^\Delta$, where $\Delta$ is the index set of the family $P$, by $\| x \| (\alpha) = p_{\alpha}(x)$ for $\alpha \in \Delta$ and $x \in X$. It is easy to verify that this mapping is a pseudo-norm on $X$ over $\mathbb{R}^\Delta$. It follows from theorem 5.2 that this mapping is a norm on $X$ over $\mathbb{R}^\Delta$ if $X$ is a $T_1$ space. These results are summarized in the following theorem due to Iseki [8].

**THEOREM 5.5.** Any locally convex topological linear space $X$ is pseudo-normable over $\mathbb{R}^\Delta$ for some set $\Delta$. If $X$ is a $T_1$ space then it is normable over $\mathbb{R}^\Delta$ for some $\Delta$.

**B. ELEMENTARY LENGTH TOPOLOGIES FOR LOCALLY CONVEX SPACES**

Let $(X, \| \| \| \|)$ be a locally convex space where $\| \| \| \|$ is a pseudo-norm or a norm over $\mathbb{R}^\Delta$.

**DEFINITION 5.4.** A length on $X$ over $\mathbb{R}^\Delta$ is a function $f : X \to \mathbb{R}^\Delta$ such that for any $\alpha \in \Delta$
N1) \[ ||x|| (\alpha) = n\lambda \Rightarrow f(x)(\alpha) = n\lambda; \]
N2) \[ ||x_1|| (\alpha) \leq ||x_2|| (\alpha) \Rightarrow f(x_1)(\alpha) \leq f(x_2)(\alpha); \]
N3) \[ f(x + ||x|| (\alpha) x)(\alpha) = f(x)(\alpha) + \lambda \text{ if } ||x|| (\alpha) > 0. \]

DEFINITION 5.5. For any real number \( r \) define \([r]\) and \( \{r\} \) as in definition G2. Then let \([\ ] : \mathbb{R}^\Delta \to \mathbb{R}^\Delta \) and \( \{\} : \mathbb{R}^\Delta \to \mathbb{R}^\Delta \) be defined by \([f](\alpha) = [f(\alpha)]\) and \(\{f\}(\alpha) = \{f(\alpha)\}\).

THEOREM 5.6. A function \( f : X \to \mathbb{R}^\Delta \) is a length on \( X \) over \( \mathbb{R}^\Delta \) if and only if \( f \) has one of the two forms

1) \[ f(x) = f_\lambda \left[ \frac{||x|| - g}{f_\lambda} \right] \text{ where } f_\lambda (\alpha) = \lambda > 0 \text{ and } 0 \leq g(\alpha) < \lambda \text{ for every } \alpha \in \Delta; \]

2) \[ f(x) = f_\lambda \left[ \frac{||x|| - g}{f_\lambda} \right] \text{ where } f_\lambda (\alpha) = \lambda > 0 \text{ and } 0 < g(\alpha) \leq \lambda \text{ for every } \alpha \in \Delta. \]

PROOF. Suppose that \( f \) is a length over \( \mathbb{R}^\Delta \) and \( \alpha \in \Delta \). Since

\[ ||x|| (\alpha) = p_\alpha (x), \text{ for each } \alpha \in \Delta \text{ the conditions N1), N2) and N3) of \]
definition 5.4 imply that \( f(x)(\alpha) \) is a length on \( X \) defined in terms of \( p_\alpha \). It then follows from theorem 3.1 that

\[ f(x)(\alpha) = \lambda \left[ \frac{p_\alpha (x) - a_\alpha}{\lambda} \right] \]

where \( \lambda > 0 \) is fixed and \( 0 \leq a_\alpha < \lambda \) or

\[ f(x)(\alpha) = \lambda \left[ \frac{p_\alpha (x) - a_\alpha}{\lambda} \right] \]

where \( 0 < a_\alpha \leq \lambda \). Now define \( f_\alpha : \mathbb{R}^\Delta \in \mathbb{R}^\Delta \) by \( f_\alpha (\alpha) = \lambda \text{ for every } \alpha \in \Delta \) and define \( g : \mathbb{R}^\Delta \in \mathbb{R}^\Delta \) by \( g(\alpha) = a_\alpha \). Then \( f \) has form 1) or form 2).

If \( f \) has form 1 or form 2 then, by theorem 3.1, for each \( \alpha \in \Delta \)

\( f(x)(\alpha) \) is a length on \( X \) defined in terms of \( ||x|| (\alpha) = p_\alpha (x) \). It

follows that N1), N2), and N3) of definition 5.4 hold.

The statement of the next theorem requires the following concepts which can be found in [7].
Let \((X, \rho)\) be a metric space over \(R^A\) and \(H\) a directed set. A mapping \(x : H \rightarrow X\) is called a sequence of type \(H\) on \(X\). The image \(x(\xi)\) of \(\xi \in H\) under \(x\) is denoted by \(x_\xi\) and the sequence is written in the usual form \(x = (x_\xi), \xi \in H\).

A sequence \(x\) of type \(H\) in \(X\) over \(R^A\) is said to converge to \(y \in X\) if the following condition is satisfied: for every neighborhood of the origin \(U \subset R^A\) there is an element \(\xi_U \in H\) such that \(\rho(x_\xi, y) \in U\) for \(\xi > \xi_U\). Convergence is denoted by writing \(x_\xi \rightarrow y\) or \(\lim_{\xi \in H} x_\xi = y\).

In other words, \(\lim_{\xi \in H} x_\xi = y\) in \(X\) if and only if \(\lim_{\xi \in H} \rho(x_\xi, y) = 0\) in \(R^A\).

Let \(A = \{f_\lambda \in R^A : \lambda \in (0,1)\text{ and } f_\lambda(\alpha) = \lambda \text{ for every } \alpha \in \Delta\}\). The set \(A\) is directed by the relation \(f_\lambda \ll f_\lambda\) if and only if \(\lambda_2 \leq \lambda_1\). Also, for each \(\lambda \in (0,1)\) and each fixed \(x \in X\), the mapping \(f_\lambda \rightarrow f(x)\), where \(f(x)\) has form 1) or form 2) of theorem 5.6, is a sequence of type \(A\) on \(R^A\).

**Theorem 5.7.** If \(f\) is a length on \(X\) over \(R^A\) for each \(f_\lambda \in A\) then
\[
\lim_{f_\lambda \in A} f(x) = ||x|| \text{ for each } x \in X.
\]

**Proof.** The set \(U = \bigcap_{i=1}^{n} U_{\alpha_i} = \{h \in R^A : h > 0, \alpha_i \in \Delta \text{ and } -\varepsilon < h(\alpha_i) < \varepsilon \text{ for } 1 \leq i \leq n\}\) is a basic open set about 0 in \(R^A\). It suffices to show that for each such \(U\) there is a \((f_\lambda)_U \in A\) such that
\[
\rho(f(x), ||x||) = \left| f(x) - ||x|| \right| \in U \text{ for all } f_\lambda \in A \text{ such that } f_\lambda \gg (f_\lambda)_U.
\]
For each \(\alpha_i, 1 \leq i \leq n\),
\[
\left| f(x) - ||x|| \right| (\alpha_i) = \left| f(x)(\alpha_i) - ||x||(\alpha_i) \right| = \left| f(x)(\alpha_i) - p_{\alpha_i}(x) \right|
\]
where \( f(x)(\alpha_i) = \lambda \left[ \frac{p_{\alpha_i}(x) - a_{\alpha_i}}{\lambda} \right] \) or \( f(x)(\alpha_i) = \lambda \left\{ \frac{p_{\alpha_i}(x) - a_{\alpha_i}}{\lambda} \right\} \).

Then, by theorem 3.1, \( f(x)(\alpha_i) \rightarrow p_{\alpha_i}(x) \) as \( \lambda \rightarrow 0 \) for \( 1 \leq i \leq n \) and \( x \in X \) so there is a \( \delta_i > 0 \) such that \( \left| f(x)(\alpha_i) - \|x\|_1(\alpha_i) \right| < \varepsilon \) for \( \lambda < \delta_i \). Choose \( \lambda_0 \in (0,1) \) such that \( \lambda_0 < \min \{ \delta_i \} \) then let \( 1 \leq i \leq n \)

\( f_{\lambda} = f_{\lambda_0} \). If \( f_{\lambda} \gg f_{\lambda} \) then \( \lambda \rightarrow \lambda_0 \) so \( f(x) - \|x\| \in U \) hence

\[
\lim_{f_{\lambda} \in A} f(x) = \|x\|.
\]

**DEFINITION 5.6.** An elementary length over \( R^A \) is a function \( f : X \rightarrow R^A \) which satisfies N1), N2) and

N3') \( f(nx) \leq nf(x) \) for every \( x \in X \) and \( n = 0, 1, 2, \cdots \).

**THEOREM 5.8.** A function \( f : X \rightarrow R^A \) is an elementary length over \( R^A \) if and only if \( f(x) = f_{\lambda} \left( \frac{\|x\|}{f_{\lambda}} \right) \).

**PROOF.** Let \( f \) be an elementary length over \( R^A \) and \( \alpha \in \Delta \). The mapping \( f_{\alpha} : X \rightarrow R^A \) defined by \( f_{\alpha}(x) = f(x)(\alpha) \) satisfies

N1) \( \|x\|_{\alpha}(\alpha) = p_{\alpha}(x) = n\lambda \Rightarrow f_{\alpha}(x) = n\lambda ; \)

N2) \( \|x_1\|_{\alpha}(\alpha) = p_{\alpha}(x_1) \leq \|x_2\|_{\alpha}(\alpha) = p_{\alpha}(x_2) \Rightarrow f_{\alpha}(x_1) \leq f_{\alpha}(x_2) ; \)

N3') \( f_{\alpha}(nx) \leq nf_{\alpha}(x) . \)

It follows from theorem 3.1 that \( f_{\alpha}(x) = f(x)(\alpha) \) is an elementary length on \( X \) defined in terms of the pseudo-norm \( p_{\alpha}(x) \) and that

\[
f_{\alpha}(x) = f(x)(\alpha) = \lambda \left[ \frac{\|x\|_{\alpha}(\alpha)}{\lambda} \right] = f_{\lambda} \left[ \frac{\|x\|}{f_{\lambda}} \right] (\alpha) \]

Thus \( f(x) = f_{\lambda} \left[ \frac{\|x\|}{f_{\lambda}} \right] . \)
If \( f(x) = f_{\lambda} \left[ \frac{||x||}{f_{\lambda}} \right] \) then \( f(x)(\alpha) \) is an elementary length on \( X \) defined in terms of \( p_{\alpha}(x) \). By theorem 3.1 \( f(x)(\alpha) \) satisfies N1), N2), and N3') above so \( f \) is an elementary length over \( R^\Delta \).

**COROLLARY 5.1.** An elementary length over \( R^\Delta \) is a length over \( R^\Delta \).

**DEFINITION 5.7.** A function \( f : X \to R^\Delta \) is sub-additive if \( f(x+y) \leq f(x) + f(y) \).

**THEOREM 5.9.** A length \( f \) over \( R^\Delta \) is an elementary length over \( R^\Delta \) if and only if it is sub-additive.

**PROOF.** If \( f \) is a length over \( R^\Delta \) which satisfies \( f(x+y) \leq f(x) + f(y) \) then, by induction, \( f(nx) \leq nf(x) \) for every \( x \in X \).

If \( f \) is an elementary length over \( R^\Delta \) then \( f(x) = f_{\lambda} \left[ \frac{||x||}{f_{\lambda}} \right] \).

\[
so \quad f(x+y) = f_{\lambda} \left[ \frac{||x+y||}{f_{\lambda}} \right] \leq f_{\lambda} \left[ \frac{||x|| + ||y||}{f_{\lambda}} \right].
\]

It is shown in [4] that for real numbers \( r,s \geq 0 \) \([r+s] \leq [r] + [s]\). Then \( f(x+y)(\alpha) \leq f_{\lambda} \left[ \frac{||x|| + ||y||}{f_{\lambda}} \right](\alpha) = \lambda \left[ \frac{||x||(\alpha) + ||y||}{\lambda} \right] \leq \lambda \left[ \frac{||x||}{\lambda} \right] + \lambda \left[ \frac{||y||}{\lambda} \right] = \left( f_{\lambda} \left[ \frac{||x||}{f_{\lambda}} \right] + f_{\lambda} \left[ \frac{||y||}{f_{\lambda}} \right] \right)(\alpha) \) so \( f(x+y) \leq f(x) + f(y) \).

**COROLLARY 5.2.** If \( X \) is a locally convex topological linear space and \( f \) is an elementary length on \( X \) over \( R^\Delta \) then \( \rho(x,y) = f(x-y) \) is a pseudo-metric on \( X \) over \( R^\Delta \). If \( X \) is a \( T_1 \) space then \( \rho(x,y) = f(x-y) \) is a metric on \( X \) over \( R^\Delta \).
PROOF. If $x = y$ then $\rho(x, y) = f_{\lambda}\left[\frac{0}{f_{\lambda}}\right] = 0$ and $\rho(x, y) = \rho(y, x)$ follows from $||-x|| = ||x||$. $\rho(x, y) = f(x-y) = f((x-z) + (z-y)) \leq \rho(x, z) + \rho(z, y)$. If $f(x-y) = f_{\lambda}\left[\frac{|x-y|}{f_{\lambda}}\right] = 0$ then $||x-y|| = 0$.

If $X$ is a $T_1$ space it follows from theorem 5.2 that $||x-y|| = 0$ if and only if $x = y$.

The natural topology for the space $(X, f(x-y))$ will now be considered. Suppose that there is an $\alpha_j \in \Delta$ such that $p_{\alpha_j} \in P$ is a norm. Let $x \in X$ and consider $\Omega(x, U)$ where $U = U_{\lambda}^{\alpha_j}$.

$$\Omega(x, U) = \left\{ y : f(x-y) = f_{\lambda}\left[\frac{|x-y|}{f_{\lambda}}\right] \in U_{\lambda}^{\alpha_j} \right\} = \left\{ y : \lambda\left[\frac{|x-y|}{\lambda}^{(\alpha_j)}\right] < \lambda \right\} = \left\{ y : \frac{p_{\alpha_j}(x-y)}{\lambda} < \lambda \right\}$$

so $y \in \Omega(x, U)$ implies that $p_{\alpha_j}(x-y) = 0$.

Hence $\Omega(x, U) = \{x\}$ so the natural topology is discrete in this case.

A further discussion of these topologies requires the following definition which is given in [5].

DEFINITION 5.8. A family $(p)_{\alpha \in \Delta}$ of pseudo-norms is saturated if for any finite sub-family $\{p_{\alpha_1}, \ldots, p_{\alpha_n}\}$ the pseudo-norm $p$ defined by $p(x) = \max_{1 \leq j \leq n} \left\{ p_{\alpha_j}(x) \right\}$ also belongs to the family.

THEOREM 5.10. If the family $(p)_{\alpha \in \Delta}$ is saturated and for each $\alpha \in \Delta$ $p_{\alpha}$ fails to be a norm then the natural topology for $(X, f(x-y))$ is not discrete.

PROOF. We show that every neighborhood $\Omega(0, U)$ of 0 contains
some $y \neq 0$. It is sufficient to take $U = \bigcap_{i=1}^{n} U_{\alpha_i}$ since sets of the form $\Omega(x,U)$ are a base at $x$ for $(X,f(x-y))$. Now,

$$\Omega(0,U) = \left\{ y : \lambda \left( \frac{p_{\alpha_i}(y)}{\lambda} \right) < \lambda \text{ for } 1 \leq i \leq n \right\} \text{ so } y \in \Omega(0,U) \text{ iff } p_{\alpha_i}(y) = 0 \text{ for } 1 \leq i \leq n. \text{ Since } p = \max_{1 \leq i \leq n} p_{\alpha_i} \text{ is in the family } (p_{\alpha_i})_{\alpha} \in \Delta, \text{ } p \text{ fails to be a norm. This means that there is some } y \in X \text{ such that } p(y) = 0 \text{ and } y \neq 0. \text{ Thus } p_{\alpha_i}(y) = 0 \text{ for } 1 \leq i \leq n \text{ so } y \in \Omega(0,U).

**THEOREM 5.11.** $(X,f(x-y))$ is a Hausdorff space if and only if $(X,|||)\!)$ is a Hausdorff space.

**PROOF.** Suppose that $(X,f(x-y))$ is a Hausdorff space and $x \in X$ such that $x \neq 0$. If $p_{\alpha}(x) = 0$ for every $\alpha \in \Delta$, then $||x|| = 0$.

Hence $f(x-0) = p(x,0) = f_{\lambda} \left( \frac{||x||}{f_{\lambda}} \right) = 0$ and therefore $p(x,0) \in U$ for every neighborhood $U$ of the origin in $R^{\Delta}$. This is a contradiction since, in this case, $\{0,x\} \in \Omega(x,U_1) \cap \Omega(0,U_2)$ for every choice of $U_1$ and $U_2$. By theorem 5.2, $(X,|||)\!)$ is Hausdorff.

Suppose that $(X,|||)\!)$ is Hausdorff and $x,y \in X$ such that $x-y \neq 0$. By theorem 5.2, there is an $\alpha_j \in \Delta$ such that $||x-y||_{(\alpha_j)} = p_{\alpha_j}(x-y) \neq 0$. Let $U = U_{\alpha_j}^{\lambda}$ and let $\Omega(x,U)$, $\Omega(y,U)$ be neighborhoods of $x$ and $y$ in $(X,f(x-y))$. Suppose that $z \in \Omega(x,U) \cap \Omega(y,U)$. Then $f(x-z)(\alpha_j) = \lambda \left( \frac{||x-z||_{(\alpha_j)}}{\lambda} \right) = 0$ and $f(z-y) = f(y-z) = \lambda \left( \frac{||z-y||_{(\alpha_j)}}{\lambda} \right) = 0$ so $||x-z||_{(\alpha_j)} = ||z-y||_{(\alpha_j)} = 0$. But this means that
\[ p_{\alpha_j}(x-y) = ||x-y||_{(\alpha_j)} \leq ||x-z||_{(\alpha_j)} + ||z-y||_{(\alpha_j)} = 0 \] which is a contradiction. Thus \( \Omega(x,U) \) and \( \Omega(y,U) \) are disjoint neighborhoods of \( x \) and \( y \) so \( (X,f(x-y)) \) is Hausdorff.

**EXAMPLE 5.1.** Let \( \Omega \neq \emptyset \) be an open subset of \( \mathbb{R}^n \) and let \( C(\Omega) \) be the linear space of all continuous functions from \( \Omega \) into the complex numbers. For every compact subset \( K \) of \( \Omega \) the function \( q_K \) defined by \( q_K(f) = \max_{x \in K} |f(x)| \) is a pseudo-norm on \( C(\Omega) \). The family \( (q_K) \) where \( K \) runs through the compact subsets of \( \Omega \), generates a locally convex topology on \( C(\Omega) \). It is easy to show that this family is saturated and we now show that each \( q_K \) fails to be a norm. If \( K \) is a compact subset of \( \Omega \) then \( \Omega \setminus K \) is an open subset of \( \Omega \). Let \( d(x,y) = |x-y| \) be the usual metric for \( \mathbb{R}^n \), \( x_0 \in \Omega \setminus K \) and \( S_{r_0}(x_0) = \{ x : |x-x_0| < r_0 \} \) where \( r_0 \) is chosen so that \( S_{r_0}(x_0) \subseteq \Omega \setminus K \). Define \( f \in C(\Omega) \) by

\[ f(x) = \begin{cases} \exp \left(-\frac{r^2_0}{r^2_0 - |x-x_0|^2}\right) & \text{if } |x-x_0| < r_0 \\ 0 & \text{if } |x-x_0| \geq r_0 \end{cases} \]

Since \( f(x) = 0 \) on \( K \), \( q_K(f) = 0 \) hence \( q_K \) fails to be a norm. If \( f \in C(\Omega) \) such that \( f(x_0) \neq 0 \) then \( q_{\{x_0\}}(f) \neq 0 \) so \( C(\Omega) \) is a Hausdorff space. Then by theorems 5.10 and 5.11 the space \( (C(\Omega), f(x-y)) \) is non-discrete and Hausdorff.

**EXAMPLE 5.2.** The space of distributions on an open subset of \( \mathbb{R}^n \), as described in [9], also satisfies the conditions of theorems
5.10 and 5.11. The verification is rather lengthy and will be omitted.

An elementary length topology which is Hausdorff does not correspond very well with the properties that might be expected of such a topology. In the normed linear space case the elementary length topologies on \( X \) are always non-\( T_0 \). A development along the same lines as the one for the normed linear space case will now be given. In what follows, a length over \( \mathbb{R}^\Delta \) or an elementary length over \( \mathbb{R}^\Delta \) may be used. For convenience, an elementary length will be chosen.

Let \( f \) be an elementary length over \( \mathbb{R}^\Delta \). The mapping \( d : X \times X \to \mathbb{R}^\Delta \) defined by \( d(x,y) = |f(x) - f(y)| \) is a pseudo-metric on \( X \) over \( \mathbb{R}^\Delta \) and the natural topology for \( X \) determined by \( d \) will now be considered. A base at \( x \in X \) consists of sets of the form \( \Omega(x,U) \) where \( U = \cap_{i=1}^n U_i \). Throughout the remainder of this chapter, elementary length topology will mean the natural topology of the pseudo-metric \( d \) on \( X \) over \( \mathbb{R}^\Delta \).

**Theorem 5.12.** The space \((X,d)\) is never \( T_0 \).

**Proof.** Let \( x \in X \) such that \( x \neq 0 \) and let \( y = \alpha x \) where \( \alpha \neq 1 \) is a scalar such that \( |\alpha| = 1 \). Now, \( d(x,y) = |f(x) - f(y)| \) and \( f(y) = f_{\lambda} \left( \frac{|\alpha x|}{f_{\lambda}} \right) = f_{\lambda} \left( \frac{|x|}{f_{\lambda}} \right) = f(x) \) so \( d(x,y) = 0 \). But then \( \{x\} \) and \( \{y\} \) have the same closure in \((X,d)\) so this space is not \( T_0 \).

If \((X,\tau)\) is a locally convex space such that \( \tau \) is generated by a family \((p_{\alpha})_{\alpha \in \Delta} \) of pseudo-norms, and \( p_{\delta} \) is a continuous pseudo-norm
on X then \((p_\alpha)_{\alpha \in \Delta} \cup \{p_\delta\}\), where \(\delta \notin \Delta\), also generates the topology \(\tau\). In other words, the topology of X may be generated by pseudo-norms over different semi-fields. It seems natural to call two such pseudo-norms equivalent and to ask if the elementary length topologies generated by equivalent pseudo-norms over semi-fields are comparable. A necessary condition for the simple case described above will now be given.

Suppose \((p_\alpha)_{\alpha \in \Delta}\) is a family of pseudo-norms which generates the topology \(\tau\) and \(p_\delta\) is a continuous pseudo-norm on \((X,\tau)\). Let \(t_1\) and \(t_2\) be the elementary length topologies generated by \((p_\alpha)_{\alpha \in \Delta}\) and \((p_\alpha)_{\alpha \in \Delta} \cup \{p_\delta\}\) respectively. Let \(S = \{x : p_\delta(x) = \lambda\}\).

**Theorem 5.13.** If \(t_2 \subset t_1\) then for each \(x \in X\) such that \(p_\delta(x) \neq 0\) there exist a finite set \(J_x \subset \Delta\) and functions \(j : S \to J_x,\ K : S \to \{1, 2, 3, \cdots\}\) such that \(P_j(w)(x) = K(w)p_\delta(x)\) where \(w = \frac{\lambda x}{p_\delta(x)} \in S\).

**Proof.** Let \(\|\|_1\) and \(\|\|_2\) be pseudo-norms over \(R^\Delta\) and \(R^{\Delta_2}\) respectively where \(\Delta_2 = \Delta \cup \{\delta\}\). The elementary length topologies \(t_1\) and \(t_2\) are generated by the pseudo-metrics

\[d_1(x,y) = f_\lambda \left[\frac{\|x\|_1}{f_\lambda}\right] - f_\lambda \left[\frac{\|y\|_1}{f_\lambda}\right]\] and
\[d_2(x,y) = f_\lambda \left[\frac{\|x\|_2}{f_\lambda}\right] - f_\lambda \left[\frac{\|y\|_2}{f_\lambda}\right].\]

If \(x \in S\) then, since \(t_2 \subset t_1\), there is a finite subset \(J_x\) of \(\Delta\) such that \(\Omega_1(x, U_1) = \{y : d_1(x,y) \in U_1\} \subset \Omega_2(x, U_2) = \{y : d_2(x,y) \in U_2\}\) where \(U_1 = \bigcap_{j \in J_x} U_\alpha^j\) and \(U_2 = U_\delta\). For each \(j \in J_x\), let \(\frac{p_\alpha^j(x)}{\lambda} = m_j\). If \(m_j = 0\) for every \(j \in J_x\) then \(p_\alpha^j(2x) = 0\) so
\[ 2x \in \Omega_1(x, U_1) = \left\{ y : \left[ \frac{p_{\alpha_j}(x)}{\lambda} \right] = \left[ \frac{p_{\alpha_j}(y)}{\lambda} \right] \text{ for every } j \in J_x \right\}. \]

But \[ p_{\delta}(2x) = 2\lambda \text{ so } 2x \notin \Omega_2(x, U_2) = \left\{ y : \left[ \frac{p_{\delta}(y)}{\lambda} \right] = \left[ \frac{p_{\delta}(x)}{\lambda} \right] = 1 \right\} \text{ which} \]

contradicts \[ \Omega_1(x, U_1) \subset \Omega_2(x, U_2). \]

Thus \[ 0 < \frac{p_{\alpha_j}(x)}{\lambda} \leq m_j \neq 0 \text{ for some } j \in J_x. \]

Suppose that \[ p_{\alpha_j}(x) < m_j \lambda \text{ for every } j \in J_x \text{ such that } p_{\alpha_j}(x) \neq 0. \]

If \[ p_{\alpha_j}(x) \neq 0 \text{ then } m_j - 1 < \frac{p_{\alpha_j}(x)}{\lambda} < m_j \text{ so } 1 < \frac{\lambda m_j}{p_{\alpha_j}(x)} \]

It then follows that \[ 1 < b = \min \left\{ \frac{\lambda m_j}{p_{\alpha_j}(x)} : j \in J_x \text{ and } p_{\alpha_j}(x) \neq 0 \right\}. \]

Let \[ 1 < \beta < b. \]

If \[ p_{\alpha_j}(x) = 0 \text{ then } p_{\alpha_j}(\beta x) = 0 \text{ and if } p_{\alpha_j}(x) \neq 0 \]

for \[ j \in J_x \text{ then } m_j - 1 < \frac{p_{\alpha_j}(x)}{\lambda} < \frac{\beta p_{\alpha_j}(x)}{\lambda} = \frac{p_{\alpha_j}(\beta x)}{\lambda} < \frac{b p_{\alpha_j}(x)}{\lambda} \]

\[ \frac{\lambda m_j}{p_{\alpha_j}(x)} = m_j \text{ so } \beta x \in \Omega_1(x, U_1). \]

But \[ p_{\delta}(\beta x) = \beta \lambda > \lambda \text{ so } \beta x \notin \Omega_2(x, U_2). \]

This is a contradiction since \[ \Omega_1(x, U_1) \subset \Omega_2(x, U_2). \]

Thus if \[ p_{\delta}(x) = \lambda \text{ then } p_{\alpha_j}(x) = m_j \lambda \text{ for some } j \in J_x. \]

Hence for \[ x \in S \]

there is a mapping \[ j : S \rightarrow J_x \subset \Delta \text{ and a mapping } K : S \rightarrow \{1, 2, 3, \cdots\} \]

such that \[ p_j(x)(x) = m_j(x) \lambda = K(x)p_{\delta}(x). \]

If \[ z \in X \text{ such that } p_{\delta}(z) \neq 0 \text{ then } w = \frac{\lambda z}{p_{\delta}(z)} \in S \text{ hence } p_j(w)\left(\frac{\lambda z}{p_{\delta}(z)}\right) = K(w)\lambda. \]

Therefore \[ p_j(w)(z) = K(w)p_{\delta}(z) \text{ where } j \text{ and } K \text{ are the functions described above.} \]
THEOREM 5.14. Each of the following conditions is sufficient for \( t_2 \subset t_1 \).

1) There is an integer \( k > 0 \) and an \( \alpha \in \Delta \) such that \( p_\alpha(x) = kp_\delta(x) \) for every \( x \in X \).

2) There is a finite subset \( J \) of \( \Delta \) such that \( p_\delta(x) = \max \{p_\alpha(x)\} \) for every \( x \in X \).

PROOF. Suppose 1) holds and let \( d_1(x,y) \) and \( d_2(x,y) \) be defined as in the proof of theorem 5.13. If \( U \) and \( V \) are neighborhoods of the origin in \( \mathbb{R}^\Delta \) then \( \Omega_2(x,U) \cap \Omega_2(x,V) = \{y : d_2(x,y) \in U\} \cap \{y : d_2(x,y) \in V\} = \Omega_2(x,UV) \). Thus if \( \Omega_2(x,U_2) = \Omega_2(x,V_2) \) it suffices to show that \( \Omega_2(x,U_2) \subset t_1 \). For if \( \Omega_2(x,V_2) \) is a \( t_2 \) basic open set about \( x \), then we may take \( V_2 = \bigcap_{\beta \in I} U_\beta^\lambda \) where \( I \) is a finite subset of \( \Delta \). If \( \delta \notin I \) then \( \Omega_2(x,V_2) \subset t_1 \) and if \( \delta \in I \) then \( \Omega_2(x,V_2) = \Omega_2(x,U_2) \cap \Omega_2(x,V_2') \) where \( V_2' = \bigcap_{\beta \in I, \beta \neq \delta} U_\beta^\lambda \). Hence if \( \Omega_2(x,U_2) \subset t_1 \) we have \( \Omega_2(x,V_2) \subset t_1 \) since \( \Omega_2(x,V_2') \subset t_1 \). Let \( \tau_\alpha \) and \( \tau_\delta \) be the elementary length topologies determined with the aid of \( p_\alpha \) and \( p_\delta \) respectively. By theorem 3.2 \( \tau_\delta \subset \tau_\alpha \). If \( U_1 = U_\alpha^\lambda \) then

\[
B_\alpha(x,\lambda) = \Omega_1(x,U_1) = \left\{ y : \left[\frac{p_\alpha(y)}{\lambda}\right] = \left[\frac{p_\alpha(x)}{\lambda}\right] \right\} \subset \Omega_2(x,U_2) = \left\{ y : \left[\frac{p_\delta(y)}{\lambda}\right] = \left[\frac{p_\delta(x)}{\lambda}\right] \right\}
\]

so \( \Omega_2(x,U_2) \subset t_1 \).

Suppose that 2) holds and let \( \Omega_2 = U_\delta^\lambda \). We show that \( \Omega_2(x,U_2) \subset t_1 \) for every \( x \in X \). Let \( U_1 = \bigcap\{U_\alpha^\lambda : \alpha \in J\} \) and

\[
\Omega_1(x,U_1) = \{y : d_1(x,y) \in U_1\} = \left\{ y : \left[\frac{p_\alpha(y)}{\lambda}\right] = \left[\frac{p_\alpha(x)}{\lambda}\right] \right\} \text{ for every } \alpha \in J.
\]
Let \( y \in \Omega_1(x, U_1) \). If \( m_\alpha = \left[ \frac{p_\alpha(x)}{\lambda} \right] \) then \( m_\alpha - 1 < \frac{p_\alpha(y)}{\lambda} \leq m_\alpha \) for every \( \alpha \in J \). Since \( p_\delta(x) = \max\{p_\alpha(x) : \alpha \in J\} \) there is a \( \beta \in J \) such that \( p_\delta(x) = p_\beta(x) \) so \( m_\beta = \left[ \frac{p_\delta(x)}{\lambda} \right] \). Thus \( \Omega_2(x, U_2) = \left\{ z : \left[ \frac{p_\delta(x)}{\lambda} \right] = \left[ \frac{p_\delta(x)}{\lambda} \right] = m_\beta \right\} \). Now, \( m_\alpha < m_\beta \) for every \( \alpha \in J \) and for every \( \gamma \in J \) such that \( m_\gamma = m_\beta \) we have \( m_\beta - 1 < \frac{p_\gamma(y)}{\lambda} \leq m_\beta \). If \( \alpha \in J \) such that \( m_\alpha \neq m_\beta \) then \( m_\alpha < m_\beta - 1 \) so \( \frac{p_\alpha(y)}{\lambda} < m_\alpha < m_\beta - 1 < \frac{p_\beta(y)}{\lambda} \leq m_\beta \). It follows that \( m_\beta - 1 < \frac{p_\delta(y)}{\lambda} \leq m_\beta \) so \( y \in \Omega_2(x, U_2) \). Then \( \Omega_1(x, U_1) \subset \Omega_2(x, U_2) \) so \( \Omega_2(x, U_2) \in t_1 \).

Let \((p_\alpha)_{\alpha \in A}\) be a family of pseudo-norms on \( X \) and let \( ||\ ||\ ) be an ordinary norm on \( X \). Let \( \tau \) be the locally convex topology generated by the family \((p_\alpha)_{\alpha \in A}\). If, for each \( \alpha \in A \), there is an integer \( k_\alpha \) such that \( k_\alpha p_\alpha(x) = ||x|| \) for every \( x \in X \), then \((X, \tau)\) is normable. Thus the next theorem shows that the elementary length topology for a non-normable locally convex space cannot be determined from Gudder's construction for normed linear spaces.

Let \( t_1 \) and \( t_2 \) be elementary length topologies on \( X \) generated with the aid of \((p_\alpha)_{\alpha \in A}\) and the ordinary norm \( ||\ ||\ ) respectively.

**THEOREM 5.15.** If there exists \( \alpha \in A \) such that for every positive integer \( k \) the ordinary norm, \( ||\ ||\ ), is not equal to \( k p_\alpha \) then \( t_1 \neq t_2 \).

**PROOF.** We prove the contrapositive of the theorem: If \( t_1 = t_2 \) then for every \( \alpha \in A \) there is a positive integer \( k_\alpha \) such that \( k_\alpha p_\alpha(x) = ||x|| \) for every \( x \in X \). If \( t_1 \subset t_2 \) then for each \( \alpha \in A \)
let $U_\alpha = U^\alpha$. Then for every $x \in X$, $\Omega(x, U_\alpha) \in t_1 = t_2$. Hence if $t_\alpha$ is the elementary length topology generated by applying the Gudder construction in terms of $p_\alpha$ we have $t_\alpha \subseteq t_2$ since 

$\Omega(x, U) = B_\alpha(x, \lambda) = \left\{ y : \left[ \frac{p_\alpha(y)}{\lambda} \right] = \left[ \frac{p_\alpha(x)}{\lambda} \right] \right\}$. Then by theorem 3.2 $||x|| = k_\alpha p_\alpha(x)$ for some positive integer $k_\alpha$ and every $x \in X$. 
VI. TOPOLOGICAL LINEAR SPACES

Let \((X,t)\) be a topological linear space. We recall a basic theorem [5].

**THEOREM 6.1.** There exists a fundamental system \(\mathcal{U}\) of neighborhoods of 0 such that:

I. Each \(U\) in \(\mathcal{U}\) is balanced and absorbing;

II. If \(U \in \mathcal{U}\) and \(\alpha \neq 0\), \(\alpha U \in \mathcal{U}\);

III. If \(U \in \mathcal{U}\), there exists \(V \in \mathcal{U}\) such that \(V + V \subset U\).

The following two theorems, due to T. L. Hicks [10], are reformulations of results given by D. H. Hyers in [11]. They make possible a unified approach to the study of topological vector spaces; that is, we can view the topology as being given by a "measuring device" over a Tikhonov semifield.

**THEOREM 6.2.** Let \(\mathcal{U}\) be as in theorem 6.1. If \(U \in \mathcal{U}\), \(\rho_U\) denotes the Minkowski functional of \(U\).

1. \(\rho_U(0) = 0\) and \(\rho_U(x) \geq 0\).

2. \(\rho_U(\alpha x) = |\alpha| \rho_U(x)\).

3. Given \(\eta > 0\) and \(U \in \mathcal{U}\), there exists \(\delta > 0\) and \(V \in \mathcal{U}\) such that \(\rho_V(x) < \delta\) and \(\rho_V(y) < \delta\) implies \(\rho_U(x+y) < \eta\).

4. \(U \supset V \ (U \subseteq V)\) implies \(\rho_U(x) \leq \rho_V(x)\).

5. If \((X,t)\) is Hausdorff, \(\rho_U(x) = 0\) for every \(U \in \mathcal{U}\) implies \(x = 0\).

**THEOREM 6.3.** Suppose \(\Delta\) is a directed set and \(\{\rho_U : U \in \Delta\}\) satisfies conditions 1 - 4 of theorem 6.2. For each \(U \in \Delta\) and \(r > 0\), let
Then there exists a unique topology \( t \) for \( X \) such that \( X \) is a topological linear space and \( \{ S_U(r) : r > 0, U \in \Delta \} \) is a fundamental system of neighborhoods of 0.

Let \( \Delta \) be a directed set and \((X, t)\) a \( T_2 \) topological linear space.

**DEFINITION 6.1.** A function \( || \cdot || : X \to R^\Delta \) is a semi-norm over \( R^\Delta \) if it satisfies

1) \( ||x|| = 0 \) if and only if \( x = 0 \);
2) \( ||\beta x|| = |\beta| ||x|| \) for every scalar \( \beta \) and every \( x \in X \);
3) if \( \alpha_1, \alpha_2 \in \Delta \) such that \( \alpha_1 > \alpha_2 \) then \( ||x||(\alpha_1) > ||x||(\alpha_2) \);
4) given \( \varepsilon > 0 \) and \( \alpha_1 \in \Delta \) there exists \( \delta > 0 \) and \( \alpha_2 \in \Delta \) such that if \( ||x||(\alpha_2) < \delta \) and \( ||y||(\alpha_2) < \delta \) then \( ||x+y||(\alpha_1) < \varepsilon \).

Let \( U \) be as in theorem 6.1. For \( U, V \in U \) define \( U \succeq V \) if and only if \( U \supseteq V \). It is easy to verify that \((U, \succeq)\) is a directed set.

If \( \Delta = U \) the mapping \( || \cdot || : X \to R^\Delta \) defined by \( ||x||(U) = \rho_U(x) \), where \( \rho_U \) is as in theorem 6.2, is a semi-norm on \( X \) over \( R^\Delta \). The next result now follows from theorem 6.3.

**THEOREM 6.4.** The topology of any \( T_2 \) topological linear space may be generated by a semi-norm over \( R^\Delta \) for some set \( \Delta \).

Theorems 5.6, 5.7 and 5.8 remain unchanged if the pseudo-norm over \( R^\Delta \) in that development is replaced by a semi-norm over \( R^\Delta \). If \( f(x) = f_\lambda \left[ \frac{||x||}{f_\lambda} \right] \) is an elementary length over \( R^\Delta \) corresponding to a semi-norm over \( R^\Delta \) then \( d(x, y) = |f(x) - f(y)| \) is a pseudo-metric on
The natural topology on $X$ determined from $d(x,y)$ will be called an elementary length topology on $X$ corresponding to the semi-norm, $|| \cdot ||$, over $R^\Delta$.

The topology of a $T_2$ topological linear space $X$ may be generated by a semi-norm over $R^\Delta$. The set $\Delta$ is a fundamental system of neighborhoods of 0 in $X$. Thus a fundamental system $\Delta_1$ might be chosen and the same topology on $X$ generated by a semi-norm over $R^{\Delta_1}$. If we suppose that $\Delta_1 = \Delta \cup \{\delta\}$, where $\delta \notin \Delta$, and that $\Delta_1$ is equivalent to $\Delta$, it is natural to ask if the elementary length topologies $\tau$ and $\tau_1$ generated with the aid of the semi-norms $|| \cdot ||$ and $|| \cdot ||_1$ over $R^\Delta$ and $R^{\Delta_1}$ respectively are comparable. If $S = \{y : ||y||_1(\delta) = \lambda\}$ we have the following necessary condition for $\tau_1 \subset \tau$.

**THEOREM 6.5.** If $\tau_1 \subset \tau$ then for each $x \in X$ such that $||x||_1(\delta) \neq 0$ there exists a finite subset $J_x$ of $\Delta$ and functions $j : S \to J_x$, $K : S \to \{1, 2, 3, \ldots\}$ such that $||x||_1(j(w)) = K(w)||x||_1(\delta)$ where $w = \frac{\lambda_x}{||x||_1(\delta)} \in S$.

**PROOF.** The proof of theorem 5.13 may be carried over without significant change.

**THEOREM 6.6.** Each of the following conditions is sufficient for $\tau_1 \subset \tau$.

1) There is an integer $K > 0$ and an $\alpha \in \Delta$ such that $||x||(\alpha) = K||x||_1(\delta)$ for every $x \in X$.

2) There is a finite subset $J$ of $\Delta$ such that $||x||_1(\delta) = \max_{\alpha \in J}||x||(\alpha)$.
PROOF. The proofs of 1) and 2) of theorem 5.14 may be modified in an obvious way to prove this theorem.

REMARK. The requirement that the topology of a topological linear space be Hausdorff may be dropped and the appropriate modifications of the results presented in this chapter remain valid. In the absence of the Hausdorff requirement, the semi-norm over $\mathbb{R}^\Delta$ described earlier may be zero for some $x \neq 0$. It seems appropriate to call such a function a pseudo-semi-norm over $\mathbb{R}^\Delta$. Then any topological linear space is pseudo-semi-normable over $\mathbb{R}^\Delta$ for some set $\Delta$ and an elementary length topology may be defined with the aid of this function.
VII. QUANTUM MECHANICS USING ELEMENTARY LENGTH TOPOLOGIES

In [3] Atkinson and Halpern gave the following quantum mechanical description of the scattering of two particles of energy and momentum $E_1, P_1$ and $E_2, P_2$. Let $K$ be the complex numbers, $X_r$, the reals with an elementary length topology and $C(X_r, K)$ the set of all bounded, continuous functions from $X_r$ into $K$. Define $D : C(X_r, K) \to C(X_r, K)$ by
\[(Df)(x) = \frac{f(x+\lambda) - f(x-\lambda)}{2\lambda}.\]
Corresponding to the momentum is the operator $P = -i\hbar D$ which is self-adjoint with respect to the inner product on $C(X_r, K)$ defined by $(f, g) = \lambda \sum_{n=-\infty}^{\infty} f^*(n\lambda)g(n\lambda)$ where $f^*$ is the complex conjugate of $f$. A position operator is defined by
\[q(x) = \lambda \left[ \frac{X^T}{X} \right].\]
Then $\psi(p', q') = g(p') \exp \left[ \frac{iq^T}{\lambda} \sin^{-1} \left( \frac{\lambda p'^T}{\hbar} \right) \right]$ is a solution of the equation $-i\hbar D\psi = p'\psi$. The function $\psi(p', q')$ is then used to draw certain conclusions concerning the scattering of the particles.

In this chapter a development similar to the one described above will be given for certain linear spaces.

Let $X$ be a topological linear space over the real or complex numbers such that the topology of $X$ is determined by a single real valued function $q$; i.e., a semi-norm, norm, $p$-norm, quasi-norm or a convex functional. If $B(x) = \left\{ y : \left[ \frac{q(y)}{\lambda} \right] = \left[ \frac{q(x)}{\lambda} \right] \right\}$ then
\[\{B(x) : x \in X\}\] is a base for an elementary length topology on $X$. Let $X_r$ denote $X$ with this topology. The following results will be used repeatedly in the remainder of this work.
PROPOSITION 7.1. \( f \in C(X, K) \) if and only if for each \( x \in X \), \( f \) is constant on \( B(x) \).

PROOF. Suppose that \( f \in C(X, K) \). For \( r > 0 \) let 
\[ S_r(f(x)) = \{ z : |z - f(x)| < r \}. \]
Then since \( f \) is continuous there is an open set \( G \) about \( x \) such that \( f(G) \subseteq S_r(f(x)) \). Since \( G \) is open \( x \in B(y) \subseteq G \) for some \( y \in X \). But by proposition 3.1 \( B(y) = B(x) \) so \( f(B(x)) \subseteq S_r(f(x)) \) for every \( r > 0 \). Suppose that there is a \( y \in B(x) \) such that \( f(y) \neq f(x) \). Let \( r_0 = \frac{|f(y) - f(x)|}{2} \). Then
\[ |f(y) - f(x)| = 2r_0 > r_0 \]
so \( f(B(x)) \subseteq S_{r_0}(f(x)) \) since \( f(y) \notin S_{r_0}(f(x)) \). Thus \( f \) is constant on \( B(x) \).

Suppose that \( f \) is constant on \( B(x) \) for every \( x \in X \) and that \( y \in X \). Let \( G \) be an open set containing \( f(y) \). If \( y \in B(x_n) \) then \( f(B(x_n)) \subseteq \{ f(y) \} \subseteq G \) so \( f \) is continuous at \( y \).

For each positive integer \( n \) let \( x_n \in X \) such that \( q(x_n) = n \lambda \) and let \( x_{-n} = -x_n \). For \( f \in C(X, K) \) the terms of the expression
\[ \sum_{n=-\infty}^{\infty} |f(x_n)|^2 \]
do not depend on the choice of \( x_n \) since, by proposition 7.1, \( f(x_n) = f(x_n') \) for any \( x_n' \) such that \( q(x_n') = q(x_n) = n \). Also, if \( y \in B(x) \) then \( -y \in B(x) \) hence, by proposition 7.1, \( f(y) = f(-y) \) and, in particular, \( f(x_{-n}) = f(-x_n) = f(x_n) \). Thus
\[ \sum_{n=-\infty}^{\infty} |f(x_n)|^2 = |f(x_0)|^2 + 2 \sum_{n=1}^{\infty} |f(x_n)|^2. \]

PROPOSITION 7.2. \( E = \left\{ f : f \in C(X, K) \text{ and } \sum_{n=-\infty}^{\infty} |f(x_n)|^2 \right\} \)
is a subspace of \( C(X, K) \).
PROOF. Let $f, g \in E$. Then

$$
\sum_{n=-\infty}^{\infty} |f(x_n) + g(x_n)|^2 = |f(x_0) + g(x_0)|^2 + 2 \sum_{n=1}^{\infty} |f(x_n) + g(x_n)|^2 
$$

$$
\leq |f(x_0)|^2 + 2|f(x_0)g(x_0)| + |g(x_0)|^2 
$$

$$
+ 2 \sum_{n=1}^{\infty} |f(x_n)|^2 + 4 \sum_{n=1}^{\infty} |f(x_n)g(x_n)| 
$$

$$
+ 2 \sum_{n=1}^{\infty} |g(x_n)|^2 
$$

$$
= \sum_{n=-\infty}^{\infty} |f(x_n)|^2 + \sum_{n=-\infty}^{\infty} |g(x_n)|^2 + 2|f(x_0)g(x_0)| 
$$

$$
+ 4 \sum_{n=1}^{\infty} |f(x_n)g(x_n)| 
$$

so it suffices to show that $\sum_{n=1}^{\infty} |f(x_n)g(x_n)| < \infty$. By Holder's inequality for series

$$
\sum_{n=1}^{\infty} |f(x_n)g(x_n)| \leq \left(\sum_{n=1}^{\infty} |f(x_n)|^2\right)^{1/2} \left(\sum_{n=1}^{\infty} |g(x_n)|^2\right)^{1/2}
$$

and this product is finite since $f, g \in E$. For $f \in E$ and $\alpha \in K$, $\alpha f \in E$ is clear.

PROPOSITION 7.3. $(\cdot, \cdot) : E \times E \to K$ defined by

$$(f,g) = \lambda \sum_{n=-\infty}^{\infty} f^*(x_n)g(x_n)$$

is an inner product on $E$.

PROOF. $\sum_{n=-\infty}^{\infty} f^*(x_n)g(x_n) = f^*(x_0)g(x_0) + 2 \sum_{n=1}^{\infty} f^*(x_n)g(x_n)$ and

$$
\sum_{n=1}^{\infty} |f^*(x_n)g(x_n)| \leq \left(\sum_{n=1}^{\infty} |f^*(x_n)|^2\right)^{1/2} \left(\sum_{n=1}^{\infty} |g(x_n)|^2\right)^{1/2} \text{ by Holder's inequality.}
$$
inequality for series. This product is finite since \( f^*, g \in E \) therefore the series \( \sum_{n=1}^{\infty} f^*(x_n)g(x_n) \) converges. Thus \((f, g) \in K\). If \((f, f) = 0\) then \(f(x_n) = 0\) for every integer \(n\). Since \(f\) is constant on \(B(x_n)\) it follows that \(f(x) = 0\) for every \(x \in X\). The remaining requirements are straightforward.

**DEFINITION 7.1.** Two normed linear spaces \(N\) and \(N'\) are isometrically isomorphic if there exists a linear transformation \(T : N \rightarrow N'\) such that \(T\) is one-to-one, onto and \(||T(x)|| = ||x||\) for every \(x \in N\).

**PROPOSITION 7.4.** \(E\) is isometrically isomorphic to \(\ell_2\).

**PROOF.** Let \(\{x_n\}_{n=0}^{\infty}\) be a sequence of vectors in \(X\) such that \(q(x_n) = n\lambda\). Define \(T : E \rightarrow \ell_2\) by \(T(f) = (\sqrt{\lambda}f(x_0), \sqrt{2\lambda}f(x_1), \sqrt{3\lambda}f(x_2), \ldots)\).

Since \(\left(\lambda |f(x_0)|^2 + \sum_{n=1}^{\infty} 2|f(x_n)|^2\right)^{\frac{1}{2}} = \left(\lambda \sum_{n=-\infty}^{\infty} |f(x_n)|^2\right)^{\frac{1}{2}} = ||f||\), \(T(f) \in \ell_2\). Suppose \(\{x'_n\}_{n=0}^{\infty}\) is a sequence such that \(q(x'_n) = n\lambda\). For each \(n\), \(x'_n \in B(x_n)\) so it follows from proposition 7.1 that \(f(x'_n) = f(x_n)\). Thus \(T(f)\) is independent of the sequence \(\{x_n\}\) and therefore \(T\) is well defined. If \(T(f_1) = T(f_2)\) then \(f_1(x_n) = f_2(x_n)\) for \(n = 0, 1, 2, \ldots\). Given \(x \in X\), \(x \in B(x_n)\) for exactly one value of \(n\). Since \(f_1, f_2 \in C(X, K)\), it now follows from proposition 7.1 that \(f_1(x) = f_2(x)\) for every \(x \in X\). Therefore \(T\) is one-to-one. If \((z_0, z_1, \ldots) \in \ell_2\) define \(f : X \rightarrow K\) by
\[
f(x) = \begin{cases} 
\frac{1}{\sqrt{\lambda}} z_0 & \text{if } x \in B(0) \\
\frac{1}{\sqrt{2\lambda}} z_n & \text{if } x \in B(x_n), \text{ where } q(x_n) = n\lambda, n = 1, 2, \ldots. 
\end{cases}
\]

By proposition 7.1, \( f \in E \) and it is clear that \( T(f) = (z_0, z_1, \ldots) \).
Therefore \( T \) is onto.

\[
T(f_1 + f_2) = (\sqrt{\lambda} f_1(x_0) + \sqrt{\lambda} f_2(x_0), \sqrt{2\lambda} f_1(x_1) + \sqrt{2\lambda} f_2(x_2), \ldots) = T(f_1) + T(f_2) \text{ and } T(\alpha f) = \alpha T(f) \text{ so } T \text{ is linear.}
\]

\[
||T(f)|| = \left( \lambda |f(x_0)|^2 + 2\lambda \sum_{n=1}^{\infty} |f(x_n)|^2 \right)^{\frac{1}{2}} = \left( \lambda \sum_{n=-\infty}^{\infty} |f(x_n)|^2 \right)^{\frac{1}{2}} = ||f|| 
\]

Therefore \( T \) is an isometric isomorphism.

**PROPOSITION 7.5.** \((E, (\ , \ ))\) is a Hilbert space.

**PROOF.** Isometric isomorphisms preserve completeness and \( \ell_2 \) is complete.

For each \( x \in X, x \in B(x_n) \) for exactly one value of \( n \). This fact makes it possible to define a function in \( C(X, K) \) by specifying its values on \( B(x_n) \) for \( n = 0, 1, 2, \ldots. \)

**DEFINITION 7.2.** For \( f \in C(X, K) \) let \( Df = g \) where, for every \( x \in B(x_n), n = 0, 1, 2, \ldots, \)
\[
g(x) = \frac{f(x_{n+1}) - f(x_{n-1})}{2\lambda}.
\]

**PROPOSITION 7.6.** \( D \) is an operator on \( E. \)

**PROOF.** \( Df = g \) is constant on \( B(x_n) \) for each \( n \) so \( g \in C(X, K) \) by proposition 7.1. Since \( f(x_1) = f(x_{-1}), g(x_0) = 0.\)

Also, \( g(x_{-n}) = g(-x_n) = g(x_n) \) so
\[
\sum_{n=-\infty}^{\infty} |g(x_n)|^2 = |g(x_0)|^2 + 2 \sum_{n=1}^{\infty} |g(x_n)|^2 = 2 \sum_{n=1}^{\infty} |g(x_n)|^2.
\]
Now,
\[ \sum_{n=1}^{\infty} |g(x_n)|^2 = \frac{1}{4\lambda^2} \sum_{n=1}^{\infty} |f(x_{n+1}) - f(x_{n-1})|^2 \]

\[ \leq \frac{1}{4\lambda^2} \left( \sum_{n=1}^{\infty} |f(x_{n+1})|^2 + 2 \sum_{n=1}^{\infty} |f(x_{n+1})f(x_{n-1})| \right) \]

\[ + \sum_{n=1}^{\infty} |f(x_{n-1})|^2 \]

\[ \leq \frac{1}{4\lambda^2} \left( \sum_{n=1}^{\infty} |f(x_{n+1})|^2 + 2 \left( \sum_{n=1}^{\infty} |f(x_{n+1})|^2 \right)^\frac{1}{2} \left( \sum_{n=1}^{\infty} |f(x_{n-1})|^2 \right)^\frac{1}{2} \right) \]

\[ + \sum_{n=1}^{\infty} |f(x_{n-1})|^2 \]

where the last inequality follows from Holder's inequality for series. Each term in the last expression is finite since \( f \in E \) so it follows that \( g \in E \). Thus \( D : E \to E \). Also,

\[ D(f_1 + f_2)(x) = \frac{f_1(x_{n+1}) + f_2(x_{n+1}) - f_2(x_{n-1}) - f_2(x_{n-1})}{2\lambda} \]

\[ = (Df_1 + Df_2)(x) \]

and \( D(\alpha f) = \alpha Df \) so \( D \) is linear.

To show that \( D \) is continuous let \( \{f_k\}_{k=1}^{\infty} \) be a sequence in \( E \) such that \( \|f_k\| = \left( \lambda \sum_{n=-\infty}^{\infty} |f_k(x_n)|^2 \right)^\frac{1}{2} \to 0 \) as \( k \to \infty. \)
\[ |Df_k| = \left( \lambda \sum_{n=\infty}^\infty |Df_k(x_n)|^2 \right)^{1/2} \]

\[ = \left( \frac{1}{4\lambda} \sum_{n=\infty}^\infty |f_k(x_{n+1}) - f_k(x_{n-1})|^2 \right)^{1/2} \]

\[ \leq \frac{1}{2\sqrt{\lambda}} \left[ \sum_{n=1}^\infty |f_k(x_{n+1})|^2 + 2\left( \sum_{n=1}^\infty |f_k(x_n)|^2 \right)^{1/2} \left( \sum_{n=1}^\infty |f_k(x_{n-1})|^2 \right)^{1/2} \right. \]

\[ + \sum_{n=1}^\infty |f_k(x_{n-1})|^2 \] 

by Holder's inequality. Now,

\[ ||f_k|| = \left( \lambda |f_k(x_0)|^2 + 2\lambda \sum_{n=1}^\infty |f_k(x_n)|^2 \right)^{1/2} \to 0 \text{ as } k \to \infty \text{ so each term of the last expression goes to zero as } k \to \infty. \text{ Thus } f_k \to 0 \Rightarrow Df_k \to 0 \text{ so } D \text{ is continuous.} \]

**PROPOSITION 7.7.** \( P = -i\hbar D \) is a self-adjoint operator on \( E \).

**PROOF.** It suffices to show that \((Pf,f)\) is real for every \( f \in E \).

\[ (Pf,f) = \frac{i\hbar}{2} \sum_{n=\infty}^\infty \{ f(x_{n+1}) - f(x_{n-1}) \}^* f(x_n) \text{ and} \]

\[ \left\{ \sum_{n=\infty}^\infty \{ f(x_{n+1}) - f(x_{n-1}) \}^* f(x_n) \right\}^* \]

\[ = \sum_{n=\infty}^\infty f(x_{n+1})f(x_n)^* - \sum_{n=\infty}^\infty f(x_{n-1})f(x_n)^* \]

\[ = \sum_{n=\infty}^\infty f(x_{n+1})f(x_n)^* \sum_{n=\infty}^\infty f(x_{n+1})f(x_n)^* \]

\[ = \sum_{n=\infty}^\infty \{ f(x_{n+1}) - f(x_{n-1}) \}^* f(x_n) \]
so \((Pf,f)^* = (Pf,f)\) hence \((Pf,f)\) is real.

Let \(q_1(x) = \lambda \left[ \frac{q(x)}{\lambda} \right] \). Then, for each real number \(p'\), the function 

\[
\psi(p',q_1(x)) = g(p') \exp \left[ i \frac{q_1(x)}{\lambda} \frac{\sin^{-1}(\lambda p'/\Pi)}{\lambda} \right]
\]

is constant on \(B(x_n)\) for \(n = 0, 1, 2, \cdots\) therefore \(\psi(p',q_1(x)) \in C(x_r,K)\).

**PROPOSITION 7.8.** For each real number \(p'\) the function \(\psi(p',q_1)\) is a solution of the equation \(\psi p' = p' \psi\).

**PROOF.** Let \(\theta = \sin^{-1}(\lambda \frac{p'}{\Pi})\). Then

\[
\psi(p',q_1) = -\frac{i}{2\lambda} g(p') \left\{ \exp \left[ i \frac{q_1(x_{n+1})}{\lambda} \theta \right] - \exp \left[ i \frac{q_1(x_{n-1})}{\lambda} \theta \right] \right\}.
\]

\[q_1(x_{n+1}) = \lambda \left[ \frac{q(x_{n+1})}{\lambda} \right] = \lambda \left[ (n+1)\frac{\lambda}{\lambda} \right] = (n+1)\lambda = q_1(x_n) + \lambda\] and 

\[q_1(x_{n-1}) = \lambda \left[ \frac{q(x_{n-1})}{\lambda} \right] = \lambda \left[ (n-1)\frac{\lambda}{\lambda} \right] = (n-1)\lambda = q_1(x_n) - \lambda\] so

\[
\psi(p',q_1) = -\frac{i}{2\lambda} g(p') \exp \left[ i \frac{q_1(x_n)\theta}{\lambda} \right] \left\{ \exp \{i\theta\} - \exp \{-i\theta\} \right\} = -\frac{i}{2\lambda} 2i \sin \theta \psi(p',q_1) = p' \psi(p',q_1).
\]

As in [3], the function \(g(p')\) is determined from the conditions

1) \[\sum_{n=-\infty}^{\infty} g^*(p') \exp \left\{ -i \sin^{-1}(\lambda \frac{p'}{\Pi}) \right\} g(p'') \exp \left\{ i \sin^{-1}(\lambda \frac{p''}{\Pi}) \right\} = \delta(p'-p'')\]

2) \[\int_{-\infty}^{\infty} g^*(p) \exp \left\{ -i \sin^{-1}(\lambda \frac{p}{\Pi}) \right\} g(p) \exp \left\{ i \sin^{-1}(\lambda \frac{p}{\Pi}) \right\} dp = \delta_{m,n}\]

which are equations (B11-a) and (B11-b) of [3].
VIII. SUMMARY, CONCLUSIONS AND FURTHER PROBLEMS

The main results obtained in this work are as follows.

1) The elementary length topologies determined with the aid of different norms are always different (theorem 3.2).

2) The "measuring devices" norms, pseudo-norms and semi-norms over Tikhonov semi-fields may be used to give a "natural" construction of elementary length topologies on $T_2$ topological linear spaces.

3) In almost all cases, (some exceptions are given in theorem 5.14) different "measuring devices" give rise to different elementary length topologies.

4) The set $C(X,\mathbb{K})$, as defined in chapter VII, has sufficient mathematical structure to allow a quantum mechanical analysis based on the assumption of the existence of an elementary length.

It is this author's opinion that further research involving the elementary length concept should be directed toward the development of a detailed theory involving the constant $\lambda$. One attempt that might prove to be worthwhile would be to solve the Schroedinger equation defined in terms of the operator $D$ defined in chapter VII.

Recent papers by Cadzow [12] and Logan [13], [14], [15] seem to indicate that some of the techniques needed for a variational approach to the development of a physical theory involving elementary length have already been explored to some extent. The extent to which the
variational approach has been successful in existing physical theories suggests that a development using variational techniques would be worthy of consideration.
REFERENCES


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VITA

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