Quasi-pseudometrics over Tikhonov semifields and fixed point theorems

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QUASI-PSEUDOMETRICS OVER
TIKHONOV SEMIFIELDS AND
FIXED POINT THEOREMS
by
RONALD EVANS SATTERWHITE, 1944 -

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ABSTRACT

It has been shown that topological spaces are characterized as quasi-pseudometric spaces over some Tikhonov semifield.

Sufficient conditions are given for a $T_1$ space to be metrizable over some Tikhonov semifield.

Completely regular (uniform) spaces are characterized as pseudometric spaces over some Tikhonov semifield.

Certain metric, pseudometric, quasi-metric, quasi-pseudometric spaces over a Tikhonov semifield are shown to be respectively metric, pseudometric, quasi-metric, quasi-pseudometric spaces in the usual sense.

Several results from fixed point theory in the metric space setting are generalized to the setting of completely regular (uniform) Hausdorff spaces.
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I. INTRODUCTION

The equivalence of completely regular Hausdorff spaces and metric spaces over Tikhonov semifields yields new machinery in the study of completely regular Hausdorff spaces. There are several results in a metric space setting that have interesting analogues in a completely regular Hausdorff space setting.

An important problem is to determine what class of spaces are pseudometrizable, quasi-metrizable, or quasi-pseudometrizable over some Tikhonov semifield, thereby obtaining new, and possibly more convenient, tools for the study of such spaces.

If $\Lambda$ is a countable set we will see that metric, quasi-metric, pseudometric, or quasi-pseudometric spaces over the Tikhonov semifield $\mathbb{R}^\Lambda$ are respectively metric, quasi-metric, pseudometric, or quasi-pseudometric spaces in the usual sense.

Although metrics over Tikhonov semifields are similar to metrics in the usual sense, one crucial difference is that $\rho(x,y)$ may be a divisor of zero where $\rho$ is a metric over a Tikhonov semifield. We show that if $C(X,\mathbb{R})$ contains a completely regular family of one-to-one functions, then $(X,t)$ can be metrized over some Tikhonov semifield by a metric $\rho$ having the property that $\rho(x,y)$ is not a divisor of zero if $x \neq y$, where $(X,t)$ is a completely regular Hausdorff space.

An interesting result which parallels a well-known
result in the area of functional analysis is the following: 
by using the Minkowski functionals, one can "norm" a locally 
convex Hausdorff linear topological space over a Tikhonov 
semifield, and the generalized norm induces a compatible met­
ric over that particular Tikhonov semifield. We will exploit 
this result in obtaining a fixed point theorem.

One of the important properties enjoyed by metrics over 
Tikhonov semifields and not by quasi-pseudometrics over 
Tikhonov semifields is continuity. We will give necessary 
and sufficient conditions for the latter to be continuous.
The concept of Tikhonov semifields was developed and initially studied by Antonovskii, Boltjanskii, and Sarymsakov in 1959. In 1966, Antonovskii and others (1) published a survey of some of the topological aspects of Tikhonov semifields. One of their major results was the characterization of completely regular Hausdorff spaces as precisely those spaces admitting a compatible metric over some Tikhonov semifield.

Quasi-pseudometric spaces have been studied by Stoltenberg (2) and Kelly (3). Conditions have been given under which a quasi-pseudometric space is a pseudometric space. A quasi-pseudometric space gives rise, in a natural way, to the concept of a bitopological space, whose study was initiated by Kelly (3) in 1963 and was studied by Fletcher (4), Lane (5), and others.

The classical Banach fixed point theorem has been the motivation for numerous fixed point theorems in the metric space setting. Edelstein (6) and Kannan (7) and (8) have made extensive studies on variations of this theorem. Iseki (9) has generalized the theorem to certain completely regular Hausdorff spaces.
III. TIKHONOV SEMIFIELDS

Let $\Delta$ be an arbitrary nonempty set. Denote by $R$ the set of real numbers. We give $R^\Delta$ the product topology.

Addition and multiplication in $R^\Delta$ are defined pointwise:

$$(f+g)(q) = f(q) + g(q), \quad (fg)(q) = f(q)g(q)$$

for $f, g \in R^\Delta, q \in \Delta$.

Addition and multiplication are continuous. Thus, $R^\Delta$ is a commutative topological ring.

For $q \in \Delta$, we define $1_q \in R^\Delta$ by

$$1_q(p) = \begin{cases} 
1, & p = q \\
0, & p \neq q, p \in \Delta.
\end{cases}$$

The function $F: \Delta \to R^\Delta$ defined by $F(q) = 1_q$ is one to one. Hence, we can regard $\Delta$ as being embedded in $R^\Delta$.

If $r \in R$, then $\bar{r} \in R^\Delta$ is defined by $\bar{r}(q) = r$ for each $q \in \Delta$. The function $\bar{1}$ is the unity element of $R^\Delta$. If $f \in R^\Delta$ satisfying $f(q) > 0$ for each $q \in \Delta$, then $f$ has a multiplicative inverse in $R^\Delta$, namely, $f^{-1} = \bar{1}_f$, where $f^{-1}(q) = \bar{1}_f(q) = \frac{1}{f(q)}$ for each $q \in \Delta$.

We can introduce an order on $R^\Delta$. Let $K^\Delta = \{ f \in R^\Delta : f(q) > 0 \text{ for each } q \in \Delta \}$. We call $K^\Delta$ the cone of strictly positive elements. The closure of $K^\Delta$, denoted by $\bar{K}^\Delta$, is the set $\{ f \in R^\Delta : f(q) \geq 0 \text{ for each } q \in \Delta \}$. For $f, g \in R^\Delta$

$$f \gg g \text{ if } (f-g) \in \bar{K}^\Delta, \text{ and}$$

$$f \gg g \text{ if } (f-g) \in K^\Delta.$$

The symbols "\gg" and "\gg" are given the obvious meanings.
If \( M \subseteq \mathbb{R}^\Delta \) and \( M \) is bounded above, then \( M \) has a least upper bound which we denote by \( \vee M \). This function is defined by

\[
(\vee M)(q) = \sup_{f \in M} f(q).
\]

If \( N \subseteq \mathbb{R}^\Delta \) and \( N \) is bounded below, then \( N \) has a greatest lower bound which we denote by \( \wedge N \). This function is defined by

\[
(\wedge N)(q) = \inf_{f \in N} f(q).
\]

The operations "\( \vee \)" and "\( \wedge \)" are continuous so that \( \mathbb{R}^\Delta \) is a topological vector lattice.

For \( a, b \in \mathbb{R} \) and \( q \in \Delta \), \( U^q_{a,b} = \{ f \in \mathbb{R}^\Delta : a < f(q) < b \} \).

We know that the collection \( \{ U^q_{a,b} : a, b \in \mathbb{R}, q \in \Delta \} \) is a subbase for the product topology on \( \mathbb{R}^\Delta \).

\( \mathbb{R}^\Delta \) is called a Tikhonov semifield. For these and other standard results in the study of Tikhonov semifields see (1).
IV. QUASI-PSEUDOMETRIC SPACES OVER $\mathbb{R}^\Delta$

Let $X$ be an arbitrary nonempty set. A mapping $\rho : X \times X \rightarrow \mathbb{R}^\Delta$ is called a metric on $X$ over $\mathbb{R}^\Delta$ provided:

1. $\rho(x,y) = \overline{0}$ if and only if $x = y$.
2. $\rho(x,y) = \rho(y,x)$
3. $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$.

The pair $(X, \rho)$ is called a metric space over $\mathbb{R}^\Delta$.

If we replace 1 by

1'. $\rho(x,x) = \overline{0}$, we have a pseudometric over $\mathbb{R}^\Delta$. If we delete 2, we have a quasi-metric over $\mathbb{R}^\Delta$, and if we also replace 1 by 1', we have a quasi-pseudometric over $\mathbb{R}^\Delta$. If $\rho$ satisfies this latter condition, we call $\rho$ a q.p. metric over $\mathbb{R}^\Delta$.

In (1) it is shown that if $(X, \rho)$ is a metric space over $\mathbb{R}^\Delta$, there is, in a natural way, a topology for $X$ induced by $\rho$. In the following theorem we extend this to the case when $\rho$ is a q.p. metric over $\mathbb{R}^\Delta$.

**THEOREM 1.** Let $(X, \rho)$ be a q.p. metric space over $\mathbb{R}^\Delta$. Let $U$ be a basic open set in $\mathbb{R}^\Delta$ such that $\overline{0} \in U$. For $x \in X$, let $\Omega(x, U) = \{ y \in X : \rho(x, y) \in U \}$. The family of sets $\Omega(x, U)$, as $x$ runs over $X$, and $U$ over basic open sets containing $\overline{0}$, is a base for a topology on $X$. We call this topology the natural topology for $X$.

We prove two lemmas which will aid in proving the theorem:
LEMMA 1. \( \Omega(x,U) \cap \Omega(x,V) = \Omega(x,U \cap V) \).

LEMMA 2. If \( y \in \Omega(x,U) \), there exists a basic open set \( V \) such that \( \emptyset \in V \) and \( \Omega(y,V) \subseteq \Omega(x,U) \).

PROOF OF LEMMA 1. Obvious!

Suppose that Lemma 2 holds if \( U \) is replaced by a subbasic open set of the form \( U_{-a,a}^q \). Then we claim that it holds for an arbitrary basic open set \( U \) where \( \emptyset \in U \). To see this, suppose \( y \in \Omega(x,U) \). Then \( \rho(x,y) \in U \), and hence there are subbasic open sets \( U_{a_i,b_i}^{q_i} \), \( i = 1, \ldots, n \) such that
\[
\rho(x,y) \in \bigcap_{i=1}^n U_{a_i,b_i}^{q_i} \subseteq U. 
\]
Clearly, for each \( i \) we can choose \( \varepsilon_i > 0 \) in such a way that \( \rho(x,y) \in U_{\varepsilon_i,b_i}^{q_i} \subseteq U_{a_i,b_i}^{q_i} \) for \( i = 1, \ldots, n \). It follows that
\[
\rho(x,y) \in \bigcap_{i=1}^n U_{\varepsilon_i,b_i}^{q_i} \subseteq U. 
\]
Thus \( y \in \Omega(x, \bigcap_{i=1}^n U_{\varepsilon_i,b_i}^{q_i}) = \bigcap_{i=1}^n \Omega(x,U_{\varepsilon_i,b_i}^{q_i}) \) by Lemma 1. By hypothesis, we can find for each \( i \), a basic open set \( V_i \), where \( \emptyset \in V_i \) for each \( i \), such that \( \Omega(y,V_i) \subseteq \Omega(x,U_{\varepsilon_i,b_i}^{q_i}) \). Hence
\[
\Omega(y, \bigcap_{i=1}^n V_i) \subseteq \bigcap_{i=1}^n \Omega(x,U_{\varepsilon_i,b_i}^{q_i}) \subseteq \Omega(x,U). 
\]
There is a basic open set \( V \) such that \( \emptyset \in V \subseteq \bigcap_{i=1}^n V_i \). Hence
\[
\Omega(y,V) \subseteq \Omega(y, \bigcap_{i=1}^n V_i) \subseteq \Omega(x,U). 
\]

PROOF OF LEMMA 2. Because of the preceding remarks, we assume that \( U = U_{-a,a}^q \). If \( y \in \Omega(x,U_{-a,a}^q) \), then we have \( 0 \leq [\rho(x,y)](q) < a \). Choose \( \varepsilon > 0 \) satisfying \( [\rho(x,y)](q) + \varepsilon < a \).

Let \( V = U_{\varepsilon,b} \), and suppose \( z \in \Omega(y,V) \). Then \( [\rho(y,z)](q) < \varepsilon \). Thus
\[
[\rho(x,z)](q) \leq [\rho(x,y)](q) + [\rho(y,z)](q) < [\rho(x,y)](q) + \varepsilon < a. 
\]
Thus \( z \in \Omega(x,U_{-a,a}^q) \), and hence \( \Omega(y,V) \subseteq \Omega(x,U_{-a,a}^q) \).

PROOF OF THEOREM 1. Suppose \( z \in \Omega(x,U) \cap \Omega(y,V) \). By
Lemma 2, there are basic open sets $Q$ and $L$ satisfying $\bar{0} \in Q \cap L$, $\Omega(z,Q) \subseteq \Omega(x,U)$, and $\Omega(z,L) \subseteq \Omega(y,V)$. It follows that $\Omega(z,Q \cap L) \subseteq \Omega(x,U) \cap \Omega(y,V)$. There is a basic open set $K$ satisfying $\bar{0} \in K$ and $K \subseteq Q \cap L$. Hence $z \in \Omega(z,K) \subseteq \Omega(z,Q \cap L) \subseteq \Omega(x,U) \cap \Omega(y,V)$. Thus the collection of sets is a basis for a topology.

We use $t_\rho$ to denote the natural topology induced by $\rho$.

A result in (1) is that if $\rho$ is a metric on $X$ over $R^A$, there is a uniform structure whose associated topology coincides with $t_\rho$. In the following theorem we get an analogous result if $\rho$ is a q.p. metric on $X$ over $R^A$.

**THEOREM 2.** Let $(X,\rho)$ be a q.p. metric space over $R^A$. Let $U$ be a basic open set containing $\bar{0}$ in $R^A$. Let $S(U) = \{(x,y) \in X \times X : \rho(x,y) \in U\}$. Then the family $T$ of all sets $S(U)$ is a base for a quasi-uniform structure on $X$, and the topology generated by $T$ coincides with $t_\rho$.

**PROOF.** We denote the diagonal by $A$, i.e. $A = \{(x,x) : x \in X\}$. Suppose $S(U) \in T$. If $x \in X$, then $\rho(x,x) = \bar{0} \in U$. Thus $(x,x) \in S(U)$ for each $x \in X$, and hence $A \subseteq S(U)$. Now suppose $S(U_1), S(U_2) \in T$ and $(x,y) \in S(U_1) \cap S(U_2)$. Hence $\rho(x,y) \in U_1 \cap U_2$. There is a basic open set $U_3$ containing $\bar{0}$ such that $U_3 \subseteq U_1 \cap U_2$ and $\rho(x,y) \in U_3$. It follows that $(x,y) \in S(U_3) \subseteq S(U_1) \cap S(U_2)$. If $S(U) \in T$, we must find $S(V) \in T$ satisfying $S(V) \circ S(V) \subseteq S(U)$. We can write $U = \bigcap_{i=1}^n q_i a_i b_i$. We can choose $\varepsilon_i > 0$, $i = 1, \ldots, n$ such that $\bigcap_{i=1}^n q_i a_i \varepsilon_i b_i \subseteq U$. Let $M = \bigcap_{i=1}^n q_i a_i \varepsilon_i$ and $M_1 = q_i a_i \varepsilon_i = \frac{q_i a_i \varepsilon_i}{2} = \frac{q_i a_i \varepsilon_i}{2}$. 


Let $K = \bigcap_{i=1}^{n} S(M_i)$. Since $S(M_i) \in T$, there is $S(V) \in T$ such that $S(V) \subseteq K$. We claim that $S(V) \circ S(V) \subseteq S(U)$. Let $x, y \in S(V) \circ S(V)$. Then there exists $z$ such that $(x, z) \in S(V)$ and $(z, y) \in S(V)$. Thus for $i = 1, \ldots, n$ we have $[(\rho(x, z))(q_i)] < \frac{\varepsilon_i}{2}$ and $[(\rho(z, y))(q_i)] < \frac{\varepsilon_i}{2}$. It follows from the triangular inequality that $[(\rho(x, y))(q_i)] < \varepsilon_i$, $i = 1, \ldots, n$. Therefore $\rho(x, y) \in M \subseteq U$, and hence $(x, y) \in S(U)$. Thus $S(V) \circ S(V) \subseteq U$. Hence $T$ is a base for a quasi-uniform structure on $X$. If $x \in X$, then $S(U)[x] = \{y: (x, y) \in S(U)\} = \{y: \rho(x, y) \in U\} = \omega(x, U)$. Hence the topology generated by $T$ coincides with $t_\rho$.

Note that if $\rho$ is a pseudometric on $X$ over $R^A$, $T$ will be a compatible uniform structure.

Suppose $(X, \rho)$ is a q.p. metric space over $R^A$. The following two results are due to Hicks (10):

**THEOREM 3.** The natural topology $t_\rho$ is $T_0$ if and only if $x \neq y$ implies $\rho(x, y) \neq 0$ or $\rho(y, x) \neq 0$.

**THEOREM 4.** The natural topology $t_\rho$ is $T_1$ if and only if $\rho(x, y) = 0$ implies $x = y$. Thus $t_\rho$ is $T_1$ if and only if $\rho$ is a quasi-metric on $X$ over $R^A$.

It was known in (1) that if $\rho$ is a metric on $X$ over $R^A$, then $t_\rho$ is Hausdorff.

A space $(X, t)$ is said to be metrizable over $R^A$ if $X$ admits a metric $\rho$ over $R^A$ such that $t = t_\rho$. In (1) it is shown that $(X, t)$ is metrizable over $R^A$ if and only if $(X, t)$ is completely regular (uniformizable) and Hausdorff. The following question naturally arises: what spaces $(X, t)$ are either quasi-metrizable over $R^A$ or quasi-pseudometrizable over $R^A$? Hicks (10) and Boltjanskii (11), working independently
have the following theorem:

**THEOREM 5:** If \((X, t)\) is any topological space, then there is a set \(~\) such that \((X, t)\) is quasi-pseudometrizable over \(R^\Lambda\).

Hicks (10) noted that if \(t\) is \(T_1\), \((X, t)\) is quasi-metrizable over \(R^\Lambda\).

In (1), the following theorem is given for metric spaces over \(R^\Lambda\) and the proof carries over.

**THEOREM 6:** If \((X, p)\) is a pseudometric space over \(R^\Lambda\), then the mapping \((x, y) \rightarrow p(x, y)\) from \(X \times X\) into \(R^\Lambda\) is continuous.

In general, there may be numerous ways to quasi-pseudometrize a given space over some Tikhonov semifield. If \(p\) is a quasi-metric over \(R^\Lambda\) or a quasi-pseudometric over \(R^\Lambda\), we can't be sure that the mapping in Theorem 6 is continuous. In fact we can cite counterexamples. We have the following results:

**THEOREM 7:** Suppose \((X, p)\) is a q.p. metric space over \(R^\Lambda\), and suppose that for each \(x_0 \in X\), the mapping \(x \rightarrow p(x_0, x)\) is continuous. Then \((X, t_p)\) is completely regular.

**PROOF.** Suppose \(x_0 \notin F\) where \(F\) is closed in \(X\). We construct a continuous function \(\phi: X \rightarrow [0,1]\) such that \(\phi(x_0) = 0\) and \(\phi(x) = 1\) for each \(x \in F\). Since \(x_0 \notin F\), there is a basic open set \(U\) in \(R^\Lambda\) such that \(\bar{0} \in U\) and \(x_0 \in \Omega(x_0, U) \subset X \setminus F\).

Since \(R^\Lambda\) is completely regular, there is a continuous function \(g: R^\Lambda \rightarrow [0,1]\) such that \(g(\bar{0}) = 0\) and \(g(x) = 1\) for each \(x \in R^\Lambda \setminus U\). Define \(\phi(x) = g[p(x_0, x)]\). Clearly \(\phi\) is continuous, and \(\phi(x_0) = g[p(x_0, x_0)] = g(\bar{0}) = 0\). If \(x \in F\), then \(x \notin \Omega(x_0, U)\), and hence \(p(x_0, x) \notin U\). Thus \(p(x_0, x) \in R^\Lambda \setminus U\). Thus \(\phi(x) =\)
\[ g[ρ(x_0, x)] = 1. \]

COROLLARY 1. If in Theorem 7 we require that \( ρ \) be a quasi-metric on \( X \) over \( R^Δ \), then \((X, t_ρ)\) is metrizable over \( R^Δ' \) for some \( Δ' \).

PROOF. The space \((X, t_ρ)\) will be completely regular and \( T_1 \), hence completely regular and Hausdorff.

Suppose \((X, ρ)\) is a q.p. metric space over \( R^Δ \). Define \( ρ': X \times X \to R^Δ \) as follows:

\[ ρ'(x, y) = ρ(y, x). \]

It is easy to see that \( ρ' \) is a q.p. metric on \( X \) over \( R^Δ \).

We will use \( t'_ρ \), to denote the natural topology induced by \( ρ' \). Also, if \( U \) is a basic open set in \( R^Δ \) such that \( 0 \in U \) we write \( Ω'(x, U) = \{ y \in X : ρ'(x, y) \in U \} \).

DEFINITION (Kelly (3)). A bitopological space is a triple \((X, t, t')\) where \( X \) is a set of points; \( t \) and \( t' \) are topologies for \( X \).

Because of Theorem 5, given any space \((X, t)\) there is an associated bitopological space, namely, \((X, t_ρ, t'_ρ)\).

Definition (Kelly (3)). A bitopological space \((X, t, t')\) is said to be pairwise Hausdorff provided the following is true: \( x, y \in X, x \neq y \) implies there exist \( U \in t, V \in t' \) such that \( x \in U, y \in V, \) and \( U \cap V = \emptyset \).

THEOREM 8. Suppose \( ρ \) is a q.p. metric on \( X \) over \( R^Δ \). Then \( ρ \) is a quasi-metric on \( X \) over \( R^Δ \) if and only if \((X, t_ρ, t'_ρ)\) is pairwise Hausdorff.

PROOF. To prove the necessity, suppose \( x, y \in X \) such that \( x \neq y \). Then \( ρ(x, y) \neq 0 \), and thus there exists \( q \in Δ \)
and \( \varepsilon > 0 \) such that \( [p(x,y)](q) = \varepsilon \). Let \( O = \Omega(x, U^q_{\frac{\varepsilon}{2}}, \frac{\varepsilon}{2}) \), and let \( O' = \Omega'(y, U^q_{\frac{\varepsilon}{2}}, \frac{\varepsilon}{2}) \). Clearly \( x \in O \), and \( 0 \in t_{p'} \). Also, \( y \in O' \), and \( 0' \in t_{p'} \). We claim that \( O \cap O' = \phi \). Suppose not, and suppose \( z \in O \cap O' \). It follows that \( [p(x,z)](q) < \frac{\varepsilon}{2} \), \( [p'(y,z)](q) < \frac{\varepsilon}{2} \), and thus \( [p(z,y)](q) < \frac{\varepsilon}{2} \). Hence by the triangular inequality we have \( [p(x,y)](q) < \varepsilon \) which is a contradiction.

To prove the sufficiency, suppose \( x, y \in X \) such that \( x \neq y \). Suppose also that \( p(x,y) = \bar{0} \). By hypothesis we can find \( 0 \in t_{p} \), \( O' \in t_{p} \), such that \( x \in O \), \( y \in O' \), and \( O \cap O' = \phi \).

There exist sets \( \{q_1, \ldots, q_n\} \subset A \) and \( \{\varepsilon_1, \ldots, \varepsilon_n\} \subset R \) such that \( \varepsilon_i > 0 \) for \( i = 1, \ldots, n \) and \( x \in \Omega(x, \bigcap_{i=1}^{n} U^q_{\varepsilon_i}, \varepsilon_i) \subset 0 \).

But \( [p(x,y)](q_i) = 0 \) for \( i = 1, \ldots, n \) implies \( y \in \Omega(x, \bigcap_{i=1}^{n} U^q_{\varepsilon_i}, \varepsilon_i) \) which is a contradiction. Thus \( p(x,y) \neq \bar{0} \), and hence \( p \) is a quasi-metric on \( X \) over \( R^A \).

Let \( \Omega(x, U^q_{[-\varepsilon, \varepsilon]}) = \{y : [p(x,y)](q) \leq \varepsilon\} \). We have the following:

**Lemma.** The set \( F = \Omega(x, U^q_{[-\varepsilon, \varepsilon]}) \) is \( t_{\rho}, -\)-closed.

**Proof.** Let \( z \) be a \( t_{\rho}, -\)-limit point of \( F \), and suppose \( z \notin F \). Then \( [p(x,z)](q) = \delta > \varepsilon \). Let \( \delta = \delta - \varepsilon \). Let \( O' = \Omega'(z, U^q_{\frac{\varepsilon}{2}}, \frac{\varepsilon}{2}) \). By hypothesis there exists \( y \) such that \( y \in F \cap O' \), and hence \( [p(x,y)](q) \leq \varepsilon \), \( [p'(z,y)](q) < \frac{\delta}{2} \). Thus \( [p(y,z)](q) < \frac{\delta}{2} \). Using the triangular inequality, we get \( [p(x,z)](q) < \varepsilon + \frac{\delta}{2} = \delta \) which is a contradiction. Hence \( z \in F \), and thus \( F \) is \( t_{\rho}, -\)-closed.

**Definition** (Kelly (3)). In a bitopological space \((X,t,t')\),
t is said to be regular with respect to $t'$ if, given $x \in X$, there exists a $t$-neighborhood base at $x$ of $t'$-closed sets. The bitopological space $(X,t,t')$ is said to be pairwise regular if $t$ is regular with respect to $t'$ and vice versa.

**Theorem 9.** $(X,t,t')$ is pairwise regular.

**Proof.** Let $x \in X$. The collection of all sets of the form $\bigcap_{i=1}^{n} \bigcup_{q_i} [-\varepsilon_i, \varepsilon_i]$ is clearly a $t$-neighborhood base at $x$ of $t'$-closed sets. Similarly, $t'$ is regular with respect to $t$.

**Definition (Stoltenberg (2)).** In a bitopological space $(X,t,t')$, $t$ is said to be locally compact with respect to $t'$ if, given $x \in X$, there exists a $t$-neighborhood $O$ of $x$ such that the $t'$-closure of $O$ is $t'$-compact. The bitopological space $(X,t,t')$ is said to be pairwise locally compact if $t$ is locally compact with respect to $t'$ and vice versa.

**Theorem 10 (Stoltenberg (2)).** Let $(X,t,t')$ be a bitopological space such that $(X,t,t')$ is pairwise Hausdorff and $t'$ locally compact with respect to $t$. Then $t \subset t'$.

If $\rho$ is a quasi-metric on $X$ over $R^\lambda$, and if $t_\rho'$ is locally compact with respect to $t_\rho$, then $t_\rho \subset t_\rho'$, since $(X,t_\rho,t_\rho')$ is pairwise Hausdorff.

**Definition (Fletcher (4)).** In a bitopological space $(X,t,t')$, $t$ is said to be completely regular with respect to $t'$ if for each $t$-closed set $C$ and each $x \not\in C$, there is a real-valued function $f$ on $X$ into $[0,1]$ such that $f(x) = 0$, $f(C) = \{1\}$, $f$ is $t$-upper semicontinuous, and $t'$-lower semicontinuous. If $t$ is completely regular with respect to $t'$
and vice versa, we say that \((X, t, t')\) is pairwise completely regular.

In (5), a bitopological space \((X, P, Q)\) is defined to be quasi-uniformizable if there exists a quasi-uniform structure \(U\) such that \(U\) generates \(P\) and \(U^{-1}\) generates \(Q\). The following result is then established:

**THEOREM.** The bitopological space \((X, P, Q)\) is quasi-uniformizable if and only if \((X, P, Q)\) is pairwise completely regular.

The proof of this theorem relies on several lemmas and quasi-uniform space theory. We prove the necessity in, we feel, a more direct manner, using the notion of a q.p. metric over \(R^\Lambda\). Note that if \((X, P, Q)\) is quasi-uniformizable, then \(P = t_\rho\) and \(Q = t_\rho\), where \(\rho\) is a suitable q.p. metric over \(R^\Lambda\) for a suitable set \(\Delta\).

**THEOREM 11.** The bitopological space \((X, t_\rho, t_\rho)\) is pairwise completely regular.

**PROOF.** It suffices to show that \(t_\rho\) is completely regular with respect to \(t_\rho\). Suppose \(x_0 \in X\), \(C\) is \(t_\rho\)-closed, and \(x_0 \not\in C\). There is a basic open set \(U\) in \(R^\Lambda\) such that \(\bar{0} \in U\) and \(x_0 \in \Omega(x_0, U) \subseteq X \setminus C\). Without loss of generality we write \(U = \bigcap_{i=1}^{n} U_{-\varepsilon, \varepsilon}^{\varepsilon}\) where \(\varepsilon > 0\). We consider the set \(\{q_1, \ldots, q_n\} \subseteq \Delta\). Define \(\psi: R^\Lambda \rightarrow \mathbb{R}\) as follows:

\[
\psi(f) = \min\{\varepsilon, \max_{1 \leq i \leq n} |f|(q_i)\}, \quad f \in R^\Lambda.
\]

It is clear that \(\psi(\bar{0}) = 0\). If \(g \in R^\Lambda \setminus U\), then there exists \(i, 1 \leq i \leq n\) such that \(|g|(q_i) \geq \varepsilon\), and thus \(\psi(g) = \varepsilon\).
Define \( f: X \to [0, \varepsilon] \) by

\[
f(x) = \psi[\rho(x_0, x)].
\]

We see that \( f(x_0) = \psi(0) = 0 \). If \( x \in C \), then \( x \not\in \Omega(x_0, U) \).
Hence \( \rho(x_0, x) \not\in U \), and thus \( \psi[\rho(x_0, x)] = \varepsilon \), i.e. \( f(x) = \varepsilon \).

Now we show that \( f \) is upper-semicontinuous with respect to \( t_\rho \). Suppose \( f(x) < r \), where without loss of generality \( 0 < r \leq \varepsilon \). Hence \( \psi[\rho(x_0, x)] < r \), and thus \( \rho(x_0, x) \in \bigcap_{i=1}^{n} \mathbb{U}_{q_i \rho, r} \).

Choose \( \beta > 0 \) satisfying \( [\rho(x_0, x)](q_i) + \beta < r \) for each \( i, 1 \leq i \leq n \). Let \( V = \bigcap_{i=1}^{n} \mathbb{U}_{q_i \rho, \beta} \). Let \( z \in \Omega(x, V) \). Then we have

\[
[\rho(x_0, z)](q_i) < [\rho(x_0, x)](q_i) + [\rho(x, z)](q_i) < [\rho(x_0, x)](q_i) + \beta < r \quad \text{for all } i, 1 \leq i \leq n.
\]

Hence \( f(z) < r \), and thus \( f \) is upper-semicontinuous with respect to \( t_\rho \).

Now we show that \( f \) is lower-semicontinuous with respect to \( t_\rho \). Suppose \( f(x) > r \), where without loss of generality \( 0 \leq r < \varepsilon \). Hence \( \psi[\rho(x_0, x)] > r \), and thus \( \max_{1 \leq i \leq n} [\rho(x_0, x)](q_i) > r \).

Thus there exists \( i^*, 1 \leq i^* \leq n \), such that

\[
[\rho(x_0, x)](q_{i^*}) > r.
\]

Let \( \Omega(x_0, \mathbb{U}_{q_{i^*} \rho, r}) = \{ y : [\rho(x_0, y)](q_{i^*}) > r \} \).

Clearly \( x \) is in this set, and the set is \( t_\rho \)-open since its complement is \( t_\rho \)-closed. If \( z \in \Omega(x_0, \mathbb{U}_{q_{i^*} \rho, r}) \), then

\[
[\rho(x_0, z)](q_{i^*}) > r, \quad \text{and thus } \max_{1 \leq i \leq n} [\rho(x_0, z)](q_i) > r \text{ from which it follows that } \psi[\rho(x_0, z)] > r, \text{ i.e. } f(z) > r.
\]

Thus \( f \) is lower-semicontinuous with respect to \( t_\rho \). It is clear that a function satisfying the requirements is \( g = \frac{r}{\varepsilon} \).

If \( \rho \) is a q.p. metric on \( X \) over \( R^\Lambda \) we define, for each \( x \in X, \rho_x: X \to R^\Lambda \) by \( \rho_x(y) = \rho(x, y) \), for each \( y \in X \).

**Theorem 12.** If \( \rho_x \) is \( t_\rho \)-continuous, then we have
t\_p -closure \( \Omega(x,u^q_{a,b}) \subseteq \Omega(x,u^q_{a,b}) \).

**PROOF.** We abbreviate \( t\_p -closure A \) by \( t\_p -cl A \). Note that \( t\_p -cl \Omega(x,u^q_{a,b}) = t\_p -cl(\rho^\neg 1_{x,a,b} u^q_{a,b}) \). Since \( \rho^\_X \) is \( t\_p \)-continuous we have \( t\_p -cl(\rho^\neg 1_{x,a,b} u^q_{a,b}) \subseteq (R^\_A -cl u^q_{a,b}) \). We claim that \( R^\_A -cl u^q_{a,b} \subseteq u^q_{a,b} \) = \( \{ f \in \mathbb{R}^A : a \leq f(q) \leq b \} \). Suppose this is not the case. Then there is a limit point of \( u^q_{a,b} \), say \( f \), such that either \( f(q) > b \) or \( f(q) < a \). Suppose \( f(q) = b = \lambda > 0 \). Consider \( u^q_{a,b} \). This is an open set containing \( f \), but it does not meet \( u^q_{a,b} \), a contradiction. Similarly, one shows that \( f(q) < a \). Now we have the following: \( t\_p -cl(\rho^\neg 1_{x,a,b} u^q_{a,b}) \subseteq (\rho^\_X -1 u^q_{a,b}) \), i.e. \( t\_p -cl \Omega(x,u^q_{a,b}) \subseteq \Omega(x,u^q_{a,b}) \).

**THEOREM 13.** In order that \( \rho^\_X \) be \( t\_p \)-continuous, it is both necessary and sufficient that the set \( \{ y : [\rho(x,y)](q) \leq a \} \) be \( t\_p \)-closed.

**PROOF.** Without loss of generality we may assume \( a \geq 0 \). Suppose that \( \rho^\_X \) is \( t\_p \)-continuous. Then \( \rho^\neg 1_{x,a,b} u^q_{a,b} \) is \( t\_p \)-open. Hence \( \{ y : a < [\rho(x,y)](q) \} \cap \{ y : [\rho(x,y)](q) < b \} \) is \( t\_p \)-open. It follows that \( \{ y : [\rho(x,y)](q) \leq a \} \cup \{ y : [\rho(x,y)](q) \geq b \} \) is \( t\_p \)-closed. We know that \( \{ y : [\rho(x,y)](q) \geq b \} \) is \( t\_p \)-closed since its complement is \( t\_p \)-open. We show that \( \{ y : [\rho(x,y)](q) \leq a \} \) is \( t\_p \)-closed. Suppose this is not the case. Then there is a limit point of the set, say \( z \), such that \( z \notin \{ y : [\rho(x,y)](q) \leq a \} \). It follows that \( z \in \{ y : [\rho(x,y)](q) \geq b \} \).

Choose \( \delta > 0 \) satisfying \( a + \delta < b \). By Theorem 12 we have \( t\_p -cl \{ y : [\rho(x,y)](q) \leq a \} \subseteq t\_p -cl \{ y : [\rho(x,y)](q) < a + \delta \} \subseteq \{ y : [\rho(x,y)](q) \leq a + \delta \} \).
Thus \([\rho(x,z)](q) \leq a + \delta < b\), which is a contradiction. Hence \(\{y:[\rho(x,y)](q) \leq a\}\) is \(\tau_\rho\)-closed.

Conversely, \(\rho_x^{-1} A_{a,b} = \{y:[\rho(x,y)](q) > a\} \cap \{y:[\rho(x,y)](q) < b\}\), both sets being \(\tau_\rho\)-open. Thus \(\rho_x\) is \(\tau_\rho\)-continuous.

**THEOREM 14.** If \(\{y:[\rho(x,y)](q) \leq a\}\) is \(\tau_\rho\)-closed, then \(\Omega(x,\Omega^q_{c,d})\) is \(\tau_\rho\)-closed.

**PROOF.** We see that \(\Omega(x,\Omega^q_{c,d}) = \{y:[\rho(x,y)](q) \leq c\} \cap \{y:[\rho(x,y)](q) > d\}\), both sets being \(\tau_\rho\)-closed.

We summarize these results in the following theorem:

**THEOREM 15.** The following are equivalent:

(a). \(\rho_x\) is \(\tau_\rho\)-continuous.

(b). \(\{y:[\rho(x,y)](q) \leq a\}\) is \(\tau_\rho\)-closed.

(c). \(\Omega(x,\Omega^q_{c,d})\) is \(\tau_\rho\)-closed.

**THEOREM 16.** In order that \(\rho_x'\) be \(\tau_\rho\)-continuous, it is both necessary and sufficient that \(\tau_\rho, \subset \tau_\rho\).

**PROOF.** Suppose \(\rho_x'\) is \(\tau_\rho\)-continuous, and consider the following subbasic open set in \(\tau_\rho', \Omega'(x,\Omega^q_{c,d,e}) = (\rho_x')^{-1} \Omega^q_{c,d,e}\), which is \(\tau_\rho\)-open. Thus \(\tau_\rho, \subset \tau_\rho\).

Conversely, consider \((\rho_x')^{-1} \Omega^q_{a,b} = \{y: \rho'(x,y) \in \Omega^q_{a,b}\} = \{y:[\rho'(x,y)](q) < b\} \cap \{y:[\rho'(x,y)](q) > a\}\), the former set being \(\tau_\rho\)-open and hence \(\tau_\rho\)-open; the latter set being \(\tau_\rho\)-open since its complement is \(\tau_\rho\)-closed. Thus \(\rho_x'\) is \(\tau_\rho\)-continuous.

Define \(d: X \times X \to K^\Lambda\) by \(d(x,y) = \max\{\rho(x,y), \rho'(x,y)\}\), i.e. \(d(x,y)](q) = \max\{[\rho(x,y)](q), [\rho'(x,y)](q)\}\). It is easy to show that \(d\) is a pseudo-metric on \(X\) over \(R^\Lambda\).

The following is a generalization of a definition in (2):
DEFINITION. A q.p. metric space \((X,\rho)\) over \(R^\Delta\) will be called a strong q.p. metric space over \(R^\Delta\) if \(\rho \subset t_{\rho}'\).

THEOREM 17. If \((X,\rho)\) is a strong q.p. metric space over \(R^\Delta\), then \((X,t_{\rho})\) is pseudo-metrizable over \(R^\Delta\).

PROOF. We know that \(d(x,y) = \max\{\rho(x,y), \rho'(x,y)\}\) defines a pseudometric on \(X\) over \(R^\Delta\). Let \(t_d\) denote the natural topology induced by \(d\). We will show that \(t_{\rho}' \subset t_d\).

To show that \(t_{\rho}' \subset t_d\), it suffices to show that \(\Omega'(x,\frac{d}{a},b)\) is \(t_d\)-open, where \(a,b > 0\). Suppose \(z \in \Omega'(x,\frac{d}{a},b)\). Then \([\rho'(x,z)](q) < b\). Choose \(\epsilon > 0\) satisfying \([\rho'(x,z)](q) + \epsilon < b\). Consider \(D = \{y: [d(z,y)](q) < \epsilon\}\). Clearly, \(z \in D\), and \(D \in t_d\). Let \(e \in D\). Then, we have \([\rho'(z,e)](q) \leq [d(z,e)](q) < \epsilon\). Thus \([\rho'(x,e)](q) \leq [\rho'(x,z)](q) + [\rho'(z,e)](q) < [\rho'(x,z)](q) + \epsilon < b\). Hence \(e \in \Omega'(x,\frac{d}{a},b)\), and thus \(D \subset \Omega'(x,\frac{d}{a},b)\). It follows that \(\Omega'(x,\frac{d}{a},b)\) is \(t_d\)-open. Let \(D = \{y: [\rho(x,y)](q) < b\}\bigcap \{y: [\rho'(x,y)](q) < b\}\), both sets being \(t_{\rho}'\)-open. Hence \(t_d \subset t_{\rho}'\).

COROLLARY. If \(\rho_x\) is \(t_{\rho}'\)-continuous, then \((X,t_{\rho})\) is pseudometrizable over \(R^\Delta\).

PROOF. Theorem 16 tells us that \(t_{\rho}' \subset t_{\rho}\). Noting that \((\rho')' = \rho\), the result follows from the previous theorem.

DEFINITION (Weston (12)). In a bitopological space \((X,t,t')\), \(t\) is said to be coupled to \(t'\) if \(t\)-cl \(U \subset t'-\text{cl} \ U\) for each \(t\)-open set \(U\).

LEMMA. Given \(x \in X\), and \(F\), a \(t_{\rho}'\)-closed set such that \(x \notin F\), there exists \(U \in t_{\rho}\), and \(V \in t_{\rho}\) such that \(x \in U\), \(F \subset V\), and \(U \cap V = \emptyset\).

PROOF. We have shown that \((X,t_{\rho}',t_{\rho}')\) is pairwise regular.
In particular, $t_\rho$, is regular with respect to $t_\rho$. Hence there exists a $t_\rho$-neighborhood base at $x$ of $t_\rho$-closed sets. Denote this base by $B$. Now $X\setminus F$ is $t_\rho$-open and $x \in X\setminus F$. Hence there exists $B \in B$ such that $x \in B \subset X\setminus F$. Furthermore since $B \in B$, $B$ is a $t_\rho$-neighborhood of $x$, and hence there exists $U \in t_\rho$, such that $x \in U \subset B \subset X\setminus F$. Now $X\setminus B \in t_\rho$. Let $V = X\setminus B$. Then $x \in U$, $F \subset V$, and $U \cap V = \emptyset$.

**Theorem 18.** If $\rho$ is a q.p. metric on $X$ over $R^\Delta$, and if $t_\rho$ is coupled to $t_\rho$, then $(X, t_\rho)$ is pseudometrizable over $R^\Delta$.

**Proof.** All we need establish is that $t_\rho \subset t_\rho$. Thus suppose $U \in t_\rho$, and $x \in U$. By the previous lemma, there exist $N \in t_\rho$ and $M \in t_\rho$, $M \cap N = \emptyset$, such that $x \in M$ and $X\setminus U \subset N$. Since $t_\rho$ is coupled to $t_\rho$, $t_\rho$-cl $N \subset t_\rho$-cl $N$, and hence $X\setminus U \subset N \subset t_\rho$-cl $N \subset t_\rho$-cl $N \subset t_\rho$-cl $(X\setminus M) = X\setminus M$. Thus $x \in M \subset X\setminus (t_\rho$-cl $N) \subset X\setminus (t_\rho$-cl $N) \subset X\setminus N \subset U$. Now, $X\setminus (t_\rho$-cl $N) \in t_\rho$. Thus $U \in t_\rho$, and therefore $t_\rho \subset t_\rho$.

**Definition.** A $T_1$ regular space is called a Moore space if it has a sequence $\{U_i\}$ of open covers such that $\{S(x, U_i)\}$ is a neighborhood base at each point $x$ where $S(x, U_i) = \bigcup \{U : x \in U \text{ and } U \in U_i\}$.

**Theorem 19.** Suppose $(X, \rho)$ is a regular strong quasi-metric space over $R^\Delta$ where $\Delta$ is finite. Then $(X, t_\rho)$ is a Moore space.

**Proof.** Let $\Delta = \{q_1, \ldots, q_n\}$. Define $U_t$ for each positive integer $t$ as follows: $U_t = \Omega(x, \bigcap_{i=1}^{n} \frac{q_i - 1}{2^t + \frac{1}{2^t} }): x \in X$. We show that $\{U_t\}_{t=1}^\infty$ is the desired sequence of open covers for
X. Let \( x_0 \in X \), and suppose \( V \) is a neighborhood of \( x_0 \). For \( k \) sufficiently large we have \( \Omega(x_0, \bigcap_{i=1}^{n} \frac{U_1}{2^k,2^k} \frac{q_i}{2^k,2^k}) \subseteq V \). Since \( t_{\rho} \subseteq t_{\rho}', \) there exists a positive integer \( m \), which without loss of generality we may choose satisfying \( m > k \), such that \( \Omega'(x_0, \bigcap_{i=1}^{n} \frac{U_1}{2^m,2^m} \frac{q_i}{2^m,2^m}) \subseteq \Omega(x_0, \bigcap_{i=1}^{n} \frac{U_1}{2^k,2^k} \frac{q_i}{2^k,2^k}) \). Now suppose that \( x_0 \in \Omega(y, \bigcap_{i=1}^{n} \frac{U_1}{2^m,2^m} \frac{q_i}{2^m,2^m}) \) and \( z \in \Omega(y, \bigcap_{i=1}^{n} \frac{U_1}{2^m,2^m} \frac{q_i}{2^m,2^m}) \). Hence \( [\rho(y,x_0)](q_i) < \frac{1}{2^m} \) for \( i = 1, \ldots, n \), and hence \( [\rho'(x_0,y)](q_i) < \frac{1}{2^m} \) for \( i = 1, \ldots, n \). Hence \( y \in \Omega'(x_0, \bigcap_{i=1}^{n} \frac{U_1}{2^m,2^m} \frac{q_i}{2^m,2^m}) \subseteq \Omega(x_0, \bigcap_{i=1}^{n} \frac{U_1}{2^k,2^k} \frac{q_i}{2^k,2^k}) \), and thus \( [\rho(x_0,y)](q_i) < \frac{1}{2^k} \) for \( i = 1, \ldots, n \). Then by the triangular inequality we have \( [\rho(x_0,z)](q_i) < \frac{1}{2^k} + \frac{1}{2^m} < \frac{1}{2^k} + \frac{1}{2^k} = \frac{1}{2^{k-1}} \) for \( i = 1, \ldots, n \). Hence we see that \( z \in \Omega(x_0, \bigcap_{i=1}^{n} \frac{U_1}{2^{k-1},2^{k-1}} \frac{q_i}{2^{k-1},2^{k-1}}) \subseteq V \). Thus \( \Omega(y, \bigcap_{i=1}^{n} \frac{U_1}{2^{k-1},2^{k-1}} \frac{q_i}{2^{k-1},2^{k-1}}) \subseteq V \). Clearly, \( \Omega(x_0, \bigcap_{i=1}^{n} \frac{U_1}{2^{k-1},2^{k-1}} \frac{q_i}{2^{k-1},2^{k-1}}) \subseteq V \) since \( m > k > k-1 \). Thus \( x_0 \in \text{cl}(x_0, \bigcup_{i=1}^{n} \frac{U_1}{2^{k-1},2^{k-1}} \frac{q_i}{2^{k-1},2^{k-1}}) \subseteq V \), and hence \( (X, t_{\rho}) \) is a Moore space.

**COROLLARY.** If \( (X, t_{\rho}) \) is a strong quasi-metric space over \( R^\Delta \) where \( \Delta \) is finite, and if \( \rho_x \) is \( t_{\rho} \)-continuous, then \( (X, t_{\rho}) \) is a Moore space.

**PROOF.** The space \( (X, t_{\rho}) \) will be \( T_1 \) since \( \rho \) is a quasi-metric on \( X \) over \( R^\Delta \). Since \( \rho_x \) is \( t_{\rho} \)-continuous, \( (X, t_{\rho}) \) will be completely regular and hence regular.

In the case of usual metric spaces, the following are equivalent:

(a). \( x \in \text{cl} M, M \subseteq X \)
However, for metric spaces over semifields this is not true. See (1) for an example. The following generalizes a definition in (1).

**DEFINITION.** Let \((X, p)\) be a q.p. metric space over \(R^\Lambda\). For \(q, q' \in \Lambda\), we write \(q \gtrdot q'\) if \([p(x,y)](q) \gtrdot [p(x,y)](q')\) for arbitrary \(x, y \in X\). It is clear that \(\gtrdot\) partially orders \(\Lambda\). If \(\Lambda\) so becomes a directed set, we say that \(p\) is directed.

The following three theorems are established in (1) for \(p\) a metric over \(R^\Lambda\). The proof for the case when \(p\) is a q.p. metric over \(R^\Lambda\) is similar, and we include it for completeness.

**THEOREM 20.** Let \((X, p)\) be a q.p. metric space over \(R^\Lambda\). For \(M \subseteq X\) and \(x \in X\), put \(p(x, M) = \inf_{m \in M} p(x, m)\). If \(p\) is directed, then \(x \in t_p^{-\text{cl} M}\) if and only if \(p(x, M) = 0\).

**PROOF.** Suppose \(p(x, M) = 0\). Choose an arbitrary neighborhood \(U\) of \(\vec{0}\) in \(R^\Lambda\). Let \(\{q_1, \ldots, q_k\} \subseteq \Lambda\) and \(\epsilon > 0\) be such that \(\bigcap_{i=1}^{k} U_{\epsilon, q_i} \subseteq U\). Since \(p\) is directed, there exists \(q \in \Lambda\) such that \(q \gtrdot q_i\), for \(i = 1, \ldots, k\). Also, since \(p(x, M) = 0\), there exists \(m \in M\) such that \([p(x, m)](q) < \epsilon\). Thus we have the following: \([p(x, m)](q_i) \leq [p(x, m)](q) < \epsilon\), for \(i = 1, \ldots, k\). Hence \(p(x, m) \in U_{\epsilon, q_1} \cap \cdots \cap U_{\epsilon, q_k} \subseteq U\). Hence \(m \in \Omega(x, U)\), and thus \(x \in t_p^{-\text{cl} M}\).

Conversely, if \(x \in t_p^{-\text{cl} M}\), then \(\Omega(x, U_{\epsilon, q})\) meets \(M\) for any \(q \in \Lambda\) and \(\epsilon > 0\), i.e. there exists \(m \in M\) such that \([p(x, m)](q) < \epsilon\). Thus \([p(x, M)](q) = 0\) for each \(q \in \Lambda\), so that \(p(x, M) = \vec{0}\).

**THEOREM 21.** Let \((X, p)\) be a q.p. metric space over \(R^\Lambda\).
Then $X$ admits a directed q.p. metric $\rho^*$ over some Tikhonov semifield $R^\Delta$ such that $t_\rho = t_{\rho^*}$.

**PROOF.** Let $\Delta^*$ denote the collection of all non-empty finite subsets of $\Delta$. Let $E = \{q_1, \ldots, q_k\} \in \Delta^*$. Define $\rho^*:X \times X \to K^{\Delta^*}$ by

$$[\rho^*(x, y)](E) = \max_{q \in E} [\rho(x, y)](q).$$

Clearly $\rho^*(x, x) = 0$ for each $x \in X$. Suppose $x, y, z \in X$.

Then, $\max_{q \in E} [\rho(x, z)](q) \leq \max_{q \in E} ( [\rho(x, y)](q) + [\rho(y, z)](q)) \leq \max_{q \in E} [\rho(x, y)](q) + \max_{q \in E} [\rho(y, z)](q)$. Thus $\rho^*(x, z) \leq \rho^*(x, y) + \rho^*(y, z)$, and hence $\rho^*$ is a q.p. metric on $X$ over $R^\Delta$. Let $E_1, \ldots, E_k \in \Delta^*$. Let $q_{i,j}$ denote a typical member of $E_i$.

Then, the following set is a typical basic open set containing $0$ in $R^\Delta : V = \bigcap_{l = 1}^{E_1} a_l \cap \ldots \cap \bigcap_{l = k}^{E_k} a_k$. The following set is a basic open set containing $0$ in $R^\Delta : W = \bigcap_{l = 1}^{E_1} a_l \cap \ldots \cap \bigcap_{l = k}^{E_k} a_k$. Recall that $S(U) = \{(x, y) \in X \times X : \rho(x, y) \in U\}$. Similarly, let $S^*(U) = \{(x, y) \in X \times X : \rho^*(x, y) \in U\}$. For the above sets, $V$ and $W$, we claim that $S^*(V) = S(W)$. To prove this, suppose $(x, y) \in S^*(V)$. Hence $\rho^*(x, y) \in V$, from which it follows that $[\rho^*(x, y)](E_i) < a_i$ for $i = 1, \ldots, k$, and thus $\max_{q \in E_i} [\rho(x, y)](q) < a_i$ for $i = 1, \ldots, k$. Hence $[\rho(x, y)](q_{i,j}) < a_i$ for $i = 1, \ldots, k$, and it follows that $(x, y) \in S(W)$. Now suppose that $(x, y) \in S(W)$, so that $\rho(x, y) \in W$. Consider $[\rho^*(x, y)](E_i)$ for $i = 1, \ldots, k$. We know that $[\rho(x, y)](q_{i,j}) < a_i$, and hence $\max_{q \in E_i} [\rho(x, y)](q) < a_i$. Therefore $[\rho^*(x, y)](E_i) < a_i$ for $i = 1, \ldots, k$. Thus $\rho^*(x, y) \in V$, so that $(x, y) \in S^*(V)$. 

Therefore we have $S^*(V) = S(W)$. Now, suppose we start with a basic open set $W$ containing 0 in $\mathbb{R}^\Delta$, say $W = \bigcap_{i=1}^k U_{a_i}^{q_i, a_i}$.

Construct $V$, a basic open set containing 0 in $\mathbb{R}^\Delta^*$, as follows: $V = U_{a_1}^{E_1} \cap \cdots \cap U_{a_k}^{E_k}$, where $E_i = \{q_i\}$, $i = 1, \ldots, k$. Clearly $S^*(V) = S(W)$. Therefore $\rho$ and $\rho^*$ generate the same natural quasi-uniform structure, and hence the same topology.

Now we must show that $\rho^*$ is directed. Suppose $E_1, E_2, E_3 \in \Delta^*$ such that $E_1 \succeq E_2$ and $E_2 \succeq E_3$. Then we have

$$\max_{q \in E_1} [\rho(x,y)(q)] \geq \max_{q \in E_2} [\rho(x,y)(q)] \geq \max_{q \in E_3} [\rho(x,y)(q)].$$

Hence $E_1 \succeq E_2$. If $E \in \Delta^*$, it is clear that $E \succeq E$. Now suppose $E_1, E_2 \in \Delta^*$. Let $E_1 = \{q_1, \ldots, q_n\}$, and $E_2 = \{q'_1, \ldots, q'_k\}$. Let $E = E_1 \cup E_2$. Clearly, $E \in \Delta^*$, $E \succeq E_1$, and $E \succeq E_2$. Thus $\Delta^*$ is directed by $\succeq$, and hence $\rho^*$ is directed.

**Definition.** Let $(X, \rho)$ be a q.p. metric space over $\mathbb{R}^\Delta$.

The q.p. metric $\rho$ is said to be bounded provided there exists a least upper bound $\bigvee_{x, y \in X} \rho(x,y)$.

**Theorem 22.** If $(X, \rho)$ is a q.p. metric space over $\mathbb{R}^\Delta$, then there exists a compatible bounded q.p. metric $\rho_1$ on $X$ over $\mathbb{R}^\Delta$.

**Proof.** Define $\rho_1:X \times X \to \overline{\mathbb{R}}^\Delta$ by $\rho_1(x,y) = \frac{\rho(x,y)}{1 + \rho(x,y)}$. Clearly $\rho_1(x,x) = 0$ for each $x \in X$. Let $x, y, z \in X$ and $q \in \Delta$.

If $[\rho(x,z)](q) = 0$, then $[\rho_1(x,z)](q) = 0$ and hence $[\rho_1(x,z)](q) \leq [\rho_1(x,y)](q) + [\rho_1(y,z)](q)$. If $[\rho(x,z)](q) \neq 0$, then by the triangular inequality we see that $[\rho(x,y)](q) + [\rho(y,z)](q) \neq 0$. Now we have the following:
\[ [\rho_1(x,y) + \rho_1(y,z)](q) = \frac{[\rho(x,y)](q)}{1 + [\rho(x,y)](q)} + \frac{[\rho(y,z)](q)}{1 + [\rho(y,z)](q)} \]

\[ \geq \frac{[\rho(x,y)](q)}{1 + [\rho(x,y) + \rho(x,z)](q)} + \frac{[\rho(y,z)](q)}{1 + [\rho(y,z) + \rho(x,y)](q)} \]

\[ = \frac{[\rho(x,y) + \rho(y,z)](q)}{1 + [\rho(x,y) + \rho(y,z)](q)} = \frac{1}{1 + \frac{1}{[\rho(x,z)](q)}} \]

\[ \geq \frac{1}{1 + \frac{1}{[\rho(x,z)](q)}} = \frac{[\rho(x,z)](q)}{1 + [\rho(x,z)](q)} = [\rho_1(x,z)](q). \]

Hence \( \rho_1 \) is a q.p. metric on \( X \) over \( R^\Delta \). Clearly \( \rho_1 \) is bounded since \( \{\rho_1(x,y) : x, y \in X\} \) is bounded above by \( 1 \in R^\Delta \) and hence has a least upper bound.

Now we show that \( t_\rho = t_{\rho_1} \). Consider the following sub-basic open sets:

\[ \Omega(x, U^q_{\epsilon, \epsilon}) = \{y : [\rho(x,y)](q) < \epsilon\} \]

\[ \Omega_1(x, U^q_{\epsilon, \epsilon}) = \{y : [\rho_1(x,y)](q) < \frac{\epsilon}{1+\epsilon}\}. \]

That these sets agree follows from the fact that if \( t > 0 \), and \( \epsilon > 0 \), then \( t < \epsilon \) if and only if \( \frac{t}{1+t} < \frac{\epsilon}{1+\epsilon} \). Note that every subbasic open set induced by \( \rho_1 \) can be written in the above form. Hence, \( t_\rho = t_{\rho_1} \).

The following is a generalization of a definition in (1):

DEFINITION. A directed and bounded q.p. metric on \( X \) over \( R^\Delta \) is said to be regular.

THEOREM 23. Suppose \( (X, \rho) \) is a q.p. metric space over \( R^\Delta \). Then \( X \) admits a compatible regular q.p. metric \( \dot{d} \) over some Tikhonov semifield \( R^\Delta^* \).
PROOF. Apply Theorem 21 and then Theorem 22.

If \((X, \rho)\) is a q.p. metric space over \(R^\Delta\), are there conditions under which \(X\) is a q.p. metric space in the usual sense? We answer the question in the affirmative.

THEOREM 24. If \((X, \rho)\) is a q.p. metric space over \(R^\Delta\) where \(\Delta\) is finite, then \((X, t_\rho)\) is quasi-pseudometrizable in the usual sense.

PROOF. As a result of Theorem 1.6 (13), it suffices to show the existence of a compatible quasi-uniform structure having a countable base. To this end suppose \(\Delta = \{1, \ldots, K\}\). Let \(U_n\) be defined for each natural number \(n\) by \(U_n = \{(x, y) \in X \times X : [\rho(x, y)](i) < \frac{1}{n}, i = 1, \ldots, K\}\). Let \(B = \{U_n : n \in N\}\). We show that \(B\) is a base for a quasi-uniform structure on \(X\). Let \(A = \{(x, x) : x \in X\}\). Clearly \(A \subseteq U_n\) for each \(n \in N\). If \(U_n, U_m \in B\), where without loss of generality \(m < n\), then \(U_n \subseteq U_m\). Hence \(U_n \subseteq U_n \cap U_m\). Also, it is easy to see that \(U_{2n} \circ U_{2n} \subseteq U_n\) for each \(n \in N\). Thus \(B\) is a base.

If \(U\) is a basic open set in \(R^\Delta\) such that \(0 \in U\), recall that \(T = \{S(U)\}\) is a base for the natural quasi-uniform structure on \(X\). We show that \(B\) and \(T\) are equivalent. Suppose \(U_n \in B\). Then \(U_n = S(\bigcap_{i=1}^{K} \frac{U_{j_i}^i}{\frac{1}{n}, n})\). Now suppose \(S(V) \in T\). There exists a set \(\{j_1, \ldots, j_t\} \subseteq \{1, \ldots, K\}\) and \(\{\epsilon_1, \ldots, \epsilon_t\} \subseteq R\) where \(\epsilon_i > 0\), \(i = 1, \ldots, t\), such that \(V = \bigcap_{i=1}^{t} \frac{U_{j_i}^i}{\epsilon_i} \). Choose \(n \in N\) satisfying \(\frac{1}{n} < \min_{i} \epsilon_i\). Then \(U_n \subseteq S(V)\). Thus \(B\) is equivalent to \(T\).

COROLLARY 1. IF \((X, \rho)\) is a quasi-metric space over \(R^\Delta\) where \(\Delta\) is finite, then \((X, t_\rho)\) is quasi-metrizable in the usual sense.
PROOF. Since $(X,t_{\rho})$ is $T_1$, by Theorem 1.6 (13) it suffices to exhibit a countable base for a compatible quasi-uniform structure. Note that $\mathcal{B}$ as defined in Theorem 24 is such a base.

COROLLARY 2. If $(X,\rho)$ is a metric space over $R^\Delta$ where $\Delta$ is finite, then $(X,t_{\rho})$ is metrizable in the usual sense.

PROOF. Note that $\mathcal{B}$ as defined in Theorem 24 will be a countable base for a compatible uniform structure, and $(X,t_{\rho})$ is Hausdorff. The result follows from a well known theorem in (14).

Now suppose $(X,\rho)$ is a q.p. metric space over $R^N$ where $N$ denotes the set of natural numbers. We will exhibit a countable base for a quasi-uniform structure which is compatible with the natural quasi-uniform structure. Let $U_n = \{(x,y) \in X \times X : [\rho(x,y)](i) < \frac{1}{n}, i = 1, \ldots, n\}$. Let $\mathcal{B} = \{U_n : n \in N\}$. Let $A = \{(x,x) : x \in X\}$. Clearly $A \subseteq U_n$ for each $n \in N$. Suppose $U_n, U_K \in \mathcal{B}$, where without loss of generality $K < n$. Then $U_n \subseteq U_K$, and hence $U_n \subseteq U_n \cap U_K$. For $n \in N$, we have $U_{2n} \cap U_{2n} \subseteq U_n$. Hence $\mathcal{B}$ is a base for a quasi-uniform structure. Now we show that $\mathcal{B}$ is equivalent to $T$, the aforementioned base for the natural quasi-uniform structure. If $U_n \in \mathcal{B}$, then $U_n = S(\bigcap_{i=1}^{n} U_{\frac{i}{n}})$. Now suppose $S(V) \in T$. Then there exists a set $\{j_1, \ldots, j_t\} \subseteq N$ and a set $\{\epsilon_1, \ldots, \epsilon_t\} \subseteq R$, $\epsilon_i > 0$ for $i = 1, \ldots, t$, such that $V = \bigcap_{i=1}^{t} U_{\epsilon_i,j_i}$. Choose $n \in N$ satisfying $n > j_i$ for $i = 1, \ldots, t$, and $1/n < \min \epsilon_i$. Thus $U_n \subseteq S(V)$. Hence $\mathcal{B}$ is equivalent to $T$. Using the argument in the proof of Theorem 24, we obtain the following:
THEOREM 25. If \((X,\rho)\) is a q.p. metric space over \(\mathbb{R}^N\), quasi-metric space over \(\mathbb{R}^N\), pseudometric space over \(\mathbb{R}^N\), metric space over \(\mathbb{R}^N\), then \((X,\rho)\) is respectively a q.p. metric space, quasi-metric space, pseudometric space, metric space in the usual sense.

The following definition with the Hausdorff separation property added is given in (15):

DEFINITION. Suppose \((X,t)\) is a Hausdorff topological space. A family \(F \subset C(X,\mathbb{R})\) is said to be a completely regular family if, given \(x \in X\), \(F\) a closed subset of \(X\) such that \(x \not\in F\), there exists \(f \in F\) such that \(f(x) \not\in \text{cl}\ f(F)\).

REMARK 1. It is easy to show that if \((X,t)\) is Hausdorff and if \(F\) is a completely regular family in \(C(X,\mathbb{R})\), then \((X,t)\) is completely regular and Hausdorff.

The following result is an exercise in (15):

LEMMA. Suppose \((X,t)\) is Hausdorff and \(F\) is a completely regular family in \(C(X,\mathbb{R})\). The collection of all sets of the form \(\{y \in X: |f(x) - f(y)| < \varepsilon \; ; \; f \in F; \; \varepsilon > 0\}\) is a base for the neighborhood system at \(x \in X\).

REMARK 2. The proof of the above lemma relies in no way on the Hausdorff property.

In (1), completely regular (uniformizable) Hausdorff spaces are characterized as those spaces admitting a compatible metric over \(\mathbb{R}^{\Delta}\) for some \(\Delta\). The proof relies on the Tikhonov embedding theorem. In the following theorem, we characterize completely regular (uniform) spaces as those spaces admitting a compatible pseudometric over \(\mathbb{R}^{\Delta}\) for some \(\Delta\). This is a slight improvement over the above-mentioned
theorem. Also, the compatible pseudometric over $\mathbb{R}^{\Lambda}$ is constructed.

**Theorem 26.** The space $(X,t)$ is completely regular (uniformizable) if and only if $(X,t)$ is pseudometrizable over $\mathbb{R}^{\Lambda}$ for some $\Lambda$.

**Proof.** To prove the necessity, there exists a completely regular family $F \subseteq C(X,\mathbb{R})$, where possibly $F = C(X,\mathbb{R})$.

Define $\rho : X \times X \to \mathbb{R}^{F}$ by $[\rho(x,y)](f) = |f(x) - f(y)|$, where $x,y \in X$, $f \in F$. It is easy to show that $\rho$ is a pseudometric on $X$ over $\mathbb{R}^{F}$. If $x \in X$, it is easy to see that $\{y \in X : |f(x) - f(y)| < \varepsilon, f \in F; \varepsilon > 0\} = \Omega(x,U_{\varepsilon}^{f}).$ Now consider $\Omega(x,U)$ where $0 \in U$ and $U$ is a basic open set in $\mathbb{R}^{F}$. There exists $f_{1},\ldots,f_{K} \in F$ and $\varepsilon_{1},\ldots,\varepsilon_{K} \in \mathbb{R}$ where $\varepsilon_{i} > 0$ for $i = 1,\ldots,K$ such that $U = \bigcap_{i=1}^{K} \bigcup_{x_{i},y_{i}] \subseteq \Omega(x,U)$. Note that $y \in \Omega(x,U)$ if and only if $[\rho(x,y)](f_{i}) < \varepsilon_{i}$ for $i = 1,\ldots,K$, and hence if and only if $|f_{i}(x) - f_{i}(y)| < \varepsilon_{i}$ for $i = 1,\ldots,K$. We have $x \in \{y \in X : |f_{i}(x) - f_{i}(y)| < \varepsilon_{i}, i = 1,\ldots,K\}$ which is $t$-open. By the previous lemma and Remark 2, there exists $h \in F$ and $\varepsilon > 0$ such that $x \in \{y \in X : |h(x) - h(y)| < \varepsilon\} \subseteq \{y \in X : |f_{i}(x) - f_{i}(y)| < \varepsilon_{i}, i = 1,\ldots,K\} = \Omega(x,U)$. Thus for arbitrary $x \in X$, we have shown the equivalence of local bases in $t$ and $t_{\rho}$. Hence $t = t_{\rho}$.

To prove the sufficiency, we have already seen that if $(X,t)$ is pseudometrizable over $\mathbb{R}^{\Lambda}$, then $(X,t)$ admits a compatible uniform structure.

**Remark 3.** Note that if in Theorem 26, $(X,t)$ is Hausdorff, then $\rho$ will be a metric over $\mathbb{R}^{\Lambda}$. Conversely, if $\rho$ is a metric.
over $\mathbb{R}^\Delta$, then $(X,t_\rho)$ will be completely regular and Hausdorff.

REMARK 4. In the previous lemma we noted that the family of all sets of the form $\{y \in X: |f(x) - f(y)| < \varepsilon; f \in F; \varepsilon > 0\}$ is a local base at $x \in X$. In fact it can be shown to be a base for the topology $t$.

THEOREM 27. A separable space $(X,t)$ is metrizable if and only if it is $T_1$ and there exists a countable completely regular family in $C(X,\mathbb{R})$.

PROOF. Suppose $(X,t)$ is metrizable. Let $d$ denote a metric for $X$. Let $\{x_1, \ldots, x_n, \ldots\}$ be a countable dense subset of $X$. We show that $\{d_{x_1}, \ldots, d_{x_n}, \ldots\}$ is a completely regular family in $C(X,\mathbb{R})$. It is well known that $d_{x_i} \in C(X,\mathbb{R})$ for each $i \in \mathbb{N}$. Suppose $x \in X$, $F$ is closed in $X$, and $x \notin F$. Hence $d(x,F) = \beta > 0$. Let $S(x,\frac{\beta}{3}) = \{y: d(x,y) < \frac{\beta}{3}\}$. Then $x \in S(x,\frac{\beta}{3}) \subset X \setminus F$. Since $S(x,\frac{\beta}{3})$ is open in $t_d$, there exists $i \in \mathbb{N}$ such that $x_i \in S(x,\frac{\beta}{3})$, and hence $d_{x_i}(x) < \frac{\beta}{3}$. We claim that $d_{x_i}(F) \geq \frac{2\beta}{3}$. Suppose not, i.e. suppose $d_{x_i}(F) < \frac{2\beta}{3}$. We have $d(x,F) \leq d(x,x_i) + d(x_i,F) < \frac{\beta}{3} + \frac{2\beta}{3} = \beta$ which is a contradiction. Thus $d_{x_i}(F) \geq \frac{2\beta}{3}$ implies that g.l.b. $d_{x_i}(y) \geq \frac{2\beta}{3}$, and hence $d_{x_i}(y) \geq \frac{2\beta}{3}$ for each $y \in F$. Therefore $d_{x_i}(x) \notin \text{cl}(d_{x_i}(F))$. Thus $\{d_{x_1}, \ldots, d_{x_n}, \ldots\}$ is a completely regular family in $C(X,\mathbb{R})$.

Conversely, the existence of a countable completely regular family $F$ in $C(X,\mathbb{R})$ implies that $(X,t)$ is pseudometrizable over $\mathbb{R}^F$ by Theorem 26. But $F$ is countable, and hence $(X,t)$ is pseudometric by Theorem 25. Since $(X,t)$ is $T_1$, it is metrizable.
THEOREM 28. Suppose \( \{f_1, \ldots, f_n, \ldots \} \) is a countable completely regular family in \( C(X, \mathbb{R}) \). Then \( (X, t) \) is second countable.

PROOF. Let \( B_n = \{ f_n^{-1}(a, b) : a, b \text{ rational, } a < b \} \). Let \( \mathcal{B} = \{ B_n : n \in \mathbb{N} \} \). We show that \( \mathcal{B} \) is a base for \( t \). Suppose \( 0 \in t \) and \( x \in 0 \). There exists \( \varepsilon > 0 \) and \( i \in \mathbb{N} \) such that \( x \in \{ y : |f_i(x) - f_i(y)| < \varepsilon \} \subset 0 \). Choose rational numbers \( a \) and \( b \) satisfying \( f_i(x) - \varepsilon < a < f_i(x) < b < f_i(x) + \varepsilon \).

Then \( x \in f_i^{-1}(a, b) \subset \{ y : |f_i(x) - f_i(y)| < \varepsilon \} \subset 0 \). Thus \( \mathcal{B} \) is a countable base for \( t \).

COROLLARY. Suppose \( (X, d) \) is a metric space. Then \( C(X, \mathbb{R}) \) has a countable completely regular family if and only if \( (X, t_d) \) is separable.

PROOF. Apply theorems 28 and 27.

DEFINITION. A space \( (X, t) \) is perfectly normal if it is normal, \( T_1 \), and every closed subset is a \( G_\delta \) set.

THEOREM 29. A space \( (X, t) \) is perfectly normal if and only if it admits a metrization over \( R^\Delta \) for some \( \Delta \) and the metric \( \rho \) has the following property: if \( A \) and \( B \) are disjoint closed subsets of \( X \), there exists \( q \in \Delta \) such that \( x \not\in A \) implies that \( [\rho(x, A)](q) > 0 \) and \( y \not\in B \) implies that \( [\rho(y, B)](q) > 0 \).

PROOF. To prove the necessity we note that \( (X, t) \) is completely regular and Hausdorff and hence can be metrized over \( R^C(X, \mathbb{R}) \) by the following function:

\[
[\rho(x, y)](f) = |f(x) - f(y)|, \quad x, y \in X, \ f \in C(X, \mathbb{R}).
\]

Suppose \( A \) and \( B \) are disjoint closed subsets of \( X \). Since
(X, t) is perfectly normal, there exists \( g \in C(X, \mathbb{R}) \) such that
\[
0 < g < 1,
\]
\( A = g^{-1}(0) \), and \( B = g^{-1}(1) \). Suppose \( x \not\in A \).

Then \( [\rho(x, A)](q) = \inf_{a \in A} |g(x) - g(a)| = g(x) > 0 \). Suppose \( x \not\in B \). Then \( [\rho(x, B)](q) = \inf_{b \in B} |g(x) - g(b)| = \inf_{b \in B} |g(x) - 1| = |g(x) - 1| > 0 \) since \( x \not\in B \) implies that \( g(x) \neq 1 \).

To prove the sufficiency, all we need to show, since \((X, t_\rho)\) is \( T_1 \), is that given any closed subset \( F \), there exists a continuous function \( f \), where \( 0 < f < 1 \), such that \( F = f^{-1}(0) \). Thus, suppose \( F \) is a closed subset of \( X \), where without loss of generality \( F \neq X \). Suppose \( z \not\in F \). Let \( q \) be an element of \( \Delta \) satisfying the hypothesis. Define \( f : X \to [0,1] \) by:
\[
f(x) = \frac{[\rho(x, F)](q)}{[\rho(x, F)](q) + [\rho(x, z)](q)}.
\]

It is clear that \( F = f^{-1}(0) \) and \( 0 < f < 1 \). For fixed \( q \in \Delta \), \( \rho_q : X \times X \to \mathbb{R} \) defined by \( \rho_q(x, y) = [\rho(x, y)](q) \) is a pseudometric. Thus, for fixed \( F \subset X \), \( x \to \rho_q(x, F) \) is continuous with respect to the topology on \( X \) generated by \( \rho_q \). The topology \( t_\rho \) is generated by the collection \( \{ t_q : q \in \Delta \} \) of pseudometric topologies. Hence \( t_q \rho \) is weaker than \( t_\rho \) for each \( q \in \Delta \). Thus \( x \to \rho_q(x, F) \) is continuous with respect to \( t_\rho \).
V. FIXED POINT THEOREMS

In this section we prove some fixed point theorems for certain types of selfmaps on metric spaces over \( R^\Lambda \) i.e. for Hausdorff uniform spaces. These theorems are the analogues of some well-known fixed point theorems in a metric space setting, and in most cases, extend the original results.

**DEFINITION 1** (Antonovskii and others (1)). Suppose \((X, \rho)\) is a metric space over \( R^\Lambda \), and suppose \( Z \) is a directed set. A mapping \( x:Z \to X \) is called a sequence of type \( Z \) on \( X \). If \( z \in Z \), we denote \( x(z) \) by \( x_z \). Thus, in the usual notation, \( x = \{x_z\} \). (This is equivalent to the notion of a net.)

**DEFINITION 2** (1). A sequence \( x \) of type \( Z \) on \((X, \rho)\) is said to converge to \( a \in X \) if given \( U, \) a neighborhood of \( \bar{0} \) in \( R^\Lambda \), there exists \( z_U \in Z \) such that \( \rho(x_z, a) < U \) for \( z > z_U \), where "\( > \)" directs \( Z \). We write \( x_z \to a \) or \( \lim_{z \in Z} x_z = a \). Hence \( x_z \to a \) if and only if \( \rho(x_z, a) \to \bar{0} \). (This is equivalent to convergence of a net.)

**DEFINITION 3** (1). A sequence \( x \) of type \( Z \) on \((X, \rho)\) is called fundamental if given \( U, \) a neighborhood of \( \bar{0} \) in \( R^\Lambda \), there exists \( z_U \in Z \) such that \( \rho(x_{z'}, x_{z''}) < U \) for \( z', z'' > z_U \).

**DEFINITION 4.** A sequence will be called a Cauchy sequence if it is fundamental of type \( N \), where \( N \) denotes the natural numbers.

In the following, denote by \( \mathfrak{U} \) a certain (arbitrary, but fixed) base for the filter of neighborhoods of \( \bar{0} \) in \( R^\Lambda \). We know that \( \mathfrak{U} \) is partially ordered and directed by set inclusion, i.e. \( U > U' \) means \( U \subset U' \) where \( U, U' \in \mathfrak{U} \).
DEFINITION 5. The space \((X, \rho)\) is called complete if every fundamental sequence of type \(\omega\) is convergent, equivalently if every Cauchy net converges.

DEFINITION 6. The space \((X, \rho)\) is called sequentially complete if every Cauchy sequence converges.

Note that \((X, \rho)\) is complete if and only if \((X, U)\) is complete as a uniform space, where \(U\) denotes the natural uniform structure induced by \(\rho\).

DEFINITION 7. Suppose \((X, \rho)\) is a metric space over \(\mathbb{R}^\Lambda\). A mapping \(T: X \rightarrow X\) is called a contraction if there exists \(r \in \mathbb{R}, 0 < r < 1\), such that \(\rho(Tx, Ty) \leq r \rho(x, y)\) for each \(x, y \in X\). The mapping \(T\) will be called nonexpansive if \(\rho(Tx, Ty) \leq \rho(x, y)\) for each \(x, y \in X\).

THEOREM 1. If \(T\) is a nonexpansive selfmap on \((X, \rho)\), a metric space over \(\mathbb{R}^\Lambda\), then \(T\) is continuous. Hence contraction mappings are continuous.

PROOF. Consider a subbasic open set \(\Omega(Tx, U_{\epsilon, \varepsilon})\). If \(y \in \Omega(x, U_{\epsilon, \varepsilon})\), then \([\rho(Tx, Ty)](q) \leq [\rho(x, y)](q) < \varepsilon\). Hence \(Ty \in \Omega(Tx, U_{\epsilon, \varepsilon})\), and \(T\) is continuous.

Iseki (9) obtained the following analogue of the classical Banach fixed point theorem:

THEOREM 2. If \(T\) is a contraction defined on a sequentially complete metric space, \((X, \rho)\), over \(\mathbb{R}^\Lambda\), then \(T\) has a unique fixed point.

The following results are similar to those obtained by Edelstein (6):

THEOREM 3. Suppose \((X, \rho)\) is a metric space over \(\mathbb{R}^\Lambda\). Suppose \(T\) is a contractive selfmap satisfying the following:
there exists \( x \in X \) such that \( \{ T^n_i \} \supset \{ T^i \} \) with \( \lim_{i \to \infty} T^n_i \in X \).

Then \( z = \lim_{i \to \infty} T^n_i \) is a unique fixed point.

**Proof.** Suppose \( Tz \neq z \). Then there exists \( q \in A \) and \( \varepsilon > 0 \) such that \( [\rho(Tz, z)](q) = \varepsilon \). Since \( T \) is a contraction, it follows easily that \( T^n_i \to Tz \). Note that \( \Omega(z, U^q_{\frac{\varepsilon}{3', 3}}) \cap \Omega(Tz, U^q_{\frac{\varepsilon}{3', 3}}) = \emptyset \). There exists a positive integer \( N_0 \) such that \( i > N_0 \) implies \( T^n_i \in \Omega(z, U^q_{\frac{\varepsilon}{3', 3}}) \) and \( T^n_i+1 \in \Omega(Tz, U^q_{\frac{\varepsilon}{3', 3}}) \). Hence

\[
i > N_0 \text{ implies } [\rho (T^n_i, T^n_i + 1)](q) \geq \frac{\varepsilon}{3}.
\]

On the other hand, we have \( [\rho (T^n_i, T^n_i + 2)](q) \leq r [\rho (T^n_i, T^n_i + 1)](q) \). A repeated use of this for \( k > j > N_0 \) gives:

\[
[\rho (T^n_k, T^n_k + 1)](q) \leq [\rho (T^n_{k-1}, T^n_{k-1} + 1)](q) \leq [\rho (T^n_{k-2}, T^n_{k-2} + 1)](q) \leq [\rho (T^n_{k-3}, T^n_{k-3} + 1)](q) \leq \cdots \leq [\rho (T^n_j, T^n_j + 1)](q) \]

\[
\leq r^k \cdot [\rho (T^n_j, T^n_j + 1)](q) \to 0 \text{ if } j \text{ is fixed and } k \to \infty,
\]

which is a contradiction since \( [\rho (T^n_k, T^n_k + 1)](q) > \frac{\varepsilon}{3} \). Therefore \( Tz = z \). If \( Ty = y \), then \( \rho (Tz, Ty) = \rho (z, y) \) in which case \( y = z \).

**Corollary 1.** If \( T \) is a contraction on \((X, \rho)\), a compact metric space over \( R^A \), then \( T \) has a unique fixed point.

**Theorem 4.** Let all assumptions of Theorem 3 hold. Suppose \( \{ T^n_i \} \), \( y \in X \), contains a convergent subsequence \( \{ T^n_j \} \).
Then \( \lim_{n \to \infty} T^n y = z \).

PROOF. By Theorem 3 we have \( \lim_{i \to \infty} T^n y^i = z \). Given \( (q_t)_{t=1}^{k} \subseteq \mathbb{R}^n \) and \( \{ \varepsilon_t \}_{t=1}^{k} \subseteq \mathbb{R}^n \), where \( \varepsilon_t > 0 \), \( t = 1, \ldots, k \), there exists a positive integer \( N_0 \) such that \( i > N_0 \) implies \( [d(z, T^n y^i)](q_t) < \varepsilon_t \), \( t = 1, \ldots, k \). If \( m = n_i + j \) (\( n_i \) fixed, \( j \) variable) is any positive integer greater than \( n_i \), then we have \( [d(z, T^n y^m)](q_t) = j^{n_i + j} \leq \varepsilon_t \), \( t = 1, \ldots, k \). Hence \( \lim_{n \to \infty} T^n y = z \).

DEFINITION (1). Suppose \((X, \rho)\) is a metric space over \( \mathbb{R}^n \). A set \( M \subseteq X \) is said to be bounded if the set consisting of the elements \( \rho(x', x'') \), where \( x', x'' \in M \) is bounded above in \( \mathbb{R}^n \). The element \( d(M) = \bigvee \{ \rho(x', x'') : x', x'' \in M \} \) is called the diameter of \( M \).

THEOREM 5. Suppose \( \{F_n\} \) is a sequence of type \( N \) of non-empty closed sets in \((X, \rho)\), a sequentially complete metric space over \( \mathbb{R}^n \). Suppose \( F_i \subseteq F_j \) for \( i > j \), and suppose \( d(F_n) \to 0 \). Then \( F = \bigcap_{n=1}^{\infty} F_n \) is a singleton.

PROOF. Clearly \( F \) cannot consist of two distinct points. Hence it suffices to show that \( F \neq \emptyset \). For each \( n \in \mathbb{N} \), let \( x_n \in F_n \). We show that \( \{x_n\} \) is Cauchy. Suppose \( \{\varepsilon_1, \ldots, \varepsilon_k\} \subseteq \mathbb{R}^n \) and \( \{q_1, \ldots, q_k\} \subseteq \Delta \) where \( \varepsilon_i > 0 \) for \( i = 1, \ldots, k \). Choose \( n' \in \mathbb{N} \) satisfying the following: \( t > n' \) implies \( [d(F_t)](q_i) < \min\{\varepsilon_1, \ldots, \varepsilon_k\} \) for \( i = 1, \ldots, k \). Now suppose \( n, t > n' \) and without loss of generality \( n \geq t \). Then \( [d(F_t)](q_i) \leq [d(F_t)](q_i) < \min\{\varepsilon_1, \ldots, \varepsilon_k\} \leq \varepsilon_i \) for \( i = 1, \ldots, k \). Thus \( \{x_n\} \) is Cauchy and therefore converges. Suppose \( x_n \to x \). We claim that \( x \in F \). Suppose this is not the case. Then there exists
j such that \( x \not\in F_j \), and hence \( x \not\in F_m \) for \( m > j \). Since \( F_m \) is closed, there exists an open set, which without loss of generality we may assume to be a basic open set, containing \( x \) and missing \( F_m \). Let \( \Omega(x, \bigcap_{i=1}^{s} U_{\varepsilon_i}^{q_i}) \) be such a set. Since \( F \) is closed, there exists an open set, which without loss of generality we may assume to be a basic open set, containing \( x \) and missing \( F \). Let \( r_2(x, U_1) \) be such a set. Since

\[
\rho(x, x_n) \in \bigcap_{i=1}^{s} U_{\varepsilon_i}^{q_i},
\]

Hence \( x_n \in \Omega(x, \bigcap_{i=1}^{s} U_{\varepsilon_i}^{q_i}) \) if \( n \) is sufficiently large, but this contradicts the fact that \( F_n \) misses \( \Omega(x, \bigcap_{i=1}^{s} U_{\varepsilon_i}^{q_i}) \) for \( n \) sufficiently large. Thus \( x \in F \).

We now obtain some results similar to those in (16) in a more general setting.

**DEFINITION.** Let \( (X, \rho) \) be a metric space over \( \mathbb{R}^{\Lambda} \), and let \( f: X \to X \). Define a function \( f^\rho: X \times X \to \mathbb{R}^{\Lambda} \) as follows:

\[
\begin{align*}
(1) & \quad f^\rho(x, x) = 0 \\
(2) & \quad f^\rho(x, y) = \max(\rho(x, f(x)), \rho(y, f(y))), \quad x \neq y.
\end{align*}
\]

**THEOREM 6.** Let \( (X, \rho) \) be a sequentially complete metric space over \( \mathbb{R}^{\Lambda} \), and let \( f: X \to X \). Suppose also that

(a) \( G_n = \{ x \in X : \rho(x, f(x)) < \frac{1}{n} \}, \) \( n \geq 1, \) \( n \in \mathbb{N}, \) is closed.

(b) \( \bigcap \{ \rho(x, f(x)) : x \in X \} = 0. \)

(c) Given \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that

\[
\rho(x, y) < \delta \text{ implies } \rho(f(x), f(y)) < \varepsilon.
\]

Then \( f \) has a unique fixed point.

**PROOF.** From (b) we see that \( G_n \neq \emptyset \). By (c), there exists \( \delta_n > 0 \) such that \( \rho(f(x), f(y)) < \frac{1}{n} \) if \( f^\rho(x, y) < \frac{1}{n} \).

Choose a positive integer \( k_n \) such that \( \frac{1}{k_n} < \min\{\delta_n, \frac{1}{n}\} \).
If \( x, y \in G_{k_n} \), then \( \rho(x, f(x)) \ll \frac{1}{k_n} \) and \( \rho(y, f(y)) \ll \frac{1}{k_n} \), and it follows that \( f^0(x, y) \ll \frac{1}{k_n} \ll \frac{1}{n} \). Hence \( \rho(f(x), f(y)) \ll \frac{1}{n} \). Therefore \( d[f(G_{k_n})] \ll \frac{1}{n} \). A result in (1) gives us that \( d[\text{cl } f(G_{k_n})] \ll \frac{1}{n} \). Without loss of generality, we may choose \( k_n < k_{n+1} \) for each \( n \in \mathbb{N} \) so that \( G_{k_n} \supset G_{k_{n+1}} \), and thus \( \text{cl } f(G_{k_n}) \supset \text{cl } f(G_{k_{n+1}}) \). By Theorem 5, \( \bigcap_{n=1}^{\infty} \text{cl } f(G_{k_n}) \) is a singleton, which we will call \( z \). We claim that for each \( n > 1 \), \( n \in \mathbb{N} \), there exists \( z_n \in G_{k_n} \) such that \( \rho(z, f(z_n)) \ll \frac{1}{n} \). Suppose this is not the case. Then for \( y \in G_{k_n} \), there exists \( q \in \Delta \) such that \( [\rho(z, f(y))] (q) > \frac{1}{n} \). Suppose \( \lambda > 0 \) satisfies \( [\rho(z, f(y))] (q) > \frac{1}{n} + \lambda \). We claim that \( \Omega(z, \frac{U_{\lambda, \frac{1}{2}, \frac{1}{2}}}{2}) \cap f(G_{k_n}) = \emptyset \). Suppose this is not true. Suppose \( s \in G_{k_n} \) such that \( f(s) \in \Omega(z, \frac{U_{\lambda, \frac{1}{2}, \frac{1}{2}}}{2}) \cap f(G_{k_n}) \). Thus \( [\rho(z, f(s))] (q) < \frac{\lambda}{2} \). We now have \( [\rho(z, f(y))] (q) < [\rho(z, f(s))] (q) + [\rho(f(s), f(y))] (q) < \frac{\lambda}{2} + \frac{1}{n} \) since \( f(s), f(y) \in f(G_{k_n}) \) and \( d[f(G_{k_n})] \ll \frac{1}{n} \). But \( [\rho(z, f(y))] (q) < \frac{\lambda}{2} + \frac{1}{n} \) is a contradiction. Hence we have \( \Omega(z, \frac{U_{\lambda, \frac{1}{2}, \frac{1}{2}}}{2}) \cap f(G_{k_n}) = \emptyset \). But this tells us that \( z \not\in \text{cl } f(G_{k_n}) \) which is a contradiction. Hence for each \( n > 1 \) there exists \( z_n \in G_{k_n} \) such that \( \rho(z, f(z_n)) \ll \frac{1}{n} \). We also know that \( z_n \in G_{k_n} \) implies that \( \rho(z_n, f(z_n)) \ll \frac{1}{k_n} \ll \frac{1}{n} \). Hence \( \rho(z_n, z) \ll \rho(z_n, f(z_n)) + \rho(f(z_n), z) \ll \frac{1}{n} \). It is clear that \( z_n \to z \). Now we show that \( z \in G_{k_n} \) for each \( n > 1 \). Suppose this is not true. In particular, suppose \( z \not\in G_{k_{k_m}} \). Hence \( z \not\in G_{k_m} \) for \( m \geq j \). Since \( G_{k_m} \) is closed, there exists
a basic open set 0 such that $z \in 0$ and $0 \cap G_{k_m} = \emptyset$ for $m > j$.

Let $0 = \Omega(z, \bigcap_{i=1}^{t} U_{\epsilon_i}^{q_i})$. Since $z_n \to z$, we have $\rho(z_n, z) \leq \bigcap_{i=1}^{t} U_{\epsilon_i}^{q_i}$ for $n$ sufficiently large. Hence $z_n \in \Omega(z, \bigcap_{i=1}^{t} U_{\epsilon_i}^{q_i})$ for $n$ sufficiently large which is a contradiction. Thus $z \in G_{k_n}$ for $n \geq 1$, and hence $\rho(z, f(z)) = 0$. Thus $f(z) = z$.

Now suppose $z_1$ and $z_2$ are distinct fixed points. Condition (c) assures us that $\rho(f(z_1), f(z_2)) = 0$ for any $\epsilon > 0$ since $f^0(z_1, z_2) = \emptyset$. Hence $\rho(f(z_1), f(z_2)) = 0$, and hence $f(z_1) = f(z_2)$. But this says that $z_1 = z_2$ which is a contradiction. Thus the fixed point is unique.

**DEFINITION.** Let $(X, \rho)$ be a metric space over $\mathbb{R}^\Delta$, and suppose $f: X \to X$. Then $\rho_f$ is the function on $X \times X$ into $\mathbb{R}^\Delta$ defined by $\rho_f(x, y) = \bigwedge_n \{ \rho(f(x), f(y)) : n \geq 1 \}$. Let $X_\rho = \{ r \in \mathbb{R} : r > 0 \}$, and for any $s > r$, there exist $x, y \in X$ and $q \in \Delta$ such that $[\rho(x, y)](q) \in [r, s]$. 

**LEMMA 1.** Let $(X, \rho)$ be a metric space over $\mathbb{R}^\Delta$, and suppose $f: X \to X$. Suppose there exists $\phi: X_\rho \to [0, \omega)$ such that $\rho_f(x, y) \leq \phi_\rho(x, y)$ for any $x, y \in X$. Furthermore, suppose $\sup_{s > r} \inf_{t \in [r, s]} (t - \phi(t)) > 0$ for $r \in X_\rho \setminus \{0\}$. Then $\rho_f(x, y) = 0$ for any $x, y \in X$. Hence $\bigwedge_{n \geq 1} \{ \rho(x, f(x)) : x \in X \} = \emptyset$.

**PROOF.** Suppose $\rho_f(x, y) \neq 0$ for some $x, y \in X$. Then there exists $q \in \Delta$ such that $[\rho_f(x, y)](q) = r > 0$, i.e.,

$\bigwedge_{n \geq 1} \{ \rho(f(x), f(y)) : n \geq 1 \} \cap [r, s] = \emptyset$. By hypothesis, there exists $s \in (r, \omega)$ such that $u = \inf_{t \in [r, s]} (t - \phi(t)) > 0$. Let $t \in (0, s-r)$. Then there exists $n \geq 1$ such that

$[\rho(f(x), f(y))](q) < r + t$. Now $[\rho(f(x), f(y))] \in [r, s]$, and hence $[\rho(f(x), f(y))](q) - \phi([\rho(f(x), f(y))] \cap [r, s]) > u$ from which
it follows that $\phi ([\rho (f(x), f(y))]) (q) \leq [\rho (f(x), f(y))] (q) - u$.

Now we have $[\rho_{f}(x, y)] (q) \leq [\rho_{f}(f(x), f(y))] (q) \leq [\rho (f(x), f(y))] (q) - u < r + t - u$.

Letting $t \to 0^+$ we have $[\rho_{f}(x, y)] (q) \leq r - u$, which is a contradiction. Hence $\rho_{f}(x, y) = \overline{0}$ for any $x, y \in X$.

**Lemma 2.** Let $(X, \rho)$ be a metric space over $R^\Delta$, and let $f : X \to X$. Suppose there exists $\phi : X_\rho \to [0, \infty)$ such that $\inf(t - \phi(t)) > 0$, for $r \in X_\rho \setminus \{0\}$. Suppose that $\phi^{-1}\{0\} = \{0\}$.

Also suppose $\rho(f(x), f(y)) \leq \phi(x, y)$ for all $x, y \in X$. Then conditions (b) and (c) of Theorem 6 are satisfied.

**Proof.** We will need the fact that $f$ is nonexpansive, so we establish that here. For any $q \in \Delta$ such that $[\rho(f(x), f(y))] (q) = 0$, it is clear that $[\rho(f(x), f(y))] (q) \leq [\rho(x, y)] (q)$. Thus suppose that $[\rho(f(x), f(y))] (q) \neq 0$. Then $\phi([\rho(x, y)] (q)) \neq 0$, and hence $[\rho(x, y)] (q) \neq 0$. Thus $[\rho(x, y)] (q) \in X_\rho \setminus \{0\}$, and therefore $[\rho(x, y)] (q) - \phi([\rho(x, y)] (q)) > 0$ so that $[\rho(x, y)] (q) > \phi([\rho(x, y)] (q)) \geq [\rho(f(x), f(y))] (q)$. Now if $x, y \in X$, $q \in \Delta$, we have $[\rho_{f}(x, y)] (q) = (\bigwedge \{\rho(f(x), f(y)): n_i \geq 1\}) (q) \leq [\rho(f(x), f(y))] (q) \leq \phi([\rho(x, y)] (q))$. Also, $\inf(t - \phi(t)) > 0$ for $r \in X_\rho \setminus \{0\}$. Hence $\sup_{s \geq r} \inf_{t \geq s} (t - \phi(t)) > 0$ for $r \in X_\rho \setminus \{0\}$.

Thus by Lemma 1, $\bigwedge \{\rho(x, f(x))\} = \overline{0}$ so that condition (b) is satisfied. Now for $x, y \in X$, $q \in \Delta$ we have the following:

$$[\rho(x, y)] (q) \leq [\rho(x, f(x))] (q) + [\rho(f(x), f(y))] (q) + [\rho(f(y), y)] (q) \leq 2 \max([\rho(x, f(x))] (q), [\rho(y, f(y))] (q)) + \phi([\rho(x, y)] (q)) = 2[\rho_{f}(x, y)] (q) + \phi([\rho(x, y)] (q))$$

Let $s(r) = \inf(t - \phi(t))$ for $r \in X_\rho \setminus \{0\}$. By hypothesis, $s(r) > 0$ for each $r \in X_\rho \setminus \{0\}$. Now suppose $[\rho(x, y)] (q) \geq r$,
Then \( p(x,y) \) and \( \psi([p(x,y)])[q_0] \geq s(r) \), and hence \( \psi([p(x,y)])[q] \leq [p(x,y)])[q] - s(r) \). Thus \( [p(x,y)])[q] \leq 2[f^\rho(x,y)](q) + [p(x,y)])[q] - s(r) \), i.e. \( s(r) \leq 2[f^\rho(x,y)](q) \). Now we claim that if \( f^\rho(x,y) \ll \frac{s(r)}{2} \), then \( \rho(x,y) \ll r \). Suppose this is not true. Then there exists \( q_0 \in \Delta \) such that \( [p(x,y)])[q_0] \geq r \). From the results above we see that \( s(r) \leq 2[f^\rho(x,y)](q_0) \) and \( [f^\rho(x,y)](q_0) < \frac{s(r)}{2} \). Thus \( s(r) < \frac{2s(r)}{2} \) which is a contradiction. Since \( f \) is nonexpansive we have \( [p(f(x),f(y))][q] \leq [p(x,y)])[q] \). Suppose \( \epsilon > 0 \) is given. If \( \epsilon \in X_\rho \setminus \{0\} \), then \( f^\rho(x,y) \ll \frac{s(\epsilon)}{2} \) implies that \( \rho(x,y) \ll \epsilon \), and hence \( \rho(f(x),f(y)) \ll \epsilon \). Suppose \( \epsilon \notin X_\rho \setminus \{0\} \) and there exists \( \lambda > 0 \) such that \( \lambda \in X_\rho \setminus \{0\} \) and \( \lambda > \epsilon \). Then \( f^\rho(x,y) \ll \frac{s(\lambda)}{2} \) implies \( \rho(f(x),f(y)) \ll \lambda \ll \epsilon \). Note that \( \bigwedge \{\rho(x,f(x)) : x \in X\} = 0 \) implies that \( X_\rho \setminus \{0\} \) contains arbitrarily small positive numbers. Thus the two possibilities for \( \epsilon \) are all we need to consider.

COROLLARY 1. Let \((X,\rho)\) be a sequentially complete metric space over \( R^\Delta \). Suppose \((x : \rho(x,f(x)) \ll \frac{1}{n})\) is closed for \( n \geq 1 \) where \( f:X \to X \). Then under the hypothesis of Lemma 2, \( f \) has a unique fixed point.

PROOF. Apply Lemma 2 and Theorem 6.

LEMA 3. Let \((Y,\rho)\) be a complete metric space over \( R^\Delta \). Let \( M \) be a nonempty bounded subset of \( Y \) for which \( d(M) \) is bounded on \( \Delta \). Let \( T:M \to M \), and suppose there is a nonnegative real-valued function \( \psi \) on \([0,\infty)\) which is increasing on \((0,\infty)\) and satisfies:

(a). \( \psi \) is upper semi-continuous from the right;
(b). \( \psi(t) < t \) for all \( t > 0 \), and \( \psi^{-1}(0) = \{0\} \);

(c). \( T \) is \( \psi \)-contractive on \( M \), i.e. \( x, y \in M \) implies
\[ \rho(T(x), T(y)) \leq \psi \rho(x, y). \]

Then:

(i). \( T \) is uniformly continuous on \( M \) and can be extended to a uniformly continuous function \( f \) on \( \text{cl} M = X \) to itself;

(ii). There is a function \( \phi \) on \([0, \infty)\) into itself such that
\[ \inf(t - \phi(t)) > 0 \text{ for } r > 0, \text{ and } f \text{ is } \phi \)-contractive on \( X \).

**PROOF.** The sets \( M \) and \( X \) are Hausdorff uniform spaces in a natural way. Also, \( M \) is dense in \( X \) and \( X \) is complete as a uniform space. The fact that \( T \) is uniformly continuous follows readily from (b) and (c). The existence of the function \( f \) is the result of a well-known theorem (Theorem 26, pg. 195 (14)). Let
\[ a = \sup\{ [d(M)](q) : q \in \Delta \}. \]
Then \( a < \infty \). We omit the simple case \( a = 0 \). Let \( \phi \) be the function on \([0, \infty)\) defined by
\[ \phi(t) = \psi(t), \text{ if } t \leq a \]
\[ \phi(t) = \psi(a), \text{ if } t > a. \]

At this point we show that \( f \) is \( \phi \)-contractive on \( X \). Recall that \( d(X) = d(M) \), so that it suffices to show that \( f \) is \( \psi \)-contractive on \( X \). Suppose this is not the case. Then there exist \( x, y \in X, q \in \Delta \) such that
\[ [\rho(f(x), f(y))] (q) > \psi([\rho(x, y)] (q)) + \varepsilon \text{ for some } \varepsilon > 0. \]
Note that $\psi$ increasing on $(0,\infty)$ and $\psi$ upper semi-continuous from the right together imply that $\psi$ is upper semi-continuous on $(0,\infty)$. Let $I$ be an open interval of real numbers satisfying the following: $[\rho(f(x),f(y))](q) \in I$ and $\psi([\rho(x,y)](q)) + \varepsilon \not\in I$. Now use the continuity of $\rho$, the uniform continuity of $f$, and the fact that $\psi$ is upper semi-continuous on $(0,\infty)$ and upper semi-continuous from the right on $[0,\infty)$ to get elements $z,w$ in $M$ satisfying the following:

$[\rho(f(z),f(w))](q) \in I$ and $\psi([\rho(z,w)](q)) < \psi([\rho(x,y)](q)) + \varepsilon$.

This is clearly a contradiction since $f$ extends $T$, and $T$ is $\psi$-contractive on $M$. Now we show that $\inf(t - \phi(t)) > 0$ for $t > 0$. Since $a > \psi(a)$ and $\psi(t) = \psi(a)$ if $t \geq a$, then

$\inf(t - \phi(t)) = a - \psi(a) > 0$. Thus, suppose $r \in (0,a)$. Since $t \geq a$

$t > \phi(t)$ for $r > 0$, then $\inf(t - \phi(t)) \geq 0$. Suppose $t \geq r$

$\inf(t - \phi(t)) = 0$. Hence, there exists a sequence $\{t_n\}$ in $[r,\infty)$ such that $\lim_{n \to \infty}(t_n - \phi(t_n)) = 0$. Note that $t_n - \phi(t_n)$ can be made arbitrarily small only if $t_n < a$. So we may as well suppose that $t_n < a$ for all $n \geq 1$. Since $[r,a]$ is compact, we may, by taking a subsequence, assume that $\{t_n\}$ is a monotone sequence which converges to some point $t_0 \in [r,a]$. Suppose $\{t_n\}$ is decreasing. Then by the upper semi-continuity of $\psi$ and the fact that $\lim_{n \to \infty}(t_n - \phi(t_n)) = 0$, it follows that $t_0 < \psi(t_0)$ which is a contradiction. If $\{t_n\}$ is increasing, then the fact that $\psi$ is increasing implies that $\psi(t_0) > \psi(t_n)$ for all $n \geq 1$. Letting $n \to \infty$ we get $\psi(t_0) > t_0$ which is a contradiction.

**COROLLARY 2.** If the hypotheses of Lemma 3 hold, and if $\{x \in X: \rho(x,f(x)) < \frac{1}{n}\}$ is closed for $n \geq 1$, then $f$ has a
unique fixed point.

PROOF. Apply Corollary 1.

It can be shown that if $X$ is a locally convex topological linear space which is Hausdorff, then $X$ is "normable" over $R^\Delta$ for some $\Delta$. Let $\rho$ denote the semifield metric derived from the semifield norm, "$|||\cdot|||"$. If $X$ is complete with respect to $\rho$, we say that $X$ is a Banach space over $R^\Delta$.

COROLLARY 3. Let $K$ be a nonempty bounded closed convex subset of $B$, a Banach space over $R^\Delta$. Suppose $d(K)$ is bounded on $\Delta$. Let $f$ be a nonexpansive selfmap on $K$. Suppose that given $\varepsilon > 0$, there exists $\delta > 0$ such that $f^\rho(x,y) << \frac{\delta}{2}$ implies that $\rho(x,y) << \varepsilon$. Also, suppose $\{x: \rho(x,f(x)) < \frac{1}{n}\}$ is closed for $n \geq 1$. Then $f$ has a unique fixed point.

PROOF. Let $z \in K$ and $t \in (0,1)$. Then the mapping $f_t$ defined by $f_t(x) = (1-t)z + tf(x)$, for $x \in K$, maps $K$ into $K$. Since $K$ is closed, it is complete. Now suppose $x,y \in K$ and $q \in \Delta$. Then we have $[\rho(f_t(x),f_t(y))](q) = t||f(x)-f(y)||_1(q) \leq t||x-y||_1(q) = t[\rho(x,y)](q)$. Theorem 2 assures us that $f_t$ has a unique fixed point $x_t \in K$. Now $[\rho(x_t,f(x_t))](q) = [\rho(f_t(x_t),f(x_t))](q) = (1-t)||z-f(x_t)||_1(q)$. Since $d(K)$ is bounded on $\Delta$, then $[\rho(x_t,f(x_t))](q)$ can be made arbitrarily small by letting $t \to 1$. Hence $\bigwedge\{\rho(x,f(x)):x \in K\} = 0$. Then by Theorem 6, $f$ has a unique fixed point.

The following definition generalizes a definition in (17):

DEFINITION. Suppose $(X,\rho)$ is a metric space over $R^\Delta$. Let $f:X \to X$. We say that $f$ is a weakly uniformly strict contraction provided the following holds: given $\varepsilon > 0$, there
exists $\delta > 0$ such that $\varepsilon \leq \rho(x,y)(q) < \varepsilon + \delta$ implies that $\rho(f(x),f(y))(q) < \varepsilon$.

**LEMMA 4.** Suppose $(X,\rho)$ is a metric space over $R^\Delta$. Suppose $\rho$ has the property that $x,y \in X$, $x \neq y$ implies that $\rho(x,y) \gg 0$. Suppose $f:X \to X$ is a weakly uniformly strict contraction. Then $f$ is a strict contraction, i.e. $x \neq y$ implies that $\rho(f(x),f(y)) \ll \rho(x,y)$.

**PROOF.** Suppose $x,y \in X$, $x \neq y$, and $q \in \Delta$. Then $\rho(x,y)(q) = r > 0$. Hence there exists $\delta > 0$ such that $r = \rho(x,y)(q) < r + \delta$ implies that $\rho(f(x),f(y))(q) < r = \rho(x,y)(q)$. Since $q$ was arbitrary, it follows that $\rho(f(x),f(y)) \ll \rho(x,y)$.

**REMARK 1.** Lemma 4 assures us that such a function $f$ is continuous and can have at most one fixed point.

**REMARK 2.** The "standard" procedure for metrizing a completely regular Hausdorff space over $R^\Delta$, in general, does not yield a metric $\rho$ for which $x \neq y$ implies that $\rho(x,y) \gg 0$. However, it is easily seen from Theorem 26 that the following condition is sufficient: $C(X,R)$ contains a completely regular family of one to one functions.

We give an example of a non-metrizable completely regular Hausdorff space in support of this condition.

**EXAMPLE.** On the reals $R$, choose a base for a topology $\mathcal{U}$ to be the family of all sets of the form $[a,b)$, $a < b$. Note that each set of the form $(a,b)$ is in $\mathcal{U}$ since $(a,b) = \bigcup \{[c,b): a < c < b\}$. Hence, the identity map from $(R,\mathcal{U})$ to $(R,\mathcal{U})$ is continuous, where $\mathcal{U}$ denotes the usual topology. In (18) it is shown that $(R,\mathcal{U})$ is non-metrizable, completely
regular, and Hausdorff. We will exhibit a completely regular family of one to one continuous real-valued functions. Suppose \( F \) is closed in \((R, t)\), \( z \in R \), and \( z \notin F \). There exist \( a, b \in R \) such that \( z \in [a, b) \) and \( [a, b) \cap F = \emptyset \). We consider two cases:

case (i): \( z = a \); Define \( f:(R, t) \to (R, \mathcal{U}) \) as follows:
\[
    f(x) = \begin{cases} 
    x, & x < a \\
    x + \frac{b-a}{2}, & x \geq a.
    \end{cases}
\]

It is routine to verify that \( f \) is continuous. Clearly \( f(z) \notin \text{cl}\{f(F)\} \).

case (ii): \( a < z < b \); the identity map separates \( z \) and \( F \).

Note that in each case, the prescribed function is one to one.

**LEMMA 5.** Suppose \((X, \rho)\) is a sequentially complete metric space over \( R^\Delta \), and suppose \( f:X \to X \) is a strict contraction. Suppose that for each \( x \in X \), \( \{f(x)\} \) is a Cauchy sequence. Then \( f \) has a unique fixed point. Moreover, for any \( x \in X \), \( \lim_{n \to \infty} f(x) = z \), the unique fixed point.

**PROOF.** Since \( X \) is sequentially complete, \( \{f(x)\} \) converges for each \( x \in X \). Suppose \( f(x) \to z \). The continuity of \( f \) implies that \( f(z) = f(\lim_{n \to \infty} f(x)) = \lim_{n \to \infty} f(x) = z \). Thus \( z \) is the unique fixed point.

**THEOREM 7.** Suppose \((X, \rho)\) is a sequentially complete metric space over \( R^\Delta \) such that \( x, y \in X \), \( x \neq y \), implies that \( \rho(x, y) \to 0 \). Suppose \( f:X \to X \) is a weakly uniformly strict
contraction. Then $f$ has a unique fixed point $z$, and $x \in X$ implies that $f(x) \rightarrow z$.

**Proof.** From the previous lemma, all we need to show is that $\{f(x)\}$ is a Cauchy sequence for any $x \in X$. We will use the following notation for $x \in X$, $x$ fixed but arbitrary:

$x_1 = f(x), \ldots, x_n = f(x), \ldots$. We show that $\bigwedge\{\rho(x_n, x_{n+1}) : n \in \mathbb{N}\} = 0$. Suppose this is not the case. We see that $\rho(f(x), f(y)) \leq \rho(x, y)$ implies that $\{\rho(x_n, x_{n+1})\}$ decreases with $n$. Suppose $\bigwedge\{\rho(x_n, x_{n+1})\} = g \neq 0$. Hence there exists $q \in \Delta$ such that $g(q) > 0$. Let $\delta > 0$ be the number associated with $g(q)$ in the definition of "weakly uniformly strict contraction". There exists $m \in \mathbb{N}$ such that $g(q) \leq [\rho(x_m, x_{m+1})](q) < g(q) + \delta(q)$. Hence we have $0 < g(q) \leq [\rho(x_m, x_{m+1})](q) < g(q) + \delta$. Thus $[\rho(x_{m+1}, x_{m+2})](q) < g(q)$ which is a contradiction. Now suppose there exists $x \in X$ such that $\{f(x)\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ such that for any positive integer $N_0$, there exist $m, n > N_0$ and $q_{N_0} \in \Delta$ such that $[\rho(x_m, x_n)](q_{N_0}) > 2\varepsilon$. Let $\delta > 0$ be that number associated with $\varepsilon$ in the definition of weakly uniformly strict contraction. Let $\delta' = \min\{\delta, \varepsilon\}$.

Let $M$ be a positive integer sufficiently large to insure that $\rho(x_M, x_{M+1}) \leq \frac{\delta'}{3}$, and choose $m, n > M$ satisfying $[\rho(x_m, x_n)](q_M) > 2\varepsilon$. Without loss of generality, suppose $m < n$. Suppose $j \in [m, n]$. It can easily be shown that $|\rho(x_m, x_j) - \rho(x_m, x_{j+1})| \leq \frac{\delta'}{3}$. We claim that there exists $j \in [m, n]$ satisfying $\varepsilon + \frac{2\delta'}{3} < [\rho(x_m, x_j)](q_M) < \varepsilon + \delta'$, since $[\rho(x_m, x_{m+1})](q_M) < \varepsilon$ and $[\rho(x_m, x_n)](q_M) > 2\varepsilon > \varepsilon + \delta'$. Suppose this is not the case. Then there exists $j' \in [m, n]$. 


such that \([\rho (x_m,x_j)](q_M) \leq \epsilon + \frac{2\delta'}{3}\) and \([\rho (x_m',x_{j+1}')] (q_M') \geq \epsilon + \delta'\). But this implies that \(|\rho (x_m,x_j') - \rho (x_m',x_{j+1}')| (q_M') \geq \frac{\delta'}{3}\) which is a contradiction. Thus there exists \(j \in [m,n]\) such that \(\epsilon + \frac{2\delta'}{3} < [\rho (x_m,x_j)](q_M) < \epsilon + \delta'\). However, for all \(m\) and \(j\) we have \(\rho (x_m,x_j) \leq \rho (x_m,x_{j+1}') + \rho (x_{m+1}',x_{j+1}) + \rho (x_{j+1}',x_j)\). Hence \([\rho (x_m,x_j)](q_M') < \frac{\delta'}{3} + \epsilon + \frac{\delta'}{3} = \epsilon + \frac{2\delta'}{3}\) which is a contradiction. Hence \([f(x)]\) is a Cauchy sequence.

The following three theorems generalize some results in (7) and (8):

THEOREM 8. Let \((X,\rho)\) be a metric space over \(\mathbb{R}^n\). Let \(T:X \rightarrow X\) satisfy the following conditions:

(i). \(\rho (Tx,Ty) \leq (f)(\rho (x,Tx) + \rho (y,Ty))\) where \(f: \Delta \rightarrow (0,1/2), x,y \in X\);

(ii). There exists \(z \in X\) such that \(T\) is continuous at \(z\);

(iii). There exists \(x \in X\) such that \(\{Tx_i\}\) has a subsequence \(\{Tx_i^n\}\) converging to \(z\).

Then \(z\) is the unique fixed point of \(T\).

PROOF. The continuity of \(T\) at \(z\) implies that \(\{Tx_i^n\}\) converges to \(Tz\). Suppose \(z \neq Tz\). Then there exists \(q \in \Delta\) and \(\epsilon > 0\) such that \([\rho(z,Tz)](q) = \epsilon\). Consider the following disjoint neighborhoods of \(z\) and \(Tz\) respectively:

\(\Omega (z,U^q_{\frac{\epsilon}{3}})\) and \(\Omega (Tz,U^q_{\frac{\epsilon}{3}})\). Since \(\{Tx_i^n\}\) converges to \(z\), and \(\{Tx_i^{n+1}\}\) converges to \(Tz\), there exists a positive integer \(N_0\) such that \(i > N_0\) implies that \(Tx_i^n \in \Omega (z,U^q_{\frac{\epsilon}{3}})\) and

\(Tx_i^{n+1} \in \Omega (Tz,U^q_{\frac{\epsilon}{3}})\). Thus \(i > N_0\) implies \([\rho(Tx_i^n,Tx_i^{n+1})](q) > \frac{\epsilon}{3}\).

On the other hand we have \([\rho(Tx_i^n,Tx_i^{n+2})](q) \leq \frac{\epsilon}{3}\)
f(q)\{\rho(T^{i},T^{i}) + \rho(T^{i},T^{i})\}(q). \text{ Hence}

\[\sum_{n=1}^{\infty} \rho(T^{n},T^{n+1}) \leq \frac{f(q)}{1-f(q)} \sum_{n=1}^{\infty} \rho(T^{n},T^{n+1})\]

Now suppose $t > j > N_0$. Then we have the following:

\[\sum_{n=1}^{\infty} \rho(T^{n},T^{n+1}) \leq \frac{f(q)}{1-f(q)} \sum_{n=1}^{\infty} \rho(T^{n},T^{n+1})\]

\[\leq \frac{f(q)}{1-f(q)} \left( \sum_{n=1}^{\infty} \rho(T^{n},T^{n+1}) \right)\]

\[\leq \ldots \leq \frac{f(q)}{1-f(q)} \left( \sum_{n=1}^{\infty} \rho(T^{n},T^{n+1}) \right)\]

for fixed $j$ and $t \to \infty$ which is a contradiction. Thus $T^{n} = z$.

Suppose there exists $y \in X$ such that $Ty = y$. Then for any $q \in \Delta$ we have

\[\rho(z,y) \leq f(q) (\rho(z,Tz) + \rho(y,Ty)) = 0.

Hence $z = y$.

THEOREM 9. Let $(X,\rho)$ be a metric space over $\mathbb{R}$. Let $T:X \to X$. Suppose there exists $x_0 \in X$ such that $T$ is continuous at $x_0$. If there exists $x \in X$ such that $\{T^n x\}$ converges to $x_0$, then $Tx_0 = x_0$. If, in addition,

\[\rho(T_{x_0},Tz) \leq (f)(\rho(x_0,z)), \quad z \in X, \quad f: \Delta \to (0,1),\]

then $x_0$ is the unique fixed point of $T$.

PROOF. Let $q \in \Delta$. Then

\[\rho(T_{x_0},T_{x_0}) \leq \rho(T_{x_0},T_{x_0}) + \rho(T_{x_0},T_{x_0})\]

\[\rho(T_{x_0},T_{x_0}) \leq \rho(T_{x_0},T_{x_0}) + \rho(T_{x_0},T_{x_0})\]

which tends to zero as $n \to \infty$ since $\{T^n x\}$ converging to $x_0$ and the continuity of $T$ at $x_0$ together imply $\{T^n x\}$ converges to $T_{x_0}$. Hence $T_{x_0} = x_0$. Now if there exists $x_1 \in X$ such that $T_{x_1} = x_1$, $x_1 \neq x_0$, then there exists $q \in \Delta$ such that $\rho(x_1,x_0)$
= \varepsilon > 0. \text{ We now have } \rho(x_1, x_0) = \rho(Tx_1, Tx_0) \leq f(q) \rho(x_1, x_0) \text{ which implies that } f(q) \geq 1 \text{ which is a contradiction. Thus the fixed point is unique.}

DEFINITION. Let \((X, \rho)\) be a metric space over \(R^\Delta\). Suppose \(T : X \rightarrow X\) satisfies the following condition:

\[\rho(Tx, Ty) \leq \frac{1}{2} \{\rho(x, Tx) + \rho(y, Ty)\}, x, y \in X.\]

Such a map \(T\) is said to have property A over \(X\).

DEFINITION. Let \((X, \rho)\) be a metric space over \(R^\Delta\). Suppose \(T: X \rightarrow X\). The map \(T\) is said to have property B on \(G \subset X\) if, given \(F \subset G, F\) closed and containing more than one element, and \(TF \subset F\), there exists \(x \in F\) such that \(\rho(x, Tx) \leq \bigvee \{\rho(y, Ty) : y \in F\}\).

THEOREM 10. Let \((X, \rho)\) be a compact metric space over \(R^\Delta\). Suppose \(T: X \rightarrow X\) such that \(T\) has properties A and B over \(X\). Also, suppose that \(F \subset X, F \neq \emptyset\), and \(TF \subset F\) imply that \(F' \subset (TF)'\), (\(F'\) denoting the derived set). Then \(T\) has a unique fixed point.

PROOF. Let \(X(K)\) denote the collection of subsets \(K_a \subset X\) such that \(K_a \neq \emptyset, K_a \text{ closed, and } TK_a \subset K_a\). Partially order this collection by set inclusion. Let \(C\) denote a chain in this collection. Clearly, \(C\) is a family of closed sets with the finite intersection property. The compactness of \(X\) implies that \(\bigcap C \neq \emptyset\). Let \(H = \bigcap C\). Then \(H\) is closed, and \(TH \subset H\). Clearly, \(H\) is a lower bound of \(C\). Hence, by Zorn's lemma, there exists a minimal member of \(X(K)\), say \(K\). If \(K\) contains only one element, then that element is a fixed point. If \(K\) contains more than one element, then, since \(T\) has
property B, there exists \( x \in K \) such that \( \rho(x,Tx) = f \cdot \bigvee \{ \rho(y,Ty) : y \in K \} \). Let \( K_1 = \{ z \in K : \rho(z,Tz) \leq f \} \). Now \( K_1 \subseteq K \) and \( K_1 \neq \emptyset \) since \( x \in K_1 \). Furthermore, if \( z \in K_1 \), then \( z \in K \), and hence \( Tz \in K \). We have the following:

\[
\rho(Tz,T^2z) \leq \frac{\rho(z,Tz)}{2} + \frac{\rho(Tz,T^2z)}{2}, \quad \text{and thus } \rho(Tz,T^2z) = \rho(Tz,T^2z) \leq \frac{\rho(z,Tz)}{2}, \quad \text{so that } \rho(Tz,T^2z) \leq \rho(z,Tz) \leq f.
\]

Hence \( Tz \in K_1 \), and thus \( TK_1 \subseteq K_1 \). Now we show that \( K_1 \) is closed. Suppose \( y \in (K_1)' \). Then, by hypothesis, \( y \in (TK_1)' \).

Hence there exists a sequence \( \{Te_\beta\} \) of type \( \omega \) in \( K_1 \), where \( \{e_\beta\} \subseteq K_1 \), such that \( \{Te_\beta\} \) converges to \( y \) (a result in \( (1) \)). Let \( q \in \Delta \) and \( \varepsilon > 0 \). There exists \( \beta_0 \in \Delta \) such that \( \rho(y,Te_\beta) \in U_{\varepsilon,q}^q_f \) for all \( \beta > \beta_0 \). Hence if \( \beta > \beta_0 \) we have the following:

\[
[p(y,Ty)](q) \leq [p(y,Te_\beta)](q) + [p(TE_\beta, Ty)](q)
\]

\[
\leq [p(y,Te_\beta)](q) + \frac{[p(e_\beta, Te_\beta)](q)}{2} + \frac{[p(y,Ty)](q)}{2}.
\]

Thus, \( [p(y,Ty)](q) \leq \frac{\rho(y,Te_\beta)}{2} + \frac{\rho(e_\beta, Te_\beta)}{2} \), so that \( [p(y,Ty)](q) \leq 2[p(y,Te_\beta)](q) + [p(e_\beta, Te_\beta)](q) \), and thus \( [p(y,Ty)](q) < 2\varepsilon + f(q) \). Since \( \varepsilon \) was arbitrary, we must have \( [p(y,Ty)](q) < f(q) \), and since \( q \) was arbitrary, it follows that \( y \in K_1 \). Thus \( K_1 \) is closed. But this contradicts the minimality of \( K \). Therefore \( K \) consists of a single point, and hence a fixed point of \( T \). The uniticity follows from the fact that if \( x = Tx \) and \( y = Ty \), then \( \rho(x,y) = \rho(Tx,Ty) \leq \frac{1}{2} [\rho(x,Tx) + \rho(y,Ty)] = 0 \). Thus \( x = y \).
VI. CONCLUSIONS AND FURTHER PROBLEMS

We have seen that the existence of a completely regular family of one to one functions in $C(X, \mathbb{R})$ is sufficient to imply that $(X, t)$ can be metrized over $\mathbb{R}^\Delta$ for some $\Delta$ by a semifield metric $\rho$ satisfying the property that $x \neq y$ implies that $\rho(x, y)$ is not a divisor of zero, where $(X, t)$ is completely regular and Hausdorff. It can also be shown that a necessary condition for a space $(X, t)$ to admit such a semifield metrization is that given $x \in X$, $\{x\}$ is a $G_\delta$ set. We would like to see a characterization of spaces $(X, t)$ admitting such a semifield metrization.

If we give $\mathbb{R}^\Delta$ the box topology, (see (14), pg. 107), we can obtain several results for metric spaces over $\mathbb{R}^\Delta$. Every metric space over $\mathbb{R}^\Delta$ is completely regular and Hausdorff. We were not able, however, to characterize those spaces admitting a metrization over $\mathbb{R}^\Delta$. It is clear that the collection of such spaces contains the metric spaces and is contained in the collection of completely regular Hausdorff spaces. It can be shown that if $(X, t)$ is homeomorphic to a subspace of $\mathbb{R}^\Delta$, then $(X, t)$ is metrizable over $\mathbb{R}^\Delta$.
REFERENCES


VITA

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